BICOMPLEX INTEGRAL TRANSFORMS AND APPLICATIONS

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Certificate

This is to certify that **Mr. Mahesh Puri Goswami** has worked under my supervision for the award of degree of **Doctor of Philosophy** in Mathematics on the topic entitled, **"BICOMPLEX INTEGRAL TRANSFORMS AND APPLICATIONS"**. The findings contained in this thesis are original and have not been submitted to any University/Institute, in part or full, for award of any degree.

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Declaration

I, **Mr. Mahesh Puri Goswami (ID 2013RMA9058)** declare that the thesis entitled **Bicomplex Integral Transforms and Applications** is my own work conducted under the supervision of **Dr. Ritu Agarwal**, Department of Mathematics, Malaviya National Institute of Technology Jaipur, Rajasthan, India.

I declare that the work does not contain any part of work that has been submitted for the award of any degree either in this University or in any other University/Institute without proper citation.

Place: Jaipur **MAHESH PURI GOSWAMI**

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(MAHESH PURI GOSWAMI)

Contents

1

Introduction to Bicomplex Numbers

Bicomplex analysis is a recent powerful mathematical tool to develop the theory of functions belonging to large class of frequency domain. The concept of bicomplex numbers play a vital role in solving problems of electromagnetism. It has great advantage of dealing both the vector fields (electric and magnetic) together as a single vector field in bicomplex space. This approach is also advantageous than quaternionic approach due to the commutative property of bicomplex numbers.

The present chapter deals with an introduction to the topic of the study as well as a brief review of the contributions made by some of the earlier workers on the subject matter presented in this thesis. Next a brief chapter by chapter summary of the thesis has been given.

1.1 History and Literature Review of Bicomplex Numbers

Beginning from the end of the first half of the 19^{th} century, particularly, in Great Britain, developed the theory on geometrical interpretation of complex numbers that led to the birth of new systems of hypercomplex numbers. In particular, the discovery in 1843 of quaternion numbers by well-known Irish mathematician *Hamilton* (see, e.g. [63]) revealed the existence of an algebraic system that had all the properties of real and complex numbers except commutativity of multiplication. It was described as physical rotations in a four-dimensional space. Also, it was as an extension of complex number concept into four dimensions [74], [76] and [108].

As a result, researches were carried out on new systems of hypercomplex numbers, leading to the discovery of octonions, theory of pluriquaternions and biquaternions. The idea of bicomplex numbers came to *James Cockle* beginning from the observation made by *Horner* on the existence of irrational equations which has neither real nor complex solutions. *Cockle* [31] assumed a new imaginary unit j s.t. $j^2 = 1$, and taking inspiration from the theory of quaternions which was defined by *Hamilton*, he defined the bicomplex number as $p = x_0 + x_1 i + x_2 j + ijx_3$, with $j^2 = 1$.

In 1892, *Segre* [132], rediscovered the algebra of bicomplex numbers and presented as the analytical representation of the points of bicomplex geometry and identified that *Hamilton* introduced the same quantities in the study of biquaternions and also described the geometrical interpretation of the algebra of bicomplex numbers. In 1928 and 1932, *Futagawa* originated the concept of holomorphic functions of a bicomplex variable in a series of papers [49], [50].

In 1934, *Dragoni* [37] gave some basic results in the theory of bicomplex holomorphic functions while *Price* [119] and *Rönn* [126] have developed the bicomplex algebra and function theory. *Price* [119] discussed the field property of bicomplex numbers and observed that the commutativity in the former is obtained if the ring of these numbers contains zero-divisors and so can not form a field. However, the property of commutativity of bicomplex numbers is later on recognized as the complex numbers with complex coefficients due to this effect there are deep similarities between the bicomplex and complex numbers by *Olariu* [112].

In 2004, *Rochon* [122] generalized a holomorphic Riemann zeta function in bicomplex form. In the same year, *Rochon* and *Tremblay* [124] generalized the Schrödinger equation from complex form to bicomplex form and obtained Born's formula for the class of bicomplex wave functions having a null hyperbolic angle. In 2006, *Rochon* and *Tremblay* [125] discussed hyperbolic and bicomplex Hilbert spaces with their properties and *Goyal* and *Goyal* [57], introduced a holomorphic bicomplex Hurwitz Zeta-function.

In recent developments efforts have been made and a number of results have been obtained using bicomplex numbers. In 2007, *Goyal* et al. [56] extended Polygamma function in bicomplex form. In 2008, *Charak* and *Rochon* [26] extended the factorization of meromorphic functions from complex variable to bicomplex variable. In 2009, *Charak* et al. [29] obtained Julia and Fatou sets in bicomplex form. In 2010, *Lavoie* et al. [90], [91] investigated bicomplex quantum harmonic oscillator with eigenvalues and eigenkets and introduced the concept of infinite dimensional bicomplex Hilbert spaces with their applications to quantum harmonic oscillator. In 2011, *Lavoie* et al. [92] discussed finite dimensional bicomplex Hilbert spaces, linear operators, orthogonal bases and self adjoint operators with their applications in quantum mechanics. In the same year, *Kumar* and *Kumar* [88] extended the Laplace transform in the bicomplex variable from their complex counterpart. In 2012, *Luna-Elizarraras* et al. [97] introduced about the algebra of bicomplex numbers and their elementary functions.

In 2013, *Charak* et al. [27], [28] discussed the Riesz-Fischer theorem and bicomplex Spectral decomposition theorem on infinite dimensional bicomplex Hilbert spaces. In the same year, *Mathein* et al. [101] obtained the analytical solution of the quantum Coulomb potential problem formulated in terms of bicomplex numbers and *Singh* and *Srivastava* [135] discussed the continuity and compactness of the bicomplex spaces and its subsets. In 2014, *Dubey* et al. [38] studied the bicomplex Orlicz spaces. Further, discussed some applications of Hahn-Banach theorem on bicomplex Banach modules. In the same year, *Banerjee* et al. [10] generalized the inversion Laplace transform in bicomplex variable. In 2015, *Banerjee* et al. [11], [12] extended the Fourier transform and its inverse to bicomplex variable. In the same year, *Kumar* and *Singh* [89] studied the basic properties of bicomplex linear operators on bicomplex Hilbert spaces and proved Littlewood's subordination principle for bicomplex Hardy space.

1.2 Bicomplex Numbers

Ordered pairs of the real numbers forms the well known field of complex numbers wherein the operations of addition and multiplication are defined as

(a) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$,

(b) $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1), \quad x_1,\, x_2,\, y_1,\, y_2 \in \mathbb{R}$

where $\mathbb{R}$ is the set of real numbers. The theory of analytic functions on domains in C_1 (set of complex numbers) has been extensively developed.

