# A STUDY OF MELLIN-BARNES TYPE INTEGRALS, GENERAL POLYNOMIALS AND INTEGRAL OPERATORS OF ARBITRARY ORDERS OF SINGLE AND MULTIPLE VARIABLES WITH APPLICATIONS

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# A STUDY OF MELLIN-BARNES TYPE INTEGRALS, GENERAL POLYNOMIALS AND INTEGRAL OPERATORS OF ARBITRARY ORDERS OF SINGLE AND MULTIPLE VARIABLES WITH APPLICATIONS

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under the supervision of

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# CERTIFICATE

This is to certify that Mr. Manish Kumar Bansal has worked under my supervision for the award of degree of **Doctor of Philosophy** in Mathematics on the topic entitled, **A STUDY OF MELLIN-BARNES TYPE INTEGRALS, GENERAL POLYNOMIALS AND INTEGRAL OPERATORS OF ARBITRARY ORDERS OF SINGLE AND MULTIPLE VARIABLES WITH APPLICATIONS**. The findings contained in this thesis are original and have not been submitted to any University/Institute, in part or full, for award of any degree.

Jaipur

December 2016

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# DECELERATION

I hereby declare that the thesis entitled **A STUDY OF MELLIN-BARNES TYPE INTEGRALS, GENERAL POLYNOMIALS AND INTEGRAL OPERATORS OF ARBITRARY ORDERS OF SINGLE AND MULTIPLE VARIABLES WITH APPLICATIONS** is my own work conducted under the supervision of **Dr. RASHMI JAIN**, Professor, Department of Mathematics, Malaviya National Institute of Technology Jaipur, Rajasthan, India.

I firmly declare that the presented work does not contain any part of any work that has been submitted for the award of any degree either in this University or in any other University/Institute without proper citation.

Jaipur **MANISH KUMAR BANSAL**

December 2016 **(2012RMA9542)**

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***(MANISH KUMAR BANSAL)***

# ABSTRACT

**Chapter 1** is intended to provide an introduction to various functions, polynomials, integral transforms and fractional integral operators studied by some of the earlier researchers. Further, we present the brief chapter by chapter summary of the thesis. Finally, we give a list of research papers which have either been published or accepted for publication in reputed journals having a bearing on subject matter of the thesis.

**In Chapter 2**, we first present an integral representation and Mellin transform of S-generalized Gauss hypergeometric function. Next, we give its complex integral representation and a relationship between S-generalized Gauss hypergeometric function and H-function of two variables. Further, we introduce a new integral transform whose kernel is the S-generalized Gauss hypergeometric function and point out its three special cases which are also believed to be new. We specify that the well-known Gauss hypergeometric function transform follows as a simple special case of our integral transforms. Next, we established an inversion formula

for above integral transform. Finally, we establish image of Fox H-function under the S-Generalized Gauss hypergeometric function transform and also obtain the images of five useful and important functions which are special cases of Fox H-function under the S-generalized Gauss hypergeometric function transform.

**In Chapter 3**, we study a pair of a general class of fractional integral operators whose kernels involve the product of a Appell Polynomial, Fox H-function and S-Generalized Gauss Hypergeometric Function. First we define and give the conditions of existence of the operators of our study and then we obtain the images of certain useful functions in them. Further, we evaluate four new integrals involving Appell's Function, Multivariate generalized Mittag-Leffler Function, generalization of the modified Bessel function and generalized hypergeometric function by the application of the images established and also gives the three unknown and two known integral of these operators. Next we develop six results wherein the first two contain the Mellin transform of these operators, the next two the corresponding inversion formulae and the last two their Mellin convolutions. Later on, we establish a theorem analogous to the well known Parseval Goldstein theorem for our unified fractional integral operators.

**In Chapter 4**, we first derive three new and interesting expressions for the composition of the two fractional integral operators, which are slight variants of the operators defined in Chapter 3. Finally, we obtain two interesting finite double integral formulae as an application of our first composition formula known results which follow as special case of our findings have also been mentioned.

**In Chapter 5**, we evaluate two unified and general finite integrals. The first integral involves the product of the Appell Polynomial $A_n(z)$, the Generalized form of the Astrophysical Thermonuclear function $I_3$ and Generalized Mittag - Leffler Function $E^{\gamma,\delta}_{\alpha,\beta,\tau,\mu,\rho,p}(z;s,r)$. Next, we give five special cases of our main integral. The second integral involves the $I_3$ and $\overline{H}-$ function. Further, we give the five special cases of our main integral

**In Chapter 6**, we find the solution of Bagley Torvik Equation using Generalized Differential Transform Method (GDTM). Since the function f(t) taken in this chapter is general in nature by specializing the function f(t) and taking different values of constants A, B and C. We can obtain a large number of special cases of Bagley Torvik Equation. Our findings match with the results obtained earlier by Ghorbani et al. [29] by He's variational iteration method. Further, we find the solution of Fractional Relaxation Oscillation Equation using GDTM. Since the function f(x) taken in this chapter is general in nature by specializing the function f(x) and taking different values of constants A. we can obtain a large number of special cases of Fractional Relaxation Oscillation Equation. Here we give eight numerical examples. Furthermore these examples are also represented graphically by using the MATHEMATICA SOFTWARE. Finally, we find the solution of Fractional Order Riccati Differential Equation using GDTM. and we give eight numerical examples. Furthermore these examples are also represented graphically by using the MATHEMATICA SOFTWARE.

# Contents

# 1

# INTRODUCTION TO THE TOPIC OF STUDY AND CHAPTER BY CHAPTER SUMMARY OF THE THESIS

# 1. INTRODUCTION TO THE TOPIC OF STUDY AND CHAPTER BY CHAPTER SUMMARY OF THE THESIS

The present chapter deals with an introduction to the topic of the study as well as a brief review of the contributions made by some of the earlier workers on the subject matter presented in this thesis. Next a brief chapter by chapter summary of the thesis has been given. At the end of this chapter, list of research paper having a bearing on subject matter has been given.

## 1.1 SPECIAL FUNCTIONS

Special functions have vast applications in all branches of engineering, applied sciences, statistics and various other fields. A large number of eminent mathematicians such as Euler, Gauss, Kummer, Ramanujan and several others worked out hard to develop the commonly used special functions like the Gamma function, the elliptic functions, Bessel functions, Whittaker functions and polynomials that go by the name of Jacobi, Legendre, Laguerre, Hermite.

The core of special functions is the Gauss hypergeometric functions ${}_2F_1$, introduced by famous mathematicians C F Gauss. It is represented by the following series:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a.b}{c} \frac{z}{1!} + \frac{a.(a+1).b.(b+1)}{c.(c+1)} \frac{z^2}{2!} + \cdots \tag{1.1.1}$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \qquad \text{for} \quad n \geq 0; (a)_0 = 1, c \neq 0, -1, -2, \cdots$$

a, b, c and z may be real or complex. Also if either of the numbers a or b is a non-positive integer, the function reduces to a polynomial, but if c is non-positive integer, the function is not defined since all but a finite number of terms of the series become infinite.

This series has a fundamental importance in the theory of special function and is known as Gauss hypergeometric series. It is usually represented by the symbol ${}_2F_1(a, b; c; z)$ the well known Gauss hypergeometric function.

In (1.1.1), if we replace z by $\frac{z}{b}$ and let $b \to \infty$ then

$$\frac{(b)_n}{b^n} z^n \to z^n$$

and we arrive at the following well known Kummer's series

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a.(a+1)}{c.(c+1)}\frac{z^2}{2!} + \cdots \tag{1.1.2}$$

It is represented by the symbol ${}_1F_1(a; c; z)$ and is known as confluent hypergeometric function.

A natural generalization of ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$, which is defined in the following manner:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{matrix} \; z\right] &= {}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \end{aligned} \tag{1.1.3}$$

where p and q are either positive integers or zero and empty product is interpreted as unity, the variable z and all the parameters $a_1, \cdots, a_p; \quad b_1, \cdots, b_q$ are real or complex numbers such that no denominator parameters is zero or a negative integer.

The conditions of convergence of the function ${}_pF_q$ are as follow:

(i) when $p \leq q$, the series on the right hand side of (1.1.3) is convergent.

(ii) when $p = q+1$, the series is convergent if $|z| < 1$ and divergent when $|z| > 1$, and on the circle $|z| = 1$, the series is

(a) absolutely convergent if $\Re(w) > 0$

(b) conditionally convergent if $-1 < \Re(w) < 0$ for $z \neq 1$

(c) divergent if $\Re(w) \leq -1$

where $w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$

(iii) when $p > q+1$, the series never converges except when z = 0 and the function is only defined when the series terminates.

A comprehensive account of the functions ${}_2F_1$, ${}_1F_1$ and ${}_pF_q$ can be found in the works of Luke [67], Slater [106], Exton [25] and Rainville [90] and their applications can be found in Mathai and Saxena [74].

### 1.1.1 S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

The S-generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ was introduced and investigated by Srivastava et al. [113, p. 350, Eq. (1.12)]. It is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \qquad (1.1.4)$$

provided that $(\Re(p) \geq 0; \quad \min\{\Re(\alpha), \Re(\beta), \Re(\tau), \Re(\mu)\} > 0; \quad \Re(c) > \Re(b) > 0)$

in terms of the classical Beta function $B(\lambda,\mu)$ and the S-generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$, which was also defined by Srivastava et al. [113, p. 350, Eq. (1.13)] as follows:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^{\tau}(1-t)^{\mu}}\right)dt \tag{1.1.5}$$

$$(\Re(p)\geq 0;\quad \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta),\Re(\tau),\Re(\mu)\}>0)$$

If we take p = 0 in (1.1.5), it reduces to classical Beta Function and $(\lambda)_n$ denotes the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by (see [115, p. 2 and pp. 4-6]; see also [114, p. 2]):

$$\begin{aligned}(\lambda)_n&=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\\&=\begin{cases}1, & (n=0)\\ \lambda(\lambda+1)...(\lambda+n-1), & (n\in\mathbb{N}\quad :=\{1,2,3,\cdots\})\end{cases}\end{aligned} \tag{1.1.6}$$

provided that the Gamma quotient exists (see, for details,[108, p. 16 et seq.] and [112, p. 22 et seq.]).

For $\tau=\mu$, the S-generalized Gauss hypergeometric function defined by (1.1.4) reduces to the following generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau)}(a,b;c;z)$ studied earlier by Parmar [83, p. 44]:

$$F_p^{(\alpha,\beta;\tau)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\tau)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\qquad(|z|<1) \tag{1.1.7}$$

$$(\Re(p)\geq 0;\quad \min\{\Re(\alpha),\Re(\beta),\Re(\tau)\}>0;\quad \Re(c)>\Re(b)>0).$$

which, in the *further* special case when $\tau = 1$, reduces to the following extension of the generalized Gauss hypergeometric function (see, e.g., [82, p. 4606, Section 3] ; see also [81, p. 39]):

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{1.1.8}$$

$$(\Re(p) \geqslant 0; \quad \min\{\Re(\alpha),\Re(\beta)\} > 0; \quad \Re(c) > \Re(b) > 0)$$

Upon setting $\alpha = \beta$ in (1.1.8), we arrive at the following Extended Gauss hypergeometric function (see [13, p. 591, Eqs. (2.1) and (2.2)]:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{1.1.9}$$

$$(\Re(p) \geqslant 0; \quad \Re(c) > \Re(b) > 0)$$

### 1.1.2 THE H- FUNCTION

The $H-$function is defined by the following Mellin-Barnes type integral [109, p. 10] with the integrand containing products and quotients of the Euler gamma functions. Such a function generalizes most of the known special functions.

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[ z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] = H_{P,Q}^{M,N}\left[ z \,\middle|\, \begin{matrix} (a_1,\alpha_1),\cdots,(a_P,\alpha_P) \\ \\ (b_1,\beta_1),\cdots,(b_Q,\beta_Q) \end{matrix} \right]$$

$$:= \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \Theta(\mathfrak{s}) z^{\mathfrak{s}} \, d\mathfrak{s}, \tag{1.1.10}$$

where $\omega = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

and

$$\Theta(\mathfrak{s}) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \mathfrak{s}) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \mathfrak{s})}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j \mathfrak{s}) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \mathfrak{s})}, \tag{1.1.11}$$

Also M, N, P and Q are non-negative integers satisfying $1 \leqq M \leqq Q$ and $0 \leqq N \leqq P$ ; $\alpha_j (j = 1, \cdots, P)$ and $\beta_j (j = 1, \cdots, Q)$ are assumed to be positive quantities for standardization purposes. The definition of the H-function given by (1.1.10) will, however, have meaning even if some of these quantities are zero. Also, $a_j (j = 1, \cdots, P)$ and $b_j (j = 1, \cdots, Q)$ are complex numbers such that none of the points

$$\mathfrak{s} = \frac{b_h + \nu}{\beta_h} \quad h = 1, \cdots, M; \nu = 0, 1, 2, \cdots \tag{1.1.12}$$

which are the poles of $\Gamma(b_h - \beta_h s), h = 1, \cdots, M$ and the points

$$\mathfrak{s} = \frac{a_i - \eta - 1}{\alpha_i} \quad i = 1, \cdots, N; \eta = 0, 1, 2, \cdots \tag{1.1.13}$$

which are the poles of $\Gamma(1 - a_i + \alpha_i s)$ coincide with one another, i.e

$$\alpha_i (b_h + \nu) \neq b_h (a_i - \eta - 1) \tag{1.1.14}$$

for $\nu, \eta = 0, 1, 2, \cdots$; $h = 1, \cdots, M$; $i = 1, \cdots, N$.

Further, the contour $\mathfrak{L}$ runs from $-\omega\infty$ to $+\omega\infty$ such that the poles $\Gamma(b_h - \beta_h s), h = 1, \cdots, M$, lie to the right left of $\mathfrak{L}$ and the poles of $\Gamma(1 - a_i + \alpha_i s), \quad i = 1, \cdots, N$ lie to the left of $\mathfrak{L}$. Such a contour is possible on account of (1.1.14). These assumptions will be adhered to throughout the present work.

## SPECIAL CASES

The following special cases of the $H-$function have been made use in this thesis:

1. **Lorenzo-Hartley G-function** [34, p. 64, Eq. (2.3)]

$$H_{1,2}^{1,1}\left[-az^q \middle| \begin{matrix} (1-r,1) \\ (0,1), \quad (1+\nu-rq,q) \end{matrix}\right] = \frac{\Gamma(r)}{z^{rq-\nu-1}} G_{q,\nu,r}[a,z] \qquad (1.1.15)$$

Here $G_{q,\nu,r}$ is the Lorenzo-Hartley G-function [66].

2. **Generalized Mittag-Leffler function** [73, p. 25, Eq. (1.137)]

$$H_{1,2}^{1,1}\left[-z \middle| \begin{matrix} (1-\gamma,1) \\ (0,1), \quad (1-\beta,\alpha) \end{matrix}\right] = \Gamma(\gamma) E_{\alpha,\beta}^{\gamma}(z) \qquad (1.1.16)$$

$$(\alpha,\beta,\gamma \in \mathbb{C}; \qquad \Re(\alpha,\beta,\gamma) > 0)$$

where $E_{\alpha,\beta}^{\gamma}$ is the generalized Mittag-Leffler function given by [87].

3. **Generalized Hypergeometric function**[109, p. 18, Eq. (2.6.3)]

$$H_{p,q+1}^{1,p}\left[z \middle| \begin{matrix} (1-a_j,1)_{1,p} \\ (0,1),(1-b_j,1)_{1,q} \end{matrix}\right] = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)} {}_pF_q[(a_p);(b_q);-z]; \qquad (1.1.17)$$

4. **Generalized Bessel Maitland Function** [73, p. 25, Eq. (1.139)]

$$H_{1,3}^{1,1}\left[\frac{z^2}{4} \middle| \begin{matrix} (\lambda+\frac{\nu}{2},1) \\ (\lambda+\frac{\nu}{2},1),(\frac{\nu}{2},1),(\mu(\lambda+\frac{\nu}{2})-\lambda-\nu,\mu) \end{matrix}\right] = J_{\nu,\lambda}^{\mu}(z) \qquad (1.1.18)$$

where $J_{\nu,\lambda}^{\mu}$ is the Generalized Bessel Maitland Function [72, p. 128, Eq. (8.2)]

5. **Wright's Generalized Bessel Function**[109, p. 19, Eq. (2.6.10)]

$$H_{0,2}^{1,0}\left[z \,\middle|\, \begin{matrix} -- \\ (0,1),(-\lambda,\nu) \end{matrix}\right] = J_{\lambda}^{\nu}(z) \tag{1.1.19}$$

6. **Krätzel Function** [73, p. 25, Eq. (1.141)]

$$H_{0,2}^{2,0}\left[z \,\middle|\, \begin{matrix} -- \\ (0,1),(\frac{\nu}{\rho},\frac{1}{\rho}) \end{matrix}\right] = \rho Z_{\rho}^{\nu}(z) \qquad z,\nu \in \mathbb{C}, \rho > 0 \tag{1.1.20}$$

where $Z_{\rho}^{\nu}$ is the Krätzel Function [62].

7. **Modified Bessel function of the third kind** [30, p. 155, Eq. (2.6)]

$$H_{1,2}^{2,0}\left[z \,\middle|\, \begin{matrix} \left(1-\frac{\sigma+1}{\beta},\frac{1}{\beta}\right) & \\ (0,1), & \left(-\gamma-\frac{\sigma}{\beta},\frac{1}{\beta}\right) \end{matrix}\right] = \lambda_{\gamma,\sigma}^{(\beta)}(z) \tag{1.1.21}$$

### 1.1.3 THE $\overline{H}$−FUNCTION

Though the H-function is sufficiently general in nature, many useful functions notably generalized Riemann Zeta function [23], the polylogarithm of complex order [23], the exact partition of the Gaussian model in statistical mechanics [50], a certain class of Feynman integrals [23] and others do not form its special cases. Inayat Hussain [50] introduced a generalization of the H-function popularly known as $\overline{H}$ -function which includes all the above mentioned functions as its special cases. This function is developing fast and stands on a firm footing through the publications of Buschman and Srivastava [10], Rathie [91], Saxena [103, 104], Gupta and Soni [42], Jain and Sharma [53], Gupta, Jain and Sharma

[41], Gupta, Jain and Agrawal [40] and several others. The $\overline{H}$-function is defined and represented in the following manner:

$$\overline{H}^{m,n}_{p,q}[z] = \overline{H}^{m,n}_{p,q}\left[ z \middle| \begin{array}{cc} (e_j, E_j; \in_j)_{1,n}, & (e_j, E_j)_{n+1,p} \\ \\ (f_j, F_j)_{1,m}, & (f_j, F_j; \Im_j)_{m+1,q} \end{array} \right]$$

$$:= \frac{1}{2\pi\omega} \int\limits_{\mathfrak{L}} \overline{\Theta}(\xi) z^{\xi} d\xi \tag{1.1.22}$$

where, $\omega = \sqrt{-1}$,$z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi) = \frac{\prod\limits_{j=1}^{m} \Gamma(f_j - F_j\xi) \prod\limits_{j=1}^{n} \{\Gamma(1 - e_j + E_j\xi)\}^{\in_j}}{\prod\limits_{j=m+1}^{q} \{\Gamma(1 - f_j + F_j\xi)\}^{\Im_j} \prod\limits_{j=n+1}^{p} \Gamma(e_j - E_j\xi)} \tag{1.1.23}$$

It may be noted that $\overline{\Theta}(\xi)$ contains fractional powers of some of the gamma functions. $m, n, p, q$ are integers such $1 \le m \le q, 0 \le n \le p, (E_j)_{1,p}, (F_j)_{1,q}$ and $(\in_j)_{1,n}, (\Im_j)_{m+1,q}$ are positive quantities for standardization purpose. The definition (1.1.22) will however have meaning even if some of these quantities are zero, giving us in turn simple transformation formulae.

$(e_j)_{1,p}$ and $(f_j)_{1,q}$ are complex numbers such that the points

$$\xi = \frac{f_j + k}{F_j} \quad j = 1, \cdots, m; \quad k = 0, 1, 2, \cdots$$

which are the poles of $\Gamma(f_j - F_j\xi)$, and the points

$$\xi = \frac{e_j - 1 - k}{E_j} \quad j = 1, \cdots, n; \quad k = 0, 1, 2, \cdots$$

which are the singularities of $\{\Gamma(1 - e_j + E_j\xi)\}^{\in_j}$, do not coincide.

We retain these assumptions throughout the thesis.

The contour $\mathfrak{L}$ is the line from $c - i\infty$ to $c + i\infty$, suitably intended to keep the poles of $\Gamma(f_j - F_j\xi) \quad j = 1, \cdots, m$ to the right of the path, and the singularities of $\{\Gamma(1 - e_j + E_j\xi)\}^{\in_j} \quad j = 1, \cdots, n$ to the left of the path.

If $\in_i = \Im_j = 1 \quad (i = 1, \cdots, n; \quad j = m+1, \cdots, q)$, the $\overline{H}-$function reduces to the familiar $H-$function.

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}-$function given by (1.1.22) have been given by Gupta, Jain and Agarwal [40]

$$\left.\begin{array}{l} (i)\, |\arg(z)| < \dfrac{1}{2}\Omega\pi \quad \text{and} \quad \Omega > 0 \\ (ii)\, |\arg(z)| = \dfrac{1}{2}\Omega\pi \quad \text{and} \quad \Omega \geq 0 \\ \quad \text{and (a)} \quad \mu \neq 0 \quad \text{and the contour} \quad \mathfrak{L} \quad \text{is so chosen that} \quad (c\mu + \lambda + 1) < 0 \\ \quad \text{(b)} \quad \mu = 0 \quad \text{and} \quad (\lambda + 1) < 0 \end{array}\right\} \tag{1.1.24}$$

where

$$\Omega = \sum_{j=1}^{m} F_j + \sum_{j=1}^{n} E_j \in_j - \sum_{j=m+1}^{q} F_j \Im_j - \sum_{j=n+1}^{p} E_j \tag{1.1.25}$$

$$\mu = \sum_{j=1}^{n} E_j \in_j + \sum_{j=n+1}^{p} E_j - \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{q} F_j \Im_j \tag{1.1.26}$$

$$\begin{aligned} \lambda = \Re \left( \sum_{j=1}^{m} f_j + \sum_{j=m+1}^{q} f_j \Im_j - \sum_{j=1}^{n} e_j \in_j - \sum_{j=n+1}^{p} e_j \right) \\ + \frac{1}{2} \left( \sum_{j=1}^{n} \in_j - \sum_{j=m+1}^{q} \Im_j + p - m - n \right) \end{aligned} \tag{1.1.27}$$

The following series representation for the $\overline{H}-$Function given by Rathie [91] and Saxena [103] has been used in the present work:

$$\overline{H}_{p,q}^{m,n}\left[z\left|\begin{array}{cc}(e_j,E_j;\in_j)_{1,n}, & (e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m}, & (f_j,F_j;\Im_j)_{m+1,q}\end{array}\right.\right]=\sum_{t=0}^{\infty}\sum_{h=1}^{m}\overline{\Theta}(\mathfrak{s}_{t,h})z^{\mathfrak{s}_{t,h}} \tag{1.1.28}$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t,h})=\frac{\prod\limits_{j=1,j\neq h}^{m}\Gamma(f_j-F_j\mathfrak{s}_{t,h})\prod\limits_{j=1}^{n}\{\Gamma(1-e_j+E_j\mathfrak{s}_{t,h})\}^{\in_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-f_j+F_j\mathfrak{s}_{t,h})\}^{\Im_j}\prod\limits_{j=n+1}^{p}\Gamma(e_j-E_j\mathfrak{s}_{t,h})}\frac{(-1)^t}{t!F_h},\quad \mathfrak{s}_{t,h}=\frac{f_h+t}{F_h} \tag{1.1.29}$$

In the Sequel, we shall also make use of the following behavior of the $\overline{H}_{p,q}^{m,n}[z]$ function for small and large value of z as recorded by Saxena et al. [102, p. 112, Eqs.(2.3) and (2.4)].

$$\overline{H}_{p,q}^{m,n}[z]=O[|z|^{\alpha}],\text{for small z, where}\quad \alpha=\min_{1\leq j\leq m}\Re\left(\frac{f_j}{F_j}\right) \tag{1.1.30}$$

$$\overline{H}_{p,q}^{m,n}[z]=O[|z|^{\beta}],\text{for large z, where}\quad \beta=\max_{1\leq j\leq n}\Re\left(\in_j\left(\frac{e_j-1}{E_j}\right)\right) \tag{1.1.31}$$

provided that either of the following conditions are satisfied:

$$\left.\begin{array}{l}(i)\quad \mu<0\quad\text{and}\quad 0<|z|<\infty\\ \\ (ii)\quad \mu=0\quad\text{and}\quad 0<|z|<\delta^{-1}\end{array}\right\} \tag{1.1.32}$$

where

$$\mu=\sum_{j=1}^{n}E_j\in_j+\sum_{j=n+1}^{p}E_j-\sum_{j=1}^{m}F_j-\sum_{j=m+1}^{q}F_j\Im_j \tag{1.1.33}$$

$$\delta=\prod_{j=1}^{n}(E_j)^{E_j\in_j}\prod_{j=n+1}^{p}(E_j)^{E_j}\prod_{j=1}^{m}(F_j)^{-F_j}\prod_{m+1}^{q}(F_j)^{-F_j\Im_j} \tag{1.1.34}$$

## SPECIAL CASES

The following special cases of the $\overline{H}-$function have been made use in this thesis:

(I) **The Polylogarithm of order** $p$ [23, p.30, §1.11, Eq. (14)] and [39, p. 315, Eq. (1.9)]

$$F(z,p) = \sum_{r=1}^{\infty} \frac{z^r}{r^p} = z\overline{H}_{1,2}^{1,1}\left[-z \left| \begin{matrix} (0,1;p+1) & \\ (0,1), & (-1,1;p) \end{matrix} \right. \right]$$

$$= -\overline{H}_{1,2}^{1,1}\left[-z \left| \begin{matrix} (1,1;p+1) & \\ (1,1), & (0,1;p) \end{matrix} \right. \right] \tag{1.1.35}$$

Here $F(z,p)$ is the polylogarithm function of order $p$ .

(II) **The Generalized Wright Hypergeometric Function** [41, p. 271, Eq. (7)]

$${}_p\overline{\Psi}_q\left( \begin{matrix} (e_j,E_j;\in_j)_{1,p}; \\ \\ (f_j,F_j;\Im_j)_{1,q}; \end{matrix} \, z \right) = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma(e_j+E_jr)\}^{\in_j}}{\prod_{j=1}^{q} \{\Gamma(f_j+F_jr)\}^{\Im_j}} \frac{z^r}{r!}$$

$$= \overline{H}_{p,q+1}^{1,p}\left[-z \left| \begin{matrix} (1-e_j,E_j;\in_j)_{1,p} \\ \\ (0,1),(1-f_j,F_j;\Im_j)_{1,q} \end{matrix} \right. \right] \tag{1.1.36}$$

${}_p\overline{\Psi}_q$ reduces to ${}_p\Psi_q$, the familiar Wright's Generalized hypergeometric function [109, p. 19, Eq. (2.6.11)], when all the exponents $(\in_j)_{1,n}$, $(\Im_j)_{m+1,q}$ take the value 1.

(III) **The Generalized Riemann Zeta Function** [23, p. 27, §1.11, Eq. (1)] and [39, pp. 314–315, Eq. (1.6) and (1.7)]

$$\phi(z,p,\eta)=\sum_{r=0}^{\infty}\frac{z^r}{(\eta+r)^p}=\overline{H}_{2,2}^{1,2}\left[-z\,\middle|\,\begin{matrix}(0,1;1), & (1-\eta,1;p)\\ (0,1), & (-\eta,1;p)\end{matrix}\right] \tag{1.1.37}$$

(IV) **Generalized Hurwitz Lerch Zeta Function** [51, pp. 147 & 151, Eqs.(6.2.5) and (6.4.2)]

$$\phi_{\alpha,\beta,\gamma}(z,p,\eta)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}\frac{z^r}{(\eta+r)^p}$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\overline{H}_{3,3}^{1,3}\left[-z\,\middle|\,\begin{matrix}(1-\eta,1;p),(1-\alpha,1;1), & (1-\beta,1;1)\\ (0,1),(1-\gamma,1;1), & (-\eta,1;p)\end{matrix}\right] \tag{1.1.38}$$

(V) **Generalized Wright Bessel Function** [41, p. 271, Eq.(8)]

$$\overline{J}_{\lambda}^{\nu,\mu}(z)=\sum_{r=0}^{\infty}\frac{(-z)^r}{r!(\Gamma(1+\lambda+\nu r))^{\mu}}$$

$$=\overline{H}_{0,2}^{1,0}\left[z\,\middle|\,\begin{matrix} \\ (0,1),(-\lambda,\nu;\mu)\end{matrix}\right] \tag{1.1.39}$$

(VI) **A Generalization of the Generalized Hypergeometric Function**

[41, p. 271, Eq. (9)]

$$
{}_p\overline{F}_q\left(\begin{matrix}(e_j,\in_j)_{1,p};\\ \\ (f_j,\Im_j)_{1,q};\end{matrix}\quad z\right)
$$

$$
=\sum_{r=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\{(e_j)_r\}^{\in_j}}{\prod\limits_{j=1}^{q}\{(f_j)_r\}^{\Im_j}}\frac{z^r}{r!}=\frac{\prod\limits_{j=1}^{q}\{\Gamma(f_j)\}^{\Im_j}}{\prod\limits_{j=1}^{p}\{\Gamma(e_j)\}^{\in_j}}\overline{H}^{1,p}_{p,q+1}\left[-z\,\middle|\,\begin{matrix}(1-e_j,1;\in_j)_{1,p}\\ \\ (0,1),(1-f_j,1;\Im_j)_{1,q}\end{matrix}\right]
$$

$$
=\frac{\prod\limits_{j=1}^{q}\{\Gamma(f_j)\}^{\Im_j}}{\prod\limits_{j=1}^{p}\{\Gamma(e_j)\}^{\in_j}}\,{}_p\overline{\Psi}_q\left(\begin{matrix}(e_j,1;\in_j)_{1,p};\\ \\ (f_j,1;\Im_j)_{1,q};\end{matrix}\quad z\right) \tag{1.1.40}
$$

The function ${}_p\overline{F}_q$ reduces to well known ${}_pF_q$ for $\in_j=1(j=1,\cdots,p)$, $\Im_j=1(j=1,\cdots,q)$ in it.

Naturally, all functions which are special cases of the $H-$function are also special cases of the $\overline{H}-$function.

### 1.1.4 THE MULTIVARIABLE $H-$FUNCTION

The multivariable $H-$function occuring in the thesis was introduced and studied by Srivastava and Panda [118, p. 130, Eq. (1.1)]. This function involves r complex variables and will be defined and represented in the following contracted form [109, pp. 251–252, Eqs. (C.1–C.3)]

$$H^{0,B:A_1,B_1;\cdots;A_r,B_r}_{C,D:C_1,D_1;\cdots;C_r,D_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,C}:(c_j^{(1)},\gamma_j^{(1)})_{1,C_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,D}:(d_j^{(1)},\delta_j^{(1)})_{1,D_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,D_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathfrak{L}_1}\cdots\int_{\mathfrak{L}_r}\psi(\xi_1,\cdots,\xi_r)\prod_{i=1}^{r}(\phi_i(\xi_i)z_i^{\xi_i})d\xi_1\cdots d\xi_r \qquad (i=1,\cdots,r) \tag{1.1.41}$$

where $\omega=\sqrt{-1}$,

$$\psi(\xi_1,\cdots,\xi_r)=\frac{\prod\limits_{j=1}^{B}\Gamma(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod\limits_{j=1}^{D}\Gamma(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod\limits_{j=B+1}^{C}\Gamma(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i)} \tag{1.1.42}$$

$$\phi_i(\xi_i)=\frac{\prod\limits_{i=1}^{A_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod\limits_{j=1}^{B_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod\limits_{j=A_i+1}^{D_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod\limits_{j=B_i+1}^{C_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)} \qquad (i=1,\cdots,r) \tag{1.1.43}$$

All the greek letters occuring on the left-hand side of (1.1.41) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable $H-$function will, however, be meaningful even if some of these quantities are zero such that

$$\Lambda_i\equiv\sum_{j=1}^{C}\alpha_j^{(i)}+\sum_{j=B_i+1}^{C_i}\gamma_j^{(i)}-\sum_{j=1}^{D}\beta_j^{(i)}-\sum_{j=1}^{D_i}\delta_j^{(i)}>0 \quad (i=1,2,\cdots,r) \tag{1.1.44}$$

$$\Omega_i\equiv-\sum_{j=B+1}^{C}\alpha_j^{(i)}+\sum_{j=1}^{B_i}\gamma_j^{(i)}-\sum_{j=B_i+1}^{C_i}\gamma_j^{(i)}-\sum_{j=1}^{D}\beta_j^{(i)}+\sum_{j=1}^{A_i}\delta_j^{(i)}-\sum_{j=A_i+1}^{D_i}\delta_j^{(i)}>0 \quad (i=1,2,\cdots,r) \tag{1.1.45}$$

where $B, C, D, A_i, B_i, C_i, D_i$ are non negative integers such that $0 \leq B \leq C$, $D \geq 0$, $\quad 0 \leq B_i \leq C_i$ and $1 \leq A_i \leq D_i$, $\quad (i = 1, \cdots, r)$.

The sequences of the parameters in (1.1.41) are such that none of the poles of the integrand coincide i.e. the poles of the integrand in (1.1.41) are simple. The contour $\mathfrak{L}_i$ in the complex $\xi_i-$ plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \quad (j = 1, \cdots, A_i)$ are separated from those of $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)}\xi_i)$ $(j = 1, \cdots, B_i)$ and $\Gamma(1 - a_j + \sum\limits_{i=1}^{r} \alpha_j^{(i)}\xi_i) \quad (i = 1, \cdots, r; j = 1, \cdots, B)$.