The question arises that, 'what happens if the above definitions are applied to pairs of complex numbers and the corresponding function theory is investigated?' The new set of ordered pairs of complex numbers allow the same definition of all fundamental operations except the division that is not possible by an ordered pair (z_1, z_2) if $z_1^2 + z_2^2 = 0$.

Precisely, the set of complex numbers generated by the field of real numbers i.e.

$$C_1 = \{x + i_1 y : x,\, y \in \mathbb{R}\} = \{(x, y) : x,\, y \in \mathbb{R}\},$$

where $i_1^2 = -1$. Thus the set C_1 is the set of ordered pairs of real numbers. Now, we apply this process to the ordered pair of complex numbers. Consequently, we require two complex planes which are denoted by $C(i_1)$ and $C(i_2)$, where $i_1^2 = -1 = i_2^2;\ i_1 i_2 = i_2 i_1 = j$.

We denote the set of all bicomplex numbers by C_2, defined by *Segre* [132] as

$$C_2 = \{\xi : \xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3;\ x_0,\, x_1,\, x_2,\, x_3 \in \mathbb{R}\}$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2 z_2;\ z_1,\, z_2 \in C_1\}.$$

$\cdot$	i_0	i_1	i_2	j
i_0	i_0	i_1	i_2	j
i_1	i_1	$-i_0$	j	$-i_2$
i_2	i_2	j	$-i_0$	$-i_1$
j	j	$-i_2$	$-i_1$	i_0

The Cayley table of the set of the set C_2 is of the form:

where $i_0 := 1$ acts as identity, and

$$i_1 i_2 = i_2 i_1 = j$$

$$i_1 j = j i_1 = -i_2$$

$$i_2 j = j i_2 = -i_1$$

$$j^2 = i_0.$$

Such an approach says that we provided the real four-dimensional linear space $\mathbb{R}^4$ with a standard basis

$$i_0 = (1, 0, 0, 0),\ i_1 = (0, 1, 0, 0),\ i_2 = (0, 0, 1, 0),\ j = (0, 0, 0, 1)$$

with the following arithmetic operations.

Let $\xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3$ and $\eta = y_0 + i_1 y_1 + i_2 y_2 + j y_3$, then the addition and multiplication of two bicomplex numbers is defined as

$$\xi + \eta := (x_0 + y_0) + i_1(x_1 + y_1) + i_2(x_2 + y_2) + j(x_3 + y_3);$$

$$\xi \cdot \eta := (x_0 y_0 - x_1 y_1 - x_2 y_2 + x_3 y_3) + i_1(x_0 y_1 + x_1 y_0 - x_2 y_3 - x_3 y_2)$$

$$+ i_2(x_0 y_2 - x_1 y_3 + x_2 y_0 - x_3 y_1) + j(x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0),$$

respectively. In C_2, multiplication is commutative, associative and distributive over addition and C_2 is a commutative algebra but not division algebra.

Three important subsets of C_2 can be specified as

$$C(i_k) = \{x + i_k y : x,\, y \in \mathbb{R}\}, \quad k = 1, 2 \text{ and } \mathbb{D} = \{x + jy : x,\, y \in \mathbb{R}\}.$$

Each of the sets $C(i_k)$ is isomorphic to the field of complex numbers and $\mathbb{D}$ is called set of hyperbolic numbers, also called duplex numbers (see, e.g. [121], [123], [137]). C_2 has zero divisors, viz. the set of points

$$\left\{z_1 + i_2 z_2 \in C_2 : z_1^2 + z_2^2 = 0\right\} = \{(i_1 \pm i_2)z : z \in C_1\}. \tag{1.1}$$

This set is called *null-cone* and denoted by $\mathcal{NC}$ or $\mathcal{O}_2$ [119].

Definition 1.1. Let $\xi \in C_2$, we denote by C_2^{-1} the set of all invertible elements defined by

$$C_2^{-1} = \left\{\xi = z_1 + i_2 z_2 : z_1^2 + z_2^2 \neq 0\right\}. \tag{1.2}$$

An important property of the invertible elements follows from the following result:

Theorem 1.1. *Let ξ, $\eta \in C_2$, if ξ and η are invertible, then $\xi\eta$ is also invertible and $(\xi\eta)^{-1} = \xi^{-1}\eta^{-1}$.*

1.3 Idempotent Representation and Cartesian Set in Bicomplex Space

Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, viz.

$$\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2, \tag{1.3}$$

(where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$).

This representation of a bicomplex number is known as *idempotent representation* of ξ. The coefficients $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called the *idempotent components* of the bicomplex number $\xi = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called *idempotent basis.*

The auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{w_1 = z_1 - i_1 z_2,\ \forall\, z_1, z_2 \in C_1\},\ A_2 = \{w_2 = z_1 + i_1 z_2,\ \forall\, z_1, z_2 \in C_1\}.$$

A Cartesian set $X_1 \times_e X_2$ determined by $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ and is defined as:

$$X_1 \times_e X_2 = \{z_1 + i_2 z_2 \in C_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2,\ w_1 \in X_1, w_2 \in X_2\}. \tag{1.4}$$

With the help of idempotent representation, we give the following definition:

Definition 1.2. The projection mappings $P_1 : C_2 \to A_1 \subseteq C_1$ and $P_2 : C_2 \to A_2 \subseteq C_1$ are defined as

$$(P_1 : z_1 + i_2 z_2) = P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1,$$
$$(P_2 : z_1 + i_2 z_2) = P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2,$$
$$\forall\, z_1 + i_2 z_2 \in C_2.$$

Remark 1.1. From equation (1.1), $\xi \in \mathcal{O}_2$ iff at least one of $P_1(\xi)$ and $P_2(\xi)$ vanishes.

Theorem 1.2. *Every bicomplex number $z_1 + i_2 z_2 \in C_2$ is uniquely expressed as*

$$z_1 + i_2 z_2 = P_1(z_1 + i_2 z_2)e_1 + P_2(z_1 + i_2 z_2)e_2. \tag{1.5}$$

1.3 Idempotent Representation and Cartesian Set in Bicomplex Space

The representation (1.5) of bicomplex numbers is useful because addition, multiplication and division can be done term by term (*Price* [119]) and it is helpful to understand the structure of functions of a bicomplex variable.