It is known that multiple Mellin-Barnes contour integral representing the multivariable $H-$ function (1.1.41) converges absolutely [119, p. 130, Eq. (1.4)] under the condition (1.1.45) when

$$|\arg(z_i)| < \frac{1}{2}\Omega_i\pi, \quad (i = 1, \cdots, r) \tag{1.1.46}$$

The point $z_i = 0 (i = 1, \cdots, r)$ and various exceptional parameter values are excluded.

## SPECIAL CASES

By suitably specializing the various parameters occuring in the multivariable $H-$ function defined by (1.1.41), it reduces to the simpler special functions of one and more variables.

Some of them which have been used in this thesis are given below:

(i) If we take $\alpha_j^{(1)} = \alpha_j^{(2)} = ... = \alpha_j^{(r)} \quad (j = 1, ..., D)$ and $\beta_j^{(1)} = \beta_j^{(2)} = ... = \beta_j^{(r)} \quad (j = 1, ..., D)$ in (1.1.41), it reduces to a special multivariable $H-$ function studied by Saxena [103].

(ii) If we take $r = 2$, in (1.1.41) , we get $H-$function of two variables defined in [109, p.82, eq.(6.1.1)].

(iii) A relation between $H-$function of two variable and the Appell function [109, p.89,Eq.(6.4.6)] is given as below:

$$H^{0,0:1,2;1,2}_{0,1:2,1;2,1}\left[\begin{array}{c} -x \\ -y \end{array}\middle| \begin{array}{ccc} - : & (1-c,1),(1-c',1); & (1-e,1),(1-e',1) \\ (1-b;1,1): & (0,1); & (0,1) \end{array}\right]$$

$$= \frac{\Gamma(c)\Gamma(c')\Gamma(e)\Gamma(e')}{\Gamma(b)} F_3(c,e,c',e';b;x,y), \quad |x|<1, |y|<1 \tag{1.1.47}$$

(iv) if we reduce Multivariable H-function into generalized hypergeometric Function [52, p.xi,Eq.(A.18)] as given below

$$H^{0,C:1,0;\cdots;1,0}_{C,D:0,1;\cdots;0,1}\left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (1-a_j;1,\cdots,1)_{1,C} : --;\cdots;-- \\ \\ \\ \\ (1-b_j;1,\cdots,1)_{1,D} : (0,1);\cdots;(0,1) \end{array}\right]$$

$$= \frac{\prod_{j=1}^{C}\Gamma(a_j)}{\prod_{j=1}^{D}\Gamma(b_j)} {}_CF_D\left[\begin{array}{c} (a_C); \\ \\ (b_D); \end{array} - (z_1+\cdots+z_r)\right] \tag{1.1.48}$$

(v) if we reduce Multivariable H-function into Multivariate generalized Mittag-Leffler Function [9, p.187,Eq.(B.27)] as given below

$$H^{0,0:1,1;\cdots;1,1}_{0,1:1,1;\cdots;1,1}\left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} -- : (1-\gamma_1,1);\cdots;(1-\gamma_r,1) \\ \\ \\ \\ (1-\lambda;\rho_1,\cdots,\rho_r) : (0,1);\cdots;(0,1) \end{array}\right]$$

$$= \prod_{j=1}^{r}\Gamma(\gamma_j) E^{(\gamma_j)}_{(\rho_j),\lambda}(z_1,\cdots,z_r) \tag{1.1.49}$$

### 1.1.5 THE APPELL POLYNOMIALS

A class of polynomials over the field of complex numbers which contains many classical polynomial systems. The Appell Polynomials were introduced by Appell [6]. The series of Appell Polynomials is defined by :

$$A_n(z) = \sum_{k=0}^{n} \frac{a_{n-k}}{k!} z^k, \qquad n = 0, 1, 2, \cdots \tag{1.1.50}$$

where $a_{n-k}$ is the complex coefficients and $a_0 \neq 0$

**SPECIAL CASES OF THE APPELL POLYNOMIALS** $A_n[z]$

On suitably specializing the coefficients $a_{n-k}$, occurring in (1.1.50), the Appell polynomials $A_n[z]$ can be reduced to various type polynomials as cited in the papers referred to above.

The following special cases of the Appell polynomials $A_n[z]$ will be required in the thesis:

(a) **Cesaro Polynomial**

If we take $a_{n-k} = \binom{\tau+n-k}{n-k} k!$

$$A_n[z] \to g_n^{(\tau)}(z) \tag{1.1.51}$$

where $g_n^{(\tau)}(z)$ is Cesaro polynomial[112, p. 449, Eq. (20)] and is given by:

$$g_n^{(\tau)}(z) = \sum_{k=0}^{n} \binom{\tau+n-k}{n-k} z^k$$

$$= \binom{\tau+n}{n} {}_2F_1\left(\begin{array}{c} -n, 1; \\ -\tau-n; \end{array} z\right) \tag{1.1.52}$$

.

(b) **Laguerre Polynomial**

On taking $a_{n-k} = (-1)^k \binom{\alpha+n}{n-k}$

$$A_n[z] \to L_n^{\alpha}(z), \tag{1.1.53}$$

where $L_n^{(\alpha)}(z)$ is the Laguerre Polynomial [121, p. 101, Eq. (5.1.6)] and is given by :

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{(-z)^k}{k!}$$

$$= \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; 1+\alpha; z) \tag{1.1.54}$$

(c) **Shively Polynomial**

If we take $a_{n-k} = \dfrac{(\lambda+n)_n(-n)_k(\alpha_1)_k\cdots(\alpha_p)_k}{n!(\lambda+n)_k(\beta_1)_k\cdots(\beta_q)_k}$

$$A_n[z] \to S_n^{\lambda}(z) \tag{1.1.55}$$

where $S_n^{\lambda}(z)$ is the Shively Polynomial [112, p. 187, Eq. (49)] and is given by :

$$S_n^{\lambda}(z) = \sum_{k=0}^{n} \frac{(\lambda+n)_n(-n)_k(\alpha_1)_k\cdots(\alpha_p)_k}{n!(\lambda+n)_k(\beta_1)_k\cdots(\beta_q)_k} \frac{z^k}{k!}$$

$$= \frac{(\lambda+n)_n}{n!} {}_{p+1}F_{q+1}\left( \begin{array}{c} -n, \alpha_1, \cdots, \alpha_p; \\ \lambda+n, \beta_1, \cdots, \beta_q; \end{array} z \right) \tag{1.1.56}$$

(d) **Bateman's Polynomial**

on taking $a_{n-k} = \dfrac{(-n)_k(n+1)_k}{(1)_k(1)_k}$

$$A_n[z] \to Z_n(z) \tag{1.1.57}$$

where $Z_n(z)$ is the Bateman's Polynomial [112, p. 183, Eq. (42)] and is given by:

$$Z_n(z) = \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{(1)_k(1)_k} \frac{z^k}{k!}$$

$$= {}_2F_2\left(\begin{matrix} -n, n+1; \\ 1, 1; \end{matrix} \; z\right) \tag{1.1.58}$$

(e) **Bessel Polynomial**

If we take $a_{n-k} = \dfrac{(-n)_k(\alpha+n-1)_k(-1)^k}{\beta^k}$

$$A_n[z] \to y_n(z, \alpha, \beta) \tag{1.1.59}$$

where $y_n(z, \alpha, \beta)$ is the Bessel Polynomial [60, p. 108, Eq. (34)] and is given by:

$$y_n(z, \alpha, \beta) = \sum_{k=0}^{n} \frac{(-n)_k(\alpha+n-1)_k}{k!} \left(\frac{-z}{\beta}\right)^k$$

$$= {}_2F_0\left[\begin{matrix} -n, \alpha+n-1; \\ -; \end{matrix} \; \frac{-z}{\beta}\right] \tag{1.1.60}$$

### 1.1.6 INTEGRAL TRANSFORM

If $f(x)$ denotes of a prescribed class of functions defined on a given interval $[a, b]$ and $K(x, s)$ denotes a definite function of $x$ in that interval for each value of $s$, a parameter whose domain is prescribed, then the linear integral transform $T[f(x); s]$ of the function $f(x)$ is defined in the following manner:

$$T[f(x); s] = \int_a^b K(x, s) f(x) dx \tag{1.1.61}$$

wherein the class of functions and the domain of parameter $s$ are so prescribed that the above integral exists. In (1.1.61), $K(x,s)$ is known as the kernel of the transform, $T[f(x);s]$ is the image of $f(x)$ in the said transform; and $f(x)$ is the original of $T[f(x);s]$.

**Inversion formula for the transform**

If an integral equation can be determined that

$$f(x) = \int_{\alpha}^{\beta} \phi(s,x)T[f(x);s]ds \tag{1.1.62}$$

then (1.1.62) is termed as the inversion formula of (1.1.61).

## LAPLACE TRANSFORM

One of the simplest and most important integral transform is the well known Laplace Transform. It has been a subject of wide and extensive study on account of its applications in applied mathematics and physics.

The Laplace Transform of a function is defined as follows:

$$L\{f(x);s\} = \int_{0}^{\infty} e^{-sx} f(x)dx \tag{1.1.63}$$

and the inversion formula is given by:

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{sx} L\{f(x);s\} ds \tag{1.1.64}$$

provided that the above integral exists.

The standard works of Doestch [18] in three volumes give the detailed and complete account of Laplace Transform.

### MELLIN TRANSFORM

The well known Mellin Transform is defined by:

$$M\{f(x);s\} = \int_0^\infty x^{s-1} f(x)dx \tag{1.1.65}$$

and the inversion formula is given by:

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} M\{f(x);s\}\, ds \tag{1.1.66}$$

provided that the above integral exists.

## 1.2 FRACTIONAL CALCULUS

The term Fractional calculus has its origin to the letter written by L'hospital in 1695 to Leibniz, wherein he enquired whether a meaning could be ascribed to $\frac{d^n f(x)}{dx^n}$ if n were a fraction. Because the answer to the questions was affirmative, various authors started working on the subject. In the initial stage of development, the order n was taken to be fraction. Although now n is taken as an arbitrary number, the subject is still known as fractional calculus. The works of Oldham and Spanier [80], Samko, Kilbas and Marichev [97], Gorenflo and Vessela [31], Kiryakova [61], McBridge [75], Miller and Ross [78], Nishimoto [79], Podlubny [84], Caputo [12] provide a comprehensive account of the development and applications in the field of fractional calculus.

The following well known Fractional integral operator has been widely studied :

(i) The **left-sided Riemann-Liouville fractional integral** or **Riemann-Liouville fractional integral** of order $\alpha$, for $\alpha > 0$, $x > a$, is defined as [58, p. 69, Eq. (2.1.1)]

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{1.2.1} $$

(ii) The **right-sided Riemann-Liouville fractional integral** of order $\alpha$, for $\alpha > 0$, $x < b$, is defined as [58, p. 69, Eq. (2.1.2)]

$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (x-t)^{\alpha-1} f(t)dt \tag{1.2.2} $$

(iii) The **left-sided Riemann-Liouville fractional Derivative** or **Riemann-Liouville fractional Derivative** of order $\alpha$, $m-1 < \alpha \le m$, $m \in \mathbb{N}$, for a real valued function f(x) defined on $R_+ = (0, \infty)$, is defined as [58, p. 70, Eq. (2.1.5)]

$$ {}_aD_x^\alpha f(x) = D^m {}_aI_x^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-m+1}} \tag{1.2.3} $$

(iv) The **right-sided Riemann-Liouville fractional Derivative** of order $\alpha$, $m-1 < \alpha \le m$, $m \in \mathbb{N}$, is given by [58, p. 70, Eq. (2.1.6)]

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dx}\right)^m \int_x^b \frac{f(t)dt}{(t-x)^{\alpha-m+1}} \tag{1.2.4} $$

(v) **Caputo fractional derivative** of order $\alpha$, $m-1 < \alpha \le m$, $m \in \mathbb{N}$, is defined as [11]

$$ {}_a^CD_x^\alpha f(x) = {}_aI_x^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{1}{(x-t)^{\alpha-m+1}} \left\{ \left(\frac{d}{dx}\right)^m f(t) \right\} dt \tag{1.2.5} $$

(vi) **Weyl Fractional Integral** of order $\alpha$, for $\Re(\alpha) > 0$ and $x > 0$ is defined as [78, p. 236, Eq. (1.1)]

$$W^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt \tag{1.2.6}$$

On account of the importance of the fractional calculus operators(FCO) in several problems of mathematical physics and applied mathematics, various generalization of the FCO defined by Riemann -Liouville and Weyl have been studied from time to time by several research workers notably: Kober [54], Sneddon [107], Kalla [55], Kalla and Saxena [56], Saxena and Kumbhat [101], Manocha [70], Koul [59], Raina and Kiryakova [89], Gupta and Soni [38], Garg [28], Garg and Purohit [27], Gupta [37], Saigo [95, 96], Gupta and Jain [36] , Gupta, Jain and Agrawal [35].

The fractional calculus finds use in many fields of science and engineering, such as fluid flow, reheology model, diffusion, potential theory, electrical transmission lines, probability, image processing, ultrasonic wave propagation electrochemistry, scattering theory, transport theory, statistics, theory of viscoelasticity, potential theory and many branches of mathematical analysis like integral and differential equations, operational calculus, univalent function theory and various other problems involving special function of mathematical physics as well as their extensions and generalizations in one and more variables.

A detailed account of various fractional integral operators studied from time to time has been given by Srivastava and Saxena [114].

In the present work we have introduced and developed a pair of unified Fractional

Integral Operators whose kernels involve the product of Appell Polynomial, Fox H-function and S-Generalized Gauss's Hypergeometric Function

$$I_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} = x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} A_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$

$$H_{P,Q}^{M,N}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ \\(h_j,H_j)_{1,Q}\end{matrix}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right] f(t)dt \tag{1.2.7}$$

provided that

$$\left.\begin{aligned}&\min_{1\le j\le M}\Re\left(\nu+\nu_2\frac{h_j}{H_j}+\zeta+1, \lambda+\lambda_2\frac{h_j}{H_j}+1\right)>0\\&\min\{\nu_1,\nu_3,\lambda_1,\lambda_3\}\ge 0\end{aligned}\right\} \tag{1.2.8}$$

$$J_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} = x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda} A_n\left[z_1\left(\frac{x}{t}\right)^{\nu_1}\left(1-\frac{x}{t}\right)^{\lambda_1}\right]$$

$$H_{P,Q}^{M,N}\left[z_2\left(\frac{x}{t}\right)^{\nu_2}\left(1-\frac{x}{t}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ \\(h_j,H_j)_{1,Q}\end{matrix}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{x}{t}\right)^{\nu_3}\left(1-\frac{x}{t}\right)^{\lambda_3}\right] f(t)dt \tag{1.2.9}$$

provided that

$$\left.\begin{aligned}&\Re(w_2)>0 \quad\text{or}\quad \Re(w_2)=0 \quad\text{and}\quad \min_{1\le j\le M}\Re\left(\nu-w_1+\nu_2\frac{h_j}{H_j}\right)>0\\&\min_{1\le j\le M}\Re\left(\lambda+\lambda_2\frac{h_j}{H_j}+1\right)>0, \min\{\nu_1,\nu_3,\lambda_1,\lambda_3\}\ge 0\end{aligned}\right\} \tag{1.2.10}$$

where $f(t)\in\Lambda$ and $\Lambda$ denotes the class of functions for which

$$f(t) := \begin{cases} O\{|t|^{\zeta}\}; & \max\{|t|\}\to 0\\ \\ O\{|t|^{w_1}e^{-w_2|t|}\}; & \min\{|t|\}\to\infty\end{cases} \tag{1.2.11}$$

## 1.3 FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional differential equations have gained considerable importance due to their application in various disciplines, such as physics, mechanics, chemistry, engineering, etc. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives (see the monographs of Samko et al.[97], Kilbas et al. [58], Miller and Ross [78], Oldham and Spanier [80], and Podlubny [84]). Numerous problems in these areas are modeled mathematically by systems of fractional differential equations.

A growing number of works in science and engineering deal with dynamical systems described by fractional order equations that involve derivatives and integrals of non-integer order [Benson et al. [8], Metzler & Klafter [77], Zaslavasky [128]]. These new models are more adequate than the previously used integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances [84]. This is the most significant advantage of the fractional order models in comparison with integer order models, in which such effects are neglected. In the context of flow in porous media, fractional space derivatives exhibit large motions through highly conductive layers or fractures, while fractional time derivatives describe particles that remain motionless for extended period of time [76]

Recent applications of fractional differential equations to a number of systems have given opportunity for physicists to study even more complicated systems.

For example, the fractional diffusion equation allow describing complex systems with anomalous behavior in much the same way as simpler systems.

### 1.3.1 BAGLEY TORVIK EQUATION

The Bagley-Torvik Equation is originally formulated to study the behavior of real material by use of fractional calculus [7, 123]. It plays important role in many engineering and applied science problems. In particular, the equation with 1/2-order derivative or 3/2-order derivative can model the frequency-dependent damping materials quite satisfactorily. It can also describe motion of real physical systems, the modeling of the motion of a rigid plate immersed in a Newtonian fluid and a gas in a fluid, respectively [84, 93]. Fractional dynamic systems have found many applications in various problems such as viscoelasticity, heat conduction, electrode-electrolyte polarization, electromagnetic waves, diffusion wave, control theory, and signal processing [3, 7, 17, 71, 84, 93, 120, 122, 123, 127].

The generic form of Bagley-Torvik equation [84, p. 229] can be written as

$$A\frac{d^2y(t)}{dt^2} + B\frac{d^{\frac{3}{2}}y(t)}{dt^{\frac{3}{2}}} + Cy(t) = f(t), \qquad t > 0 \tag{1.3.1}$$

Subject to initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0 \tag{1.3.2}$$

Where y(t) is the solution of the equation, $A \neq 0$, B, and C are constant coefficient's and f(t) is a given function from I into R, I is the interval [0,T]. The analytic results on existence and uniqueness of solutions to fractional differential equations have been investigated by many authors [84, 92].

### 1.3.2 FRACTIONAL RELAXATION–OSCILLATION EQUATION

The relaxation-oscillation equation is a fractional differential equation with initial conditions. There are many relaxation-oscillation models such as fractional derivative [14, 15, 68, 69, 126]. The relaxation-oscillation equation is the primary equation of relaxation and oscillation processes. The fractional derivatives are employed in the relaxation and oscillation models to represent slow relaxation and damped oscillation [68, 69].

Fractional Relaxation-Oscillation [44, p. 5928] model can be depicted as

$$D^{\beta}y(t) + Ay(t) = f(t), \qquad t > 0 \tag{1.3.3}$$

$$y(0) = a \qquad \text{if} \qquad 0 < \beta \leq 1 \tag{1.3.4}$$

or

$$y(0) = \lambda \qquad \text{and} \qquad y'(0) = \mu \qquad \text{if} \qquad 1 < \beta \leq 2 \tag{1.3.5}$$

where A is a positive constant. For $0 < \beta \leq 2$, the above equation is called the fractional relaxation-oscillation equation. When $0 < \beta \leq 1$, the model describes the relaxation with the power law attenuation. When $1 < \beta \leq 2$, the model depicts the damped oscillation with viscoelastic intrinsic damping of oscillator [16, 124].

This model has been applied in electrical model of the heart, signal processing, modeling cardiac pacemakers, predator-prey system, spruce-budworm interactions etc. [4, 16, 43, 94, 124, 125].

### 1.3.3 FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION

The general form of Fractional Order Riccati differential equation [57] is

$$D^{\beta}y(t) = P(t)y^2(t) + Q(t)y(t) + R(t), \qquad t > 0 \tag{1.3.6}$$

Subject to initial condition

$$y(0) = B \tag{1.3.7}$$

where P(t), Q(t) and R(t) are known functions. For $\beta = 1$, the fractional-order Riccati differential equation converts into the classical Riccati differential equation.

## 1.4 METHODS OF SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

Finding accurate and efficient methods for solving fractional differential equations has been an active research undertaking. In the last decade, various analytical and numerical methods have been employed to solve linear and non-linear problems. For example the **matrix method** [85, 86] which is a numerical method. Unlike other numerical methods used for solving fractional partial differential equations in which the solution is obtained step-by-step by moving from the previous time layer to the next one, here in matrix method, we consider the whole time interval. This allows us to create a net of discretization nodes. The values of the unknown function in inner nodes are to be found. The values at the boundaries are known and these are used in construction the system of algebraic equations. **Adomian decomposition method** (ADM), introduced and developed by Adomian [1, 2], attacks the problem in a direct way and in straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion, **Homotopy perturbation method** (HPM), introduced by He [45, 46, 47, 48, 49], consider the solution as the sum of an infinite series which converges rapidly to the accurate solutions. **Variational iteration method** (VIM), established by He [49] gives rapidly convergent successive approximations of the exact solution if such a solution exists. The VIM does not require specific treatments for non-linear problems as in Adomian decomposition method, perturbation techniques, etc. **Homotopy analysis method** (HAM) introduced by Liao [63, 64, 65], is a method based on homotopy, a fundamental concept in topology and differential geometry. It is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. The method introduces the solution in the form of a convergent fractional series with elegantly computable terms. **Generalized differential transform method** (GDTM) developed by Ertuk, Momani and Odibat [22] , for solving two dimensional linear and non-linear partial differential equations of fractional order is a generalization of differential transform method, it was proposed by Zhou [129] to solve linear and non-linear initial value problem in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions and takes long time in computation, whereas the differential transform is an iterative procedure for obtaining analytic Taylor series solution.

In Chapter 6, we apply FDTM for solving Bagley Torvik Equation, Fractional Relaxation Oscillation Equation and Fractional Order Riccati Differential Equation.

## 1.5 BRIEF CHAPTER BY CHAPTER SUMMARY OF THE THESIS

Now we present a brief summary of the work carried out in Chapter 2 to 6.
In **Chapter–2**, First of all we give definition of the S-generalized Gauss hypergeometric function and S-generalized Beta function which was recently introduced by Srivastava et al. [113].

Next, we first present an integral representation and Mellin transform of S-generalized Gauss hypergeometric function. Next, we give its complex integral representation and a relationship between S-generalized Gauss hypergeometric function and H-function of two variables. Further, we introduce a new integral transform whose kernel is the S-generalized Gauss hypergeometric function and point out its three special cases which are also believed to be new. We specify that the well-known Gauss hypergeometric function transform follows as a simple special case of our integral transforms. Next, we established an inversion formula for above integral transform in theorem form. Finally, we establish image of Fox H-function under the S-Generalized Gauss hypergeometric function transform and also obtain the images of five useful and important functions which are special cases of Fox H-function (Generalized Bessel function, Gauss Hpergeometric function, Generalized Mittag-Leffler Function, Krätzel Function and Lorenzo Hartley G-function) under the S-generalized Gauss hypergeometric function transform. Which are also believed to be new.

In **Chapter–3**, we study a pair of a general class of fractional integral operators involving the Appell Polynomial, Fox H-function and S-Generalized Gauss Hypergeometric Function. First we define and give the conditions of existence of the operators of our study and then we obtain the images of certain useful functions in them. Further, we evaluate four new integrals involving Appell's Function, Multivariate generalized Mittag-Lefflet Function, generalization of the modified Bessel function and generalized hypergeometric function by the application of the images established and also gives the three unknown and two known integral of these operators. Next we develop six results wherein the first two contain the Mellin transform of these operators, the next two the corresponding inversion formulae and the last two their Mellin convolutions. Later on, we establish a theorem analogous to the well known Parseval Goldstein theorem for our unified fractional integral operators.

In **Chapter–4**, we first derive three new and interesting expressions for the composition of the two fractional integral operators, which are slight variants of the operators defined in Chapter 3. The operators of our study are quite general in nature and may be considered as extensions of a number of simpler fractional integral operators studied from time to time by several authors. By suitably specializing the coefficients and the parameters in these functions we can get a large number of (new and known) interesting expressions for the composition of fractional integral operators involving simpler special functions. Finally, we obtain two interesting finite double integral formulae as an application of our first composition formula known results which follow as special case of our findings have also been mentioned.

In **Chapter–5**, we evaluate two unified and general finite integrals. The first integral involves the product of the Appell Polynomial $A_n(z)$, the Generalized form of the Astrophysical Thermonuclear function $I_3$ and Generalized Mittag - Leffler Function $E^{\gamma,\delta}_{\alpha,\beta,\tau,\mu,\rho,p}(z;s,r)$. The arguments of the functions occurring in the integral involve the product of factors of the form $x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^l)^{-\rho}$. We also obtain five new special cases of our main Integral which are of interest by themselves and are believed to be new.

The second integral involves the Generalized form of the Astrophysical Thermonuclear function $I_3$ and $\overline{H}-$ function. The arguments of the function occurring in the integral involve the product of factors of the form

$t^{\lambda-1}(1-t)^{\sigma-1}(1-ut^{\ell})^{-\gamma}(1+vt^m)^{-\beta}$.

We also obtain five new special cases of our main Integral which are of interest by themselves and are believed to be new.

In **Chapter–6**, The object of this chapter is to find solutions of the Bagley Torvik

Equation, Fractional Relaxation Oscillation Equation and Fractional Order Riccati Differential Equation. We make use of generalized differential transform method (GDTM) to solve the equations. First of all we give definition of a Caputo fractional derivative of order $\alpha$ which was introduced and investigated by Caputo [12]. Then, we give the generalized differential transform method and inverse generalized differential transform which was introduce and investigated by Ertuk et al.[22] and some basic properties of GDTM. Next, we find solutions to three different fractional differential equations using GDTM technique.

In section 6.2 we find the solution of Bagley Torvik Equation using GDTM. Since the function f(t) taken in this chapter is general in nature by specializing the function f(t) and taking different values of constants A, B and C. We can obtain a large number of special cases of Bagley Torvik Equation. Here we give two numerical examples. Our findings match with the results obtained earlier by Ghorbani et al. [29] by He's variational iteration method.

In section 6.3 we find the solution of Fractional Relaxation Oscillation Equation using GDTM. Since the function f(x) taken in this chapter is general in nature by specializing the function f(x) and taking different values of constants A, we can obtain a large number of special cases of Fractional Relaxation Oscillation Equation. Here we give eight numerical examples. Furthermore these examples are also represented graphically by using the MATHEMATICA SOFTWARE.

In section 6.4, Again we find the solution of Fractional Order Riccati Differential Equation using GDTM. Since order of Fractional Riccati Differential equation is $\beta$ and all function are general in nature, by specializing parameters and functions, we can obtain a large number of special cases of Fractional Relaxation Oscillation Equation. Here we give eight numerical examples. Furthermore these examples are also represented graphically by using the MATHEMATICA SOFTWARE.

## 1.6 LIST OF RESEARCH PAPERS CONTRIBUTED BY THE AUTHOR

1. **M. K. BANSAL and R. JAIN** (2017). A STUDY OF UNIFIED FRACTIONAL INTEGRAL OPERATORS INVOLVING S-GENERALIZED GAUSS'S HYPERGEOMETRIC FUNCTION AS ITS KERNEL, *Palestine Journal of Mathematics*, **6(1)**, 142–152 .

2. **M. K. BANSAL and R. JAIN** (2016). ANALYTICAL SOLUTION OF BAGLEY TORVIK EQUATION BY GENERALIZED DIFFERENTIAL TRANSFORM, *International Journal of Pure and Applied Mathematics*, **110(2)**, 265–273.

3. **M. K. BANSAL and R. JAIN** (2016). CERTAIN NEW RESULTS OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM, *South East Asian J. of Math. & Math. Sci.*, **12(2)**, 115–124 .

4. **M. K. BANSAL and R. JAIN** (2016).COMPOSITION FORMULAE FOR UNIFIED FRACTIONAL INTEGRAL OPERATORS, *European Journal of Advances in Engineering and Technology.* **3(4)**, 1–6.

5. **H. M. SRIVASTAVA, R. JAIN, M. K. BANSAL** (2015). A STUDY OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION AND ITS ASSOCIATED INTEGRAL TRANSFORMS, *Turkish Journal of Analysis and Number Theory*, **3(5)**, 116–119.

6. **M. K. BANSAL and R. JAIN** (2015). APPLICATION OF GENERALIZED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL RELAXATION OSCILLATION EQUATION, *Antarctica J. Math.*, **12(1)**, 85–95.

7. **M. K. BANSAL and R. JAIN** (2015). APPLICATION OF GENERALIZED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION AND NUMERICAL RESULT, *International Journal of Pure and Applied Mathematics*, **99(3)**, 355–366.

8. **M. K. BANSAL and R. JAIN** (2015). A STUDY OF FINITE INTEGRAL INVOLVING GENERALIZED FORM OF THE ASTROPHYSICAL THERMONUCLEAR FUNCTION, *Journal of Rajasthan Academy of Physical Sciences*, **14(1)**, 51–55.

9. **M. K. BANSAL and R. JAIN** (2014). ON UNIFIED FINITE INTEGRALS INVOLVING I3 FUNCTION, GENERALIZED MITTAG-LEFFLER FUNCTION AND APPELL POLYNOMIALS, *The Journal of the Indian Academy of Mathematics*, **36(2)**, 211–219.

# 2

# S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION AND AN INTEGRAL TRANSFORM ASSOCIATED WITH IT

The main findings of this chapter have been published as detailed below:

1. **H. M. SRIVASTAVA, R. JAIN, M. K. BANSAL** (2015). A STUDY OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION AND ITS ASSOCIATED INTEGRAL TRANSFORMS, *Turkish Journal of Analysis and Number Theory*, **3(5)**, 116–119.

2. **M. K. BANSAL and R. JAIN** (2016). CERTAIN NEW RESULTS OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM, *South East Asian J. of Math. & Math. Sci.*, **12(2)**, 115–124.

## 2. S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION AND AN INTEGRAL TRANSFORM ASSOCIATED WITH IT

In this chapter, we first define all the necessary function which is used in this chapter. First of all we give definition of the S-generalized Gauss hypergeometric function and S-generalized Beta function which was recently introduced by Srivastava et al. [113]. Since S-generalized Gauss hypergeometric function is in general nature, by specializing the parameters we can obtain a number of special cases which are studied earlier by Parmar [83], Özergin [81] and [82], Chaudhry et al. [13].

Next, we present an integral representation and Mellin transform of S-generalized Gauss hypergeometric function in the theorem form 2.2.1 and 2.2.2 respectively and its complex integral representation is also discussed in theorem 2.2.3 and also we establish a relationship between S-generalized Gauss hypergeometric function and H-function of two variables and also we introduce a new integral transform whose kernel is the S-generalized Gauss hypergeometric function and point out its three special cases which are also believed to be new. We specify that the well-known Gauss hypergeometric function transform follows as a simple special case of our integral transforms. Next, we established an inversion formula for above integral transform in theorem 2.3.1. Finally, we establish image of Fox H-function under the S-Generalized Gauss hypergeometric function transform and also obtain the images of five useful and important functions which are special cases of Fox H-function (Generalized Bessel function, Gauss Hpergeometric function, Generalized Mittag-Leffler Function, Krätzel Function and Lorenzo Hartley G-function) under the S-generalized Gauss hypergeometric function transform. Which are also believed to be new.

# 2.1 INTRODUCTION

## S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

The S-generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ was introduced and investigated by Srivastava et al. [113, p. 350, Eq. (1.12)]. It is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \tag{2.1.1}$$

provided that $\quad (\Re(p) \geq 0; \quad \min\{\Re(\alpha),\Re(\beta),\Re(\tau),\Re(\mu)\} > 0; \quad \Re(c) > \Re(b) > 0)$

in terms of the classical Beta function $B(\lambda,\mu)$ and the S-generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$, which was also defined by Srivastava et al. [113, p. 350, Eq.(1.13)] as follows:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^{\tau}(1-t)^{\mu}}\right)dt \tag{2.1.2}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta),\Re(\tau),\Re(\mu)\} > 0)$$

If we take p = 0 in (2.1.2), it reduces to classical Beta Function and $(\lambda)_n$ denotes the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [115, p. 2 and pp. 4-6]; see also [114, p. 2]):

$$\begin{aligned}(\lambda)_n &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1, & (n=0) \\ \lambda(\lambda+1)...(\lambda+n-1), & (n \in \mathbb{N} \quad := \{1,2,3,\cdots\}) \end{cases}\end{aligned} \tag{2.1.3}$$

provided that the Gamma quotient exists (see, for details,[108, p. 16 et seq.] and [112, p. 22 et seq.]).

For $\tau = \mu$, the S-generalized Gauss hypergeometric function defined by (2.1.1) reduces to the following generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau)}(a,b;c;z)$ studied earlier by Parmar [83, p.44]:

$$F_p^{(\alpha,\beta;\tau)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\tau)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{2.1.4}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(\alpha),\Re(\beta),\Re(\tau)\} > 0; \quad \Re(c) > \Re(b) > 0).$$

which, in the *further* special case when $\tau = 1$, reduces to the following extension of the generalized Gauss hypergeometric function (see, e.g., [82, p.4606, Section 3] ; see also [81, p. 39]):

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{2.1.5}$$

$$(\Re(p) \geqslant 0; \quad \min\{\Re(\alpha),\Re(\beta)\} > 0; \quad \Re(c) > \Re(b) > 0)$$

Upon setting $\alpha = \beta$ in (2.1.5), we arrive at the following Extended Gauss hypergeometric function (see [13, p.591, Eqs. (2.1) and (2.2)]:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{2.1.6}$$

$$(\Re(p) \geqslant 0; \quad \Re(c) > \Re(b) > 0)$$

## Fox H-Function

A single Mellin-Barnes contour integral, occurring in the present work, is now popularly known as the $H$-function of Charles Fox (1897-1977). It will be defined and represented here in the following manner (see, for example, [109, p. 10]):

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,P} \\ \\ (b_j, \beta_j)_{1,Q} \end{matrix}\right] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_P, \alpha_P) \\ \\ (b_1, \beta_1), \cdots, (b_Q, \beta_Q) \end{matrix}\right]$$

$$:= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \Theta(\mathfrak{s}) z^{\mathfrak{s}} \, d\mathfrak{s}, \tag{2.1.7}$$

where $\mathrm{i} = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\Theta(\mathfrak{s}) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \mathfrak{s}) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \mathfrak{s})}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j \mathfrak{s}) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \mathfrak{s})}, \tag{2.1.8}$$

and

$$1 \leqq M \leqq Q \qquad \text{and} \qquad 0 \leqq N \leqq P$$

$$(M, Q \in \mathbb{N} = \{1, 2, 3, \cdots\}; \; N, P \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{2.1.9}$$

an empty product being interpreted to be 1. Here $\mathfrak{L}$ is a Mellin-Barnes type contour in the complex $\mathfrak{s}$-plane with appropriate indentations in order to separate the two sets of poles of the integrand $\Theta(\mathfrak{s})$ (see, for details, [58] and [109]).

# 2.2 MAIN RESULTS

In this section, we first give the integral representation, Mellin Transform and complex integral representation of S-generalized gauss hypergeometric function.

Next, we establish a relationship between S-generalized gauss hypergeometric function and H-function of two variables.

## INTEGRAL REPRESENTATION OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

**Theorem 2.2.1.** *Suppose that* $\Re(p) \geq 0$, $|arg(1-z)| < \pi$, $\min\{\Re(\tau), \Re(\mu), \Re(b+\tau\alpha), \Re(c-b+\mu\alpha)\} > 0$ *and* $\Re(c) > \Re(b) > 0$. *Then the following integral representation holds true* :

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\tau(1-t)^\mu}\right) dt \quad (2.2.1)$$

*where the S-generalized Gauss hypergeometric function* $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ *is given by (2.1.1).*

*Proof.* Using Eq. (2.1.1) on the left hand side of (2.2.1), we have

$$\begin{aligned}
&F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) \\
&= \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \\
&= \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n \int_0^1 t^{b+n-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\tau(1-t)^\mu}\right) \frac{z^n}{n!} dt \\
&= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\tau(1-t)^\mu}\right) \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} dt \\
&= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\tau(1-t)^\mu}\right) dt
\end{aligned}$$

which proves Theorem (2.2.1) □

## THE MELLIN TRANSFORM OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

As usual, the Mellin transform of a function f(t) is defined by (see, for example,[19, p. 340, Eq. (8.2.5)])

$$\mathfrak{M}[f(z)](s) = \int_0^\infty z^{s-1} f(z)dz \qquad \Re(s) > 0 \tag{2.2.2}$$

provided that the improper integral exists.

**Theorem 2.2.2.** *If* $\Re(p) \geq 0, \quad \min\{\Re(\tau), \Re(\mu), \Re(b+\tau\alpha), \Re(c-b+\mu\alpha)\} > 0$ *and* $\Re(c) > \Re(s) < \min\{\Re(a), \Re(b)\}$,
*then*

$$\mathfrak{M}[F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)](s) = (-1)^s \frac{B(s,a-s)B_p^{(\alpha,\beta;\tau,\mu)}(b-s,c-b)}{B(b,c-b)} \tag{2.2.3}$$

*Proof.* : In order to prove the assertion (2.2.3), by taking the Mellin transform of (2.2.1), we obtain

$$\Delta(s) := \int_0^\infty z^{s-1} \left[\frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\tau(1-t)^\mu}\right)dt\right]dz$$

Upon interchanging the order of t and z-integrals (which is permissible under the conditions stated), if we evaluate the resulting z-integral first, we get

$$\Delta(s) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t^\tau(1-t)^\mu}\right) \frac{\Gamma(s)\Gamma(a-s)}{(-t)^s\Gamma(a)}dt$$

Now with the help of (2.1.2), we get the desired result (2.2.3) after a little simplification. □

## A COMPLEX INTEGRAL REPRESENTATION OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

**Theorem 2.2.3.** *If* $\Re(p) \geq 0$, $\min\{\Re(\tau), \Re(\mu), \Re(b+\tau\alpha), \Re(c-b+\mu\alpha)\} > 0$ *and* $\Re(c) > \Re(b) > 0$ *then complex integral representation for the s-generalized*

*Gauss Hypergeometric function* $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ *is defined in the following manner:*

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$$
$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}(-z)^{-s}\frac{B(s,a-s)B_p^{(\alpha,\beta;\tau,\mu)}(b-s,c-b)}{B(b,c-b)}ds \qquad (2.2.4)$$

*Proof.* If we take the inverse Mellin transform of (2.2.3), we easily arrive the desired result □

## RELATIONSHIP BETWEEN S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION AND H-FUNCTION OF TWO VARIABLES

we give the following representation of $F_p^{(\alpha,\beta;\tau,\gamma)}(a,b;c,z)$ in terms of H-function of two variables:

$$F_p^{(\alpha,\beta;\tau,\gamma)}(a,b;c,z) = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}(-z)^{-\xi}\frac{B(\xi,a-\xi)B_p^{(\alpha,\beta;\tau,\mu)}(b-\xi,c-b)}{B(b,c-b)}d\xi$$
$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(a)B(b,c-b)}H_{1,1:3,1;1,1}^{0,1:1,2;1,1}\left[\begin{array}{c|c} p^{-1} & (1-b;\tau,1):A^* \\ -z & (1-c;\tau+\mu,1):B^* \end{array}\right] \qquad (2.2.5)$$

where

$A^* = (1,1),(1-c+b,\mu),(\beta,1);(1-a,1) \qquad B^* = (\alpha,1);(0,1)$

provided that the existence conditions in (2.1.1) for the S-generalized Gauss hypergeometric function.

*Proof.* To evaluate the contour integral (2.2.5), we first express the term $B_p^{(\alpha,\beta;\tau,\mu)}(b-\xi,c-b)$ occurring in it's integrand in terms of integral with the help of equation (2.1.2). Thus left hand side of (2.2.5) takes the following form (say$\Delta$):

$$\Delta = \frac{1}{2\pi i}\int_{\mathfrak{L}_1} (-z)^{-\xi}\frac{B(\xi, a-\xi)}{B(b, c-b)}\left[\int_0^1 t^{b-\xi-1}(1-t)^{c-b-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^{\tau}(1-t)^{\mu}}\right)dt\right]d\xi \tag{2.2.6}$$

Now we convert ${}_1F_1$ function into its contour integral form, then we change the order of contour $\eta-$ integral with t-integral (which is permissible under the conditions stated). Thus right hand side of (2.2.6) takes the following form :

$$\Delta = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(a)B(b,c-b)}\frac{1}{(2\pi i)^2}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\frac{\Gamma(\xi)\Gamma(a-\xi)\Gamma(-\eta)\Gamma(\alpha+\eta)}{\Gamma(\beta+\eta)}$$
$$\left[\int_0^1 t^{b-\xi-\tau\eta-1}(1-t)^{c-b-\mu\eta-1}dt\right](-z)^{-\xi}d\xi d\eta \tag{2.2.7}$$

Further, we evaluate the t-integral occurring in (2.2.7) with the help of well known Beta function. Thus we get the following equation

$$\Delta = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(a)B(b,c-b)}$$
$$\frac{1}{(2\pi i)^2}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\frac{\Gamma(\xi)\Gamma(a-\xi)\Gamma(-\eta)\Gamma(\alpha+\eta)\Gamma(b-\xi-\tau\eta)\Gamma(c-b-\mu\eta)}{\Gamma(\beta+\eta)\Gamma(c-\xi-(\tau+\mu)\eta)}(-z)^{-\xi}p^{\eta}d\xi d\eta \tag{2.2.8}$$

finally reinterpret the result thus obtained in terms of H-function of two variable. We easily arrive at the right hand side of (2.2.5) after a little simplification. □

## 2.3 THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM

We define the S-generalized Gauss hypergeometric transform by the following equation (see also a recent work [110] dealing with several new families of integral transforms):

$$\tilde{\mathfrak{S}}[f(z); s] = \varphi(s) = \int_0^\infty F_p^{(\alpha,\beta;\tau,\mu)}(a, b; c; sz) f(z) dz \tag{2.3.1}$$

where $f(z) \in \Lambda$, and $\Lambda$ denotes the class of functions for which

$$f(z) = \begin{cases} O\{z^\zeta\}, & (z \to 0) \\ O\{z^{w_1} e^{-w_2 z}\}, & (|z| \to \infty) \end{cases} \tag{2.3.2}$$

provided that the existence conditions in (2.1.1) for the S- generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(.)$ are satisfied and

$$\left.\begin{aligned} &\Re(\zeta) + 1 > 0 \\ &\Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \Re(w_1 - a + 1) < 0 \end{aligned}\right\} \tag{2.3.3}$$

## SPECIAL CASES

In this section, we give three special cases of our integral transform defined by (2.3.1).

### (i) GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM

If we put $\tau = \mu$ in (2.3.1), the transform in (2.3.1) reduces to the generalized Gauss hypergeometric function transform given by

$$\varphi_1(s) = \int_0^\infty F_p^{(\alpha,\beta;\tau,\tau)}(a, b; c; sz) f(z) dz \tag{2.3.4}$$

### (ii) EXTENSION OF THE GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM

By taking $\tau = \mu = 1$ in (2.3.4), we get the following extension of the generalized gauss hypergeometric function transform :

$$\varphi_2(s) = \int_0^\infty F_p^{(\alpha,\beta)}(a, b; c; sz) f(z) dz \tag{2.3.5}$$

### (iii) EXTENDED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM

Moreover, if we take $\alpha = \beta$ in (2.3.5), it reduces to the extended Gauss hypergeometric function transform given below:

$$\varphi_3(s) = \int_0^\infty F_p(a, b; c; sz) f(z) dz \tag{2.3.6}$$

if we set p = 0 in the integral transforms defined by (2.3.4), (2.3.5) and (2.3.6), we easily get the Gauss hypergeometric transform (see, for details, [109]).

## INVERSION FORMULA FOR THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM

**Theorem 2.3.1.** *If $y^{\kappa-1} f(y) \in L(0, \infty)$, the function f(y) is of bounded variation in the neighborhood of the point $y = z$, and*

$$\varphi(s) = \tilde{\mathfrak{S}}[f(z); s] = \int_0^\infty F_p^{(\alpha,\beta;\tau,\mu)}(a, b; c; sz) f(z) dz \tag{2.3.7}$$

*then*

$$\begin{aligned} &\frac{1}{2}\{f(t+0) + f(t-0)\} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \frac{(-1)^{\kappa-1} B(b, c-b)}{B(1-\kappa, a+\kappa-1) B_p^{(\alpha,\beta;\tau,\mu)}(b+\kappa-1, c-b)} z^{-\kappa} \Omega(\kappa) d\kappa \end{aligned} \tag{2.3.8}$$

*where*

$$\Omega(\kappa) = \int_0^\infty s^{-\kappa}\varphi(s)ds \tag{2.3.9}$$

*provided that existence conditions for the S-generalized Gauss hypergeometric function* $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ *given by (2.1.1) are satisfied, the S-generalized Gauss hypergeometric function transform of* $|f(z)|$ *exists, and*

$$\Re(1-\kappa) > 0, \qquad \Re(1-a-\kappa) < 0$$

*Proof.* : In order to prove the inversion formula (2.3.8), we substitute the value of $\varphi(s)$ from (2.3.7) in the right hand side of (2.3.9), We thus find that

$$\begin{aligned}\Omega(\kappa) :&= \int_0^\infty s^{-\kappa}\varphi(s)ds \\ &= \int_0^\infty s^{-\kappa}\left(\int_0^\infty F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz)f(z)dz\right)ds\end{aligned} \tag{2.3.10}$$

Upon interchanging the order of the z and s- integrals in (2.3.10)(which is permissible under the given conditions), if we evaluate the s-integral by using (2.2.3), we obtain

$$\Omega(\kappa) = \int_0^\infty \frac{B(1-\kappa, a+\kappa-1)B_p^{(\alpha,\beta;\tau,\mu)}(b+\kappa-1, c-b)}{B(b,c-b)} f(z)(-z)^{\kappa-1}dz \tag{2.3.11}$$

Finally, by applying the Mellin Inversion Formula to the above integral (2.3.11), we get the desired result (2.3.8), after a little simplification. □

## 2.4 THE S-GENERALIZED GAUSS HYPERGEOMETRIC TRANSFORM OF THE H-FUNCTION

The S-Generalized Gauss hypergeometric Transform (2.3.1) of Fox H-function (2.1.7) defined as follows :

$$\tilde{\mathfrak{S}}\left[z^{\kappa}H_{P,Q}^{M,N}\left[Az^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right];s\right]=\int_0^{\infty}F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz)z^{\kappa}H_{P,Q}^{M,N}\left[Az^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]dz$$

$$=\frac{\Gamma(\beta)A^{-\frac{\kappa+1}{\sigma}}}{\sigma\Gamma(\alpha)\Gamma(a)B(b,c-b)}H_{1,1:1+Q,1+P;3,1}^{0,1:1+N,1+M;1,2}\left[\begin{matrix}-\frac{s}{A^{\sigma}}\\ \\ \frac{1}{p}\end{matrix}\left|\begin{matrix}(1-b;1,\tau):A^{*}\\ \\ (1-c;1,\tau+\mu):B^{*}\end{matrix}\right.\right] \tag{2.4.1}$$

where

$A^{*}=(1-a,1),(1-b_j-\beta_j\frac{(\kappa+1)}{\sigma},\frac{\beta_j}{\sigma})_{1,Q};(1,1),(1-c+b,\mu),(\beta,1)$

$B^{*}=(0,1),(1-a_j-\alpha_j\frac{(\kappa+1)}{\sigma},\frac{\alpha_j}{\sigma})_{1,P};(\alpha,1)$

provided that the existence conditions in (2.1.1) for the S-Generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ are satisfied and

(i) $\min\limits_{1\le j\le M}\Re\left(\kappa+\frac{\sigma b_j}{\beta_j}\right)+1>0$ (ii) $\max\limits_{1\le j\le N}\Re\left(\kappa-a+\frac{\sigma(a_j-1)}{\alpha_j}\right)+1<0$

*Proof.* To prove the result (2.4.1), we first write the complex integral representation of S-generalized Gauss hypergeometric function defined in (2.2.4) and then change the order of $\xi-$integral with z-integral (which is permissible under the conditions stated), we obtain (say $\Delta$)

$$\Delta=\frac{1}{2\pi\mathrm{i}}\int\limits_{\mathfrak{L}}\frac{(-s)^{-u}B(u,a-u)B_p^{(\alpha,\beta;\tau,\mu)}(b-u,c-b)}{B(b,c-b)}\left\{\int\limits_0^{\infty}z^{\kappa-u}H_{P,Q}^{M,N}[Az^{\sigma}]dz\right\}du \tag{2.4.2}$$

Now we evaluate the z-integral involved in (2.4.2) with the help of [109, p.15, Eq. (2.4.1)], we have

$$\Delta = \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \frac{(-s)^{-u} B(u, a-u) B_p^{(\alpha,\beta;\tau,\mu)}(b-u, c-b)}{B(b, c-b)}$$
$$\frac{1}{\sigma} A^{-(\frac{\kappa-u+1}{\sigma})} \frac{\prod_{j=1}^{M} \Gamma(b_j + \beta_j \frac{(\kappa+1)}{\sigma} - \frac{\beta_j}{\sigma} u) \prod_{j=1}^{N} \Gamma(1 - a_j - \alpha_j \frac{(\kappa+1)}{\sigma} + \frac{\alpha_j}{\sigma} u)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j - \beta_j \frac{(\kappa+1)}{\sigma} + \frac{\beta_j}{\sigma} u) \prod_{j=N+1}^{Q} \Gamma(a_j + \alpha_j \frac{(\kappa+1)}{\sigma} - \frac{\alpha_j}{\sigma} u)} du \tag{2.4.3}$$

Next, we express S-generalized Beta function in terms of complex integral form. Finally, we get the right hand side of (2.4.1) by reinterpreting the result in terms of H-function of two variables.

### 2.4.1 SPECIAL CASES

Here we give S-generalized Gauss hypergeometric function Transform of the some important special cases of Fox H-Function involving Generalized Bessel function, Gauss Hpergeometric function, Generalized Mittag-Leffler Function, Krätzel Function and Lorenzo Hartley G-function.

1. **S-generalized Gauss hypergeometric function Transform of Generalized Bessel Function:** In (2.4.1), if we reduce Fox H-Function to the Generalized Bessel function [109, p.19, Eq.(2.6.10)] by taking M = 1, N = P = 0, Q = 2, $b_1 = 0, \beta_1 = 1, b_2 = -\lambda, \beta_2 = \rho$, we can easily get the following S-generalized Gauss hypergeometric function Transform

of Generalized Bessel Function after a little simplification.

$$\tilde{\mathfrak{S}}\left[z^{\kappa}J_{\lambda}^{\rho}[Az^{\sigma}];s\right]=\int_{0}^{\infty}F_{p}^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz)z^{\kappa}J_{\lambda}^{\rho}[Az^{\sigma}]dz$$
$$=\frac{A^{-(\frac{\kappa+1}{\sigma})}\Gamma(\beta)}{\sigma\Gamma(\alpha)\Gamma(a)B(b,c-b)}H_{1,1:3,1;3,1}^{0,1:1,2;1,2}\left[\begin{array}{c|c} -s & (1-b;1,\tau):A^{*} \\ \frac{1}{p} & (1-c;1,\tau+\mu):B^{*}\end{array}\right] \tag{2.4.4}$$

where

$A^{*}=(1-a,1),(1-(\frac{\kappa+1}{\sigma}),\frac{1}{\sigma}),(1+\lambda-\rho(\frac{\kappa+1}{\sigma}),\frac{\rho}{\sigma});(1,1),(1-c+b,\mu),(\beta,1)$

$B^{*}=(0,1);(\alpha,1)$

provided that the conditions are easily obtainable from the existing conditions of (2.4.1) are satisfied.

2. **S-generalized Gauss hypergeometric function Transform of Gauss Hypergeometric Function:** Next, if we reduce Fox H-Function to the Gauss Hpergeometric function [109, p.19, Eq.(2.6.8)] by taking M = 1, N = P = Q = 2, $a_1 = 1-u, a_2 = 1-v, b_1 = 0, b_2 = 1-w, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ in (2.4.1), we can easily get the following S-generalized Gauss hypergeometric function Transform of Gauss Hpergeometric Function after a little simplification.

$$\tilde{\mathfrak{S}}\left[z^{\kappa}{}_2F_1[u,v;w;-Az^{\sigma}];s\right]=\int_{0}^{\infty}F_{p}^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz)z^{\kappa}{}_2F_1[u,v;w;-Az^{\sigma}]dz$$
$$=\frac{A^{-(\frac{\kappa+1}{\sigma})}\Gamma(w)\Gamma(\beta)}{\sigma\Gamma(u)\Gamma(v)\Gamma(\alpha)\Gamma(a)B(b,c-b)}H_{1,1:3,3;3,1}^{0,1:3,2;1,2}\left[\begin{array}{c|c} -s & (1-b;1,\tau):A^{*} \\ \frac{1}{p} & (1-c;1,\tau+\mu):B^{*}\end{array}\right] \tag{2.4.5}$$

where

$A^{*}=(1-a,1),(1-(\frac{\kappa+1}{\sigma}),\frac{1}{\sigma}),(w-(\frac{\kappa+1}{\sigma}),\frac{1}{\sigma});(1,1),(1-c+b,\mu),(\beta,1)$

$B^* = (0,1), (u - (\frac{\kappa+1}{\sigma}), \frac{1}{\sigma}), (v - (\frac{\kappa+1}{\sigma}); \frac{1}{\sigma}); (\alpha, 1)$

provided that the conditions are easily obtainable from the existing conditions of (2.4.1) are satisfied.

3. **S-generalized Gauss hypergeometric function Transform of Generalized Mittag-Leffler Function :** Again, if we reduce Fox H-Function to the Generalized Mittag-Leffler function [73, p.25, Eq.(1.137)] by taking M = N = P = 1, Q = 2 and $a_1 = 1-\delta, b_1 = 0, b_2 = 1-\sigma, \alpha_1 = \beta_1 = 1, \beta_2 = \rho$ in (2.4.1), we can easily get the following S-generalized Gauss hypergeometric function Transform of Generalized Mittag-Leffler Function after a little simplification.

$$\tilde{\mathfrak{S}}\left[z^{\kappa} E^{\delta}_{\rho,\gamma}[Az^{\sigma}]; s\right] = \int_0^{\infty} F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz) z^{\kappa} E^{\delta}_{\rho,\gamma}[Az^{\sigma}] dz$$
$$= \frac{A^{-(\frac{\kappa+1}{\sigma})}\Gamma(\beta)}{\sigma\Gamma(\alpha)\Gamma(a)B(b,c-b)} H^{0,1:2,2;1,2}_{1,1:3,2;3,1}\left[\begin{array}{c|c} -s & (1-b;1,\tau) : A^* \\ \frac{1}{p} & (1-c;1,\tau+\mu) : B^* \end{array}\right] \tag{2.4.6}$$

where

$A^* = (1-a,1), (-\kappa,1), (\gamma - \rho(\kappa+1), \rho); (1,1), (1-c+b, \mu), (\beta, 1)$

$B^* = (0,1), (\delta - \kappa - 1, 1); (\alpha, 1)$

provided that the conditions are easily obtainable from the existing conditions of (2.4.1) are satisfied.

4. **S-generalized Gauss hypergeometric function Transform of Krätzel Function:** In (2.4.1), if we reduce Fox H-Function to the Krätzel function [73, p.25, Eq.(1.141)] by taking M = Q = 2, N = P = 0, $b_1 = 0, \beta_1 = 1, b_2 =$

$\frac{\nu}{\rho}, \beta_2 = \frac{1}{\rho}$, we can easily get the following S-generalized Gauss hypergeometric function Transform of Krätzel Function after a little simplification.

$$\tilde{\mathfrak{S}}\left[z^{\kappa} Z_{\rho}^{\nu}(Az^{\sigma}); s\right] = \int_0^{\infty} F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz) z^{\kappa} Z_{\rho}^{\nu}(Az^{\sigma}) dz$$
$$= \frac{A^{-(\frac{\kappa+1}{\sigma})}\Gamma(\beta)}{\sigma\rho\Gamma(\alpha)\Gamma(a)B(b,c-b)} H_{1,1:3,1;3,1}^{0,1:1,3;1,2}\left[\begin{array}{c|c} -s & (1-b;1,\tau):A^* \\ \frac{1}{p} & (1-c;1,\tau+\mu):B^* \end{array}\right] \tag{2.4.7}$$

where

$A^* = (1-a,1), (1-(\frac{\kappa+1}{\sigma}), \frac{1}{\sigma}), (1-\frac{\nu}{\rho}-(\frac{\kappa+1}{\sigma\rho}), \frac{1}{\sigma\rho}); (1,1), (1-c+b,\mu), (\beta,1)$

$B^* = (0,1); (\alpha,1)$

provided that the conditions are easily obtainable from the existing conditions of (2.4.1) are satisfied.

5. **S-generalized Gauss hypergeometric function Transform of Lorenzo Hartley G-function :** In (2.4.1), if we reduce Fox H-Function to the Lorenzo Hartley G-function [34, p.64, Eq.(2.3)] by taking M = N = P = 1, Q = 2, $a_1 = 1-r, \alpha_1 = 1, b_1 = 0, \beta_1 = \beta_2 = 1, b_2 = 1+\nu-r$, we can easily get the following S-generalized Gauss hypergeometric function Transform of Lorenzo Hartley G-function after a little simplification.

$$\tilde{\mathfrak{S}}\left[z^{\kappa} G_{\sigma,\nu,r}[-A,t]; s\right] = \int_0^{\infty} F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;sz) z^{\kappa} G_{\sigma,\nu,r}[-A,t] dz$$
$$= \frac{A^{-(\frac{\kappa-\nu+r\sigma}{\sigma})}\Gamma(\beta)}{\sigma\Gamma(r)\Gamma(\alpha)\Gamma(a)B(b,c-b)} H_{1,1:3,2;3,1}^{0,1:2,2;1,2}\left[\begin{array}{c|c} -\frac{s}{A^{\sigma}} & (1-b;1,\tau):A^* \\ \frac{1}{p} & (1-c;1,\tau+\mu):B^* \end{array}\right] \tag{2.4.8}$$

where

$A^* = (1-a,1), (\frac{\sigma-\kappa+\nu-r\sigma}{\sigma}, \frac{1}{\sigma}), (-\kappa,1); (1,1), (1-c+b,\mu), (\beta,1)$

$B^{*} = (0,1), (\frac{\nu-\kappa}{\sigma}, \frac{1}{\sigma}); (\alpha, 1)$

provided that the conditions are easily obtainable from the existing conditions of (2.4.1) are satisfied.

# 3

# FRACTIONAL INTEGRAL OPERATORS WITH APPLICATIONS

The main findings of this chapter have been published as detailed below:

1. **M. K. BANSAL and R. JAIN** (2017). A STUDY OF UNIFIED FRACTIONAL INTEGRAL OPERATORS INVOLVING S-GENERALIZED GAUSS'S HYPERGEOMETRIC FUNCTION AS ITS KERNEL, *Palestine Journal of Mathematics*, **6(1)**, 142–152 .

In this chapter we study a pair of a general class of fractional integral operators involving the Appell Polynomial, Fox H-function and S-Generalized Gauss Hypergeometric Function. First we define and give the conditions of existence of the operators of our study and then we obtain the images of certain useful functions in them. Further, we evaluate four new integrals involving Appell's Function, Multivariate generalized Mittag-Lefflet Function, generalization of the modified Bessel function and generalized hypergeometric function by the application of the images established and also gives the three unknown and two known integral of these operators. Next we develop six results wherein the first two contain the Mellin transform of these operators, the next two the corresponding inversion formulae and the last two their Mellin convolutions. Later on, we establish a theorem analogous to the well known Parseval Goldstein theorem for our unified fractional integral operators.

## 3.1 INTRODUCTION

### APPELL POLYNOMIAL

A class of polynomials over the field of complex numbers which contains many classical polynomial systems. The Appell Polynomials were introduced by Appell [6]. The series of Appell Polynomials is defined by :

$$A_n(z) = \sum_{k=0}^{n} \frac{a_{n-k}}{k!} z^k \qquad n = 0, 1, 2, ... \tag{3.1.1}$$

where $a_{n-k}$ is the complex coefficient and $a_0 \neq 0$.

## FOX H-FUNCTION

A single Mellin-Barnes contour integral, occurring in the present work, is now popularly known as the $H$-function of Charles Fox (1897-1977). It will be defined and represented here in the following manner [109]:

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (g_j, G_j)_{1,P} \\ \\ (h_j, H_j)_{1,Q} \end{matrix}\right] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (g_1, G_1), \cdots, (g_P, G_P) \\ \\ (h_1, H_1), \cdots, (h_Q, H_Q) \end{matrix}\right]$$
$$:= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \Theta(\xi) z^{\xi} \, d\xi, \qquad (3.1.2)$$

where $\mathrm{i} = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\Theta(\xi) = \frac{\prod_{j=1}^{M} \Gamma(h_j - H_j\xi) \prod_{j=1}^{N} \Gamma(1 - g_j + G_j\xi)}{\prod_{j=M+1}^{Q} \Gamma(1 - h_j + H_j\xi) \prod_{j=N+1}^{P} \Gamma(g_j - G_j\xi)}, \qquad (3.1.3)$$

M, N, P and Q are non-negative integers satisfying $1 \leqq M \leqq Q$, $0 \leqq N \leqq P; G_j(j = 1, \cdots, P)$ and $H_j(j = 1, \cdots, Q)$ are assumed to be positive quantities for standardization purpose.

The definition of the H-function given by (3.1.2) will, However, have meaning even if some of these quantities are zero, giving us in turn simple transformation formulas.

The nature of contour $\mathfrak{L}$ in (3.1.2), a set of sufficient conditions for the convergence of this integral, the asymptotic expansion of the H-function, some of its properties and special cases can be referred to in a book by Srivastava et al.[109]

## THE MULTIVARIABLE H-FUNCTION

The Multivariable H-Function is defined and represented in the following manner [109, p. 251–250, Eqs. (C.1–C.3)]:

$$H^{0,B:A_1,B_1;...;A_r,B_r}_{C,D:C_1,D_1;...;C_r,D_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,C} : (c_j^{(1)},\gamma_j^{(1)})_{1,C_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r} \\ . & \\ . & \\ . & \\ z_r & (b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,D} : (d_j^{(1)},\delta_j^{(1)})_{1,D_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,D_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}...\int_{\mathfrak{L}_r}\Phi(\xi_1,\xi_2,...,\xi_r)\prod_{i=1}^{r}\Theta_i(\xi_i)z^{\xi_i}\,d\xi_1 d\xi_2,...,d\xi_r, \tag{3.1.4}$$

where $\omega = \sqrt{-1}$,

$$\Phi(\xi_1,\xi_2,...,\xi_r) = \frac{\prod_{j=1}^{B}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=1}^{D}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod_{j=B+1}^{C}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}$$

$$\Theta_i(\xi_i) = \frac{\prod_{j=1}^{A_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{B_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=B_i+1}^{C_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)\prod_{j=A_i+1}^{D_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)}(i=1,2,...,r), \tag{3.1.5}$$

All the Greek letters occurring on the left and side of (3.1.4) are assumed to be positive real numbers for standardization purposes. The definition of the multivariable H-function will however be meaningful even if some of these quantities are zero. The details about the nature of contour $\mathfrak{L}_1,...,\mathfrak{L}_r$, conditions of convergence of the integral given by (3.1.4). Throughout the work it is assumed that this function always satisfied its appropriate conditions of convergence [109, p. 251, Eqs. (C.4–C.6)].

## S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

The S-generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ was introduced and investigated by Srivastava et al. [113, p. 350, Eq. (1.12)]. It is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \tag{3.1.6}$$

provided that $\quad (\Re(p) \geq 0; \quad \min\{\Re(\alpha),\Re(\beta),\Re(\tau),\Re(\mu)\} > 0; \quad \Re(c) > \Re(b) > 0)$

in terms of the classical Beta function $B(\lambda,\mu)$ and the S-generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$ was also defined by Srivastava et al. [113, p. 350, Eq.(1.13)] as follows:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^{\tau}(1-t)^{\mu}}\right)dt \tag{3.1.7}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta),\Re(\tau),\Re(\mu)\} > 0)$$

If we take p = 0 in (3.1.7), it reduces to Classical Beta Function and $(\lambda)_n$ denotes the pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [115, p. 2 and pp. 4–6]; see also [114, p. 2]):

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n=0) \\ \lambda(\lambda+1)...(\lambda+n-1), & (n \in \mathbb{N} \quad := \{1,2,3\}) \end{cases} \tag{3.1.8}$$

provided that the Gamma quotient exists (see, for details,[108, et seq.] and [112, p. 22 et seq.]).

## COMPLEX INTEGRAL REPRESENTATION OF S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

A complex integral representation of S-Generalized Gauss Hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c,z)$ was introduced and investigated by Srivastava et al.[111]. It is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c,z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} (-z)^{-\xi} \frac{B(\xi, a-\xi) B_p^{(\alpha,\beta;\tau,\mu)}(b-\xi, c-b)}{B(b,c-b)} d\xi \tag{3.1.9}$$

Now we give the following representation of $F_p^{(\alpha,\beta;\tau,\gamma)}(a,b;c,z)$ in terms of H-function of two variables:

$$\begin{aligned} F_p^{(\alpha,\beta;\tau,\gamma)}(a,b;c,z) &= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} (-z)^{-\xi} \frac{B(\xi, a-\xi) B_p^{(\alpha,\beta;\tau,\mu)}(b-\xi, c-b)}{B(b,c-b)} d\xi \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(a)B(b,c-b)} H^{0,1:1,2;1,1}_{1,1:3,1;1,1} \left[ \begin{array}{c|c} p^{-1} & (1-b;\tau,1) : A^* \\ & \\ -z & (1-c;\tau+\mu,1) : B^* \end{array} \right] \end{aligned} \tag{3.1.10}$$

where

$A^* = (1,1), (1-c+b,\mu), (\beta,1); (1-a,1) \qquad B^* = (\alpha,1); (0,1)$

## GENERALIZED INCOMPLETE HYPERGEOMETRIC FUNCTION

The generalized incomplete hypergeometric function introduced and defined by Srivastava et al. [116, p.675, Eq.(4.1)] is represented in the following manner:

$$\begin{aligned} {}_{\mathfrak{p}}\gamma_{\mathfrak{q}} \left[ \begin{array}{c} (E_{\mathfrak{p}};\sigma); \\ \\ F_{\mathfrak{q}}; \end{array} z \right] &= {}_{\mathfrak{p}}\gamma_{\mathfrak{q}} \left[ \begin{array}{c} (e_1;\sigma), e_2, ..., e_{\mathfrak{p}}; \\ \\ f_1, ..., f_{\mathfrak{q}}; \end{array} z \right] \\ &:= \sum_{n=0}^{\infty} \frac{(E_{\mathfrak{p}};\sigma)_n}{(F_{\mathfrak{q}};0)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(e_1;\sigma)_n, (e_2)_n, ..., (e_{\mathfrak{p}})_n}{(f_1)_n, (f_2)_n, ..., (f_{\mathfrak{q}})_n} \frac{z^n}{n!} \end{aligned} \tag{3.1.11}$$

where the incomplete Pochhammer symbols are defined as follows:

$$(e;\sigma)_n = \frac{\gamma(e+n,\sigma)}{\Gamma(e)} \qquad (a, n \in \mathbb{C}; x \geq 0) \tag{3.1.12}$$

and the familiar incomplete gamma function $\gamma(s,x)$ is

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt \qquad (\Re(s) > 0; x \geq 0) \tag{3.1.13}$$

provided that the defining of infinite series in each case is absolutely convergent.