Let $\xi = \xi_1 e_1 + \xi_2 e_2 \in C_2$ and $\eta = \eta_1 e_1 + \eta_2 e_2 \in C_2$, where $\xi_1, \xi_2, \eta_1, \eta_2 \in C_1$. Idempotent representation of some of the basic bicomplex functions are as follows:

1. $e^{\xi} = e^{\xi_1 e_1 + \xi_2 e_2} = e^{\xi_1} e_1 + e^{\xi_2} e_2$

2. $\cos \xi = \cos(\xi_1 e_1 + \xi_2 e_2) = (\cos \xi_1) e_1 + (\cos \xi_2) e_2$

3. $\sin \xi = \sin(\xi_1 e_1 + \xi_2 e_2) = (\sin \xi_1) e_1 + (\sin \xi_2) e_2$

4. $\xi^n = (\xi_1 e_1 + \xi_2 e_2)^n = \xi_1^n e_1 + \xi_2^n e_2$

5. $(\xi - \eta)^n = (\xi_1 e_1 + \xi_2 e_2 - \eta_1 e_1 - \eta_2 e_2)^n$

   $= [(\xi_1 - \eta_1) e_1 + (\xi_2 - \eta_2) e_2]^n = (\xi_1 - \eta_1)^n e_1 + (\xi_2 - \eta_2)^n$

6. $\frac{\xi}{\eta} = \frac{\xi_1 e_1 + \xi_2 e_2}{\eta_1 e_1 + \eta_2 e_2} = \frac{\xi_1}{\eta_1} e_1 + \frac{\xi_2}{\eta_2} e_2;\ \eta \notin \mathcal{O}_2$

7. $\xi * \eta = (\xi_1 e_1 + \xi_2 e_2)(\eta_1 e_1 + \eta_2 e_2) = \xi_1 \eta_1 e_1 + \xi_2 \eta_2 e_2$

8. $\xi^n + \eta^n = \xi_1^n e_1 + \xi_2^n e_2 + \eta_1^n e_1 + \eta_2^n e_2 = (\xi_1^n + \eta_1^n) e_1 + (\xi_2^n + \eta_2^n) e_2$

9. $\int_D f(\xi) d\xi = \int_{D_1} f_{e_1}(\xi_1) d\xi_1\, e_1 + \int_{D_2} f_{e_2}(\xi_2) d\xi_2\, e_2;$

   Here $P_1 : D \to D_1,\ P_2 : D \to D_2$

10. $\frac{d}{d\xi} f(\xi) = \frac{d}{d\xi_1} f_{e_1}(\xi_1) e_1 + \frac{d}{d\xi_2} f_{e_2}(\xi_2) e_2.$

In the following theorem, *Price* discussed the convergence of bicomplex function with respect to its idempotent complex component functions. This theorem is useful in proving our results.

Theorem 1.3 (Price [119]). *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq C_2$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$ under projection mappings $P_1 : D \to D_1 \subseteq C_1$ and $P_2 : D \to D_2 \subseteq C_1$ are convergent in domains D_1 and D_2, respectively.*

1.4 The Conjugations in Bicomplex Numbers

The complex conjugation plays an important role for both algebraic and geometric properties of C_1 and for analysis of complex functions. Three types of conjugations have been defined for bicomplex numbers:

Definition 1.3 (Bicomplex conjugation w.r.t. i_1 or 1st kind of conjugation)**.** It is defined as

$$(z_1 + i_2 z_2)^{\dagger_1} = \bar{z}_1 + i_2 \bar{z}_2, \quad \forall\, z_1, z_2 \in C_1$$

where $\bar{z}_1$, $\bar{z}_2$ are the complex conjugations of complex numbers z_1, z_2 respectively.

Definition 1.4 (Bicomplex conjugation w.r.t. i_2 or 2nd kind of conjugation)**.** It is defined by

$$(z_1 + i_2 z_2)^{\dagger_2} = z_1 - i_2 z_2, \quad \forall\, z_1, z_2 \in C_1.$$

Definition 1.5 (3rd kind of conjugation)**.** It is the composition of the above two conjugations and it is defined by

$$(z_1 + i_2 z_2)^* = \left((z_1 + i_2 z_2)^{\dagger_1}\right)^{\dagger_2} = \left((z_1 + i_2 z_2)^{\dagger_2}\right)^{\dagger_1} = \bar{z}_1 - i_2 \bar{z}_2,\ \forall\, z_1, z_2 \in C_1.$$

We can easily verify that each of these conjugates can be expressed in terms of two others, such that $\xi^* = \left(\xi^{\dagger_1}\right)^{\dagger_2} = \left(\xi^{\dagger_2}\right)^{\dagger_1}$, etc. Precisely, the conjugates form the following Klein group, under the composition:

$\circ$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$*$
$\dagger_0$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$*$
$\dagger_1$	$\dagger_1$	$\dagger_0$	$*$	$\dagger_2$
$\dagger_2$	$\dagger_2$	$*$	$\dagger_0$	$\dagger_1$
$*$	$*$	$\dagger_2$	$\dagger_1$	$\dagger_0$

where $\xi^{\dagger_0} := \xi,\ \forall\ \xi \in C_2$. All three types of conjugations have the standard properties of conjugations,

$$(\xi + \eta)^{\dagger_k} = \xi^{\dagger_k} + \eta^{\dagger_k} \tag{1.6}$$

$$\left(\xi^{\dagger_k}\right)^{\dagger_k} = \xi \tag{1.7}$$

$$(\xi \cdot \eta)^{\dagger_k} = \xi^{\dagger_k} \cdot \eta^{\dagger_k} \tag{1.8}$$

for ξ, $\eta \in C_2$, $k = 1$, 2 and $\dagger_k \equiv *$. Anyway, let us illustrate the proof for the last property in case of first kind conjugation. Let $\xi = x_1 + i_2 z_2$ and $\eta = z_3 + i_2 z_4$ with z_1, z_2, z_3, $z_4 \in C_1$. Then

$$\begin{aligned}
(\xi \cdot \eta)^{\dagger_1} &= [(z_1 z_3 - z_2 z_4) + i_2(z_1 z_4 + z_2 z_3)]^{\dagger_1} \\
&= \overline{(z_1 z_3 - z_2 z_4)} + i_2 \overline{(z_1 z_4 + z_2 z_3)} \\
&= \overline{z_1 z_3} - \overline{z_2 z_4} + i_2\left(\overline{z_1 z_4} + \overline{z_2 z_3}\right) \\
&= \bar{z}_1 \bar{z}_3 - \bar{z}_2 \bar{z}_4 + i_2(\bar{z}_1 \bar{z}_4 + \bar{z}_2 \bar{z}_3) \\
&= (\bar{z}_1 + i_2 \bar{z}_2)(\bar{z}_3 + i_2 \bar{z}_4) \\
&= \xi^{\dagger_1} \cdot \eta^{\dagger_1}.
\end{aligned}$$

Remark 1.2. Let $\xi \in C_2$, ξ is invertible iff $\xi^{\dagger_2}$ is also invertible; besides $\left(\xi^{\dagger_2}\right)^{-1} = \left(\xi^{-1}\right)^{\dagger_2}$.