## FRACTIONAL INTEGRAL OPERATORS

We study two unified fractional integral operators involving the Appell Polynomial, Fox H-function and S-Generalized Gauss's Hypergeometric Function having general arguments defined and represented in the following manner:

$$I_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} = x^{-\nu-\lambda-1} \int_0^x t^{\nu}(x-t)^{\lambda} A_n \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1} \right]$$

$$H_{P,Q}^{M,N} \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \middle| \begin{matrix} (g_j, G_j)_{1,P} \\ (h_j, H_j)_{1,Q} \end{matrix} \right] F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1-\frac{t}{x}\right)^{\lambda_3} \right] f(t) dt \tag{3.1.14}$$

where $f(t) \in \Lambda$ and $\Lambda$ denotes the class of functions for which

$$f(t) := \begin{cases} O\{|t|^{\zeta}\}; & \max\{|t|\} \to 0 \\ O\{|t|^{w_1} e^{-w_2|t|}\}; & \min\{|t|\} \to \infty \end{cases} \tag{3.1.15}$$

provided that

$$\left. \begin{matrix} \min\limits_{1 \leq j \leq M} \Re \left( \nu + \nu_2 \frac{h_j}{H_j} + \zeta + 1, \lambda + \lambda_2 \frac{h_j}{H_j} + 1 \right) > 0 \\ \min\{\nu_1, \nu_3, \lambda_1, \lambda_3\} \geq 0 \end{matrix} \right\} \tag{3.1.16}$$

$$J_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} = x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda}A_n\left[z_1\left(\frac{x}{t}\right)^{\nu_1}\left(1-\frac{x}{t}\right)^{\lambda_1}\right]$$

$$H_{P,Q}^{M,N}\left[z_2\left(\frac{x}{t}\right)^{\nu_2}\left(1-\frac{x}{t}\right)^{\lambda_2}\left|\begin{matrix}(g_j,G_j)_{1,P}\\ \\ (h_j,H_j)_{1,Q}\end{matrix}\right.\right]F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{x}{t}\right)^{\nu_3}\left(1-\frac{x}{t}\right)^{\lambda_3}\right]f(t)dt \tag{3.1.17}$$

provided that

$$\left.\begin{matrix}\Re(w_2)>0 \quad \text{or} \quad \Re(w_2)=0 \quad \text{and} \quad \min\limits_{1\le j\le M}\Re\left(\nu-w_1+\nu_2\dfrac{h_j}{H_j}\right)>0\\ \min\limits_{1\le j\le M}\Re\left(\lambda+\lambda_2\dfrac{h_j}{H_j}+1\right)>0, \min\{\nu_1,\nu_3,\lambda_1,\lambda_3\}\ge 0\end{matrix}\right\} \tag{3.1.18}$$

## 3.2 IMAGES

In this section, we will find the images of some useful functions under the our operators define by (3.1.14) and (3.1.17). We have:

**(i)**

$$I_x^{\nu,\lambda}\left[A_n, H, F_p; t^{\rho}H_{C,D;C_1,D_1;\ldots;C_r,D_r}^{0,B;A_1,B_1;\ldots;A_r,B_r}\left[\begin{matrix}z^{(1)}t^{\nu^{(1)}}(x-t)^{\lambda^{(1)}}\\ .\\ .\\ .\\ z^{(r)}t^{\nu^{(r)}}(x-t)^{\lambda^{(r)}}\end{matrix}\right]\right] = \frac{\Gamma(\beta)x^{\rho}}{\Gamma(a)\Gamma(\alpha)B(b,c-b)}$$

$$\sum_{k=0}^{n}\frac{a_{n-k}z_1^k}{k!}H_{C+3,D+2;C_1,D_1;\ldots;C_r,D_r;3,1;P,Q;1,1}^{0,B+3;A_1,B_1;\ldots;A_r,B_r;1,2;M,N;1,1}\left[\begin{matrix}z^{(1)}x^{\nu^{(1)}+\lambda^{(1)}}\\ .\\ .\\ z^{(r)}x^{\nu^{(r)}+\lambda^{(r)}}\\ p^{-1}\\ z_2\\ z_3\end{matrix}\left|\begin{matrix}A^*:C^*\\ \\ \\ \\ \\ \\ B^*:D^*\end{matrix}\right.\right] \tag{3.2.1}$$

where

$$\left.\begin{array}{l} A^* = (-\rho-\nu-\nu_1 k;\nu^{(1)},\cdots,\nu^{(r)},\nu_2,\nu_3,0),(-\lambda-\lambda_1 k;\lambda^{(1)},\cdots,\lambda^{(r)},\lambda_2,\lambda_3,0), \\ (1-b;\underbrace{0,\cdots,0,}_{r+1}\tau,1),(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},0,0,0)_{1,C} \\ B^* = (-1-\rho-(\lambda+\nu)-(\lambda_1+\nu_1)k;(\lambda^{(1)}+\nu^{(1)}),\cdots,(\lambda^{(r)}+\nu^{(r)}),(\lambda_2+\nu_2),(\lambda_3+\nu_3),0), \\ (1-c;\underbrace{0,\cdots,0,}_{r+1}\tau+\mu,1),(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)},0,0,0)_{1,D} \\ C^* = (c_j^{(1)},\gamma_j^{(1)})_{1,C_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r};(1,1),(1-c+b,\mu),(\beta,1);(g_j,G_j)_{1,P};(1-a,1) \\ D^* = (d_j^{(1)},\delta_j^{(1)})_{1,D_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,D_r};(\alpha,1);(h_j,H_j)_{1,Q};(0,1) \end{array}\right\} \tag{3.2.2}$$

provided that conditions given by (3.1.16) are satisfied.

**(ii)**

$$J_x^{\nu,\lambda}\left[A_n,H,F_p;t^\rho H^{0,B;A_1,B_1;\ldots;A_r,B_r}_{C,D;C_1,D_1;\ldots;C_r,D_r}\left[\begin{array}{c} z^{(1)}t^{-\nu^{(1)}}(1-\frac{x}{t})^{\lambda^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z^{(r)}t^{-\nu^{(r)}}(1-\frac{x}{t})^{\lambda^{(r)}} \end{array}\right]\right] = \frac{\Gamma(\beta)x^\rho}{\Gamma(a)\Gamma(\alpha)B(b,c-b)}$$

$$\sum_{k=0}^{n}\frac{a_{n-k}z_1^k}{k!}H^{0,B+3;A_1,B_1;\ldots;A_r,B_r;1,2;M,N;1,1}_{C+3,D+2;C_1,D_1;\ldots;C_r,D_r;3,1;P,Q;1,1}\left[\begin{array}{c|c} z^{(1)}x^{-\nu^{(1)}} & A^{**}:C^* \\ \cdot & \\ \cdot & \\ z^{(r)}x^{-\nu^{(r)}} & \\ p^{-1} & \\ z_2 & \\ z_3 & B^{**}:D^* \end{array}\right] \tag{3.2.3}$$

where $A^{**}$ and $B^{**}$ can be obtained from $A^*$ and $B^*$ defined in (3.2.2) by replacing $\rho$ by $-1-\rho$, and provided that conditions given by (3.1.18) are satisfied.

**(iii)**

$$I_x^{\nu,\lambda}\left[A_n,H,F_p;t^\rho\,{}_{\mathfrak{p}}\gamma_{\mathfrak{q}}\left[\begin{array}{c} (E_{\mathfrak{p}};\sigma); \\ \\ F_{\mathfrak{q}}; \end{array} z_4\left(\frac{t}{x}\right)^{\nu_4}\left(1-\frac{t}{x}\right)^{\lambda_4}\right]\right] =$$

$$\frac{\Gamma(\beta)x^{\rho}}{\Gamma(a)\Gamma(\alpha)B(b,c-b)}\sum_{i=0}^{\infty}\sum_{k=0}^{n}\frac{a_{n-k}}{k!}\frac{(e_1;\sigma)_i,(e_2)_i,...,(e_{\mathfrak{p}})_i}{(f_1)_i,(f_2)_i,...,(f_{\mathfrak{q}})_i i!}z_1^k z_4^i$$

$$H^{0,3:M,N;1,1;1,2}_{3,2:P,Q;1,1;3,1}\left[\begin{array}{c|l} z_2 & E^*;\quad (g_j,G_j)_{1,P};\quad (1-a,1);\quad (1,1),(1-c+b,\mu),(\beta,1) \\ z_3 & \\ p^{-1} & F^*;\quad (h_j,H_j)_{1,Q};\quad (0,1);\quad (\alpha,1) \end{array}\right] \tag{3.2.4}$$

$$\left.\begin{array}{l} E^*=(-\rho-\nu-\nu_1 k-\nu_4 i,\nu_2,\nu_3,0),(-\lambda-\lambda_1 k-\lambda_4 i,\lambda_2,\lambda_3,0),(1-b;0,1,\tau) \\ F^*=(-1-\rho-(\lambda+\nu))-(\lambda_1+\nu_1)k-(\lambda_4+\nu_4)i,(\lambda_2+\nu_2),(\lambda_3+\nu_3),0),(1-c;0,1,\tau+\mu) \end{array}\right\} \tag{3.2.5}$$

provided that the conditions given by (3.1.16) are satisfied.

**(iv)**

$$J_x^{\nu,\lambda}\left[A_n,H,F_p;t^{\rho}{}_{\mathfrak{p}}\gamma_{\mathfrak{q}}\left[\begin{array}{c} (E_{\mathfrak{p}};\sigma); \\ F_{\mathfrak{q}}; \end{array} z_4 t^{-\nu_4}\left(1-\frac{x}{t}\right)^{\lambda_4}\right]\right]=$$

$$\frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b,c-b)}\sum_{i=0}^{\infty}\sum_{k=0}^{n}\frac{a_{n-k}}{k!}\frac{(e_1;\sigma)_i,(e_2)_i,...,(e_{\mathfrak{p}})_i}{(f_1)_i,(f_2)_i,...,(f_{\mathfrak{q}})_i i!}z_1^k z_4^i x^{\rho-\nu_4 i}$$

$$H^{0,3:M,N;1,1;1,2}_{3,2:P,Q;1,1;3,1}\left[\begin{array}{c|l} z_2 & E^{**};\quad (g_j,G_j)_{1,P};\quad (1-a,1);\quad (1,1),(1-c+b,\mu),(\beta,1) \\ z_3 & \\ p^{-1} & F^{**};\quad (h_j,H_j)_{1,Q};\quad (0,1);\quad (\alpha,1) \end{array}\right] \tag{3.2.6}$$

where $E^{**}$ and $F^{**}$ can be obtained from $E^*$ and $F^*$ defined in (3.2.5) by replacing $\rho$ by $-1-\rho$, and provided that conditions given by (3.1.18) are satisfied.

**Proof:**

To prove (3.2.1), first of all express the I-operator involved in its left hand side, in the integral form with the help of equation (3.1.14), we have

$$I_x^{\nu,\lambda}\left[A_n, H, F_p; t^\rho H_{C,D;C_1,D_1;\ldots;C_r,D_r}^{0,B;A_1,B_1;\ldots;A_r,B_r}\left[\begin{array}{c} z^{(1)}t^{\nu^{(1)}}(x-t)^{\lambda^{(1)}} \\ . \\ . \\ . \\ z^{(r)}t^{\nu^{(r)}}(x-t)^{\lambda^{(r)}} \end{array}\right]\right] = x^{-\nu-\lambda-1}$$

$$\int_0^x t^{\nu+\rho}(x-t)^\lambda A_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right] H_{P,Q}^{M,N}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\middle| \begin{array}{c}(g_j, G_j)_{1,P} \\ (h_j, H_j)_{1,Q}\end{array}\right]$$

$$F_p^{(\alpha,\beta;\tau,\mu)}\left[a, b; c; z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right] H_{C,D;C_1,D_1;\ldots;C_r,D_r}^{0,B;A_1,B_1;\ldots;A_r,B_r}\left[\begin{array}{c} z^{(1)}t^{\nu^{(1)}}(x-t)^{\lambda^{(1)}} \\ . \\ . \\ . \\ z^{(r)}t^{\nu^{(r)}}(x-t)^{\lambda^{(r)}} \end{array}\right] dt \tag{3.2.7}$$

Now, we replace the Appell polynomial occurring in the above expression in terms of its series with the help of equation (3.1.1) and change the order of the series and the t-integral. Next, we express the S-Generalized Gauss Hypergeometric Function in terms of H-function of two variable with the help of (3.1.10). Further, we express Fox H-function, H-function of two variable and Multivariable H-function in terms of the Mellin-Barnes type contour integrals with the help of equation (3.1.2) and (3.1.4) respectively. Then changing the order of $\xi_j$- and t-integrals (j= 1,2,3,...,r + 3) (which is permissible under the conditions stated), we have

$$I_x^{\nu,\lambda}\left[A_n, H, F_p; t^\rho H_{C,D;C_1,D_1;\ldots;C_r,D_r}^{0,B;A_1,B_1;\ldots;A_r,B_r}\left[\begin{array}{c} z^{(1)}t^{\nu^{(1)}}(x-t)^{\lambda^{(1)}} \\ . \\ . \\ . \\ z^{(r)}t^{\nu^{(r)}}(x-t)^{\lambda^{(r)}} \end{array}\right]\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(a)B(b,c-b)}$$

$$\sum_{k=0}^{n} \frac{a_{n-k}}{k!} z_1^k x^{\rho-(\nu+\lambda)-(\nu_1+\lambda_1)k-(\nu_2+\lambda_2)\xi_{r+1}-(\nu_3+\lambda_3)\xi_{r+2}-1} \frac{1}{(2\pi\omega)^{r+3}} \int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\cdots\int_{\mathfrak{L}_{r+3}} \Phi(\xi_1,\xi_2,...,\xi_r)$$

$$\prod_{i=1}^{r}\Theta_i(\xi_i)\varphi(\xi_{r+2},\xi_{r+3}) \prod_{j=r+1}^{r+3}\Theta_j(\xi_j) z^{(i)\xi_i} z_2^{\xi_{r+1}} z_3^{\xi_{r+2}} p^{-\xi_{r+3}}$$

$$\int_0^x t^{\rho+\nu+\nu_1 k+\sum_{i=1}^{r}\nu^{(i)}\xi_i+\nu_2\xi_{r+1}+\nu_3\xi_{r+2}} (x-t)^{\lambda+\lambda_1 k+\sum_{i=1}^{r}\lambda^{(i)}\xi_i+\lambda_2\xi_{r+1}+\nu_3\xi_{r+2}} dt\, d\xi_1 d\xi_2,...,d\xi_{r+3}$$

(3.2.8)

Finally, evaluating the t-integral and reinterpreting the result thus obtained in terms of the H-function of r+3 variables, we easily arrive at the desired result (3.2.1) after a little simplification.

**The proof of (3.2.3), (3.2.4) and (3.2.6) can be obtained by proceeding on similar lines to those given above.**

## 3.3 APPLICATIONS

Now we shall establish four interesting integrals with the application of images:

**(I)** In the image (i), if we reduce Multivariable H-function into Appell's Function [109, p.89,Eq.(6.4.6)] and take $\rho = 0$. We obtain the following integral after a little simplification:

$$\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)}{\Gamma(d_1)\Gamma(d_2)} \frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\nu}\left(1-\frac{x}{t}\right)^{\lambda} A_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$

$$H_{P,Q}^{M,N}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ \\(h_j,H_j)_{1,Q}\end{matrix}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right]$$

$$F_3\left[c_1,c_2,c_3,c_4;b_1;z^{(1)}t^{\nu^{(1)}}(x-t)^{\lambda^{(1)}},z^{(2)}t^{\nu^{(2)}}(x-t)^{\lambda^{(2)}}\right]dt \qquad (3.3.1)$$

$$= \frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b,c-b)} \sum_{k=0}^{n} \frac{a_{n-k} z_1^k}{k!} H^{0,3;1,2;1,2;1,2;M,N;1,1}_{3,3;2,1;2,1;3,1;P,Q;1,1} \left[ \begin{array}{c|c} \begin{array}{c} -z^{(1)} x^{\nu^{(1)}+\lambda^{(1)}} \\ -z^{(2)} x^{\nu^{(2)}+\lambda^{(2)}} \\ p^{-1} \\ z_2 \\ z_3 \end{array} & \begin{array}{c} A^* : C^* \\ \\ \\ \\ B^* : D^* \end{array} \end{array} \right] \tag{3.3.2}$$

where

$$\left.\begin{array}{l} A^* = (-\nu - \nu_1 k; \nu^{(1)}, \nu^{(2)}, \nu_2, \nu_3, 0), (-\lambda - \lambda_1 k; \lambda^{(1)}, \lambda^{(2)}, \lambda_2, \lambda_3, 0), (1-b; 0,0,0,\tau,1) \\ B^* = (-1-(\lambda+\nu)-(\lambda_1+\nu_1)k; (\lambda^{(1)}+\nu^{(1)}), (\lambda^{(2)}+\nu^{(2)}), (\lambda_2+\nu_2), (\lambda_3+\nu_3), 0), \\ (1-c; 0,0,0,\tau+\mu,1), (1-b_1; 1,1,0,0,0) \\ C^* = (1-c_1,1), (1-c_2,1); (1-c_3,1), (1-c_4,1); (1,1), (1-c+b,\mu), (\beta,1); \\ (g_j, G_j)_{1,P}; (1-a,1) \\ D^* = (0,1); (0,1); (\alpha,1); (h_j, H_j)_{1,Q}; (0,1) \end{array}\right\}$$

**(II)** On taking in image (ii), if we reduce Multivariable H-function into Multivariate generalized Mittag-Leffler Function [99, p.5,Eq.(2.1)] and take $\rho = 0$. We easily get the following integral after a little simplification:

$$\prod_{j=1}^{r} \Gamma(\gamma_j) \int_x^{\infty} \frac{1}{t} \left(\frac{x}{t}\right)^{\nu} \left(1-\frac{x}{t}\right)^{\lambda} A_n \left[ z_1 \left(\frac{x}{t}\right)^{\nu_1} \left(1-\frac{x}{t}\right)^{\lambda_1} \right]$$
$$H^{M,N}_{P,Q} \left[ z_2 \left(\frac{x}{t}\right)^{\nu_2} \left(1-\frac{x}{t}\right)^{\lambda_2} \middle| \begin{array}{c} (g_j, G_j)_{1,P} \\ (h_j, H_j)_{1,Q} \end{array} \right] F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{x}{t}\right)^{\nu_3} \left(1-\frac{x}{t}\right)^{\lambda_3} \right]$$
$$E^{\gamma_j}_{\beta_j, b_1} \left( -z^{(1)} t^{-\nu^{(1)}} (1-\frac{x}{t})^{\lambda^{(1)}}, ..., -z^{(r)} t^{-\nu^{(r)}} (1-\frac{x}{t})^{\lambda^{(r)}} \right) dt \tag{3.3.3}$$

$$= \frac{\Gamma(\beta)x^{-1}}{\Gamma(a)\Gamma(\alpha)B(b,c-b)} \sum_{k=0}^{n} \frac{a_{n-k}z_1^k}{k!} H^{0,3;1,1;...;1,1;1,2;M,N;1,1}_{3,3;1,1;...;1,1;3,1;P,Q;1,1} \left[ \begin{array}{c|c} z^{(1)}x^{-\nu^{(1)}} & E^*:G^* \\ \cdot & \\ \cdot & \\ z^{(r)}x^{-\nu^{(r)}} & \\ p^{-1} & \\ z_2 & \\ z_3 & F^*:H^* \end{array} \right] \tag{3.3.4}$$

$$\left.\begin{array}{l} E^* = (1-\nu-\nu_1 k;\nu^{(1)},...,\nu^{(r)},\nu_2,\nu_3,0),(-\lambda-\lambda_1 k;\lambda^{(1)},...,\lambda^{(r)},\lambda_2,\lambda_3,0),(1-b;\underbrace{0,\cdots,0}_{r+1},\tau,1) \\ F^* = (-(\lambda+\nu))-(\lambda_1+\nu_1)k;(\lambda^{(1)}+\nu^{(1)}),...,(\lambda^{(r)}+\nu^{(r)}),(\lambda_2+\nu_2),(\lambda_3+\nu_3),0), \\ (1-c;\underbrace{0,...,0}_{r+1},\tau+\mu,1),(1-b_1;\beta_1,...,\beta_r,0,0,0) \\ G^* = (1-c_1,1);...;(1-c_r,1);(1,1),(1-c+b,\mu),(\beta,1);(g_j,G_j)_{1,P};(1-a,1) \\ H^* = (0,1);...;(0,1);(\alpha,1);(h_j,H_j)_{1,Q};(0,1) \end{array}\right\} \tag{3.3.5}$$

**(III)** Further, in the image (i), if we reduce Multivariable H-function into generalization of the modified Bessel function of the third kind [30, p.155,Eq.(2.6)] and take $\rho = 0$. We arrive at the following integral after a little simplification:

$$\frac{1}{x}\int_0^x \left(\frac{t}{x}\right)^{\nu}\left(1-\frac{t}{x}\right)^{\lambda} A_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right] H^{M,N}_{P,Q}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2} \middle| \begin{array}{c} (g_j,G_j)_{1,P} \\ (h_j,H_j)_{1,Q} \end{array}\right]$$
$$F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right]\lambda^{(\eta)}_{\gamma,\sigma}\left[z^{(1)}t^{\nu^{(1)}}(x-t)^{\lambda^{(1)}}\right]dt \tag{3.3.6}$$

$$= \frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b,c-b)} \sum_{k=0}^{n} \frac{a_{n-k}z_1^k}{k!} H^{0,3;2,0;1,2;M,N;1,1}_{3,2;1,2;3,1;P,Q;1,1} \left[ \begin{array}{c|c} & M^*:P^* \\ z^{(1)}x^{\nu^{(1)}+\lambda^{(1)}} & \\ p^{-1} & \\ z_2 & \\ z_3 & \\ & N^*:Q^* \end{array} \right] \tag{3.3.7}$$

where

$$\left.\begin{array}{l} M^{*}=(-\nu-\nu_{1}k;\nu^{(1)},\nu_{2},\nu_{3},0),(-\lambda-\lambda_{1}k;\lambda^{(1)},\lambda_{2},\lambda_{3},0),(1-b;0,0,\tau,1) \\ N^{*}=(-1-(\lambda+\nu)-(\lambda_{1}+\nu_{1})k;(\lambda^{(1)}+\nu^{(1)}),(\lambda_{2}+\nu_{2}),(\lambda_{3}+\nu_{3}),0), \\ (1-c;0,0,\tau+\mu,1) \\ P^{*}=(1-(\sigma+1)/\eta;1/\eta);(1,1),(1-c+b,\mu),(\beta,1);(g_{j},G_{j})_{1,P};(1-a,1) \\ Q^{*}=(0,1),(-\gamma-\sigma/\eta,1/\eta);(\alpha,1);(h_{j},H_{j})_{1,Q};(0,1) \end{array}\right\} \tag{3.3.8}$$

**(IV)** Again if we take in image (ii) $\rho=0$ and reduce Multivariable H-function into generalized hypergeometric Function [52, p.xi,Eq.(A.18)]. We get the following integral after a little simplification:

$$\frac{\prod_{j=1}^{p'}\Gamma(1-a_{j})}{\prod_{j=1}^{q'}\Gamma(1-b_{j})}\int_{x}^{\infty}\frac{1}{t}\left(\frac{x}{t}\right)^{\nu}\left(1-\frac{x}{t}\right)^{\lambda}A_{n}\left[z_{1}\left(\frac{x}{t}\right)^{\nu_{1}}\left(1-\frac{x}{t}\right)^{\lambda_{1}}\right]$$
$$H_{P,Q}^{M,N}\left[z_{2}\left(\frac{x}{t}\right)^{\nu_{2}}\left(1-\frac{x}{t}\right)^{\lambda_{2}}\left|\begin{array}{c}(g_{j},G_{j})_{1,P}\\ \\ (h_{j},H_{j})_{1,Q}\end{array}\right.\right]F_{p}^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_{3}\left(\frac{x}{t}\right)^{\nu_{3}}\left(1-\frac{x}{t}\right)^{\lambda_{3}}\right]$$
$${}_{p'}F_{q'}\left[\begin{array}{c}(1-a_{p'});\\(1-b_{p});\end{array}\;-\left(z^{(1)}t^{-\nu^{(1)}}\left(1-\frac{x}{t}\right)^{\lambda^{(1)}}+...+z^{(r)}t^{-\nu^{(r)}}\left(1-\frac{x}{t}\right)^{\lambda^{(r)}}\right)\right]dt \tag{3.3.9}$$

$$=\frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b,c-b)}\sum_{k=0}^{n}\frac{a_{n-k}z_{1}^{k}}{k!}H_{p'+3,q'+2;0,1;...;0,1;3,1;P,Q;1,1}^{0,p'+3;1,0;...;1,0;1,2;M,N;1,1}\left[\begin{array}{c|c} z^{(1)}x^{-\nu^{(1)}} & S^{*}:U^{*} \\ . & \\ . & \\ z^{(r)}x^{-\nu^{(r)}} & \\ p^{-1} & \\ z_{2} & \\ z_{3} & T^{*}:V^{*} \end{array}\right] \tag{3.3.10}$$

$$\left.\begin{aligned}
&S^* = (1-\nu-\nu_1 k;\nu^{(1)},...,\nu^{(r)},\nu_2,\nu_3,0),(-\lambda-\lambda_1 k;\lambda^{(1)},...,\lambda^{(r)},\lambda_2,\lambda_3,0),\\
&(1-b;\underbrace{0,...,0}_{r+1},\tau,1),(1-a_j;\underbrace{1,...,1}_{r},0,0,0)_{1,p'}\\
&T^* = (-(\lambda+\nu)-(\lambda_1+\nu_1)k;(\lambda^{(1)}+\nu^{(1)}),...,(\lambda^{(r)}+\nu^{(r)}),(\lambda_2+\nu_2),(\lambda_3+\nu_3),0),\\
&(1-c;\underbrace{0,...,0}_{r+1},\tau+\mu,1),(1-b_j;\underbrace{1,...,1}_{r},0,0,0)_{1,q'}\\
&U^* = \underbrace{-;...;-}_{r};(1,1),(1-c+b,\mu),(\beta,1);(g_j,G_j)_{1,P};(1-a,1)\\
&V^* = \underbrace{(0,1);...;(0,1)}_{r};(\alpha,1);(h_j,H_j)_{1,Q};(0,1)
\end{aligned}\right\} \tag{3.3.11}$$

# 3.4 SPECIAL CASES OF FRACTIONAL INTEGRAL OPERATORS

(i) In (3.1.14) and (3.1.17) , if we reduce Appell polynomial to Laguerre polynomial [121, p.101, Eq.(5.1.6)] and Fox H-function reduced to Lorenzo Hartley G-function [34, p.64,Eq.(2.3)], we obtain the following integral:

$$I_x^{\nu,\lambda}\{L_n,G_{q,\sigma,r},F_p;f(t)\} = x^{-\nu-\lambda-1}\int_0^x t^\nu(x-t)^\lambda L_n^{(\rho)}\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$
$$G_{q,\sigma,r}\left[z_2,\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\right]F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right]f(t)dt \tag{3.4.1}$$

and

$$J_x^{\nu,\lambda}\{L_n,G_{q,\sigma,r},F_p;f(t)\} = x^{\nu}\int_x^\infty t^{-\nu-\lambda-1}(t-x)^\lambda L_n^{(\rho)}\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$
$$G_{q,\sigma,r}\left[z_2,\left(\frac{x}{t}\right)^{\nu_2}\left(1-\frac{x}{t}\right)^{\lambda_2}\right]F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{x}{t}\right)^{\nu_3}\left(1-\frac{x}{t}\right)^{\lambda_3}\right]f(t)dt \tag{3.4.2}$$

(ii) In (3.1.14) and (3.1.17) , if we reduce Appell polynomial to Bessel polynomial [60, p.108, Eq.(34)] and Fox H-function reduced to Generalized Mittag Leffler function [73, p.25,Eq.(1.137)], we obtain the following integral:

$$I_x^{\nu,\lambda}\{y_n, E^{\eta}_{\gamma,\delta}, F_p; f(t)\} = x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} y_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}, \rho, \sigma\right]$$
$$E^{\eta}_{\gamma,\delta}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right] f(t)dt \tag{3.4.3}$$

and

$$J_x^{\nu,\lambda}\{y_n, E^{\eta}_{\gamma,\delta}, F_p; f(t)\} = x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda} y_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}, \rho, \sigma\right]$$
$$E^{\eta}_{\gamma,\delta}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{x}{t}\right)^{\nu_3}\left(1-\frac{x}{t}\right)^{\lambda_3}\right] f(t)dt \tag{3.4.4}$$

(iii) In (3.1.14) and (3.1.17) , if we reduce Appell polynomial to Cesaro polynomial [112, p.449, Eq.(20)] and Fox H-function reduced to Bessel Maitland function [73, p.25,Eq.(1.139)], we obtain the following integral

$$I_x^{\nu,\lambda}\{g_n^{\rho}, J^{\eta}_{\gamma,\delta}, F_p; f(t)\} = x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} g_n^{(\rho)}\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$
$$J^{\eta}_{\gamma,\delta}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{t}{x}\right)^{\nu_3}\left(1-\frac{t}{x}\right)^{\lambda_3}\right] f(t)dt \tag{3.4.5}$$

and

$$J_x^{\nu,\lambda}\{g_n^{\rho}, J^{\eta}_{\gamma,\delta}, F_p; f(t)\} = x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda} g_n^{(\rho)}\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$
$$J^{\eta}_{\gamma,\delta}\left[z_2\left(\frac{t}{x}\right)^{\nu_2}\left(1-\frac{t}{x}\right)^{\lambda_2}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{x}{t}\right)^{\nu_3}\left(1-\frac{x}{t}\right)^{\lambda_3}\right] f(t)dt \tag{3.4.6}$$

If we reduce Appell polynomial $A_n(z)$ and Fox's H-function to unity, S-generalized hypergeometric function into gauss hypergeometric function and $\lambda_3 = 1, \nu_3 = 0$ in our fractional integral operators defined by (3.1.14) and (3.1.17), we easily arrive at the results which are same in essence as those obtained by Saxena and Kumbhat [98].

If we reduce Appell polynomial $A_n(z)$ and Fox's H-function to unity, S-generalized hypergeometric function into gauss hypergeometric function and $\lambda_3 = 1, \nu_3 = \nu = 0$ in our fractional integral operators defined by (3.1.14) and (3.1.17), we easily arrive at the results which are same in essence as those obtained by Saigo [96].

## 3.5 MELLIN TRANSFORM, INVERSION FORMULAS AND MELLIN CONVOLUTIONS

The Mellin transform of a function f(t) is defined as usual by the following equation:

$$\mathfrak{M}\{f(t);\mathfrak{s}\} = \int_0^\infty t^{\mathfrak{s}-1} f(t)dt \tag{3.5.1}$$

provided that the integral on the right hand side exists.

Now we obtain the following results which give the Mellin transforms of the fractional integral operators defined by (3.1.14) and (3.1.17), their corresponding inversion formulae and convolutions.

**Theorem 3.5.1.** *If* $\mathfrak{M}\{[f(t)];\mathfrak{s}\}$, $\mathfrak{M}\{I_x^{\nu,\lambda}[A_n, H, F_p; f(t)];\mathfrak{s}\}$ *exist,* $\Re(1+\lambda) > 0$, $\Re(1+\nu-s) > 0$ *and the conditions of the existence of the* *operator* $I_x^{\nu,\lambda}[A_n, H, F_p; f(t)]$ *are satisfied then*

$$\mathfrak{M}\{I_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} = \mathfrak{M}\{f(t); \mathfrak{s}\}G(\mathfrak{s}) \tag{3.5.2}$$

**Theorem 3.5.2.** *If* $\mathfrak{M}\{[f(t)]; \mathfrak{s}\}$, $\mathfrak{M}\{J_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\}$ *exist,* $\Re(1+\lambda) > 0$, $\Re(\nu + s) > 0$ *and the conditions of the existence of the operator* $J_x^{\nu,\lambda}[A_n, H, F_p; f(t)]$ *are satisfied then*

$$\mathfrak{M}\{J_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} = \mathfrak{M}\{f(t); \mathfrak{s}\}G((1-\mathfrak{s})) \tag{3.5.3}$$

*where*

$$G(\mathfrak{s}) = \frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b, c-b)} \sum_{k=0}^{n} \frac{a_{n-k} z_1^k}{k!}$$

$$H^{0,3:M,N;1,1;1,2}_{3,2:P,Q;1,1;3,1}\left[\begin{array}{c|llll} z_2 & G^*; & (g_j, G_j)_{1,P}; & (1-a,1); & (1,1),(1-c+b,\mu),(\beta,1) \\ z_3 & & & & \\ p^{-1} & H^*; & (h_j, H_j)_{1,Q}; & (0,1); & (\alpha,1) \end{array}\right]$$

$$\left.\begin{array}{l} G^* = (-\lambda - \lambda_1 k; \lambda_2, \lambda_3, 0), (\mathfrak{s} - \nu - \nu_1 k; \nu_2, \nu_3, 0), (1-b; 0, 1, \tau) \\ H^* = (\mathfrak{s} - (\nu+\lambda) - (\nu_1 + \lambda_1)k - 1; (\nu_2 + \lambda_2), (\nu_3 + \lambda_3), 0), (1-c; 0, 1, \tau+\mu) \end{array}\right\} \tag{3.5.4}$$

provided that conditions given by (3.1.16)and (3.1.18)are satisfied.

**Proof:** To prove Theorem 3.5.1, first we write the Mellin transform of the I-operator defined by (3.1.14) with the help of (3.5.1)

$$\mathfrak{M}\{I_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} = \int_0^\infty x^{\mathfrak{s}-1} I_x^{\nu,\lambda}[f(t)]dx$$

$$= \int_0^\infty x^{\mathfrak{s}-1} x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda A_n \left[z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$

$$H^{M,N}_{P,Q}\left[z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \middle| \begin{array}{c} (g_j, G_j)_{1,P} \\ (h_j, H_j)_{1,Q} \end{array}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a, b; c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1-\frac{t}{x}\right)^{\lambda_3}\right] f(t)dtdx$$

Next, we change the order of x- and t-integrals. Now, we replace the Fox H-function and S-generalized hypergeometric function occuring in it in terms of Mellin Barnes Contour integral with the help of equation (3.1.2) and (3.1.10) respectively and Appell polynomial in terms of series with the help of equation (3.1.1) and interchange the order of summation and integration in the result thus obtained. Next we evaluate the t-integral and interpret the result in terms of multivariable H-function and finally with the help of (3.5.1), we easily arrive at the desired result (3.5.2) after a little simplification.