1.5 Bicomplex Moduli

We know that the product of a standard complex number and its conjugate is the square of the Euclidean metric in $\mathbb{R} \times \mathbb{R}$. Following are the bicomplex analogue of this fact. Let $z_1,\, z_2 \in C_1$ and $\xi = z_1 + i_2 z_2 \in C_2$. Then we have following results by *Rochon* and *Shapiro* [123, p. 80]

$$|\xi|_{i_1}^2 = \xi \cdot \xi^{\dagger_2} = z_1^2 + z_2^2 \in C_1, \tag{1.9}$$

$$|\xi|_{i_2}^2 = \xi \cdot \xi^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\mathrm{Re}(z_1 \bar{z}_2) i_2 \in C(i_2), \tag{1.10}$$

$$|\xi|_j^2 = \xi \cdot \xi^{*} = \left(|z_1|^2 + |z_2|^2\right) - 2\mathrm{Im}(z_1 \bar{z}_2) j \in \mathbb{D}. \tag{1.11}$$

The norm of $\xi \in C_2$ can be defined as

$$\|\xi\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}\left(|\xi|_j^2\right)} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \tag{1.12}$$

where $z_1 = x_1 + i_1 x_2$ and $z_2 = x_3 + i_1 x_4$.

Theorem 1.4 (Rochon and Shapiro [123]). *Let $\xi, \eta \in C_2$, then*

$$\|\xi \cdot \eta\| \leq \sqrt{2}\|\xi\|\|\eta\|. \tag{1.13}$$

Remark 1.3. Since $\|e_i \cdot e_i\| = \|e_i\| = \frac{\sqrt{2}}{2} = \sqrt{2}\|e_i\|\|e_i\|$, $i = 1,\, 2$, the constant $\sqrt{2}$ is the best possibility in Theorem 1.4.

The norm in bicomplex space C_2 as defined as

Definition 1.6. Let the function $\|\cdot\| : C_2 \to \mathbb{R}$ is a norm on the real space $\mathbb{R}^4 \cong C_2$, i.e. $\forall\, \xi, \eta \in C_2$ and $a \in \mathbb{R}$

(i) $\|\xi\| \geq 0$,

(ii) $\|\xi\| = 0$ iff $\xi = 0$,

(iii) $\|a\xi\| = |a|\|\xi\|$,

(iv) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$.

Bicomplex function spaces:

In 2014, *Dubey* et al. [38] discussed about the bicomplex function spaces, which defined in the following way:

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ- finite complete measure space. If $f = f_1e_1 + f_2e_2$, where f_1 and f_2 are complex-valued measurable functions on $\Omega = (\Omega, \Sigma, \mu)$, then f is bicomplex-valued measurable function Ω. Therefore, for any complex-valued function space $(F(\Omega), \|\cdot\|_\Omega)$ the bicomplex function space $(F(\Omega, C_2), \|\cdot\|_{C_2})$ defined for consisting of all functions of the type $f = f_1e_1 + f_2e_2$, where f_1 and f_2 are in $(F(\Omega), \|\cdot\|_\Omega)$ and

$$\|f\|_{C_2} = \frac{1}{\sqrt{2}} \left(\|f_1\|^2 + \|f_2\|^2\right). \tag{1.14}$$

In particular, if $\xi = s_1e_1 + s_2e_2$, then

$$\|\xi\| = \frac{1}{\sqrt{2}} \left(|s_1|^2 + |s_2|^2\right)^{\frac{1}{2}} \tag{1.15}$$

The addition and scalar multiplication on $(F(\Omega, C_2), \|\cdot\|_{C_2})$ is defined as

$$\begin{aligned} f + g &= (f_1e_1 + f_2e_2) + (g_1e_1 + g_2e_2) \\ &= (f_1 + g_1)e_1 + (f_2 + g_2)e_2 \end{aligned}$$

and

$$\begin{aligned} af &= (a_1e_1 + a_2e_2)(f_1e_1 + f_2e_2) \\ &= (a_1f_1)e_1 + (a_2f_2)e_2, \end{aligned}$$

where $f, g \in F(\Omega, C_2)$ and $a \in C_2$. Now, L^p and L^∞ spaces in C_2 defined as

Definition 1.7. Let $L^p(\Omega)$ denotes the linear space of all equivalence classes of complex-valued Σ- measurable essentially bounded functions of Ω and for any two functions that are equal μ- almost everywhere on Ω are identified. Then the corresponding bicomplex measurable function space $L^p(C_2)$ consists of all functions of the type $f = f_1 e_1 + f_2 e_2$, where $f_1, f_2 \in L^p(\Omega)$. Also,

$$\begin{aligned} \|f\|_{p,C_2} &= \|f_1 e_1 + f_2 e_2\|_{p,C_2} \\ &= \frac{1}{\sqrt{2}} \left(\|f_1\|_p^2 + \|f_2\|_p^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{1.16}$$

Definition 1.8. Let $L^\infty(\Omega)$ denotes the linear space of all equivalence classes of complex-valued Σ- measurable essentially bounded functions of Ω and for any two functions that are equal μ- almost everywhere on Ω are identified. Then the corresponding bicomplex measurable function space $L^\infty(C_2)$ consists of all functions of the type $f = f_1 e_1 + f_2 e_2$, where $f_1, f_2 \in L^\infty(\Omega)$. Also,

$$\begin{aligned} \|f\|_{\infty,C_2} &= \|f_1 e_1 + f_2 e_2\|_{\infty,C_2} \\ &= \frac{1}{\sqrt{2}} \left(\|f_1\|_\infty^2 + \|f_2\|_\infty^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{1.17}$$

1.6 Differentiation of Bicomplex Functions

The derivative of a bicomplex function definition is isomorphic to the corresponding definition in C_1 because bicomplex operations are isomorphic to the complex ones. But bicomplex numbers do not form a field due to the lack of inverses of singular numbers. Thus, the definition of the bicomplex derivative (see, e.g. *Rönn* [126, Definition 4.1]) as

Definition 1.9. Let f be a bicomplex function whose domain of definitions contains a neighborhood of the point ξ. The derivative of f at the point ξ is defined as

$$f'(\xi) = \lim_{\substack{\triangle\xi \to 0 \\ \triangle\xi \notin \mathcal{O}_2}} \frac{f(\xi + \triangle\xi) - f(\xi)}{\triangle\xi} \tag{1.18}$$

provided the limit exists.

A bicomplex function which has a derivative at the point ξ is said to be differentiable of holomorphic at ξ. If the function is holomorphic at all points of a domain $D \subseteq C_2$ it is said to be holomorphic in D.

Normal techniques for computing derivatives of sums, production, quotients and composition of functions as follows:

(i) $(f+g)'(\xi) = f'(\xi) + g'(\xi)$

(ii) $(f \cdot g)'(\xi) = f'(\xi) \cdot g(\xi) + f(\xi) \cdot g'(\xi)$

(iii) $\left(\frac{f}{g}\right)'(\xi) = \frac{f'(\xi)\cdot g(\xi) - f(\xi)\cdot g'(\xi)}{[g(\xi)]^2}$

(iv) $(f \circ g)'(\xi) = f'(g(\xi)) \cdot g'(\xi)$.

1.7 Bicomplex Integration

Consider the bicomplex function of the form

$$\begin{aligned} f(\xi) &= (f_1(z_1, z_2), f_2(z_1, z_2)) \\ &= (\phi_1(x_1, y_1, x_2, y_2) + i_1\phi_2(x_1, y_1, x_2, y_2), \phi_3(x_1, y_1, x_2, y_2) + i_1\phi_4(x_1, y_1, x_2, y_2)) \end{aligned} \tag{1.19}$$

where $\xi = (z_1, z_2), z_1 = x_1 + i_1y_1$ and $z_2 = x_2 + i_1y_2$. We assume that f_1 and f_2 are analytic function in z_1 and z_2, thereby ensuring that $\phi_i,\ i = 1, 2, 3, 4$ are continuous. The basic bicomplex integral is necessarily isomorphic to the complex integral. The *line integral* w.r.t. some four dimensional curve Γ in C_2 defined as

$$\int_{\Gamma} f(\xi) \cdot d\xi, \quad d\xi = (dz_1, dz_2)$$

Henceforth, we shall choose the curve Γ, so that it is piecewise continuously differentiable in C_2 and has the parametric equation

$$\Gamma : \xi = \xi(t), \quad \xi(t) = (z_1(t), z_2(t)) \text{ for } a \leq t \leq b \tag{1.20}$$

where $t \in \mathbb{R}$. Γ is a curve made up of two curves Γ_1 and Γ_2 in C_1 i.e.

$$\Gamma = (\Gamma_1, \Gamma_2) \tag{1.21}$$

whose parametric equations are

$$\Gamma_1 : z_1 = z_1(t), \quad z_1(t) = x_1(t) + i_1 y_1(t) \text{ for } a \leq t \leq b$$

$$\Gamma_2 : z_2 = z_2(t), \quad z_2(t) = x_2(t) + i_1 y_2(t) \text{ for } a \leq t \leq b.$$

Then the line integral of $f(\xi)$ over the curve Γ is

$$\int_{\Gamma} f(\xi) \cdot d\xi = \int_a^b f(\xi(t)) \cdot \xi'(t) dt. \tag{1.22}$$

Since f is continuous, $f(\xi(t))$ at the right-hand side is also continuous. If $\xi'(t)$ is discontinuous at some points the integration has to be taken in subintervals of $[a, b]$.

The Cauchy's theorem of a bicomplex function in bicomplex space as follows:

Theorem 1.5. *Let the bicomplex function* $f(\xi) = (f_1(z_1, z_2), f_2(z_1, z_2))$, $\xi = (z_1, z_2)$, *be holomorphic in domain* $D \subseteq C_2$. *Then*

$$\int_{\Gamma} f(\xi) \cdot d\xi = 0$$

for any closed curve Γ *that is the boundary of an orientable surface* $S(\Gamma)$ *in* D.

Further details of bicomplex differentiation and integration can be seen in *Rönn* [126].

1.8 Summary of the Thesis

Now, we present a brief summary of the work carried out in Chapter 2 to 7.

In **Chapter–2**, we prove the inversion formula for bicomplex Laplace transform, some of its properties and convolution theorem for complexified Laplace transform to bicomplex variables that is capable of transferring signals from real-valued (t) domain to bicomplex frequency (ξ) domain. The bicomplex inverse Laplace transform of a convolution function has been illustrated with the help of an example. Physical applications of bicomplex Laplace transform in finding solution of bicomplex Maxwell's equation and bicomplex Schrödinger equation for free particle are given.

Motivated by the work of *Eltayeb* and *Killicman* in this chapter we also, generalize complex double Laplace transform to bicomplex double Laplace transform. Also, we derive some of its basic properties and inversion theorem in bicomplex space. Applications of bicomplex double Laplace transform have been discussed in finding the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle by using two different approaches.

In **Chapter–3**, we define Sumudu transform with convergence conditions in bicomplex space. Also, we derive some of its basic properties and its inverse. Applications of bicomplex Sumudu transform have been illustrated to find the solution of differential equation of bicomplex-valued functions and find the solution for Cartesian transverse electric magnetic (TEM) waves in homogeneous space.

In **Chapter–4**, we define the formula for bicomplex version of Stieltjes transform, its inverse and relationship with bicomplex Laplace transform. We have discussed some of its basic operational properties and convolution theorem. Applications of bicomplex Stieltjes transform in finding the solution of singular integral equation, probability distribution theory and spectral analysis of random matrices in signal processing are given.

Further, we also define the bicomplex version of Laplace-Stieltjes transform. Also, we derive some useful properties and Tauberian theorem for Laplace-Stieltjes transform in the bicomplex variable. Applications of bicomplex Laplace-Stieltjes transform in exponential decay of tail probability and bicomplex Dirichlet series are given.

In **Chapter–5**, we define bicomplex Fourier-Stieltjes transform which is more generalized form of bicomplex Fourier transform. Also, we define some basic properties of class of bicomplex Bochner functions and generalize the classical Bochner theorem in the framework of bicomplex analysis. Applications of bicomplex Fourier transform in finding the solution of initial value heat equation in bicomplex algebra and algebraic reduction of complicated bicomplex linear time-invariant systems in easy form have been discussed. Illustrations have been given to find the solution of bicomplex heat equation and check the unboundedness condition of non-homogeneous bicomplex-valued wave equation.

The concept of bicomplex numbers is introduced in Electro-magnetics, with direct applications to the solution of Maxwell's equations. Here, we discuss the technique to find the analytic solution of the electromagnetic wave equation in vacuum with the help of bicomplex analysis as tool. Also, we find the solution of Gaussian pulse wave using bicomplex vector field.

In **Chapter–6**, motivated by the work of *Zemanian* we generalize the complex Hankel transform to bicomplex Hankel transform and derive some of its basic properties. A table of bicomplex Hankel transform is given for some functions of importance. It has found applications in solving partial differential equation of bicomplex-valued functions, signal processing, optics, electromagnetic field theory and other related problems. The application of Hankel transform has been illustrated by solving bicomplex Cauchy problem.