**The proof of Theorem 3.5.2 can be developed on similar lines**.

## INVERSION FORMULAE

On making use of the well-known inversion theorem for the Mellin Transform, (3.5.1), we easily arrive at the following inversion formulae for our fractional integral operators defined by (3.1.14) and (3.1.17):

**Formula 1**

$$\frac{f(t-0)+f(t+0)}{2} = \frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{L}} \frac{t^{-\mathfrak{s}}}{G(\mathfrak{s})}\mathfrak{M}\{I_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\}d\mathfrak{s} \tag{3.5.5}$$

**Formula 2**

$$\frac{f(t-0)+f(t+0)}{2} = \frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{L}} \frac{t^{-\mathfrak{s}}}{G(-1-\mathfrak{s})}\mathfrak{M}\{J_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\}d\mathfrak{s} \tag{3.5.6}$$

where $G(\mathfrak{s})$ is given by (3.5.4)

## MELLIN CONVOLUTION

The Mellin convolution of two functions f(t) and g(t) will be defined by

$$(f*g)(x) = (g*f)(x) = \int_0^\infty t^{-1} g\left(\frac{x}{t}\right) f(t)dt \tag{3.5.7}$$

provided that the integral involved in (3.5.7) exists.

The fractional integral operators defined by (3.1.14) and (3.1.17) can readily be expressed as Mellin convolutions. we have the following interesting results involving the Mellin convolutions :

**Result 1**

$$I_x^{\nu,\lambda}[A_n, H, F_p; f(t)] = (g*f)(x) \tag{3.5.8}$$

where

$$g(x) = x^{-\nu-\lambda-1}(x-1)^{\lambda} A_n\left[z_1 x^{-\nu_1-\lambda_1}(x-1)^{\lambda_1}\right] H_{P,Q}^{M,N}\left[z_2 x^{-\nu_2-\lambda_2}(x-1)^{\lambda_2} \middle| \begin{matrix} (g_j, G_j)_{1,P} \\ (h_j, H_j)_{1,Q} \end{matrix}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a, b; c; z_3 x^{-\nu_3-\lambda_3}(x-1)^{\lambda_3}\right] U(x-1) \tag{3.5.9}$$

**Result 2**

$$J_x^{\nu,\lambda}[A_n, H, F_p; f(t)] = (h*f)(x) \tag{3.5.10}$$

where

$$h(x) = x^{\nu}(1-x)^{\lambda} A_n\left[z_1 x^{\nu_1}(1-x)^{\lambda_1}\right] H_{P,Q}^{M,N}\left[z_2 x^{\nu_2}(1-x)^{\lambda_2} \middle| \begin{matrix} (g_j, G_j)_{1,P} \\ (h_j, H_j)_{1,Q} \end{matrix}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a, b; c; z_3 x^{\nu_3}(1-x)^{\lambda_3}\right] U(1-x) \tag{3.5.11}$$

U(x) being the Heaviside's unit function.

**Proof:** To prove Result 1 we first write the $I_x^{\nu,\lambda}$-operator defined by (3.1.14) in the following form using the definition of Heaviside's unit function:

$$I_x^{\nu,\lambda}[f(t)] = \int_0^\infty t^{-1}\left(\frac{x}{t}\right)^{-\nu-\lambda-1}\left(\frac{x}{t}-1\right)^{\lambda} H_{P,Q}^{M,N}\left[z_2\left(\frac{x}{t}\right)^{-\nu_2-\lambda_2}\left(\frac{x}{t}-1\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ (h_j,H_j)_{1,Q}\end{matrix}\right]$$
$$A_n\left[z_1\left(\frac{x}{t}\right)^{-\nu_1-\lambda_1}\left(\frac{x}{t}-1\right)^{\lambda_1}\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(\frac{x}{t}\right)^{-\nu_3-\lambda_3}\left(\frac{x}{t}-1\right)^{\lambda_3}\right] U\left(\frac{x}{t}-1\right) f(t)dt \tag{3.5.12}$$

Now making use of the equation (3.5.9) and the definition of the Mellin convolution given by (3.5.7) in the above equation, we easily arrive at the required Result 1 after a little simplification.

**The proof of Result 2 can be developed on similar lines.**

## 3.6 ANALOGUS OF PARSEVAL GOLDSTEIN THEOREM

If

$$\phi_1(x) = I_x^{\nu,\lambda}[A_n, H, F_p; f(t)] \tag{3.6.1}$$

and

$$\phi_2(x) = J_x^{\nu,\lambda}[A_n, H, F_p; f(t)] \tag{3.6.2}$$

then

$$\int_0^\infty \phi_1(x) f_2(x) dx = \int_0^\infty \phi_2(x) f_1(x) dx \tag{3.6.3}$$

provided that the integral involved in (3.6.1), (3.6.2) and (3.6.3) exists.

**Proof:** To prove the above theorem, we substitute the value of $\phi_1(x)$ from (3.6.1) in the left hand side of (3.6.3) and expressing the I-operator in its integral form by using (3.1.14). Now interchange the order of x and t-integrals (which is permissible under given conditions) and interpret the expression thus obtained in term of J-operator with the help of (3.1.17), we arrive at the desired result by (3.6.2) after a little simplification.

# 4

# COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS

The main findings of this chapter have been published as detailed below:

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# 4. COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS

In this chapter, we first derive three new and interesting expressions for the composition of the two fractional integral operators, which are slight variants of the operators defined in Chapter 3. The operators of our study are quite general in nature and may be considered as extensions of a number of simpler fractional integral operators studied from time to time by several authors. By suitably specializing the coefficients and the parameters in these functions we can get a large number of (new and known) interesting expressions for the composition of fractional integral operators involving simpler special functions. Finally, we obtain two interesting finite double integral formulae as an application of our first composition formula known results which follow as special case of our findings have also been mentioned.

## 4.1 INTRODUCTION

In line with chapter 3, $\Lambda$ will denote the class of function f(t) for which

$$f(t) = \begin{cases} O\{|t|^{\varsigma}\}; & \max\{|t|\} \to 0 \\ \\ O\{|t|^{w_1} e^{-w_2|t|}\}; & \min\{|t|\} \to \infty \end{cases} \tag{4.1.1}$$

For the sake of completeness, we would like to give in this chapter also, the definitions, notations and conditions of existence of operators of our study. The unified fractional integral operators studied in the present chapter will be defined and represented in the following manner:

$$I_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} = x^{-\nu-\lambda-1}\int_0^x t^{\nu}(x-t)^{\lambda} A_n\left[z_1\left(\frac{t}{x}\right)^{\nu_1}\left(1-\frac{t}{x}\right)^{\lambda_1}\right]$$
$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{t}{x}\right)^{\lambda_2}\left|\begin{matrix}(g_j,G_j)_{1,P}\\ \\(h_j,H_j)_{1,Q}\end{matrix}\right.\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{t}{x}\right)^{\lambda_3}\right] f(t)dt \tag{4.1.2}$$

where $f(t)\in\Lambda$ and

$$\min_{1\le j\le M}\Re\left(\nu+\zeta+1, \lambda+\lambda_2\frac{h_j}{H_j}+1\right)>0 \quad \text{and} \quad \min\{\nu_1,\lambda_1,\lambda_3\}\ge 0 \tag{4.1.3}$$

Where $A_n, H_{P,Q}^{M,N}[z]$ and $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ occurring in (4.1.2) stands for the Appell Polynomial, Fox H-function and S-generalized gauss hypergeometric function defined by (3.1.1), (3.1.2) and (3.1.6) respectively.

$$J_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} = x^{\nu}\int_x^{\infty} t^{-\nu-\lambda-1}(t-x)^{\lambda} A_n\left[z_1\left(\frac{x}{t}\right)^{\nu_1}\left(1-\frac{x}{t}\right)^{\lambda_1}\right]$$
$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{x}{t}\right)^{\lambda_2}\left|\begin{matrix}(g_j,G_j)_{1,P}\\ \\(h_j,H_j)_{1,Q}\end{matrix}\right.\right] F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{x}{t}\right)^{\lambda_3}\right] f(t)dt \tag{4.1.4}$$

provided that

$$\left.\begin{matrix}\Re(w_2)>0 \quad \text{or} \quad \Re(w_2)=0 \quad \text{and} \quad \Re(\nu-w_1)>0\\ \min\limits_{1\le j\le M}\Re\left(\lambda+\lambda_2\frac{h_j}{H_j}+1\right)>0, \min\{\nu_1,\lambda_1,\lambda_3\}\ge 0\end{matrix}\right\} \tag{4.1.5}$$

# 4.2 COMPOSITION FORMULAE FOR THE FRACTIONAL INTEGRAL OPERATORS

**RESULT 1**

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'}, H', F'_{p'}; I_{x_1}^{\nu,\lambda}\left[A_n, H, F_p; f(t)\right]\right] = \frac{1}{x_2}\int_0^{x_2} G\left(\frac{t}{x_2}\right) f(t)dt \tag{4.2.1}$$

where

$$G(X) = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(a)\Gamma(a')\Gamma(\alpha)\Gamma(\alpha')B(b, c-b)B(b', c'-b')}$$

$$\sum_{k=0}^{n}\sum_{k'=0}^{n'} \frac{a_{n-k}a_{n'-k'}}{k!k'!} z_1^k z_1'^{k'} X^{\nu+\nu_1 k}(1-X)^{\lambda+\lambda_1 k+\lambda'+\lambda_1' k'+1}$$

$$H^{0,5:1,2;1,2;M',N';1,1;M,N;1,1;1,0}_{5,4:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1}\left[\begin{array}{c|c} p'^{-1} & A^*:\ \ C^* \\ p^{-1} & \\ z_2'(1-X) & \\ -z_3'(1-X) & \\ z_2(1-X) & \\ -z_3(1-X) & \\ -(1-X) & B^*:\ \ D^* \end{array}\right] \tag{4.2.2}$$

where

$$\left.\begin{array}{l} A^* = (-\lambda-\lambda_1 k-\nu-\nu_1 k+\nu'+\nu_1'k';0,0,0,0,\lambda_2,\lambda_3,1),(-\lambda'-\lambda_1'k';0,0,\lambda_2',\lambda_3',0,0,1), \\ (-\lambda-\lambda_1 k;0,0,0,0,\lambda_2,\lambda_3,0),(1-b';\tau',0,0,1,0,0,0),(1-b;0,\tau,0,0,0,1,0) \\ B^* = (-\lambda-\lambda_1 k-\nu-\nu_1 k+\nu'+\nu_1'k';0,0,0,0,\lambda_2,\lambda_3,0),(1-c;0,(\tau+\mu),0,0,0,1,0), \\ (-1-\lambda-\lambda_1 k-\lambda'-\lambda_1'k';0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1),(1-c';(\tau'+\mu'),0,0,1,0,0,0) \\ C^* = (1,1),(1-c'+b',\mu'),(\beta',1);(1,1),(1-c+b,\mu),(\beta,1);(g_j',G_j')_{1,P'}; \\ (1-a',1);(g_j,G_j)_{1,P};(1-a,1);- \\ D^* = (\alpha',1);(\alpha,1);(h_j',H_j')_{1,Q'};(0,1);(h_j,H_j)_{1,Q};(0,1);(0,1) \end{array}\right\} \tag{4.2.3}$$

and following conditions are satisfied

$$\left.\begin{array}{l} f(t)\in\Lambda \\ \Re(\nu'+\nu+\zeta) > -2, \min\limits_{1\le j\le M}\Re\left(\lambda'+\lambda_2'\frac{h_j'}{H_j'}+\lambda+\lambda_2\frac{h_j}{H_j}\right) > -2 \\ \min\{\nu_1,\nu_1',\lambda_1,\lambda_1',\lambda_3,\lambda_3'\}\ge 0 \end{array}\right\} \tag{4.2.4}$$

**RESULT 2**

$$J_{x_2}^{\nu,\lambda}\left[A_n, H, F_p; J_{x_1}^{\nu',\lambda'}\left[A_{n'}, H', F_{p'}; f(t)\right]\right] = \int_{x_2}^{\infty} \frac{1}{t} G\left(\frac{x_2}{t}\right) f(t)dt \tag{4.2.5}$$

provided that

$$\left.\begin{array}{r} \Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \Re\left(\nu + \nu' - w_1\right) > 0; \\ \min\limits_{1\le j\le M} \Re\left(\lambda + \lambda_2 \dfrac{h_j}{H_j} + \lambda' + \lambda_2' \dfrac{h_j'}{H_j'}\right) > -2, \min\{\nu_1, \lambda_1, \lambda_3, \nu_1', \lambda_1', \lambda_3'\} \ge 0 \end{array}\right\} \tag{4.2.6}$$

and G(X) is given by (4.2.2), $f(t) \in \Lambda$, the composite operator defined by the LHS of (4.2.5) exists.

**RESULT 3**

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'}, H', F_{p'}'; J_{x_1}^{\nu,\lambda}\left[A_n, H, F_p; f(t)\right]\right] = \frac{1}{x_2}\int_0^{x_2} K\left(\frac{t}{x_2}\right) f(t)dt + \int_{x_2}^{\infty} \frac{1}{t} K^*\left(\frac{x_2}{t}\right) f(t)dt \tag{4.2.7}$$

where

$$K(T) = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(a)\Gamma(a')\Gamma(\alpha)\Gamma(\alpha')B(b, c-b)B(b', c'-b')}$$
$$\sum_{k=0}^{n}\sum_{k'=0}^{n'} \frac{a_{n-k}a_{n'-k'}\Gamma(\nu+\nu'+\nu_1 k+\nu_1' k'+1)}{k!k'!} z_1^k z_1^{,k'} T^{\nu'+\nu_1' k'}(1-T)^{\lambda+\lambda_1 k+\lambda'+\lambda_1' k'+1}$$

$$H^{0,4:1,2;1,2;M',N';1,1;M,N;1,1;1,0}_{4,4:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1}\left[\begin{array}{c|cc} p'^{-1} & A^* : & C^* \\ p^{-1} & & \\ z_2'(1-T)^{\lambda_2'} & & \\ -z_3'(1-T)^{\lambda_3'} & & \\ z_2(1-T)^{\lambda_2} & & \\ -z_3(1-T)^{\lambda_3} & & \\ -T & B^* : & D^* \end{array}\right] \tag{4.2.8}$$

where

$$\left.\begin{array}{l}
A^* = (-1-\nu-\nu'-\nu_1 k-\nu_1' k'-\lambda'-\lambda_1' k'-\lambda-\lambda_1 k; 0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1),\\
(-\lambda-\lambda_1 k; 0,0,0,0,\lambda_2,\lambda_3,1), (1-b'; \tau', 0,0,1,0,0,0), (1-b; 0,\tau,0,0,0,1,0)\\
B^* = (-1-\lambda-\lambda_1 k-\lambda'-\lambda_1' k'-\nu-\nu'-\nu_1 k-\nu_1' k'; 0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,0),\\
(-1-\lambda-\lambda_1 k-\nu-\nu'-\nu_1 k-\nu_1' k'; 0,0,0,0,\lambda_2,\lambda_3,1)(1-c'; (\tau'+\mu'), 0,,0,1,0,0,0),\\
(1-c; 0, (\tau+\mu), 0,0,0,1,0)\\
C^* = (1,1), (1-c'+b',\mu'), (\beta',1); (1,1), (1-c+b,\mu), (\beta,1); (g_j', G_j')_{1,P'};\\
(1-a',1); (g_j,G_j)_{1,P}; (1-a,1); -\\
D^* = (\alpha',1); (\alpha,1); (h_j', H_j')_{1,Q'}; (0,1); (h_j,H_j)_{1,Q}; (0,1); (0,1)
\end{array}\right\} \tag{4.2.9}$$

and $K^*(T)$ can be obtained from K(T) by interchanging the parameters with dashes with those without dashes and following conditions are satisfied

$$\left.\begin{array}{l}
f(t) \in \Lambda\\
\Re(\nu'+\nu+\zeta) > -2, \min_{1\le j\le M} \Re\left(\lambda'+\lambda_2'\frac{h_j'}{H_j'}+\lambda+\lambda_2\frac{h_j}{H_j}\right) > -2\\
\Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \Re(\nu-w_1) > 0
\end{array}\right\} \tag{4.2.10}$$

**Proof**

To prove **Result 1**, we first express both the I-operators involved in its left hand side, in the integral form with the help of equation (4.1.2). Next we interchange the order of t-and $x_1$-integrals (which is permissible under the conditions stated), we easily have after a little simplification.

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'}, H', F_{p'}'; I_{x_1}^{\nu,\lambda}\left[A_n, H, F_p; f(t)\right]\right] = x_2^{-\nu'-\lambda'-1}\int_0^{x_2} t^{\nu}\Delta \quad f(t)dt \tag{4.2.11}$$

where

$$\Delta = \int_t^{x_2} x_1^{\nu'-\nu-\lambda-1}(x_2-x_1)^{\lambda'}(x_1-t)^{\lambda} A_n\left[z_1\left(\frac{t}{x_1}\right)^{\nu_1}\left(1-\frac{t}{x_1}\right)^{\lambda_1}\right] A_{n'}\left[z_1'\left(\frac{x_1}{x_2}\right)^{\nu_1'}\left(1-\frac{x_1}{x_2}\right)^{\lambda_1'}\right]$$
$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{t}{x_1}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ (h_j,H_j)_{1,Q}\end{matrix}\right] H_{P',Q'}^{M',N'}\left[z_2'\left(1-\frac{x_1}{x_2}\right)^{\lambda_2'}\middle|\begin{matrix}(g_j',G_j')_{1,P'}\\ (h_j',H_j')_{1,Q'}\end{matrix}\right]$$
$$F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{t}{x_1}\right)^{\lambda_3}\right] F_{p'}^{(\alpha',\beta';\tau',\mu')}\left[a',b';c';z_3'\left(1-\frac{x_1}{x_2}\right)^{\lambda_3'}\right] dx_1 \tag{4.2.12}$$

To evaluate $\Delta$, we first express both the Fox's H-Functions and S-Generalized gauss hypergeometric functions in terms of their respective contour integral forms with the help of (3.1.2) and (3.1.10) respectively, next both the Appell polynomail are expressed in terms of the series with the help of (3.1.1). Further, we interchange the order of summations and contour integral and get:

$$\Delta = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(a)\Gamma(a')\Gamma(\alpha)\Gamma(\alpha')B(b,c-b)B(b',c'-b')}\sum_{k=0}^{n}\sum_{k'=0}^{n'}\frac{a_{n-k}a_{n'-k'}}{k!k'!}z_1^k z_1'^{k'} t^{\nu_1 k}$$
$$\frac{1}{(2\pi\omega)^6}\int_{\mathfrak{L}_1}\cdots\int_{\mathfrak{L}_6}\theta(\xi_2)\theta_1(\xi_2')\psi(\xi_1,\xi_3)\psi(\xi_1',\xi_3')(p^{-1})^{\xi_1}(p'^{-1})^{\xi_1'}(z_2)^{\xi_2}(z_2')^{\xi_2'}(-z_3)^{\xi_3}$$
$$(-z_3')^{\xi_3'} x_2^{-((\nu_1'+\lambda_1')k'+\lambda_2'\xi_2'+\lambda_3'\xi_3')}\int_t^{x_2} x_1^{(\nu'+\nu_1'k'-\nu-\lambda-(\nu_1+\lambda_1)k-\lambda_2\xi_2-\lambda_3\xi_3)-1}$$
$$(x_2-x_1)^{(\lambda'+\lambda_1'k'+\lambda_2'\xi_2'+\lambda_3'\xi_3')}(x_1-t)^{(\lambda+\lambda_1k+\lambda_2\xi_2+\lambda_3\xi_3)}dx_1 d\xi_1,\cdots,d\xi_6 \tag{4.2.13}$$

Now, we substitute $u = \left(\frac{x_2-x_1}{x_2-t}\right)$ in (4.2.13) and evaluate the u integral with the help of known result[90, p. 47, Eq.(16)]. Finally, re-interpreting the result in terms of the Multivariable H-function and substituting the values of $\Delta$ in (4.2.11), we get the right hand side of required result (4.2.1) after some simplification.

**The proof of (4.2.5) can be developed proceeding on lines similar to those indicated in the proof of the Result 1**.

To prove Result 3 we first express both the I- and J- operators involved on the left hand side of (4.2.7) in the integral form with the help of equation (4.1.2) and (4.1.4), we have

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'},H',F_{p'};J_{x_1}^{\nu,\lambda}\left[A_n,H,F_p;f(t)\right]\right]=x_2^{-\nu'-\lambda'-1}\int_0^{x_2}x_1^{\nu'}(x_2-x_1)^{\lambda'}$$
$$A_{n'}\left[z_1'\left(\frac{x_1}{x_2}\right)^{\nu_1'}\left(1-\frac{x_1}{x_2}\right)^{\lambda_1'}\right]H_{P',Q'}^{M',N'}\left[z_2'\left(1-\frac{x_1}{x_2}\right)^{\lambda_2'}\middle|\begin{matrix}(g_j',G_j')_{1,P'}\\(h_j',H_j')_{1,Q'}\end{matrix}\right]$$
$$F_{p'}^{(\alpha',\beta';\tau',\mu')}\left[a',b';c';z_3'\left(1-\frac{x_1}{x_2}\right)^{\lambda_3'}\right]x_1^{\nu}\int_{x_1}^{\infty}t^{-\nu-\lambda-1}(t-x_1)^{\lambda}A_n\left[z_1\left(\frac{x_1}{t}\right)^{\nu_1}\left(1-\frac{x_1}{t}\right)^{\lambda_1}\right]$$
$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{x_1}{t}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\(h_j,H_j)_{1,Q}\end{matrix}\right]F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{x_1}{t}\right)^{\lambda_3}\right]f(t)dtdx_1 \tag{4.2.14}$$

Next, we change the order of t- and $x_1$- integrals, (which is permissible under the conditions stated) and get :

$$I_{x_2}^{\nu',\lambda'}\left[A_{n'},H',F_{p'};J_{x_1}^{\nu,\lambda}\left[A_n,H,F_p;f(t)\right]\right]=\int_0^{x_2}f(t)\int_0^{t}g(x_1,x_2,t)dx_1dt+\int_{x_2}^{\infty}f(t)\int_0^{x_2}g(x_1,x_2,t)dx_1dt$$
$$=\int_0^{x_2}f(t)I_1dt+\int_{x_2}^{\infty}f(t)I_2dt \tag{4.2.15}$$

where

$$g(x_1,x_2,t) = x_2^{-\nu'-\lambda'-1}x_1^{\nu+\nu'}(x_2-x_1)^{\lambda'}t^{-\nu-\lambda-1}(t-x_1)^{\lambda}$$
$$A_{n'}\left[z_1'\left(\frac{x_1}{x_2}\right)^{\nu_1'}\left(1-\frac{x_1}{x_2}\right)^{\lambda_1'}\right]A_n\left[z_1\left(\frac{x_1}{t}\right)^{\nu_1}\left(1-\frac{x_1}{t}\right)^{\lambda_1}\right]$$
$$H_{P',Q'}^{M',N'}\left[z_2'\left(1-\frac{x_1}{x_2}\right)^{\lambda_2'}\middle|\begin{matrix}(g_j',G_j')_{1,P'}\\(h_j',H_j')_{1,Q'}\end{matrix}\right]H_{P,Q}^{M,N}\left[z_2\left(1-\frac{x_1}{t}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\(h_j,H_j)_{1,Q}\end{matrix}\right]$$
$$F_{p'}^{(\alpha',\beta';\tau',\mu')}\left[a',b';c';z_3'\left(1-\frac{x_1}{x_2}\right)^{\lambda_3'}\right]F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{x_1}{t}\right)^{\lambda_3}\right] \quad (4.2.16)$$

To evaluate the $I_1$ involved in the first integral on the right hand side of (4.2.15), we first express the Appell polynomials involved in series form with the help of (3.1.1) . The Fox H-functions and S-generalized gauss hypergeometric functions in terms of their respective well known Mellin-Barnes contour integrals with the help of equations (3.1.2) and (3.1.10). Now, interchanging the order of summations, the contour integral $\xi_i$ (i = 1,$\cdots$,6) and $x_1$- integrals (which is permissible under the conditions stated), we get:

$$\int_0^t g(x_1,x_2,t)dx_1 = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(a)\Gamma(a')B(b,c-b)B(b',c'-b')}\sum_{k=0}^{n}\sum_{k'=0}^{n'}\frac{a_{n-k}a_{n'-k'}}{k!k'!}z_1^k z_1'^{k'}$$
$$\frac{1}{(2\pi \mathrm{i})^6}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\cdots\int_{\mathfrak{L}_6}\phi_1(\xi_3,\xi_4)\phi_2(\xi_5,\xi_6)\prod_{i=1}^{6}\theta_i(\xi_i)(z_2')^{\xi_1}(z_2)^{\xi_2}(z_3')^{\xi_3}p^{-\xi_4}(z_3)^{\xi_5}p^{-\xi_6}$$
$$x_2^{-\nu'-\lambda'-(\nu_1'+\lambda_1')k'-(\nu_2'+\lambda_2')\xi_1-(\nu_3'+\lambda_3')\xi_3-1}t^{-\nu-\lambda-(\nu_1+\lambda_1)k-(\nu_2+\lambda_2)\xi_2-(\nu_3+\lambda_3)\xi_5-1}$$
$$\int_0^t x_1^{\nu+\nu'+\nu_1'k'+\nu_1k+\nu_2'\xi_1+\nu_2\xi_2+\nu_3'\xi_3+\nu_3\xi_5}(x_2-x_1)^{\lambda'+\lambda_1'k'+\lambda_2'\xi_1+\lambda_3'\xi_3}$$
$$(t-x_1)^{\lambda+\lambda_1k+\lambda_2\xi_2+\lambda_3\xi_5}dx_1d\xi_1d\xi_2...d\xi_6 \quad (4.2.17)$$

Next we substitute $x_1 = tu$ in the right hand side of (4.2.17) and integrate it with the help of known result [90, p. 47, Eq.(16)]. We get the following integral after a little simplification:

$$\int_0^t g(x_1,x_2,t)dx_1 = \frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(a)\Gamma(a')B(b,c-b)B(b',c'-b')}\sum_{k=0}^{n}\sum_{k'=0}^{n'}\frac{a_{n-k}a_{n'-k'}}{k!k'!}z_1^k z_1'^{k'}$$
$$\left(\frac{t}{x_2}\right)^{\nu'+\nu_1'k'}\left(1-\frac{t}{x_2}\right)^{\lambda+\lambda'+\lambda_1 k+\lambda_1'k'\lambda_2\xi_2+\lambda_3\xi_5+\lambda_2'\xi_1+\lambda_3'\xi_3+1}$$
$$\frac{1}{x_2}\frac{1}{(2\pi \mathrm{i})^6}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\cdots\int_{\mathfrak{L}_6}\phi_1(\xi_3,\xi_4)\phi_2(\xi_5,\xi_6)\prod_{i=1}^{6}\theta_i(\xi_i)(z_2')^{\xi_1}(z_2)^{\xi_2}(z_3')^{\xi_3}p^{-\xi_4}(z_3)^{\xi_5}$$
$$p^{-\xi_6}\frac{\Gamma(1+\nu+\nu'+\nu_1'k'+\nu_1 k)\Gamma(1+\lambda+\lambda_1 k+\lambda_2\xi_2+\lambda_3\xi_5)}{\Gamma(2+\nu+\nu'+\nu_1'k'+\nu_1 k+\lambda+\lambda_1 k+\lambda_2\xi_2+\lambda_3\xi_5)}$$
$${}_2F_1\left[\begin{array}{c}-\lambda'-\lambda_1'k'-\lambda_2'\xi_1-\lambda_3'\xi_3, 1+\nu+\nu'+\nu_1'k'+\nu_1 k;\\ 2+\nu+\nu'+\nu_1'k'+\nu_1 k+\lambda+\lambda_1 k+\lambda_2\xi_2+\lambda_3\xi_5;\end{array}\frac{t}{x_2}\right] \quad (4.2.18)$$

Then we transform the expression on the right hand side of (4.2.18) with the help of [90, p. 60, Eq.(5)]. Further, we express ${}_2F_1$ in terms of contour integral and reinterpret the result, thus obtained in terms of multivariable H-function we get $I_1$.

Again to calculate $I_2 = \int_0^{x_2} g(x_1,x_2,t)dx_1$ , we proceed on similar lines to those mentioned above $I_1$ with the difference that we substitute $x_1 = x_2 u$ in the corresponding appropriate expression to (4.2.17). On substituting the values of $I_1$ and $I_2$ in (4.2.15), we get the required result (4.2.7).

## 4.3 APPLICATIONS AND SPECIAL CASES

As our composition formulae involve the Appell Polynomial, Fox H-function and S-Generalized Gauss's Hypergeometric Function, a large number of other composition formulae involving simpler functions and polynomials, can be obtained by specializing the functions involved in our composition formulae.

We mention below two such composition formulae cum double integrals.

(i) If in the Result 1, we reduce Appell polynomial to unity and f(t) = $e^{\omega t}$, we get the following double integral which is believed to be new.

$$x_2^{-\nu'-\lambda'-1}\int_0^{x_2}\int_0^{x_1} t^{\nu} x_1^{\nu'-\nu-\lambda-1}(x_2-x_1)^{\lambda'}(x_1-t)^{\lambda}$$
$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{t}{x_1}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ \\(h_j,H_j)_{1,Q}\end{matrix}\right] H_{P',Q'}^{M',N'}\left[z_2'\left(1-\frac{x_1}{x_2}\right)^{\lambda_2'}\middle|\begin{matrix}(g_j',G_j')_{1,P'}\\ \\(h_j',H_j')_{1,Q'}\end{matrix}\right]$$
$$F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{t}{x_1}\right)^{\lambda_3}\right] F_{p'}^{(\alpha',\beta';\tau',\mu')}\left[a',b';c';z_3'\left(1-\frac{x_1}{x_2}\right)^{\lambda_3'}\right] e^{\omega t}dtdx_1 \tag{4.3.1}$$

$$=\frac{\Gamma(\beta)\Gamma(\beta')}{\Gamma(a)\Gamma(a')\Gamma(\alpha)\Gamma(\alpha')B(b,c-b)B(b',c'-b')}$$
$$H_{6,5:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1;1,1}^{0,6:1,2;1,2;M',N';1,1;M,N;1,1;1,0;1,1}\left[\begin{matrix}p'^{-1}\\p^{-1}\\z_2'\\-z_3'\\z_2\\-z_3\\-1\\-(x\omega)\end{matrix}\middle|\begin{matrix}A^*: & C^*\\ \\ \\ \\ \\ \\ \\B^*: & D^*\end{matrix}\right] \tag{4.3.2}$$

where

$$A^* = (-\lambda-\nu+\nu'; 0,0,0,0,\lambda_2,\lambda_3,1,0), (-\lambda'; 0,0,\lambda_2',\lambda_3',0,0,1,0), (1-b'; \tau',0,0,1,0,0,0,0),$$

$$(-\lambda; 0,0,0,0,\lambda_2,\lambda_3,0,0), (1-b; 0,\tau,0,0,0,1,0,0), (-1-\lambda-\lambda'; 0,0,1,1,1,1,1,0)$$

$$B^* = (-\lambda-\nu+\nu'; 0,0,0,0,\lambda_2,\lambda_3,0,0), (1-c; 0,(\tau+\mu),0,0,0,1,0,0),$$

$$(-1-\lambda-\lambda'; 0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1,0), (1-c'; (\tau'+\mu'),0,0,1,0,0,0,0)$$

$$(-2-\nu-\lambda-\lambda'; 0,0,1,1,1,1,1,1)$$

$$C^* = (1,1), (1-c'+b',\mu'), (\beta',1); (1,1), (1-c+b,\mu), (\beta,1); (g_j',G_j')_{1,P'};$$

$$(1-a',1); (g_j,G_j)_{1,P}; (1-a,1); -; (-\nu,1)$$

$$D^* = (\alpha',1); (\alpha,1); (h_j',H_j')_{1,Q'}; (0,1); (h_j,H_j)_{1,Q}; (0,1); (0,1); (0,1)$$

provided that the conditions easily obtainable from those mentioned in (4.2.4) are satisfied.

(ii) If we take f(t) = 1 and reduce Appell polynomial to unity in the Result 2, we obtain the following integral valid under the conditions derivable from the conditions stated with (4.2.6):

$$x_2^{\nu}\int_{x_2}^{\infty}\int_{x_1}^{\infty} t^{-\nu'-\lambda'-1} x_1^{\nu'-\nu-\lambda-1}(x_1-x_2)^{\lambda}(t-x_1)^{\lambda'}$$
$$H_{P,Q}^{M,N}\left[z_2\left(1-\frac{x_2}{x_1}\right)^{\lambda_2}\middle|\begin{matrix}(g_j,G_j)_{1,P}\\ (h_j,H_j)_{1,Q}\end{matrix}\right] H_{P',Q'}^{M',N'}\left[z_2'\left(1-\frac{x_1}{t}\right)^{\lambda_2'}\middle|\begin{matrix}(g_j',G_j')_{1,P'}\\ (h_j',H_j')_{1,Q'}\end{matrix}\right]$$
$$F_p^{(\alpha,\beta;\tau,\mu)}\left[a,b;c;z_3\left(1-\frac{x_2}{x_1}\right)^{\lambda_3}\right] F_{p'}^{(\alpha',\beta';\tau',\mu')}\left[a',b';c';z_3'\left(1-\frac{x_1}{t}\right)^{\lambda_3'}\right] dtdx_1 \tag{4.3.3}$$

$$= \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\nu)}{\Gamma(a)\Gamma(a')\Gamma(\alpha)\Gamma(\alpha')B(b,c-b)B(b',c'-b')} H^{0,6:1,2;1,2;M',N';1,1;M,N;1,1;1,0}_{6,5:3,1;3,1;P',Q';1,1;P,Q;1,1;0,1} \left[ \begin{array}{c|cc} p'^{-1} & A^*: & C^* \\ p^{-1} & & \\ z_2' & & \\ -z_3' & & \\ z_2 & & \\ -z_3 & & \\ -1 & B^*: & D^* \end{array} \right] \tag{4.3.4}$$

where

$A^* = (-\lambda - \nu + \nu'; 0,0,0,0,\lambda_2,\lambda_3,1), (-\lambda'; 0,0,\lambda_2',\lambda_3',0,0,1), (-\lambda; 0,0,0,0,\lambda_2,\lambda_3,0),$

$(1-b'; \tau',0,0,1,0,0,0), (1-b; 0,\tau,0,0,0,1,0), (-1-\lambda-\lambda'; 0,0,1,1,1,1,1)$

$B^* = (-\lambda - \nu + \nu'; 0,0,0,0,\lambda_2,\lambda_3,0), (1-c; 0,(\tau+\mu),0,0,0,1,0),$

$(-1-\lambda-\lambda'; 0,0,\lambda_2',\lambda_3',\lambda_2,\lambda_3,1), (1-c'; (\tau'+\mu'),0,0,1,0,0,0)$

$(-1-\nu-\lambda-\lambda'; 0,0,1,1,1,1,1)$

$C^* = (1,1), (1-c'+b',\mu'), (\beta',1); (1,1), (1-c+b,\mu), (\beta,1); (g_j', G_j')_{1,P'};$

$(1-a',1); (g_j, G_j)_{1,P}; (1-a,1); -$

$D^* = (\alpha',1); (\alpha,1); (h_j', H_j')_{1,Q'}; (0,1); (h_j, H_j)_{1,Q}; (0,1); (0,1)$

Our results also unify and extend the findings of several authors, we give below exact reference to two such results. Thus, if in these composition formulae, If we take $\lambda_2 = \lambda_2' = 0$ in (4.2.1), (4.2.5) and (4.2.7) H-Functions reduce to exponential function. Further, reducing exponential function to unity, reducing all the Appell Polynomials to unity and S-Generalized Gauss's Hypergeometric Function reducing to Gauss Hypergeometric Function by putting p = 0 thus obtained Gauss Hypergeometric Function reducing to unity, we get the corresponding expressions which are in essence the same as those given by Erdélyi [24, p. 166, Eq.

(6.1) and (6.2); p. 167, Eq. (6.3)].

Also, if we take p = 0 in (4.2.1), (4.2.5) and (4.2.7) S-Generalized Gauss's Hypergeometric Function reduce to Gauss Hypergeometric Function. Further reducing these Gauss Hypergeometric function to unity, reducing all the Appell Polynomials to unity and reducing all the H-functions to generalized hypergeometric functions, we get the corresponding expressions which are in essence the same as those given by Goyal and Jain [32, p. 253, Eq.(2.4); p. 254, Eq. (2.7); p. 255, Eq. (2.12)].

# 5

# A STUDY OF UNIFIED FINITE INTEGRALS INVOLVING ASSOCIATED $I_3$ FUNCTION, GENERALIZED MITTAG LEFFLER FUNCTION, $\overline{H}$ FUNCTION AND APPELL POLYNOMIAL

The main findings of this chapter have been published as detailed below:

1. **M. K. BANSAL and R. JAIN** (2015). A STUDY OF FINITE INTEGRAL INVOLVING GENERALIZED FORM OF THE ASTROPHYSICAL THERMONUCLEAR FUNCTION, *Journal of Rajasthan Academy of Physical Sciences*, **14(1)**, 51-55.

2. **M. K. BANSAL and R. JAIN** (2014). ON UNIFIED FINITE INTEGRALS INVOLVING I3 FUNCTION, GENERALIZED MITTAG-LEFFLER FUNCTION AND APPELL POLYNOMIALS, *The Journal of the Indian Academy of Mathematics*, **36(2)**, 211-219.

## 5. A STUDY OF UNIFIED FINITE INTEGRALS INVOLVING ASSOCIATED $I_3$ FUNCTION, GENERALIZED MITTAG LEFFLER FUNCTION, $\overline{H}$ FUNCTION AND APPELL POLYNOMIAL

In this chapter, we first define the various polynomial and functions and the results required to establish our main integrals. Next, we evaluate two unified and general finite integrals. The first integral involves the product of the Appell Polynomial $A_n(z)$, the Generalized form of the Astrophysical Thermonuclear function $I_3$ and Generalized Mittag - Leffler Function $E^{\gamma,\delta}_{\alpha,\beta,\tau,\mu,\rho,p}(z;s,r)$. The arguments of the functions occurring in the integral involve the product of factors of the form $x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^l)^{-\rho}$. On account of the most general nature of the functions occurring in the integrand of our main integral, a large number of new integrals can easily be obtained from it merely by specializing the functions and parameters. Next, we give the five special cases of our main integral involving several polynomials and functions notably Cesaro Polynomial, Laguerre Polynomial, Shively Polynomial, Bateman's Polynomial, Bessel Polynomial, Generalized Hypergeometric Function, Bessel Maitland Function, Mittag-Leffler Function, Struve Function, Generalized Bessel Maitland Function, Steady State Function which are new, sufficiently general, and of interest in themselves.

The second integral involves the Generalized form of the Astrophysical Thermonuclear function $I_3$ and $\overline{H}-$ function. The arguments of the function occurring in the integral involve the product of factors of the form $t^{\lambda-1}(1-t)^{\sigma-1}(1-ut^{\ell})^{-\gamma}(1+vt^m)^{-\beta}$. On account of the most general nature of the functions occurring in the integrand of our main integral, a large number of new integrals can easily be obtained from it merely by specializing the function and parameters. Next, we give the five special cases of our main integral involving several functions notably Generalized Wright Bessel Function, Generalized

Riemann Zeta Function, Generalized Hurwitz-Lerch Zeta Function, Polylogarithm of order p, Generalized Hypergeometric Function which are new, sufficiently general, and of interest in themselves.

## 5.1 INTRODUCTION

We shall first give a brief description of the polynomial and functions occurring as integrand in our main integral.

### THE APPELL POLYNOMIALS

A class of polynomials over the field of complex numbers which contains many classical polynomial systems. The Appell Polynomials were introduced by Appell [6]. The series of Appell Polynomials is defined by :

$$A_n(z) = \sum_{k=0}^{n} \frac{a_{n-k}}{k!} z^k, \quad n = 0, 1, 2, \cdots \tag{5.1.1}$$

where $a_{n-k}$ is the complex coefficients and $a_0 \neq 0$

### THE MULTIVARIABLE H-FUNCTION

The Multivariable H-Function is defined and represented in the following manner [109, pp. 251–252, Eqs. (C.1–C.3)]:

$$H^{0,B:A_1,B_1;\cdots;A_r,B_r}_{C,D:C_1,D_1;\cdots;C_r,D_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,C} : (c_j^{(1)},\gamma_j^{(1)})_{1,C_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,D} : (d_j^{(1)},\delta_j^{(1)})_{1,D_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,D_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int\limits_{\mathfrak{L}_1} \int\limits_{\mathfrak{L}_2} \cdots \int\limits_{\mathfrak{L}_r} \Phi(\xi_1, \xi_2, \cdots, \xi_\mathfrak{r}) \prod_{i=1}^{r} \Theta_i(\xi_i) z^{\xi_i} \; d\xi_1 d\xi_2, \cdots, d\xi_\mathfrak{r}, \tag{5.1.2}$$

where $\omega = \sqrt{-1}$,

$$\Phi(\xi_1, \xi_2, \cdots, \xi_\mathfrak{r}) = \frac{\prod\limits_{j=1}^{B} \Gamma(1 - a_j + \sum\limits_{i=1}^{r} \alpha_j^{(i)} \xi_i)}{\prod\limits_{j=1}^{D} \Gamma(1 - b_j + \sum\limits_{i=1}^{r} \beta_j^{(i)} \xi_i) \prod\limits_{j=B+1}^{C} \Gamma(a_j - \sum\limits_{i=1}^{r} \alpha_j^{(i)} \xi_i)} \tag{5.1.3}$$

$$\Theta_i(\xi_i) = \frac{\prod\limits_{j=1}^{A_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod\limits_{j=1}^{B_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod\limits_{j=B_i+1}^{C_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i) \prod\limits_{j=A_i+1}^{D_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i)} (i = 1, 2, \cdots, r), \tag{5.1.4}$$

All the Greek letters occurring on the left and side of (5.1.2) are assumed to be positive real numbers for standardization purposes. The definition of the multivariable H-function will however be meaningful even if some of these quantities are zero. The details about the nature of contour $\mathfrak{L}_1, \cdots, \mathfrak{L}_\mathfrak{r}$, conditions of convergence of the integral given by (5.1.2). Throughout the work it is assumed that this function always satisfied its appropriate conditions of convergence [109, p. 252–253, Eqs. (C.4–C.6)].

## THE $\overline{H}$-FUNCTION

The following series representation of the $\overline{H}$-function is defined and represented in the following manner [50]:

$$\overline{H}_{p,q}^{m,n} \left[ z \left| \begin{array}{cc} (e_j, E_j; \in_j)_{1,n}, & (e_j, E_j)_{n+1,p} \\ (f_j, F_j)_{1,m}, & (f_j, F_j; \Im_j)_{m+1,q} \end{array} \right. \right] = \sum_{t=0}^{\infty} \sum_{h=1}^{m} \overline{\Theta}(\mathfrak{s}_{t,h}) z^{\mathfrak{s}_{t,h}} \tag{5.1.5}$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t,h}) = \frac{\prod\limits_{j=1,j\neq h}^{m} \Gamma(f_j - F_j\mathfrak{s}_{t,h}) \prod\limits_{j=1}^{n} \{\Gamma(1-e_j+E_j\mathfrak{s}_{t,h})\}^{\in_j}}{\prod\limits_{j=m+1}^{q} \{\Gamma(1-f_j+F_j\mathfrak{s}_{t,h})\}^{\Im_j} \prod\limits_{j=n+1}^{p} \Gamma(e_j - E_j\mathfrak{s}_{t,h})} \frac{(-1)^t}{t!F_h}, \quad \mathfrak{s}_{t,h} = \frac{f_h+t}{F_h} \tag{5.1.6}$$

In the Sequel, we shall also make use of the following behavior of the $\overline{H}_{p,q}^{m,n}[z]$ function for small and large value of z as recorded by Saxena et al. [102, p. 112, Eqs.(2.3) and (2.4)].

$$\overline{H}_{p,q}^{m,n}[z] = O[|z|^{\alpha}], \text{for small z, where} \quad \alpha = \min_{1\leq j\leq m} \Re\left(\frac{f_j}{F_j}\right) \tag{5.1.7}$$

$$\overline{H}_{p,q}^{m,n}[z] = O[|z|^{\beta}], \text{for large z, where} \quad \beta = \max_{1\leq j\leq n} \Re\left(\in_j \left(\frac{e_j-1}{E_j}\right)\right) \tag{5.1.8}$$

## GENERALIZED FORM OF THE ASTROPHYSICAL THERMONUCLEAR FUNCTION

The generalized form of the astrophysical thermonuclear function is given by Saxena [100] in the following form :

$$I_3(z,t,\nu,\mu,\alpha) = \int\limits_0^{\infty} y^{\nu-1} e^{-z(y+t)^{-\mu}} [1+(\alpha-1)y]^{-\frac{1}{\alpha-1}} dy$$

where $\Re(\nu) > 0, \Re(z) > 0, \alpha > 1$ and $\mu > 0$.

**We give the following representation of $I_3$ in terms of H-function of two variable:**

$$I_3(z,t,\nu,\mu,\alpha) = \int_0^\infty y^{\nu-1} e^{-z(y+t)^{-\mu}} [1+(\alpha-1)y]^{-\frac{1}{\alpha-1}} dy$$

$$= \frac{t^\nu}{\Gamma(\frac{1}{\alpha-1})} H^{0,1:1,0;2,1}_{1,0:0,2;1,2} \left[ \begin{array}{c} \frac{z}{t^\mu} \\ \\ \frac{1}{(\alpha-1)t} \end{array} \middle| \begin{array}{l} (1+\nu;\mu,1) : -; \qquad (1,1) \\ \\ - : (0,1),(1,\mu); \quad (\frac{1}{\alpha-1},1),(\nu,1) \end{array} \right] \tag{5.1.9}$$

*Proof.* :To evaluate the integral (5.1.9), we first express the terms $[1+(\alpha-1)y]^{-\frac{1}{\alpha-1}}$ and $e^{-z(y+t)^{-\mu}}$ occurring in it's integrand in terms of well known Mellin-Barnes contour integral [109, p. 18, Eqs.(2.6.2) and (2.6.4)] and then change the order of $\xi$ - integral with y-integral (which is permissible under the conditions stated). Thus the left hand side of (5.1.9) takes the following form (say $\Delta$).

$$\Delta = \frac{1}{(2\pi\omega)^2} \int_{\mathfrak{L}_1} \int_{\mathfrak{L}_2} \frac{\Gamma(-\xi_1)\Gamma(-\xi_2)\Gamma(\frac{1}{\alpha-1}+\xi_2)(\alpha-1)^{\xi_2} z^{\xi_1}}{\Gamma(\frac{1}{\alpha-1})}$$

$$\left[ \int_0^\infty y^{\nu+\xi_2-1}(y+t)^{-\mu\xi_1} dy \right] d\xi_1 d\xi_2 \tag{5.1.10}$$

Now we evaluate the y-integral occurring in (5.1.10) with the help of well known Beta function, Thus we get the following equation

$$\Delta = \frac{t^\nu}{\Gamma(\frac{1}{\alpha-1})} \frac{1}{(2\pi\omega)^2} \int_{\mathfrak{L}_1} \int_{\mathfrak{L}_2} \frac{\Gamma(-\xi_1)\Gamma(\xi_2)\Gamma(\frac{1}{\alpha-1}-\xi_2)\Gamma(\nu-\xi_2)}{\Gamma(\mu\xi_1)}$$

$$\frac{\Gamma(-\nu+\mu\xi_1+\xi_2) z^{\xi_1} t^{-\mu\xi_1-\xi_2}}{(\alpha-1)^{\xi_2}} d\xi_1 d\xi_2 \tag{5.1.11}$$

finally reinterpret the result thus obtained in terms of H-function of two variable. We easily arrive at the right hand side of (5.1.9) after a little simplification. □

## GENERALIZED MITTAG LEFFLER FUNCTION

The Generalized Mittag-Leffler function is defined and represented in the following form [88, p. 127, Eq.(1.2)]:

$$E^{\gamma,\delta}_{\alpha,\beta,\tau,\mu,\rho,p}(z;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}z^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\tau)_{\mu n}]^{r}(\rho)_{pn}}$$

provided that $(\alpha,\beta,\gamma,\tau,\rho)\in\mathbb{C}$, $\quad\Re(\alpha,\beta,\gamma,\tau,\rho)>0,(\delta,\mu,\rho)>0$ and $\overline{H}$-Function is defined in (5.1.5).

Next, We give the five special cases and representation of Generalized Mittag-Leffler function in terms of $\overline{H}$-function:

$$E^{\gamma,\delta}_{\alpha,\beta,\tau,\mu,\rho,p}(z;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}z^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\tau)_{\mu n}]^{r}(\rho)_{pn}}$$

$$=\frac{\Gamma(\rho)(\Gamma\tau)^{r}z^{\rho-1}}{(\Gamma\gamma)^{s}}\overline{H}^{1,2}_{2,4}\left[-z^{p}\left|\begin{matrix}(0,1), & (1-\gamma,\delta;s)\\ (0,1), & (1-\rho,p),(1-\alpha(\rho-1)-\beta,\alpha p),(1-\tau,\mu;r)\end{matrix}\right.\right] \tag{5.1.12}$$

## SPECIAL CASES OF GENERALIZED MITTAG LEFFLER FUNCTION

1. **Generalized Hypergeometric Function :**If we take $\delta=p=\mu=\alpha=1$ in (5.1.12), we get

$$E^{\gamma,1}_{1,\beta,\tau,1,\rho,1}(z;s,r)=\frac{z^{\rho-1}}{\Gamma(\rho-1+\beta)}{}_{2}\overline{F}_{3}\left[\begin{matrix}1,\gamma;s\\ \rho,\rho+\beta-1,\lambda;r\end{matrix}\left|z\right.\right] \tag{5.1.13}$$

where ${}_{p}\overline{F}_{q}$ is defined by Gupta et al.[41]

2. **Bessel–Maitland Function :** Again, If we take $s = r = 0, p = \rho = 1$, $\beta = 1 + \nu$ and z is replaced by (-z) in (5.1.12), we get

$$E^{\gamma,\delta}_{\alpha,1+\nu,\tau,\mu,1,1}(z;0,0) = J^{\alpha}_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n + \nu + 1)n!} \tag{5.1.14}$$

where $J^{\alpha}_{\nu}$ is defined in [73].

3. **Generalization of the Mittag- Leffler Function :** Further, If we take $s = 1, r = 0, p = \rho = 1$ in (5.1.12), we get

$$E^{\gamma,\delta}_{\alpha,\beta,\tau,\mu,1,1}(z;1,0) = E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n} z^n}{\Gamma(\alpha n + \beta)n!} \tag{5.1.15}$$

where $E^{\gamma,\delta}_{\alpha,\beta}$ is defined in [105].

4. **Generalized Bessel–Maitland function :** If we take $s = r = p = \rho = \mu = \gamma = \delta = 1, \alpha = \mu, \lambda = \lambda + 1, \beta = \nu + \lambda + 1, z = -\frac{z^2}{4}$ in (5.1.12), we get

$$E^{1,1}_{\mu,1+\lambda+\nu,1+\lambda,1,1,1}(z;1,1) = J^{\mu}_{\nu,\lambda}(z)$$
$$= \frac{\Gamma(\lambda+1)}{\left(\frac{z^2}{4}\right)^{\nu+2\lambda}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\mu n + \nu + \lambda + 1)\Gamma(\lambda + 1 + n)} \left(\frac{z}{2}\right)^{\nu+2\lambda+2n} \tag{5.1.16}$$

where $J^{\mu}_{\nu,\lambda}(Z)$ is defined in [88].

5. **Struve function :** If we take $s = r = p = \rho = \alpha = \mu = \gamma = \delta = 1, \beta = \frac{3}{2}, \tau = \frac{3}{2} + \nu, z = -\frac{z^2}{4}$ in (5.1.12), we get

$$E^{1,1}_{1,\frac{3}{2},\frac{3}{2}+\nu,1,1,1}(z;1,1) = {}_1F_2\left[\begin{array}{c|c} 1 & \\ & \left(\frac{-z^2}{4}\right) \\ 3/2, 3/2+\nu & \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1)_n}{(3/2)_n(3/2+\nu)_n n!} \left(\frac{-z^2}{4}\right)^n \tag{5.1.17}$$

## 5.2 FIRST INTEGRAL

$$\int_0^a x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^{\ell})^{-\rho}A_n[z_2x^{\lambda_2}(a-x)^{\sigma_2}]I_3[z,z_1x^{-\lambda_1}(a-x)^{-\sigma_1},\nu_1,\mu_1,\alpha_1]$$

$$E^{\gamma_3,\delta_3}_{\alpha_3,\beta_3,\tau_3,\mu_3,\eta_3,p}[z_3x^{\lambda_3}(a-x)^{\sigma_3}(1-ux^{\ell})^{-\rho_3};s,r]dx$$

$$=\sum_{k=0}^{n}\sum_{t=0}^{\infty}\frac{a_{n-k}}{k!}\frac{[\Gamma(\tau_3)]^r\Gamma(\eta_3)[\Gamma(\gamma_3+\delta_3t)]^s}{\Gamma(\frac{1}{\alpha_1-1})[\Gamma(\gamma_3)]^s\Gamma(\eta_3+pt)[\Gamma(\tau_3+\mu_3t)]^r}$$

$$\frac{z_1^{\nu_1}z_2^{k}z_3^{\eta_3+pt-1}a^{\lambda+\sigma-(\lambda_1+\sigma_1)\nu_1+(\lambda_2+\sigma_2)k+(\sigma_3+\lambda_3)(\eta_3+pt-1)-1}}{\Gamma(\alpha_3\eta_3-\alpha_3+\beta_3+\alpha_3pt)\Gamma(\rho+\rho_3(\eta_3+pt-1))}$$

$$H^{0,3:1,0;2,1;1,1}_{3,1:0,2;1,2;1,1}\left[\begin{array}{c|c} \frac{za^{\mu_1(\lambda_1+\sigma_1)}}{z_1^{\mu_1}} & A^*:-;(1,1);(1-\rho-\rho_3(\eta_3+pt-1),1) \\ \frac{a^{(\lambda_1+\sigma_1)}}{z_1(\alpha_1-1)} & \\ -ua^{\ell} & B^*:(0,1)(1,\mu_1);(\frac{1}{\alpha_1-1},1)(\nu_1,1);(0,1) \end{array}\right] \tag{5.2.1}$$

where

$A^*=(1+\nu_1;\mu_1,1,0),(1-\sigma+\sigma_1\nu_1-\sigma_2k-\sigma_3(\eta_3+pt-1);\sigma_1\mu_1,\sigma_1,0),$

$(1-\lambda+\lambda_1\nu_1-\lambda_2k-\lambda_3(\eta_3+pt-1);\lambda_1\mu_1,\lambda_1,\ell)$

$B^*=(1-(\lambda+\sigma)+(\lambda_1+\sigma_1)\nu_1-(\lambda_2+\sigma_2)k-(\lambda_3+\sigma_3)(\eta_3+pt-1);\mu_1(\lambda_1+\sigma_1),$

$(\lambda_1+\sigma_1),\ell)$

The above result is valid under the following conditions :

(i) $\Re\left(\lambda+\lambda_1\min\{\frac{1}{\alpha_1-1},\nu\}\right)>0$ (ii) $\Re\left(\sigma+\sigma_1\min\{\frac{1}{\alpha_1-1},\nu\}\right)>0$

(iii) $\min\Re\{\nu_1,\sigma,\sigma_3,\eta_3,p,\lambda,\lambda_3,\rho,\rho_3\}\geq 0$

(iv) $\min(\ell,\sigma_1,\mu_1,\lambda_1)\geq 0$ not all zero simultaneously.

*Proof.* To prove the main integral, we first express $A_n[x]$, $I_3(z, x, \nu_1, \mu_1, \alpha_1)$ and $E^{\gamma_3,\delta_3}_{\alpha_3,\beta_3,\tau_3,\mu_3,\eta_3,p}(z; s, r)$ function in the left hand side of (5.2.1) in their respective series form and contour integral form with the help of equations (5.1.1), (5.1.9) and (5.1.12) respectively. Next we change the order of summation and $\xi_1, \xi_2$-integrals with the x-integral (which is permissible under the conditions stated). Thus left hand side of (5.2.1) takes the following form (say $\Delta$):

$$\Delta = \sum_{k=0}^{n}\sum_{t=0}^{\infty}\frac{a_{n-k}}{k!}\frac{[\Gamma(\tau_3)]^r\Gamma(\eta_3)[\Gamma(\gamma_3+\delta_3 t)]^s z_2^k z_1^{\nu_1} z_3^{\eta_3+pt-1}}{\Gamma(\frac{1}{\alpha_1-1})(\Gamma\gamma_3)^s\Gamma(\eta_3+pt)\Gamma(\alpha_3\eta_3-\alpha_3+\beta_3+\alpha_3 pt)[\Gamma(\tau_3+\mu_3 t)]^r}$$

$$\times\frac{1}{(2\pi\omega)^2}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\frac{\Gamma(-\xi_1)\Gamma(\xi_2)\Gamma(\frac{1}{\alpha_1-1}-\xi_2)\Gamma(\nu_1-\xi_2)\Gamma(-\nu_1+\mu_1\xi_1+\xi_2)z^{\xi_1}z_1^{-\mu_1\xi_1-\xi_2}}{\Gamma(\mu_1\xi_1)(\alpha_1-1)^{\xi_2}}$$

$$\Big[\int_0^a x^{\lambda-\lambda_1\nu_1+\lambda_1\mu_1\xi_1+\lambda_1\xi_2+\lambda_2 k+\lambda_3(\eta_3+pt-1)-1}(a-x)^{\sigma-\sigma_1\nu_1+\sigma_1\mu_1\xi_1+\sigma_1\xi_2+\sigma_2 k+\sigma_3(\eta_3+pt-1)-1}$$
$$(1-ux^{\ell})^{-(\rho+\rho_3(\eta_3+pt-1))}dx\Big]\, d\xi_1 d\xi_2 \tag{5.2.2}$$

Now we evaluate the x-integral occurring in (5.2.2) with the help of [52, p. 47, Eq. (1.3.3)]. Next we express the Fox H-function involved in above result in terms of contour integral, and finally reinterpret the result thus obtained in terms of multi variable H-function, we easily arrive at the right hand side of (5.2.1) after a little simplification. □

## 5.3 SPECIAL CASES OF THE FIRST INTEGRAL

(I) In the main integral, if we reduce Appell polynomials $A_n[x]$ to the Cesaro polynomial [112, p. 449, Eq. (20)] by taking $a_{n-k} = {}^{\tau_2+n-k}C_{n-k}k!, \lambda_2 = z_2 = 1, \sigma_2 = 0$, $I_3$ function to the steady state function [5, p. 53, Eq.(2.6)] by taking $\alpha_1 \to 1, \mu_1 = \frac{1}{2}$ and Generalized Mittag Leffler function to the generalized hypergeometric function defined in (5.1.13) by replacing $\delta_3 = p = \alpha_3 = \mu_3 = \sigma_3 = 1, \lambda_3 = \rho_3 = 0$ , we can easily get the following interesting integral after a little simplification.

$$\int_0^a x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^\ell)^{-\rho} g_n^{(\tau_2)}(x) I_3[z, z_1 x^{-\lambda_1}(a-x)^{-\sigma_1}, \nu_1]$$

$$\frac{[z_3(a-x)]^{\eta_3-1}}{\Gamma(\eta_3+\beta_3-1)} {}_2\overline{F}_3 \left[ \begin{array}{c} 1, \gamma_3; s \\ \\ \eta_3, \eta_3+\beta_3-1, \tau_3; r \end{array} \middle| z_3(a-x) \right] dx$$

$$= \sum_{k=0}^{n} \sum_{t=0}^{\infty} \frac{(\Gamma(\tau_3))^r \Gamma(\eta_3)[\Gamma(\gamma_3+t)]^s \Gamma(\tau_2+n-k+1) z_1^{\nu_1} z_3^{\eta_3+t-1} a^{\lambda+\sigma-(\lambda_1+\sigma_1)\nu_1+k+\eta_3+t-2}}{[\Gamma(\gamma_3)]^s \Gamma(\eta_3+t)\Gamma(\eta_3+\beta_3-1+t)[\Gamma(\tau_3+t)]^r \Gamma(\rho)\Gamma(n-k+1)\Gamma(\tau_2+1)}$$

$$H^{0,3:1,0;2,1;1,1}_{3,1:0,2;1,2;1,1} \left[ \begin{array}{c} \frac{z a^{\frac{1}{2}(\lambda_1+\sigma_1)}}{z_1^{\frac{1}{2}}} \\ \\ \frac{a^{(\lambda_1+\sigma_1)}}{z_1} \\ \\ -ua^\ell \end{array} \middle| \begin{array}{llll} A^*: & -; & (1,1); & (1-\rho,1) \\ & & & \\ & & & \\ B^*: & (0,1),(1,\frac{1}{2}); & (0,1),(\nu_1,1); & (0,1) \end{array} \right] \tag{5.3.1}$$

where

$A^* = (1+\nu_1; \frac{1}{2}, 1, 0), (1-\sigma+\sigma_1\nu_1-(\eta_3+t-1); \frac{\sigma_1}{2}, \sigma_1, 0), (1-\lambda+\lambda_1\nu_1-k; \frac{\lambda_1}{2}, \lambda_1, \ell)$

$B^* = (1-(\lambda+\sigma)+(\lambda_1+\sigma_1)\nu_1-k-(\eta_3+t-1); \frac{1}{2}(\lambda_1+\sigma_1), (\lambda_1+\sigma_1), \ell)$

provided that the conditions that are easily obtainable from the existing conditions of (5.2.1) are satisfied.

(II) Next, if we reduce the Appell polynomials $A_n[x]$ to the Laguerre Polynomial [121, p. 101, Eq. (5.1.6)] by taking $a_{n-k} = (-1)^k \; {}^{\alpha_2+n}C_{n-k}, \lambda_2 = z_2 = 1, \sigma_2 = 0$ and Generalized Mittag Leffler function to the Bessel Maitland

Function defined in (5.1.14) by replacing $\beta_3 = 1 + \nu_3, \alpha_3 = \mu_3, p = \sigma_3 = \eta_3 = 1, \lambda_3 = \rho_3 = s = r = 0, z_3 = -1$ , we can easily get the following new integral after a little simplification.

$$\int_0^a x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^{\ell})^{-\rho} L_n^{(\alpha_2)}(x) I_3[z, z_1 x^{-\lambda_1}(a-x)^{-\sigma_1}, \nu_1, \mu_1, \alpha_1] J_{\nu_3}^{\mu_3}(a-x)dx$$

$$= \sum_{k=0}^{n}\sum_{t=0}^{\infty} \frac{(-1)^t(-1)^k\Gamma(\alpha_2+n+1)}{k!\Gamma(n-k+1)\Gamma(\alpha_2+k+1)} \frac{1}{\Gamma(\frac{1}{\alpha_1-1})} z_1^{\nu_1} \frac{a^{\lambda+\sigma-(\lambda_1+\sigma_1)\nu_1+k+t-1}}{t!\Gamma(\mu_3 t+\nu_3+1)\Gamma(\rho)}$$

$$H_{3,1:0,2;1,2;1,1}^{0,3:1,0;2,1;1,1}\left[\begin{array}{c|l} \frac{za^{\mu_1(\lambda_1+\sigma_1)}}{z_1^{\mu_1}} & A^*: \quad -; \quad (1,1); \quad (1-\rho,1) \\ \frac{a^{(\lambda_1+\sigma_1)}}{z_1(\alpha_1-1)} & \\ -ua^{\ell} & B^*: \quad (0,1),(1,\mu_1); \quad (\frac{1}{\alpha_1-1},1),(\nu_1,1); \quad (0,1) \end{array}\right] \tag{5.3.2}$$

where

$A^* = (1+\nu_1; \mu_1, 1, 0), (1-\sigma+\sigma_1\nu_1-t; \sigma_1\mu_1, \sigma_1, 0), (1-\lambda+\lambda_1\nu_1-k; \lambda_1\mu_1, \lambda_1, \ell)$

$B^* = (1-(\lambda+\sigma)+(\lambda_1+\sigma_1)\nu_1-k-t; \mu_1(\lambda_1+\sigma_1), (\lambda_1+\sigma_1), \ell)$

provided that the conditions that are easily obtainable from the existing conditions of (5.2.1) are satisfied.

(III) Again in the main integral, if we reduce Appell polynomial $A_n[x]$ to the Shively Polynomial [112, p. 187, Eq. (49)] by taking $a_{n-k} = \frac{(\lambda_2'+n)_n(-n)_k(\alpha_1')_k\cdots(\alpha_p')_k}{n!(\lambda_2'+n)_k(\beta_1')_k\cdots(\beta_q')_k}$, $\lambda_2 = 1, \sigma_2 = 0$ and Generalized Mittag Leffler function to the generalization of the Mittag-Leffler Function defined in (5.1.15) by replacing $\eta_3 = p = \sigma_3 = s = 1, \rho_3 = r = \lambda_3 = 0$, we can easily get the following interesting result after a little simplification.

$$\int_0^a x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^{\ell})^{-\rho}S_n^{(\lambda_2')}(z_2x)I_3[z,z_1x^{-\lambda_1}(a-x)^{-\sigma_1},\nu_1,\mu_1,\alpha_1]$$
$$E_{\alpha_3,\beta_3}^{\gamma_3,\delta_3}[z_3(a-x)]dx$$

$$=\sum_{k=0}^{n}\sum_{t=0}^{\infty}\frac{(\lambda_2'+n)_n(-n)_k(\alpha_1')_k...(\alpha_p')_k}{n!(\lambda_2'+n)_k(\beta_1')_k...(\beta_q')_k k!}\frac{\Gamma(\gamma_3+\delta_3 t)z_1^{\nu_1}z_2^k z_3^t a^{\lambda+\sigma-(\lambda_1+\sigma_1)\nu_1+k+t-1}}{\Gamma(\gamma_3)\Gamma(\frac{1}{\alpha_1-1})\Gamma(\beta_3+\alpha_3 t)\Gamma(\rho)\Gamma(1+t)}$$

$$H_{3,1:0,2;1,2;1,1}^{0,3:1,0;2,1;1,1}\left[\begin{array}{c|ccc} \frac{za^{\mu_1(\lambda_1+\sigma_1)}}{z_1^{\mu_1}} & A^*: & -; & (1,1); & (1-\rho,1) \\ \frac{a^{(\lambda_1+\sigma_1)}}{z_1(\alpha_1-1)} & & & & \\ -ua^{\ell} & B^*: & (0,1),(1,\mu_1); & (\frac{1}{\alpha_1-1},1),(1,\nu_1); & (0,1) \end{array}\right] \tag{5.3.3}$$

where

$$A^*=(1+\nu_1;\mu_1,1,0),(1-\sigma+\sigma_1\nu_1-t;\sigma_1\mu_1,\sigma_1,0),(1-\lambda+\lambda_1\nu_1-k;\lambda_1\mu_1,\lambda_1,\ell)$$

$$B^*=(1-(\lambda+\sigma)+(\lambda_1+\sigma_1)\nu_1-k-t;\mu_1(\lambda_1+\sigma_1),(\lambda_1+\sigma_1),\ell)$$

provided that the conditions that are easily obtainable from the existing conditions of (5.2.1) are satisfied.