In **Chapter–7**, motivated by the recent applications of bicomplex theory to the study of functions of large class, we define bicomplex Mellin transform of bicomplex-valued functions. Also, we derive some of its basic properties and inversion theorem in bicomplex space.

Further, we also obtain the bicomplex Mellin transform of Riemann-Liouville fractional integral and Caputo fractional derivative of order $\alpha(\geq 0)$ of certain functions and some of their properties. Applications of bicomplex Mellin transform in networks with time-varying parameters problem and solution of differential equation involving fractional derivatives of bicomplex-valued functions have been illustrated.

2

Laplace Transform in Bicomplex Space and Applications

The main findings of this chapter have been published as:

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2014), Convolution theorem and applications of bicomplex Laplace transform, *Advances in Mathematical Sciences and Applications*, **24**(1), 113-127.

2. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Double Laplace transform in bicomplex space with applications, *CUBO: A Mathematical Journal*, **18**(2), (In press).

In this chapter, one of our concern is to extend the convolution theorem for complexified Laplace transform to bicomplex variables. Also, we extend the complex double Laplace transform to bicomplex double Laplace transform in two bicomplex variables. Applications have been discussed in finding the solution of bicomplex Schrödinger equations and bicomplex Maxwell's equation.

2.1 Introduction

The Laplace transform is widely used in physics and engineering. It is named after a mathematician and astronomer *Pierre-Simon Laplace* (1749-1827), who introduced a similar transform (now known as z transform), in his work on probability theory. The use of transforms came after second world war although in 19^{th} century it had been used by *Abel, Lerch, Heaviside* and *Bromwich.* Laplace transform is a transformation, where inputs and outputs are functions of time, to the frequency domain, in which the same inputs and outputs are functions of complex angular frequency, which measures in radians per unit time.

The Laplace transform (see, e.g. *Davies* [33, Chapter 2]) of complex-valued function $f(t)$ of exponential order $K \in \mathbb{R}$ as

$$L[f(t);\xi] = \int_0^\infty e^{-st} f(t)dt = F(s), \quad s \in C_1 \tag{2.1}$$

where $F(s)$ exists and convergent for $\mathrm{Re}(s) > K$.

The inversion formula for Laplace transform (see, e.g. *Davies* [33, Chapter 3]) is given by the following theorem

Theorem 2.1. *Let $F(s)$ be the Laplace transform of $f(t)$, analytic in the half plane $Re(s) > K$ then,*

$$f(t) = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\gamma} e^{st} F(s) ds \tag{2.2}$$

where γ is the contour taken in left of $Re(s) > K$ along the vertical line joining two points $a - i_1 r$ and $a + i_1 r$ with $a > K$ in the complex plane.

In 1936, *Van der Pol* [143] introduced about the double Laplace transform. This has been used by *Humbert* [68] in the study of hypergeometric functions; by *Jaeger* [72] to solve boundary value problems in heat conduction. In 1951, *Estrin* et al. [45] extended the complex double Laplace transform to multiple Laplace transform in n independent complex variables. In 2008, *Elatayeb and Kilicman* [40] used double Laplace transform for solving a second-order partial differential equations. In 2010, *Kilicman* and *Gaddin* [81] discussed relationship between double Laplace transform and double Sumudu transform. In 2013, *Kashuri* et al. [75] used double Laplace transform in solving partial differential equation.

In two recent developments *Kumar* and *Kumar* [88] and *Banerjee* et al. [10] have studied bicomplexified version of the Laplace transform and its inverse from its complexified form. In this procedure idempotent representation of bicomplex numbers play a vital role. Bicomplex Laplace transform is a powerful mathematical tool applied in physics, electric circuit theory, power system load frequency control, control engineering, communication, signal analysis and design, system analysis and solving differential equations.

2.2 Bicomplex Laplace Transform

The bicomplex Laplace transform and its properties are discussed by *Kumar* and *Kumar* [88] and is defined as

Definition 2.1. Let $f(t)$ be a bicomplex-valued piecewise continuous function of exponential order $K \in \mathbb{R}$. Then the bicomplex Laplace Transform of $f(t)$ for $t \geq 0$ can be defined as

$$L[f(t);\xi] = \int_0^\infty f(t)e^{-\xi t}dt = F(\xi). \tag{2.3}$$

Here $F(\xi)$ exists and is convergent for all $\xi \in D = D_1 \cup D_2 \cup D_3$
where

$$\begin{aligned} D &= \{\xi : H_\rho(\xi) \text{ represent a Right half plane } a_0 > K + |a_3|\}, \\ &= \{\xi \in C_2 : \operatorname{Re}(\xi) > K + |\operatorname{Im}_j(\xi)|\}, \\ &= \{\xi = s_1e_1 + s_2e_2 \in C_2 : \operatorname{Re}(s_1) > K \text{ and } \operatorname{Re}(s_2) > K\} \\ &= \{\xi = s_1e_1 + s_2e_2 \in C_2 : \operatorname{Re}(P_1 : \xi) > K \text{ and } \operatorname{Re}(P_2 : \xi) > K\}. \quad (2.4) \\ D_1 &= \{\xi = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2 : a_0 > K,\ a_3 = 0\}, \\ D_2 &= \{\xi = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2 : a_0 > K + a_3,\ a_3 > 0\}, \\ \text{and } D_3 &= \{\xi = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2 : a_0 > K - a_3,\ a_3 < 0\}. \end{aligned}$$

The domain D contains infinite number of points ξ which have same H_ρ hyperbolic projection because a_1 and a_2 are free from restriction.

Some of the results given by *Kumar* and *Kumar* [88] have been mentioned here for the ready reference.

Theorem 2.2 (Linearity Property). *Let $F(\xi)$ and $G(\xi)$ be the bicomplex Laplace transforms of continuous functions $f(t)$ and $g(t)$, respectively. Then*

$$L[af(t) + bg(t);\xi] = aF(\xi) + bG(\xi), \quad \xi \in D \tag{2.5}$$

where a, b are constants in the region of convergence D given by (2.4).

Theorem 2.3. *Let $f(t)$ and $f'(t)$ be continuous functions of exponential order K for $t \geq 0$, then*

$$L[f'(t);\xi] = \xi F(\xi) - f(0), \quad \xi \in D \tag{2.6}$$

where $F(\xi) = L[f(t);\xi]$ and D defined in (2.4).

Theorem 2.4. *Let $F(\xi)$ be the bicomplex Laplace transform of a continuous function $f(t)$ of exponential order K. Then*

$$L\left[\int_0^t f(u)du;\xi\right] = \frac{F(\xi)}{\xi}, \quad \xi \in D \text{ and } \xi \notin \mathcal{O}_2 \tag{2.7}$$

where D defined in (2.4).