(IV) Next, if we reduce the Appell polynomials $A_n[x]$ to the Bateman's Polynomial [112, p. 183, Eq. (42)] by taking $a_{n-k}=\frac{(-n)_k(n+1)_k}{(1)_k(1)_k}, \lambda_2=\sigma_2=0$ and Generalized Mittag Leffler function to the Struve function defined in (5.1.17) by replacing $s=r=p=\sigma_3=\eta_3=\alpha_3=\mu_3=\gamma_3=\delta_3=1, \beta_3=\frac{3}{2}, \tau_3=\frac{3}{2}+\nu_3, \lambda_3=\rho_3=0, z_3=-\frac{z_3^2}{4}$, we can easily get the following new integral after a little simplification.

$$\int_0^a x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^{\ell})^{-\rho} Z_n[z_2] I_3[z, z_1 x^{-\lambda_1}(a-x)^{-\sigma_1}, \nu_1, \mu_1, \alpha_1] \; {}_1F_2\left[\begin{array}{c} 1 \\ \\ \frac{3}{2}, \frac{3}{2}+\nu_3 \end{array} \middle| \frac{-z_3^2}{4}(a-x)\right] dx$$

$$= \sum_{k=0}^{n}\sum_{t=0}^{\infty} \frac{(-n)_k (n+1)_k}{(1)_k (1)_k k!} \frac{\Gamma(\frac{3}{2}+\nu_3) z_1^{\nu_1} z_2^k (\frac{-z_3^2}{4})^t a^{\lambda+\sigma-(\lambda_1+\sigma_1)\nu_1+t-1}}{\Gamma(\frac{1}{\alpha_1-1})\Gamma(\frac{3}{2}+t)\Gamma(\frac{3}{2}+\nu_3+t)\Gamma(\rho)}$$

$$H^{0,3:1,0;2,1;1,1}_{3,1:0,2;1,2;1,1}\left[\begin{array}{c} \frac{z a^{\mu_1(\lambda_1+\sigma_1)}}{z_1^{\mu_1}} \\ \\ \frac{a^{(\lambda_1+\sigma_1)}}{z_1(\alpha_1-1)} \\ \\ -ua^{\ell} \end{array} \middle| \begin{array}{llll} A^*: & -; & (1,1); & (1-\rho,1) \\ \\ \\ \\ B^*: & (0,1),(1,\mu_1); & (\frac{1}{\alpha_1-1},1),(\nu_1,1); & (0,1) \end{array}\right] \quad (5.3.4)$$

where

$A^* = (1+\nu_1;\mu_1,1,0),(1-\sigma+\sigma_1\nu_1-t;\sigma_1\mu_1,\sigma_1,0),(1-\lambda+\lambda_1\nu_1;\lambda_1\mu_1,\lambda_1,\ell)$

$B^* = (1-(\lambda+\sigma)+(\lambda_1+\sigma_1)\nu_1-t;\mu_1(\lambda_1+\sigma_1),(\lambda_1+\sigma_1),\ell)$

provided that the conditions that are easily obtainable from the existing conditions of (5.2.1) are satisfied.

(V) Again in the main integral, if we reduce Appell polynomial $A_n[x]$ to the Bessel Polynomial [60, p. 108, Eq. (34)] by taking $a_{n-k} = \frac{(-n)_k(\alpha_2+n-1)_k(-1)^k}{\beta_2^k}$, $\lambda_2 = \sigma_2 = 0$ and Generalized Mittag Leffler function to the Generalized Bessel Maitland function defined in (5.1.16) by replacing $\lambda_3 = \rho_3 = 0, s = r = p = \sigma_3 = \eta_3 = \mu_3 = \gamma_3 = \delta_3 = 1, \alpha_3 = \mu_3, \tau_3 = \tau_3+1, \beta_3 =$

$\nu_3+\tau_3+1, z_3=-\frac{z_3^2}{4}$ we can easily get the following interesting result after a little simplification.

$$\int_0^a x^{\lambda-1}(a-x)^{\sigma-1}(1-ux^{\ell})^{-\rho}y_n[z_2,\alpha_2,\beta_2]I_3[z,z_1x^{-\lambda_1}(a-x)^{-\sigma_1},\nu_1,\mu_1,\alpha_1]$$
$$J_{\nu_3,\tau_3}^{\mu_3}\left(-(a-x)\frac{z_3^2}{4}\right)dx$$

$$=\sum_{k=0}^{n}\sum_{t=0}^{\infty}\frac{(-n)_k(\alpha_2+n+1)_k(-1)^k}{\beta_2^k k!}\frac{\Gamma(\tau_3+1)(-1)^t z_1^{\nu_1}z_2^k(\frac{z_3}{2})^{2t}a^{\lambda+\sigma-(\lambda_1+\sigma_1)\nu_1+t-1}}{\Gamma(\frac{1}{\alpha_1-1})\Gamma(\rho)\Gamma(\nu_3+\tau_3+1+\mu_3 t)\Gamma(\tau_3+1+t)}$$

$$H_{3,1:0,2;1,2;1,1}^{0,3:1,0;2,1;1,1}\left[\begin{array}{c}\frac{za^{\mu_1(\lambda_1+\sigma_1)}}{z_1^{\mu_1}}\\ \\ \frac{a^{(\lambda_1+\sigma_1)}}{z_1(\alpha_1-1)}\\ \\ -ua^{\ell}\end{array}\middle|\begin{array}{llll}A^*: & -; & (1,1); & (1-\rho,1)\\ \\ \\ \\ B^*: & (0,1),(1,\mu_1); & (\frac{1}{\alpha_1-1},1),(\nu_1,1); & (0,1)\end{array}\right] \tag{5.3.5}$$

where

$A^*=(1+\nu_1;\mu_1,1,0),(1-\sigma+\sigma_1\nu_1-t;\sigma_1\mu_1,\sigma_1,0),(1-\lambda+\lambda_1\nu_1;\lambda_1\mu_1,\lambda_1,\ell)$

$B^*=(1-(\lambda+\sigma)+(\lambda_1+\sigma_1)\nu_1-t;\mu_1(\lambda_1+\sigma_1),(\lambda_1+\sigma_1),\ell)$

provided that the conditions that are easily obtainable from the existing conditions of (5.2.1) are satisfied.

## 5.4 SECOND INTEGRAL

$$\int_0^1 x^{\lambda-1}(1-x)^{\sigma-1}(1-ux^a)^{-\gamma}(1+vx^b)^{-\beta}I_3(z,x^{-\lambda_1}(1-x)^{-\sigma_1},\nu_1,\rho_1,\mu_1,\alpha_1)$$
$$\overline{H}_{p,q}^{m,n}\left[x^{\lambda_2}(1-x)^{\sigma_2}(1-ux^a)^{-\gamma_2}(1+vx^b)^{-\beta_2}\middle|\begin{array}{ll}(e_j,E_j;\in_j)_{1,n}, & (e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m}, & (f_j,F_j;\Im_j)_{m+1,q}\end{array}\right]dx$$

$$= \frac{1}{\Gamma(\gamma+\gamma_2\mathfrak{s}_{t,h})\Gamma(\beta+\beta_2\mathfrak{s}_{t,h})\Gamma(\frac{1}{\alpha_1-1})}\sum_{t=0}^{\infty}\sum_{h=1}^{m}\overline{\Theta}(\mathfrak{s}_{t,h})H^{0,3;1,0;2,1;1,1;1,1}_{3,1;0,2;1,2;1,1;1,1}\left[\begin{array}{c|c} \frac{1}{z^{\mu}} & A^*:C^* \\ \frac{1}{(\alpha_1-1)} & \\ -u & \\ v & B^*:D^* \end{array}\right] \tag{5.4.1}$$

where the remaining parameters of $\overline{H}^{m,n}_{p,q}[z]$ occurring in (5.4.1) are given by left hand side of (5.1.6) and

$A^* = (1-\lambda+\lambda_1\nu_1-\lambda_2\mathfrak{s}_{t,h};\lambda_1\mu_1,\lambda_1,a,0),(1-\sigma+\sigma_1\nu_1-\sigma_2\mathfrak{s}_{t,h};\sigma_1\mu_1,\sigma_1,0,b),$

$(1+\nu_1;0,0,\mu_1,1).$

$B^* = (1-(\sigma+\lambda)+(\sigma_1+\lambda_1)\nu_1-(\sigma_2+\lambda_2)\mathfrak{s}_{t,h};(\sigma_1+\lambda_1)\mu_1,(\sigma_1+\lambda_1),a,b)$

$C^* = -;(1,1);(1-\gamma-\gamma_2\mathfrak{s}_{t,h},1);(1-\beta-\beta_2\mathfrak{s}_{t,h},1)$

$D^* = (0,1),(1,\mu_1);(\frac{1}{\alpha_1-1},1),(\nu_1,1);(0,1);(0,1)$

and $\quad \mathfrak{s}_{t,h} = \frac{f_h+t}{F_h}$

The above result is valid under the following :

(i) $\Re\left(\lambda+\lambda_1\min\{\frac{1}{\alpha_1-1},\nu_1\}+\lambda_2\min_{1\le j\le m}\left(\frac{f_j}{F_j}\right)\right)>0.$

(ii) $\Re\left(\sigma+\sigma_1\min\{\frac{1}{\alpha_1-1},\nu_1\}+\sigma_2\min_{1\le j\le m}\left(\frac{f_j}{F_j}\right)\right)>0.$

(iii) $\min\Re\{\gamma,\gamma_2,\beta,\beta_2,\lambda,\lambda_2,\sigma,\sigma_2,\nu_1\}\ge 0;\quad \min(\mu_1,\sigma_1,\lambda_1,a,b)\ge 0$

not all zero simultaneously.

*Proof.* : To evaluate the above integral we first express $\overline{H}-$ function in series form and astrophysical thermonuclear function $I_3(z,x^{-\lambda_1}(1-x)^{-\sigma_1},\nu_1,\rho_1,\mu_1,\alpha_1)$ in terms of Mellin-Barnes contour integral with help of (5.1.5) and (5.1.9) respectively. Then we change the order of summations and $\xi_1,\xi_2$-integral with x-integral (which is permissible under the conditions stated). Thus the left hand side of (5.4.1) takes the following form (say $\Delta$):

$$\Delta = \sum_{t=0}^{\infty}\sum_{h=1}^{m}\frac{\overline{\Theta}(\mathfrak{s}_{\mathfrak{t},\mathfrak{h}})}{\Gamma(\frac{1}{\alpha_1-1})}\frac{1}{(2\pi\omega)^2}\int\limits_{\mathfrak{L}_1}\int\limits_{\mathfrak{L}_2}\frac{\Gamma(-\nu_1+\mu_1\xi_1+\xi_2)\Gamma(-\xi_1)\Gamma(\xi_2)\Gamma(\frac{1}{\alpha_1-1}-\xi_2)\Gamma(\nu_1-\rho\xi_2)}{\Gamma(\mu_1\xi_1)(\alpha_1-1)^{\xi_2}}$$

$$\times z^{-\mu_1\xi_1}\int\limits_0^1 x^{\lambda-\lambda_1\nu_1+\lambda_1\mu_1\xi_1+\lambda_1\xi_2+\lambda_2\mathfrak{s}_{t,h}-1}(1-x)^{\sigma-\sigma_1\nu_1+\sigma_1\mu_1\xi_1+\sigma_1\xi_2+\sigma_2\mathfrak{s}_{t,h}-1}$$

$$(1-ux^a)^{-\gamma-\gamma_2\mathfrak{s}_{t,h}}(1+vx^b)^{-\beta-\beta_2\mathfrak{s}_{t,h}}dxd\xi_1d\xi_2 \quad (5.4.2)$$

Finally, we evaluate the x-integral occurring in (5.4.2) with the help of result [33, p. 287, Eq.(3.211)] and reinterpreting the result thus obtained in terms of the multivariable H-function, we easily arrive at the right hand side of (5.4.1), after a little simplification. □

## 5.5 SPECIAL CASES OF THE SECOND INTEGRAL

(I.) In the main integral (5.4.1) , if we reduce the $\overline{H}$−function to Generalized Wright Bessel Function [41, p. 271, Eq.(8)] and take $\gamma_2 = \beta_2 = 0$, we easily get the following integral after a little simplification:

$$\int\limits_0^1 x^{\lambda-1}(1-x)^{\sigma-1}(1-ux^a)^{-\gamma}(1+vx^b)^{-\beta}I_3(z, x^{-\lambda_1}(1-x)^{-\sigma_1}, \nu_1, \rho_1, \mu_1, \alpha_1)$$

$$\overline{J}_{\tau_2}^{\nu_2,\mu_2}(x^{\lambda_2}(1-x)^{\sigma_2})dx$$

$$= \frac{1}{\Gamma(\gamma)\Gamma(\beta)\Gamma(\frac{1}{\alpha_1-1})} \sum_{t=0}^{\infty} \frac{(-1)^t}{t!\{\Gamma(1+\tau_2+\nu_2 t)\}^{\mu_2}} H^{0,3;1,0;2,1;1,1;1,1}_{3,1;0,2;1,2;1,1;1,1} \left[ \begin{array}{c|c} \frac{1}{z^{\mu}} & A^* : C^* \\ \frac{1}{(\alpha_1-1)} & \\ -u & \\ v & B^* : D^* \end{array} \right] \tag{5.5.1}$$

where

$A^* = (1-\lambda+\lambda_1\nu_1-\lambda_2 t; \lambda_1\mu_1, \lambda_1, a, 0), (1-\sigma+\sigma_1\nu_1-\sigma_2 t; \sigma_1\mu_1, \sigma_1, 0, b),$

$(1+\nu_1; 0, 0, \mu_1, 1).$

$B^* = (1-(\sigma+\lambda)+(\sigma_1+\lambda_1)\nu_1-(\lambda_2+\sigma_2)t; (\sigma_1+\lambda_1)\mu_1, (\sigma_1+\lambda_1), a, b)$

$C^* = -; (1,1); (1-\gamma, 1); (1-\beta, 1)$

$D^* = (0,1), (1, \mu_1); (\frac{1}{\alpha_1-1}, , 1), (\nu_1, 1)(0,1); (0,1).$

provided that the conditions easily obtainable from those mentioned with conditions (5.4.1) are satisfied.

(II.) On taking in (5.4.1), $\lambda_1 = \lambda_2 = \sigma_2 = 0$ and reducing $\overline{H}-$function to the generalized Riemann Zeta Function [23, p. 27, §1.11, Eq.(1)], we easily get the following integral after a little simplification:

$$\int_0^1 x^{\lambda-1}(1-x)^{\sigma-1}(1-ux^a)^{-\gamma}(1+vx^b)^{-\beta} I_3(z, (1-x)^{-\sigma_1}, \nu_1, \rho_1, \mu_1, \alpha_1)$$
$$\phi((1-ux^a)^{-\gamma_2}(1+vx^b)^{-\beta_2}, p, \eta)dx$$

$$= \frac{1}{\Gamma(\gamma+\gamma_2 t)\Gamma(\beta+\beta_2 t)\Gamma(\frac{1}{\alpha_1-1})} \sum_{t=0}^{\infty} \frac{1}{(\eta+t)^p} H^{0,2;1,0;2,1;1,2;1,1}_{2,1;0,2;1,2;2,1;1,1} \left[ \begin{array}{c|c} \frac{1}{z^{\mu}} & A^* : C^* \\ \frac{1}{(\alpha_1-1)} & \\ -u & \\ v & B^* : D^* \end{array} \right] \tag{5.5.2}$$

where

$A^* = (1-\sigma+\sigma_1\nu_1; \sigma_1\mu_1, \sigma_1, 0, b), (1+\nu_1; 0, 0, \mu_1, 1).$

$B^* = (1-(\sigma+\lambda)+\sigma_1\nu_1; \sigma_1\mu_1, \sigma_1, a, b)$

$C^* = --; (1,1); (1-\gamma-\gamma_2 t, 1), (1-\lambda, a); (1-\beta-\beta_2 t, 1)$

$D^* = (0,1), (1,\mu_1); (\frac{1}{\alpha_1-1}, 1), (\nu_1, 1); (0,1); (0,1).$

provided that the conditions that are easily obtainable from the existing conditions (5.4.1) are satisfied.

(III.) Further in the main integral (5.4.1), if we take $\lambda_1 = \sigma_1 = \lambda_2 = \sigma_2 = 0$ and reduce $\overline{H}-$function to the generalized Hurwitz-Lerch Zeta Function [51, pp. 147 & 151, Eqs.(6.2.5) and (6.4.2)], we arrive at the following interesting integral after a little simplification which is believed to be new:

$$\int_0^1 x^{\lambda-1}(1-x)^{\sigma-1}(1-ux^a)^{-\gamma}(1+vx^b)^{-\beta} I_3(z,1,\nu_1,\rho_1,\mu_1,\alpha_1)$$
$$\phi_{\nu_2,\mu_2,\tau_2}((1-ux^a)^{-\gamma_2}(1+vx^b)^{-\beta_2}, p, \eta)dx$$

$$= \frac{1}{\Gamma(\gamma+\gamma_2 t)\Gamma(\beta+\beta_2 t)\Gamma(\frac{1}{\alpha_1-1})} \sum_{t=0}^{\infty} \frac{(\nu_2)_t(\mu_2)_t}{(\tau_2)_t t!(\eta+t)^p} H^{0,1;1,0;2,1;1,2;1,2}_{1,1;0,2;1,2;2,1;2,1} \left[ \begin{array}{c|c} \frac{1}{z^{\mu}} & A^*:C^* \\ \frac{1}{(\alpha_1-1)} & \\ -u & \\ v & B^*:D^* \end{array} \right] \tag{5.5.3}$$

where

$A^* = (1+\nu_1; 0, 0, \mu_1, 1) \qquad B^* = (1-(\sigma+\lambda); 0, 0, a, b)$

$C^* = -; (1,1); (1-\gamma-\gamma_2 t, 1), (1-\lambda, a); (1-\beta-\beta_2 t, 1), (1-\sigma, b)$

$D^* = (0,1), (1,\mu_1); (\frac{1}{\alpha_1-1}, 1), (\nu_1, 1); (0,1); (0,1).$

(IV.) Again if we take in (5.4.1), $\lambda_1 = \gamma_2 = \beta_2 = 0$ and reduce $\overline{H}-$function to Polylogarithm of order p [23, p. 31 §1.11.1, Eq.(22) and (6.4.2)], we get an interesting integral after a little simplification which is believed to be new:

$$\int_0^1 x^{\lambda-1}(1-x)^{\sigma-1}(1-ux^a)^{-\gamma}(1+vx^b)^{-\beta} I_3(z,(1-x)^{-\sigma_1},\nu_1,\rho_1,\mu_1,\alpha_1)$$
$$F(x^{\lambda_2}(1-x)^{\sigma_2},p)dx$$

$$= \frac{1}{\Gamma(\gamma)\Gamma(\beta)\Gamma(\frac{1}{\alpha_1-1})} \sum_{t=0}^{\infty} \frac{(-1)^{t+1}}{t^p} H^{0,2;1,0;2,1;1,2;1,1}_{2,1;0,2;1,2;2,1;1,1} \left[ \begin{array}{c|c} \frac{1}{z^{\mu}} & A^*:C^* \\ \frac{1}{(\alpha_1-1)} & \\ -u & \\ v & B^*:D^* \end{array} \right] \tag{5.5.4}$$

where

$A^* = (1-\sigma+\sigma_1\nu_1-\sigma_2(1+t); \sigma_1\mu_1, \sigma_1, 0, b), (1+\nu_1; 0, 0, \mu_1, 1).$

$B^* = (1-(\sigma+\lambda)+\sigma_1\nu_1-(\lambda_2+\sigma_2)(1+t);\sigma_1\mu_1,\sigma_1,a,b)$

$C^* = -;(1,1);(1-\gamma,1),(1-\lambda-\lambda_2(1+t),a);(1-\beta,1)$

$D^* = (0,1),(1,\mu_1);(\frac{1}{\alpha_1-1},1),(\nu_1,1);(0,1);(0,1).$

(V.) Finally, if we reduce the $\overline{H}-$function to generalized hypergeometric function ${}_pF_q$ [41, p. 271, Eq.(9)] and take $\lambda_1=\sigma_1=\gamma_2=\beta_2=0$ in the main integral (5.4.1), we obtain the following integral after a little simplification:

$$\int_0^1 x^{\lambda-1}(1-x)^{\sigma-1}(1-ux^a)^{-\gamma}(1+vx^b)^{-\beta}I_3(z,1,\nu_1,\rho_1,\mu_1,\alpha_1)$$
$$ {}_pF_q\left[x^{\lambda_2}(1-x)^{\sigma_2}\,\middle|\,\begin{matrix}(e_j,1;\rho_j)_{1,p}\\ \\(f_j,1;\eta_j)_{1,q}\end{matrix}\right]dx$$

$$=\frac{1}{\Gamma(\gamma)\Gamma(\beta)\Gamma(\frac{1}{\alpha_1-1})}\sum_{t=0}^{\infty}\frac{\prod_{j=1}^{p}\{(e_j)_t\}^{\rho_j}}{\prod_{j=1}^{q}\{(f_j)_t\}^{\eta_j}t!}H^{0,1;1,0;2,1;1,2;1,2}_{1,1;0,2;1,2;2,1;2,1}\left[\begin{matrix}\frac{1}{z^\mu}\\ \frac{1}{(\alpha_1-1)}\\ -u\\ v\end{matrix}\,\middle|\,\begin{matrix}A^*:C^*\\ \\ \\ B^*:D^*\end{matrix}\right] \tag{5.5.5}$$

where

$A^* = (1+\nu_1;0,0,\mu_1,1)\qquad B^* = (1-(\sigma+\lambda)-(\lambda_2+\sigma_2)t;0,0,a,b)$

$C^* = -;(1,1);(1-\gamma,1),(1-\lambda-\lambda_2 t,a);(1-\beta,1),(1-\sigma-\sigma_2 t,b)$

$D^* = (0,1),(1,\mu_1);(\frac{1}{\alpha_1-1},1),(\nu_1,1);(0,1);(0,1).$

provided that the conditions that are easily obtainable from the existing conditions (5.4.1) are satisfied.

# 6

# SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION BY GENERALIZED DIFFERENTIAL TRANSFORM METHOD (GDTM)

The main findings of this chapter have been published as detailed below:

1. **M. K. BANSAL and R. JAIN** (2015). APPLICATION OF GENERALIZED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL RELAXATION OSCILLATION EQUATION, *Antarctica J. Math.*, **12(1)**, 85–95.

2. **M. K. BANSAL and R. JAIN** (2015). ANALYTICAL SOLUTION OF BAGLEY TORVIK EQUATION BY GENERALIZED DIFFERENTIAL TRANSFORM, *International Journal of Pure and Applied Mathematics*, **110(2)**, 265–273.

3. **M. K. BANSAL and R. JAIN** (2015). APPLICATION OF GENERALIZED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION AND NUMERICAL RESULT, *Inter-*

*national Journal of Pure and Applied Mathematics*, **99(3)**, 355–366.

The object of this chapter is to find solutions of the Bagley Torvik Equation, Fractional Relaxation Oscillation Equation and Fractional Order Riccati Differential Equation. We make use of generalized differential transform method (GDTM) to solve the equations. First of all we give definition of a Caputo fractional derivative of order $\alpha$ which was introduced and investigated by Caputo [12]. Then, we give the generalized differential transform method and inverse generalized differential transform which was introduce and investigated by Ertuk et al. [22] and some basic properties of GDTM. Next, we find solutions to three different fractional differential equations using GDTM technique.

In section 6.2 we find the solution of Bagley Torvik Equation using GDTM. Since all constant coefficients and function f(t) are general in nature, by specializing the constant coefficients and function f(t) we can obtain a large number of special cases of Bagley Torvik Equation. Here we give two numerical examples.

In section 6.3 we find the solution of Fractional Relaxation Oscillation Equation using GDTM. Since order of Fractional Relaxation Oscillation Equation is $\beta$, A is constant coefficient and function f(x) are general in nature, by specializing parameters and function f(x), we can obtain a large number of special cases of Fractional Relaxation Oscillation Equation. Here we give eight numerical examples. Furthermore these examples are also represented graphically by using the MATHEMATICA SOFTWARE.

In section 6.4, Again we find the solution of Fractional Order Riccati Differential Equation using GDTM. Since order of Fractional Riccati Differential equation is $\beta$ and all function are general in nature, by specializing parameters and functions, we can obtain a large number of special cases of Fractional Relaxation Oscillation Equation. Here we give eight numerical examples. Furthermore these examples are also represented graphically by using the MATHEMATICA SOFTWARE.

# 6.1 INTRODUCTION

## 6.1.1 CAPUTO FRACTIONAL DERIVATIVE OF ORDER $\alpha$

The Caputo fractional derivative of order $\alpha$ was introduced and investigate by Caputo [11] in the following manner

$$
{}_a^C D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int\limits_a^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi & \text{if} \quad (m-1<\alpha<m), m \in \mathbb{N} \\ \\ f^{(m)}(x) & \text{if} \quad \alpha = m \end{cases} \tag{6.1.1}
$$

## 6.1.2 KNOWN BASIC THEOREMS

The following theorems will be required to obtain our main findings :

**Theorem 6.1.1** ([22] Generalized Taylor Formula)**.** *Suppose that* $({}_a^C D_x^\alpha)^k f(x) \in C(a,b]$ *for k = 0, 1, 2,....n + 1, where* $0 < \alpha \leq 1$*, then we have*

$$
f(x) = \sum_{\mathrm{i}=0}^{n} \frac{(x-a)^{\mathrm{i}\alpha}}{\Gamma(\mathrm{i}\alpha+1)}(({}_a^C D_x^\alpha)^{\mathrm{i}} f)(a) + \frac{(({}_a^C D_x^\alpha)^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)}.(x-a)^{(n+1)\alpha} \tag{6.1.2}
$$

*with* $a \leq \xi \leq x, \forall \quad x \in (a,b]$.

**Theorem 6.1.2.** *[26] Suppose that f(x) =* $(x-x_0)^\lambda g(x)$*, where* $x_0, \lambda > 0$ *and g(x) has the generalized power series expansion* $g(x) = \sum\limits_{n=0}^{\infty} a_n (x-x_0)^{n\alpha}$ *with radius of* *convergence* $R > 0$, $0 < \alpha \leq 1$. *Then*

$$ {}_a^C D_x^{\gamma} \quad {}_a^C D_x^{\beta} f(x) = {}_a^C D_x^{\gamma+\beta} f(x) \tag{6.1.3} $$

*for all* $(x - x_0) \in (0, R)$, *the coefficients* $a_n = 0$ *for* $n$ *given by* $n\alpha + \lambda - \beta = 0$ *and either*
*(a)* $\lambda > \mu$, $\mu = max(\beta + [\gamma], [\beta + \gamma])$
*or*
*(b)* $\lambda \leq \mu$, $a_k = 0$ *for* $k$ = *0,1,2,...,*$[\frac{\mu-\lambda}{\alpha}]$
*where [x] denotes the greatest integer less than or equal to x.*

### 6.1.3 GENERALIZED DIFFERENTIAL TRANSFORM METHOD (GDTM)

The generalized differential transform of the $K^{th}$ derivative of function f(x) in one variable was introduced and investigated by Ertuk et al. [22, p. 1646] in the following manner:

$$ F_\alpha(k) = \frac{1}{\Gamma(\alpha k + 1)} [({}_a^C D_x^{\alpha})^k f(x)]_{x=x_0} \tag{6.1.4} $$

where $0 < \alpha \leq 1, ({}_a^C D_x^{\alpha})^k = {}_a^C D_x^{\alpha} . {}_a^C D_x^{\alpha} .... {}_a^C D_x^{\alpha} (k - times)$, ${}_a^C D_x^{\alpha}$ is defined by (6.1.1) and $F_\alpha(k)$ is the transformed function.

The Inverse Generalized Differential Transform of $F_\alpha(k)$ is defined in the following manner [22, p. 1647]

$$ f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x - x_0)^{\alpha k} \tag{6.1.5} $$

### 6.1.4 SOME BASIC PROPERTIES OF THE GENERALIZED DIFFERENTIAL TRANSFORM

If $F_\alpha(k), G_\alpha(k)$ and $H_\alpha(k)$ are the generalized differential transforms of the functions f(x), g(x) and h(x) respectively, then

(a) If f(x) = g(x) ± h(x), then $F_\alpha(k) = G_\alpha(k) \pm H_\alpha(k)$

(b) If f(x) = ag(x), then $F_\alpha(k) = aG_\alpha(k)$, where a is a constant.

(c) If f(x) = g(x)h(x), then $F_\alpha(k) = \sum_{l=0}^{k} G_\alpha(l)H_\alpha(k-l)$

(d) If f(x) = ${}_a^C D_x^\alpha g(x)$, then $F_\alpha(k) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} G_\alpha(k+1)$

(e) If f(x) = $(x-x_0)^\gamma, \gamma = n\alpha, n \in \mathbb{Z}$, then $F_\alpha(k) = \delta(k-\gamma/\alpha)$,

where

$$\delta(k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \end{cases}$$

(f) If $f(x) = {}_b^C D_x^\beta g(x), m-1 < \beta \leq m$ and the function g(x) satisfies the condition in Theorem 6.1.2, then

$$F_\alpha(k) = \frac{\Gamma(\alpha k + \beta + 1)}{\Gamma(\alpha k + 1)} G_\alpha\left(k + \frac{\beta}{\alpha}\right)$$

## 6.2 APPLICATION OF GDTM TO BAGLEY TORVIK EQUATION

In this section we consider Bagley Torvik Equation in the following form:

$$AD^2y(t) + BD^{\frac{3}{2}}y(t) + Cy(t) = f(t) \qquad \text{where} \qquad t > 0 \tag{6.2.1}$$

Subject to initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0 \tag{6.2.2}$$

where $A \neq 0$, B and C are constant coefficients.

**SOLUTION:** Applying generalized differential transform (6.1.4) on the both sides of (6.2.1) by using the property of GDTM (f) with a = b = 0, we easily arrive at the following result

$$A\frac{\Gamma(\alpha k+3)}{\Gamma(\alpha k+1)}Y_\alpha\left(k+\frac{2}{\alpha}\right)+B\frac{\Gamma(\alpha k+\frac{3}{2}+1)}{\Gamma(\alpha k+1)}Y_\alpha\left(k+\frac{3}{2\alpha}\right)+CY_\alpha(k)=F_\alpha(k) \tag{6.2.3}$$

or

$$Y_\alpha\left(k+\frac{2}{\alpha}\right)=\frac{F_\alpha(k)-CY_\alpha(k)-B\frac{\Gamma(\alpha k+\frac{5}{2})}{\Gamma(\alpha k+1)}Y_\alpha(k+\frac{3}{2\alpha})}{A(\alpha k+2)(\alpha k+1)} \tag{6.2.4}$$

Where $Y_\alpha(k)$ be the Generalized Differential Transform Function of y(t).

The generalized differential transform of the initial conditions in Eq. (6.2.2) takes the following form

$$[{}_a^C D_{t_0}^{\alpha k} y(t)]_{t=0}=0 \qquad \text{for } k=0,1,2,3,... \tag{6.2.5}$$

Utilizing the recurrence relation (6.2.4) and the transformed initial conditions (6.2.5) we calculated the value of $Y_\alpha\left(k+\frac{2}{\alpha}\right)$ for k = 1, 2, 3, $\cdots$. Then taking the inverse generalized differential transform (6.1.5), we obtain the desire solution after a little simplification.

**EXAMPLE 1:** Consider the following special case of Bagley-Torvik equation given by (6.2.1) as investigated earlier [29, 92]

$$D^2y(t)+D^{\frac{3}{2}}y(t)+y(t)=2+4\sqrt{\frac{t}{\pi}}+t^2 \tag{6.2.6}$$

Subject to initial conditions

$$y(0)=0 \qquad \text{and} \qquad y'(0)=0 \tag{6.2.7}$$

**SOLUTION:** Applying generalized differential transform (6.1.4) on the both sides of (6.2.6) by using the property of GDTM (f) with a = b = 0 and setting $t_0 = 0, \alpha = 1/2$, we easily arrive at the following result

$$Y_{\frac{1}{2}}(k+4) = \frac{2\delta(k) + \frac{4}{\sqrt{\pi}}\delta(k-1) + \delta(k-4) - Y_{\frac{1}{2}}(k) - \frac{\Gamma(\frac{k}{2}+\frac{5}{2})}{\Gamma(\frac{k}{2}+1)}Y_{\frac{1}{2}}(k+3)}{(\frac{k}{2}+2)(\frac{k}{2}+1)} \tag{6.2.8}$$

and initial conditions (6.2.7) takes the following form

$$Y_{\frac{1}{2}}(0) = 0; \qquad Y_{\frac{1}{2}}(1) = 0; \qquad Y_{\frac{1}{2}}(2) = 0; \qquad Y_{\frac{1}{2}}(3) = 0 \tag{6.2.9}$$

Utilizing the recurrence relation (6.2.8) and the transformed initial conditions (6.2.9) we calculated the value of $Y_{\frac{1}{2}}(k)$, for k = 1, 2,$\cdots$. Then we can easily obtain the following result after a little simplification

$$Y_{\frac{1}{2}}(k) = \begin{cases} 1 & \text{if} \quad k = 4 \\ 0 & \text{if} \quad k \neq 4 \end{cases} \tag{6.2.10}$$

Now, from (6.1.5), and f(x) replaced by y(t) then it takes the following form

$$y(t) = \sum_{k=0}^{\infty} Y_{\frac{1}{2}}(k) t^{\frac{k}{2}} \tag{6.2.11}$$

Next, using the values of $Y_{\frac{1}{2}}(k)$ from (6.2.10) in (6.2.11) then we get the exact solution of Bagley-Torvik equation (6.2.6) is

$$y(t) = t^2 \tag{6.2.12}$$

which is same as obtained by Ghorbani and Alavi [29] by using He's Variational Iteration Method.