Theorem 2.5. *Let $F(\xi)$ be the bicomplex Laplace transform of a continuous function $f(t)$ of exponential order K. Then*

$$L[tf(t);\xi] = -\frac{d}{d\xi}F(\xi), \quad \xi \in D \tag{2.8}$$

where D defined in (2.4).

Theorem 2.6. *Let $F(\xi)$ be the bicomplex Laplace transform of a continuous function $f(t)$ of exponential order K. If $\lim_{t\to 0} \frac{f(t)}{t}$ exists, then*

$$L\left[\frac{f(t)}{t};\xi\right] = \int_\xi^\infty F(\eta)d\eta, \quad \xi \in D \tag{2.9}$$

where D defined in (2.4).

Theorem 2.7 (First Shifting Theorem)**.** *Let $F(\xi)$ be the bicomplex Laplace transform of a continuous function $f(t)$ of exponential order K. Then*

$$L\left[e^{at}f(t);\xi\right] = F(\xi - a), \quad (\xi - a) \in D \tag{2.10}$$

where D defined in (2.4).

Theorem 2.8 (Second Shifting Theorem)**.** *Let $F(\xi)$ be the bicomplex Laplace transform of a continuous function $f(t)$ of exponential order K. Then*

$$L\left[U_a(t)f(t-a);\xi\right] = e^{-a\xi}F(\xi), \quad \xi \in D \tag{2.11}$$

where D defined in (2.4) and $U_a(t)$ is the unit step function.

Further, detailed proof of the above theorems can be found in *Kumar* and *Kumar* [88]. For solving the large class of bicomplex partial differential equations, we need integral transforms defined for large class. The bicomplex integral transforms are capable of transferring the signals from real-valued time domain to bicomplexified frequency domain.

2.3 Inverse Bicomplex Laplace Transform

Motivated by the work of *Kumar* and *Kumar* [88] and Theorem 2.1, we derive here the formula for inverse Laplace transform for bicomplex functions. An alternative proof for the same can be seen in *Banerjee* et al. [10].

Theorem 2.9. *Let* $F(\xi)$ *be the Bicomplex Laplace transform of* $f(t)$, *analytic in* $Re(P_1 : \xi) > K$ *and* $Re(P_2 : \xi) > K$ *then,*

$$f(t) = \lim_{r_1, r_2 \to \infty} \frac{1}{2\pi i_1} \int_{\Gamma} e^{\xi t} F(\xi) d\xi, \quad \xi \in D \subset C_2 \tag{2.12}$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ *is piecewise continuous differentiable closed contour in Bicomplex space and* Γ_1, Γ_2 *are closed contours in left of* $Re(s_1) > K$ *and* $Re(s_2) > K$, *along the vertical lines joining two points* $a_k - i_1 r_k$ *and* $a_k + i_1 r_k$, $k = 1, 2$, *respectively in complex plane.*

Proof. The Bicomplex Laplace Transform of $f(t)$ is defined as:

$$F(\xi) = \int_0^\infty e^{-\xi x} f(x) dx, \quad \xi \in C_2 \tag{2.13}$$

Multiplying (2.13) by $e^{\xi t}$, we obtain

$$e^{\xi t} F(\xi) = e^{\xi t} \int_0^\infty e^{-\xi x} f(x) dx \tag{2.14}$$

Let $\xi = s_1e_1 + s_2e_2$, where $\xi \in C_2$ and $s_1, s_2 \in C_1$. Then (2.14) becomes

$$e^{s_1t}F_1(s_1)e_1 + e^{s_2t}F_2(s_2)e_2 = e^{s_1t}\int_0^\infty e^{s_1x}f(x)dx\ e_1 + e^{s_2t}\int_0^\infty e^{s_2x}f(x)dx\ e_2 \tag{2.15}$$

Integrating coefficient of e_1 w.r.t. 's_1' and coefficient of 'e_2' w.r.t. s_2 between the limits $a_1 + i_1r_1$ and $a_2 + i_1r_2$ respectively, we have

$$\left\{\int_{a_1-i_1r_1}^{a_1+i_1r_1} e^{s_1t}F_1(s_1)ds_1\right\}e_1 + \left\{\int_{a_2-i_1r_2}^{a_2+i_1r_2} e^{s_2t}F_2(s_2)ds_2\right\}e_2$$
$$= \left\{\int_{a_1-i_1r_1}^{a_1+i_1r_1} e^{s_1t}ds_1\int_0^\infty e^{s_1x}f(x)dx\right\}e_1 + \left\{\int_{a_2-i_1r_2}^{a_2+i_1r_2} e^{s_2t}ds_2\int_0^\infty e^{s_2x}f(x)dx\right\}e_2$$

Putting $s_1 = a_1 - i_1p_1,\ ds_1 = -i_1dp_1$ and $s_2 = a_2 - i_1p_2,\ ds_2 = -i_1dp_2$

$$\begin{aligned}
&\left\{\int_{a_1-i_1r_1}^{a_1+i_1r_1} e^{s_1t}F_1(s_1)ds_1\right\}e_1 + \left\{\int_{a_2-i_1r_2}^{a_2+i_1r_2} e^{s_2t}F_2(s_2)ds_2\right\}e_2\\
&= \left\{i_1\int_{-r_1}^{r_1} e^{t(a_1-i_1p_1)}\int_0^\infty f(x)e^{-(a_1-i_1p_1)x}dx\,dp_1\right\}e_1\\
&\quad + \left\{i_1\int_{-r_2}^{r_2} e^{t(a_2-i_1p_2)}\int_0^\infty f(x)e^{-(a_2-i_1p_2)x}dx\,dp_2\right\}e_2\\
&= \left\{i_1e^{a_1t}\int_{-r_1}^{r_1} e^{-i_1p_1t}dp_1\int_0^\infty f(x)e^{-a_1t}e^{i_1p_1x}dx\right\}e_1\\
&\quad + \left\{i_1e^{a_2t}\int_{-r_2}^{r_2} e^{-i_1p_2t}dp_2\int_0^\infty f(x)e^{-a_2t}e^{i_1p_2x}dx\right\}e_2
\end{aligned} \tag{2.16}$$

Let us define $\phi_1(t)$ and $\phi_2(t)$ as

$$\phi_1(t) = \begin{cases} e^{-a_1t}f(t) & \text{when } t \geq 0\\ 0 & \text{when } t < 0\end{cases}$$

and

$$\phi_2(t) = \begin{cases} e^{-a_2t}f(t) & \text{when } t \geq 0\\ 0 & \text{when } t < 0\end{cases}$$