We can obtain same solution for all values of $\alpha$ such that $0 < \alpha \leq 1$.

**Example 2:** We consider the following special case of Bagley-Torvik equation given by (6.2.1) as investigated earlier [20, 21]

$$D^2 y(t) + D^{\frac{3}{2}} y(t) + y(t) = t + 1 \tag{6.2.13}$$

subject to initial condition

$$y(0) = 1 \qquad \text{and} \qquad y'(0) = 1 \tag{6.2.14}$$

**SOLUTION:** Applying generalized differential transform (6.1.4) on the both sides of (6.2.13) by using the property of GDTM (f) with a = b = 0 and setting $t_0 = 0, \alpha = 1/2$, we easily arrive at the following result

$$Y_{\frac{1}{2}}(k+4) = \frac{\delta(k) + \delta(k-2) - Y_{\frac{1}{2}}(k) - \dfrac{\Gamma(\frac{k}{2}+\frac{5}{2})}{\Gamma(\frac{k}{2}+1)} Y_{\frac{1}{2}}(k+3)}{(\frac{k}{2}+2)(\frac{k}{2}+1)} \tag{6.2.15}$$

and initial conditions (6.2.14) takes the following form

$$Y_{\frac{1}{2}}(0) = 1; \quad Y_{\frac{1}{2}}(1) = 0; \quad Y_{\frac{1}{2}}(2) = 1; \quad Y_{\frac{1}{2}}(3) = 0 \tag{6.2.16}$$

Utilizing the recurrence relation (6.2.15) and the transformed initial conditions (6.2.16) we calculated the value of $Y_{\frac{1}{2}}(k)$, for k = 1, 2,$\cdots$. Then we can easily obtain the following result after a little simplification

$$Y_{\frac{1}{2}}(k) = \begin{cases} 1 & \text{if} \qquad k = 0, 2 \\ 0 & \text{if} \qquad k \neq 0, 2 \end{cases} \tag{6.2.17}$$

Now, from (6.1.5), and f(x) replaced by y(t) then it takes the following form

$$y(x) = \sum_{k=0}^{\infty} Y_{\frac{1}{2}}(k) x^{\frac{k}{2}} \tag{6.2.18}$$

Using the values of $Y_{\frac{1}{2}}(k)$ from (6.2.17) in (6.2.18), the exact solution of Bagley-Torvik equation (6.2.13) is obtained as

$$y(t) = 1 + t \tag{6.2.19}$$

which is same as obtained by El-Sayed at el. [20] by using Adomian decomposition method (ADM) and proposed numerical method (PNM).

We can obtain same solution for all values of $\alpha$ such that $0 < \alpha \leq 1$.

## 6.3 APPLICATION OF GDTM TO FRACTIONAL RELAXATION OSCILLATION EQUATION

In this section, we shall apply GDTM for solving Fractional Relaxation Oscillation Equation:

Consider the Fractional Relaxation Oscillation Equation in the following form:

$$D^{\beta}y(x) + Ay(x) = f(x), \qquad x > 0 \tag{6.3.1}$$

Subject to initial conditions

$$y(0) = \lambda \qquad \text{and} \qquad y'(0) = \mu \tag{6.3.2}$$

where A is positive constant and $0 < \beta \leq 2$.

**SOLUTION:** Applying generalized differential transform (6.1.4) on the both sides of (6.3.1) by using the property of GDTM (f) with a = b = 0, we easily arrive at the following result

$$\frac{\Gamma(\alpha k + \beta + 1)}{\Gamma(\alpha k + 1)} Y_{\alpha}\left(k + \frac{\beta}{\alpha}\right) + AY_{\alpha}(k) = F_{\alpha}(k) \tag{6.3.3}$$

or

$$Y_\alpha\left(k+\frac{\beta}{\alpha}\right)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}[F_\alpha(k)-AY_\alpha(k)] \tag{6.3.4}$$

Where $Y_\alpha(k)$ be the Generalized Differential Transform function of y(t).

The generalized differential transform of the initial conditions in Eq. (6.3.2) takes the following form

$$Y_\alpha(k)=\frac{1}{\Gamma(\alpha k+1)}[({}_a^C D_{x_0}^\alpha)^k y(x)]_{x=0} \qquad \text{for} \qquad k=0,1,2,... \tag{6.3.5}$$

Utilizing the recurrence relation (6.3.4) and the transformed initial conditions (6.3.5) we calculated the value of $Y_\alpha\left(k+\frac{\beta}{\alpha}\right)$ for k = 1, 2, 3, $\cdots$. Then taking the inverse generalized differential transform (6.1.5), we obtain the desire solution after a little simplification.

**Example 3:** If we take $\beta=\frac{1}{4}$ , A = 1 and f(x) = 0 in (6.3.1), it reduces to Relaxation-Oscillation Equation [44]

$$D^{\frac{1}{4}}y(x)+y(x)=0 \tag{6.3.6}$$

with initial condition

$$y(0)=1 \tag{6.3.7}$$

**SOLUTION:** Applying generalized differential transform (6.1.4) on the both sides of (6.3.6) by using the property of GDTM (f) with a = b = 0 and setting $x_0=0,\alpha=1/4$, we easily arrive at the following result

$$Y_{\frac{1}{4}}(k+1)=-\frac{\Gamma(\frac{k}{4}+1)}{\Gamma(\frac{k}{4}+\frac{5}{4})}Y_{\frac{1}{4}}(k) \tag{6.3.8}$$

and initial conditions (6.3.7) takes the following form

$$Y_{\frac{1}{4}}(0) = 1 \tag{6.3.9}$$

Utilizing the recurrence relation (6.3.8) and the transformed initial conditions (6.3.9) we calculated the value of $Y_{\frac{1}{2}}(k+1)$, for k = 1, 2,$\cdots$. Then we can easily obtain the following result after a little simplification

$$\left.\begin{aligned} Y_{\frac{1}{4}}(1) = -1.1033; \quad Y_{\frac{1}{4}}(2) = 1.1284; \quad Y_{\frac{1}{4}}(3) = -1.0881; \\ Y_{\frac{1}{4}}(4) = 1; \quad Y_{\frac{1}{4}}(5) = -0.8826; \quad Y_{\frac{1}{4}}(6) = 0.7523; \\ Y_{\frac{1}{4}}(7) = -0.62175; \quad Y_{\frac{1}{4}}(8) = 0.5 \end{aligned}\right\} \tag{6.3.10}$$

Similarly we can find the value of $Y_{\frac{1}{4}}(k)$ for k = 7, 8, 9 $\cdots$ .

Now, from (6.1.5), and f(x) replaced by y(x) then it takes the following form

$$y(x) = \sum_{k=0}^{\infty} Y_{\frac{1}{4}}(k) x^{\frac{k}{4}} \tag{6.3.11}$$

Using the values of $Y_{\frac{1}{4}}(k)$ from (6.3.10) in (6.3.11), the exact solution of Fractional Relaxation Oscillation equation (6.3.6) is obtained as

$$\begin{aligned} y(x) =& 1 - 1.1033x^{\frac{1}{4}} + 1.1284x^{\frac{1}{2}} - 1.0881x^{\frac{3}{4}} + x - 0.8826x^{\frac{5}{4}} + 0.7523x^{\frac{3}{2}} \\ & - 0.62175x^{\frac{7}{4}} + 0.5x^2 \cdots \end{aligned} \tag{6.3.12}$$

**Example 4:** If we take $\beta = \frac{1}{2}$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = 1 - 1.1284x^{\frac{1}{2}} + x - 0.7523x^{\frac{3}{2}} + 0.5x^2 \cdots \tag{6.3.13}$$

**Example 5:** If we take $\beta = \frac{3}{4}$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = 1 - 1.0881x^{\frac{3}{4}} + 0.7523x^{\frac{3}{2}} - 0.3923x^{\frac{9}{4}} + 0.1667x^{3} \cdots \tag{6.3.14}$$

**Example 6:** If we take $\beta = 1$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = \exp(-x) \tag{6.3.15}$$

**Example 7:** If we take $\beta = \frac{5}{4}$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = 1 - 0.8826x^{\frac{5}{4}} + 0.3009x^{\frac{5}{2}} - 0.0603x^{\frac{15}{4}} + 0.0083x^{5} \cdots \tag{6.3.16}$$

**Example 8:** If we take $\beta = \frac{3}{2}$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = 1 - 0.7523x^{\frac{3}{2}} + 0.1667x^{3} - 0.0191x^{\frac{9}{2}} + 0.0014x^{6} \cdots \tag{6.3.17}$$

**Example 9:** If we take $\beta = \frac{7}{4}$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = 1 + x - 0.62175x^{\frac{7}{4}} - 0.22609x^{\frac{11}{4}} + 0.085972x^{\frac{7}{2}} \cdots \tag{6.3.18}$$

**Example 10:** If we take $\beta = 2$ , A = 1 and f(x) = 0 in (6.3.1) and follow the method given in Example 3, we easily arrive at the following result

$$y(x) = \cos x \tag{6.3.19}$$

**The graph given below demonstrates have been represented Eq.(6.3.12) to Eq. (6.3.19) graphically by making use of "MATHEMATICA SOFTWARE" as given below**

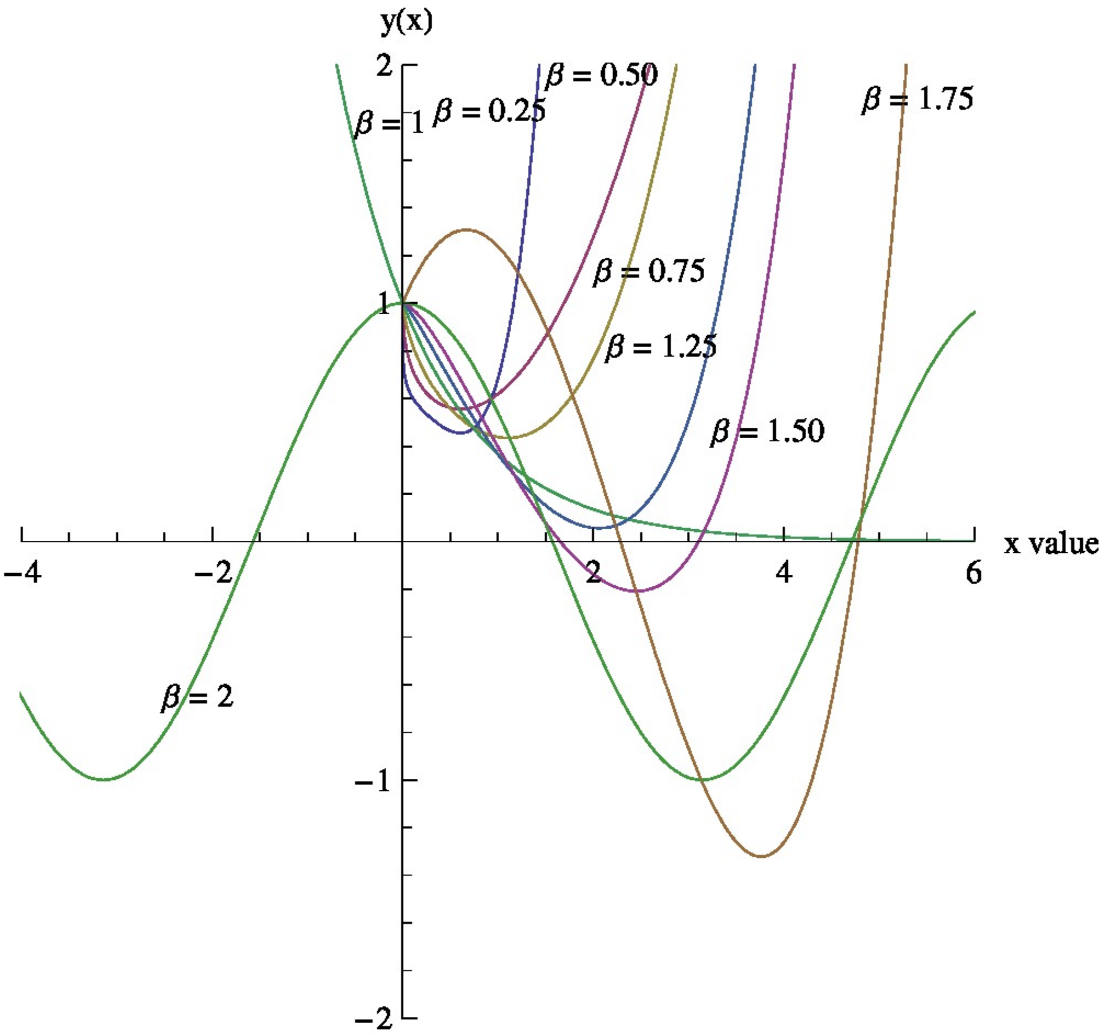


**Figure : 1. Approxiamte solution of example 3 to 10**

## 6.4 APPLICATION OF GDTM TO FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION AND NUMERICAL RESULT

In this section we consider the Fractional Order Riccati Differential Equation in the following form :

$$D^{\beta}y(t) = P(t)y^2(t) + Q(t)y(t) + R(t), \qquad t > 0, \qquad 0 < \beta \leq 1 \tag{6.4.1}$$

Subject to initial condition

$$y(0) = B \tag{6.4.2}$$

where P(t), Q(t) and R(t) are known functions.

**SOLUTION:** Applying generalized differential transform (6.1.4) on the both sides of (6.4.1) by using the property of GDTM (f) with a = b = 0, we easily arrive at the following result

$$\begin{aligned}\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} Y_{\alpha}\left(k+\frac{\beta}{\alpha}\right) &= \sum_{k=0}^{h}\sum_{l=0}^{k} P_{\alpha}(l)Y_{\alpha}(k-l)Y_{\alpha}(h-k) \\ &\quad + \sum_{l=0}^{k} Q_{\alpha}(l)Y_{\alpha}(k-l) + R_{\alpha}(k)\end{aligned} \tag{6.4.3}$$

or

$$Y_{\alpha}\left(k+\frac{\beta}{\alpha}\right) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}\left[\sum_{k=0}^{h}\sum_{l=0}^{k} P_{\alpha}(l)Y_{\alpha}(k-l)Y_{\alpha}(h-k) + \sum_{l=0}^{k} Q_{\alpha}(l)Y_{\alpha}(k-l) + R_{\alpha}(k)\right] \tag{6.4.4}$$

where $Y_{\alpha}(k)$ is the Generalized Differential Transform function of y(t).

The generalized differential transform of the initial conditions in Eq. (6.4.2) takes the following form

$$Y_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)}[({}_a^C D_{t_0}^\alpha)^k y(t)]_{t=0} \qquad \text{for} \qquad k=0,1,2,\cdots \tag{6.4.5}$$

Utilizing the recurrence relation (6.4.4) and the transformed initial conditions (6.4.5) we calculated the value of $Y_\alpha\left(k+\frac{\beta}{\alpha}\right)$ for k = 1, 2, 3, $\cdots$. Then taking the inverse generalized differential transform (6.1.5), we obtain the desire solution after a little simplification.

**Example 11:** If we take $\beta = \frac{1}{4}$ , P(t) = 1, Q(t)= 2 and R(t) = $t^2$ in (6.4.1), it reduces to

$$D^{\frac{1}{4}}y(t) = t^2 + 2y(t) + y^2(t) \tag{6.4.6}$$

Subject to initial condition

$$y(0) = 0 \tag{6.4.7}$$

**SOLUTION :** Applying generalized differential transform (6.1.4) on the both sides of (6.4.6) by using the property of GDTM (f) with a = b = 0 and setting $t_0 = 0, \alpha = 1/4$, we easily arrive at the following result

$$Y_{\frac{1}{4}}(k+1) = \frac{\Gamma(\frac{k}{4}+1)}{\Gamma(\frac{k}{4}+\frac{5}{4})}\left[\delta(k-8) + 2Y_{\frac{1}{4}}(k) + \sum_{l=0}^{k} Y_{\frac{1}{4}}(l)Y_{\frac{1}{4}}(k-l)\right] \tag{6.4.8}$$

and initial conditions (6.4.7) takes the following form

$$Y_{\frac{1}{4}}(0) = 0 \tag{6.4.9}$$

Utilizing the recurrence relation (6.4.8) and the transformed initial conditions (6.4.9) we calculated the value of $Y_{\frac{1}{4}}(k+1)$, for k = 1, 2,$\cdots$. Then we can easily obtain the following result after a little simplification

$$\left.\begin{array}{llll} Y_{\frac{1}{4}}(1)=0; & Y_{\frac{1}{4}}(2)=0; & Y_{\frac{1}{4}}(3)=0; & Y_{\frac{1}{4}}(4)=0; \\ Y_{\frac{1}{4}}(5)=0; & Y_{\frac{1}{4}}(6)=0; & Y_{\frac{1}{4}}(7)=0; & Y_{\frac{1}{4}}(8)=0; \\ Y_{\frac{1}{4}}(9)=0.6018; & Y_{\frac{1}{4}}(10)=1.204; & Y_{\frac{1}{4}}(11)=1.809; & Y_{\frac{1}{4}}(12)=2.667; \\ Y_{\frac{1}{4}}(13)=3.862; & Y_{\frac{1}{4}}(14)=5.502 & Y_{\frac{1}{4}}(15)=7.717; & Y_{\frac{1}{4}}(16)=10.67 \end{array}\right\} \tag{6.4.10}$$

Similarly we can find the value of $Y_{\frac{1}{4}}(k)$ for k = 17, 18, 19 $\cdots$ .

Now, from (6.1.5), and f(t) replaced by y(t) then it takes the following form

$$y(t)=\sum_{k=0}^{\infty} Y_{\frac{1}{4}}(k)t^{\frac{k}{4}} \tag{6.4.11}$$

Using the values of $Y_{\frac{1}{4}}(k)$ from (6.4.10) in (6.4.11), the exact solution of Fractional Order Reccati Differential Equation (6.4.6) is obtained as

$$\begin{aligned} y(t) =& 0.6018t^{\frac{9}{4}} + 1.204t^{\frac{10}{4}} + 1.809t^{\frac{11}{4}} + 2.667t^3 + 3.862t^{\frac{13}{4}} + 5.502t^{\frac{14}{4}} \\ &+ 7.717t^{\frac{15}{4}} + 10.67t^4 \cdots \end{aligned} \tag{6.4.12}$$

**Example 12:** If we take $\beta = \frac{1}{2}$ , P(t) = 1, Q(t)= 2 and R(t) = $t^2$ in (6.4.1) and follow the method given in Example 11, we easily arrive at the following result

$$\begin{aligned} y(t) =& 0.6018t^{\frac{5}{2}} + 0.6667t^3 + 0.6878t^{\frac{7}{2}} + 0.6667t^4 + 0.6114t^{\frac{9}{2}} + 0.5334t^5 \\ &+ 0.6599t^{\frac{11}{2}} + 0.8485t^6 \cdots \end{aligned} \tag{6.4.13}$$

**Example 13:**If we take $\beta = \frac{3}{4}$ , P(t) = 1, Q(t)= 2 and R(t) = $t^2$ in (6.4.1) and follow the method given in Example 11, we easily arrive at the following result

$$y(t) = 0.4522t^{\frac{11}{4}} + 0.3439t^{\frac{14}{4}} + 0.2272t^{\frac{17}{4}} + 0.133t^5 \cdots \tag{6.4.14}$$

**Example 14:** If we take $\beta = 1$ , P(t) = 1, Q(t)= 2 and R(t) = $t^2$ in (6.4.1) and follow the method given in Example 11, we easily arrive at the following result

$$y(t) = \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15} + \frac{t^6}{45} + \frac{t^7}{45} \cdots \tag{6.4.15}$$

**The graph given below demonstrates have been represented eq.(6.4.12) to eq. (6.4.15) graphically by making use of "MATHEMATICA SOFTWARE" as given below.**

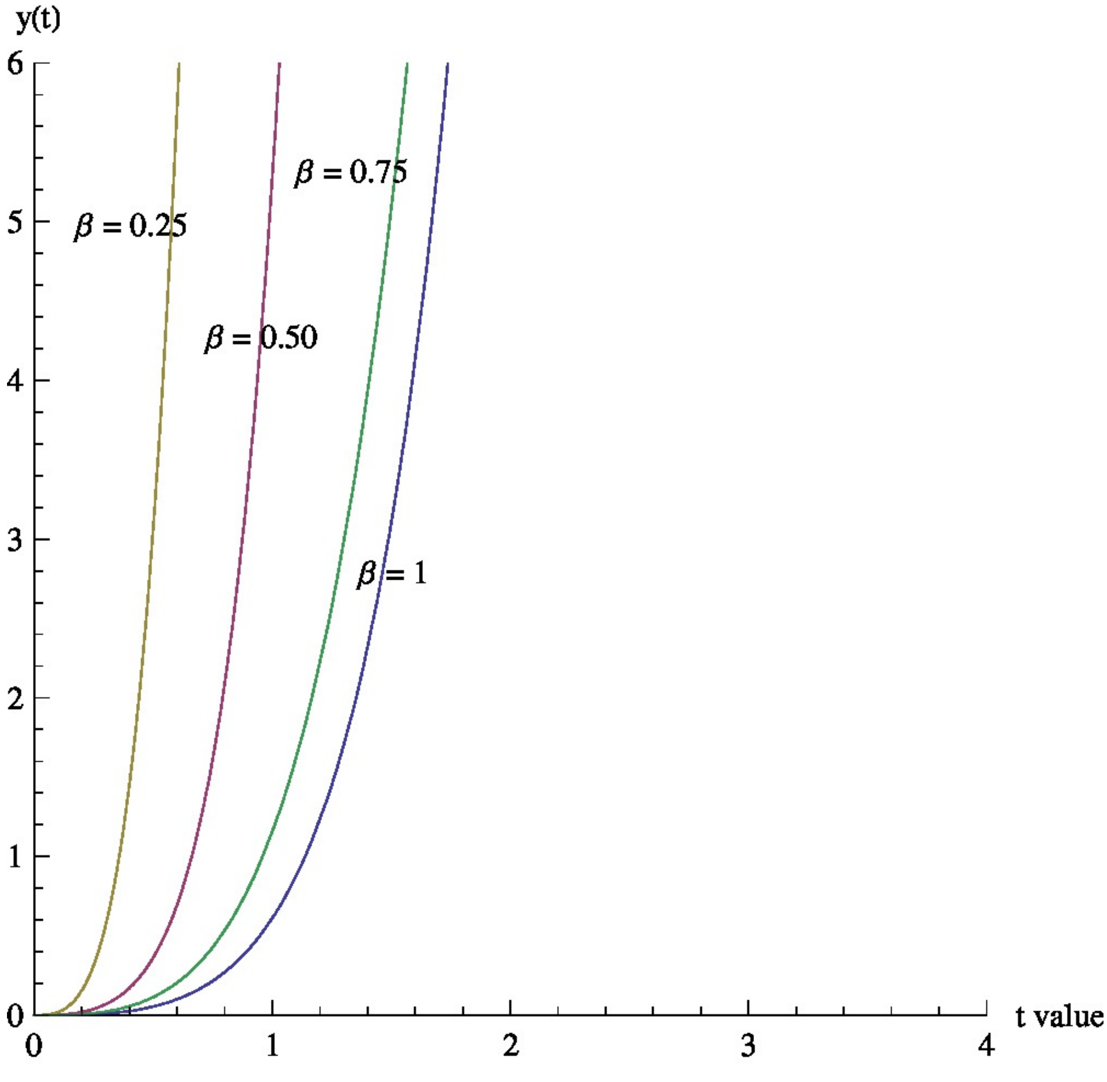


**Figure : 2. Approxiamte solution of example 11 to 14**

**Example 15:** If we take $\beta = \frac{1}{4}$ , P(t) = -1, Q(t)= 3 and R(t) = t in equation (6.4.1), it reduces to

$$D^{\frac{1}{4}}y(t) = t + 3y(t) - y^2(t) \tag{6.4.16}$$

Subject to initial condition

$$y(0) = 1 \tag{6.4.17}$$

**SOLUTION :** Applying generalized differential transform (6.1.4) on the both sides of (6.4.16) by using the property of GDTM (f) with a = b = 0 and setting $t_0 = 0, \alpha = 1/4$, we easily arrive at the following result

$$Y_{\frac{1}{4}}(k+1) = \frac{\Gamma(\frac{k}{4}+1)}{\Gamma(\frac{k}{4}+\frac{5}{4})}\left[\delta(k-4) + 3Y_{\frac{1}{4}}(k) - \sum_{l=0}^{k} Y_{\frac{1}{4}}(l)Y_{\frac{1}{4}}(k-l)\right] \tag{6.4.18}$$

and initial conditions (6.4.17) takes the following form

$$Y_{\frac{1}{4}}(0) = 1 \tag{6.4.19}$$

Utilizing the recurrence relation (6.4.18) and the transformed initial conditions (6.4.19) we calculated the value of $Y_{\frac{1}{4}}(k+1)$, for k = 1, 2,$\cdots$. Then we can easily obtain the following result after a little simplification

$$\left.\begin{array}{lll} Y_{\frac{1}{4}}(1) = 2.207; & Y_{\frac{1}{4}}(2) = 2.2578; & Y_{\frac{1}{4}}(3) = -2.519; \\ Y_{\frac{1}{4}}(4) = -11.4756; & Y_{\frac{1}{4}}(5) = -3.9283; & Y_{\frac{1}{4}}(6) = 49.5215 \end{array}\right\} \tag{6.4.20}$$

Similarly we can find the value of $Y_{\frac{1}{4}}(k)$ for k = 7, 8, 9 $\cdots$ .

Now, from (6.1.5), and f(t) replaced by y(t) then it takes the following form

$$y(t) = \sum_{k=0}^{\infty} Y_{\frac{1}{4}}(k)t^{\frac{k}{4}} \tag{6.4.21}$$

Using the values of $Y_{\frac{1}{4}}(k)$ from (6.4.20) in (6.4.21), the exact solution of Fractional Order Reccati Differential Equation (6.4.16) is obtained as

$$y(t) = 1 + 2.207t^{\frac{1}{2}} + 2.2578t - 2.5198t^{\frac{3}{2}} - 11.4756t^2 - 3.9283t^{\frac{5}{2}} + 49.5215t^3 \cdots \quad (6.4.22)$$

**Example 16:** If we take $\beta = \frac{1}{2}$ , P(t) = -1, Q(t)= 3 and R(t) = t in (6.4.16) and follow the method given in Example 15, we easily arrive at the following result

$$y(t) = 1 + 2.257t^{\frac{1}{2}} + 2t - 1.5753t^{\frac{3}{2}} - 7.048t^2 - 2.3694t^{\frac{5}{2}} + 19.7999t^3 \cdots \quad (6.4.23)$$

**Example 17:**If we take $\beta = \frac{3}{4}$ , P(t) = -1, Q(t)= 3 and R(t) = t in (6.4.16) and follow the method given in Example 15, we easily arrive at the following result

$$y(t) = 1 + 2.176t^{\frac{3}{4}} + 1.5045t^{\frac{6}{4}} - 1.6847t^{\frac{9}{4}} + 0.3009t^{\frac{10}{4}} \cdots \quad (6.4.24)$$

**Example 18:** If we take $\beta = 1$ , P(t) = -1, Q(t)= 3 and R(t) = t in (6.4.16) and follow the method given in Example 15, we easily arrive at the following result

$$y(t) = 1 + 2t + 1.5t^2 - 0.8333t^3 - 0.2083t^4 + 0.1749t^5 \cdots \quad (6.4.25)$$

**The graph given below demonstrates have been represented eq.(6.4.22) to eq. (6.4.25) graphically by making use of "MATHEMATICA SOFTWARE" as given below**

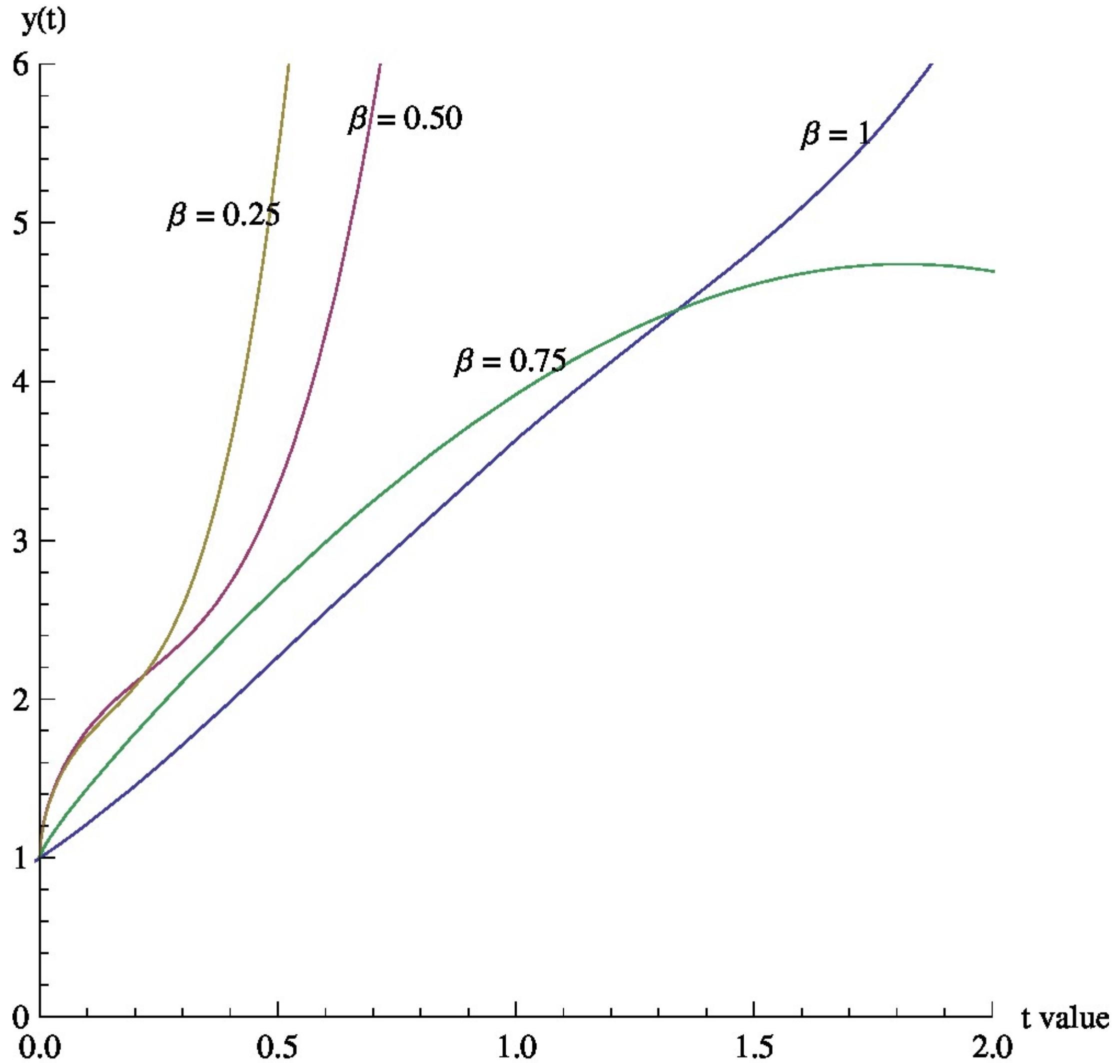


**Figure : 3. Approxiamte solution of example 15 to 18**

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Publications

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2. *ANALYTICAL SOLUTION OF BAGLEY TORVIK EQUATION BY GENERALIZED DIFFERENTIAL TRANSFORM*, International Journal of Pure and Applied Mathematics, **110(2)**, (2016), 265–273, (coauthor Rashmi Jain)

3. *CERTAIN NEW RESULTS OF THE S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION TRANSFORM*, South East Asian J. of Math. & Math. Sci.,**12(2)** (2016), 115–124 (coauthor Rashmi Jain).

4. *COMPOSITION FORMULAE FOR UNIFIED FRACTIONAL INTEGRAL OPERATORS*, European Journal of Advances in Engineering and Technology. **3(4)** (2016), 1–6, (coauthor Rashmi Jain).

5. *APPLICATION OF GENERALIZED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION AND NUMERICAL RESULT*, International Journal of Pure and Applied Mathematics, **99(3)** (2015), 355–366, (coauthor Rashmi Jain).

6. *A STUDY OF FINITE INTEGRAL INVOLVING GENERALIZED FORM OF THE ASTROPHYSICAL THERMONUCLEAR FUNCTION*, Journal of Rajasthan Academy of Physical Sciences, **14(1)** (2015), 51–55, (coauthor Rashmi Jain).

7. *APPLICATION OF GENERALIZED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL RELAXATION OSCILLATION EQUATION*, Antarctica Journal of Mathematics, **12(1)** (2015), 85–95, (coauthor Rashmi Jain).

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9. *ON UNIFIED FINITE INTEGRALS INVOLVING $I_3$ FUNCTION, GENERALIZED MITTAG-LEFFLER FUNCTION AND APPELL POLYNO-*

*MIALS*, The Journal of the Indian Academy of Mathematics, **36(2)** (2014), 211–219, (coauthor Rashmi Jain).

10. *CERTAIN NEW RESULTS INVOLVING THE M-TRANSFORM*, Journal of Mathematics and Statistics ( Accepted).

# LIST OF RESEARCH PAPERS


1. **M. K. BANSAL and R. JAIN** (2017). A STUDY OF UNIFIED FRACTIONAL INTEGRAL OPERATORS INVOLVING S-GENERALIZED GAUSS'S HYPERGEOMETRIC FUNCTION AS ITS KERNEL, *Palestine Journal of Mathematics*, **6(1)**, 142–152 .

2. **M. K. BANSAL and R. JAIN** (2016). ANALYTICAL SOLUTION OF BAGLEY TORVIK EQUATION BY GENERALIZED DIFFERENTIAL TRANSFORM, *International Journal of Pure and Applied Mathematics*, **110(2)**, 265–273.

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