The Fourier complex integral of $\phi_1(t)$ and $\phi_2(t)$ are

$$\phi_1(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i_1p_1t}\int_{-\infty}^\infty \phi(x)e^{i_1p_1x}dx\,dp_1$$

$$e^{-a_1t}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i_1p_1t}\int_0^{\infty}[e^{-a_1x}f(x)]e^{i_1p_1x}dx\,dp_1 \tag{2.17}$$

and

$$\phi_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i_1p_2t}\int_{-\infty}^{\infty}\phi(x)e^{i_1p_2x}dx\,dp_2$$

$$e^{-a_2t}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i_1p_2t}\int_0^{\infty}[e^{-a_2x}f(x)]e^{i_1p_2x}dx\,dp_2 \tag{2.18}$$

In the limiting case when $r_1, r_2 \to \infty$, (2.16) becomes

$$\begin{aligned}&\left\{\int_{a_1-i_1\infty}^{a_1+i_1\infty} e^{s_1t}F_1(s_1)ds_1\right\}e_1 + \left\{\int_{a_2-i_1\infty}^{a_2+i_1\infty} e^{s_2t}F_2(s_2)ds_2\right\}e_2\\ &= \left\{i_1e^{a_1t}\int_{-\infty}^{\infty} e^{-i_1p_1t}dp_1\int_0^{\infty} f(x)e^{-a_1x}e^{i_1p_1x}dx\right\}e_1\\ &\quad + \left\{i_1e^{a_2t}\int_{-\infty}^{\infty} e^{-i_1p_2t}dp_2\int_0^{\infty} f(x)e^{-a_2x}e^{i_1p_2x}dx\right\}e_2\end{aligned} \tag{2.19}$$

Substituting the values of the integrals from (2.17) and (2.18) in (2.19), we have

$$\begin{aligned}&\left\{\int_{a_1-i_1\infty}^{a_1+i_1\infty} e^{s_1t}F_1(s_1)ds_1\right\}e_1 + \left\{\int_{a_2-i_1\infty}^{a_2+i_1\infty} e^{s_2t}F_2(s_2)ds_2\right\}e_2\\ &= i_1e^{a_1t}\{2\pi e^{-a_1t}f(t)\}e_1 + i_1e^{a_2t}\{2\pi e^{-a_2t}f(t)\}e_2\\ &= 2\pi i_1 f(t)\end{aligned}$$

$$f(t) = \frac{1}{2\pi i_1}\left\{\int_{a_1-i_1\infty}^{a_1+i_1\infty} e^{s_1t}F_1(s_1)ds_1\ e_1 + \int_{a_2-i_1\infty}^{a_2+i_1\infty} e^{s_2t}F_2(s_2)ds_2\ e_2\right\} \tag{2.20}$$

Equation (2.20) is the inversion formula for the Bicomplex Laplace transform.

Further, let Γ_1 and Γ_2 be closed contours taken in left of $Re(P_1 : \xi) > K$ and $Re(P_2 : \xi) > K$ joining two points $a_k - i_1r_k$ and $a_k + i_1r_k,\ k = 1, 2$, respectively. From (2.20)

$$\begin{aligned}f(t) &= \lim_{r_1,r_2\to\infty}\frac{1}{2\pi i_1}\left[\int_{\Gamma_1} e^{s_1t}F_1(s_1)ds_1\ e_1 + \int_{\Gamma_2} e^{s_2t}F_2(s_2)ds_2\ e_2\right]\\ &= \lim_{r_1,r_2\to\infty}\frac{1}{2\pi i_1}\int_{(\Gamma_1,\Gamma_2)} e^{(s_1e_1+s_2e_2)t}F(s_1e_1+s_2e_2)(ds_1e_1+ds_2e_2)\\ &= \lim_{r_1,r_2\to\infty}\frac{1}{2\pi i_1}\int_{\Gamma} e^{\xi t}F(\xi)d\xi,\end{aligned}$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ is piecewise continuous differentiable closed contour in bicomplex space as discussed in section 1.7. □

2.4 Properties of Bicomplex Laplace Transform

In this section, we are discussing some properties of bicomplex Laplace transform viz. bicomplex Laplace transform of periodic function, change of scale property, initial value theorem, final value theorem and relationship between bilateral bicomplex Laplace transform and bicomplex Fourier Transform.

Theorem 2.10. *Let $f(t)$ be a periodic function with period T and bicomplex Laplace transform is $F(\xi)$, then*

$$F(\xi) = \frac{\int_0^T e^{-\xi t} f(t)dt}{1 - e^{-\xi t}}, \qquad \xi = s_1e_1 + s_2e_2 \in C_2 \text{ and } Re(s_1) > 0, Re(s_2) > 0.$$

Proof. Let $f(t)$ be a periodic function of exponential order K with period T, then for $s_1 \in C_1$ and $\mathrm{Re}(s_1) > 0$, (see, e.g. *Schiff* [130]).

$$F(s_1) = \frac{\int_0^T e^{-s_1 t} f(t)dt}{1 - e^{-s_1 t}}.$$

Taking another $s_2 \in C_1$ and $\mathrm{Re}(s_2) > 0$, we have

$$F(s_2) = \frac{\int_0^T e^{-s_2 t} f(t)dt}{1 - e^{-s_2 t}}.$$

Since $F(s_1)$and $F(s_2)$ are analytic for $\mathrm{Re}(s_1) > 0, \mathrm{Re}(s_2) > 0$ respectively, then

$$F(s_1)e_1 + F(s_2)e_2 = \frac{\int_0^T e^{-s_1 t} f(t)dt}{1 - e^{-s_1 t}} e_1 + \frac{\int_0^T e^{-s_2 t} f(t)dt}{1 - e^{-s_2 t}} e_2$$

By application of Theorem 1.3 (Price [119]), we get

$$F(s_1e_1 + s_2e_2) = \frac{\int_0^T e^{-(s_1e_1+s_2e_2)t} f(t)dt}{1 - e^{-(s_1e_1+s_2e_2)t}}$$

$$\Rightarrow F(\xi) = \frac{\int_0^T e^{-\xi t} f(t)dt}{1 - e^{-\xi t}}, \quad \text{where } \xi = s_1e_1 + s_2e_2 \in C_2.$$

□

Theorem 2.11 (Change of Scale Property)**.** *Let the function $f(t)$ of exponential order $K \in \mathbb{R}$ has bicomplex Laplace transform $F(\xi)$, then for $a > 0$*

$$L[f(at);\xi] = \frac{1}{a}\left(\frac{\xi}{a}\right), \qquad \xi = s_1e_1 + s_2e_2 \in C_2 \text{ and } Re(s_1) > K, Re(s_2) > K.$$

Proof. The bicomplex Laplace transform of $f(at)$ is given by

$$L[f(at);\xi] = \int_0^\infty e^{-\xi t} f(at)dt$$

$$\text{Put } at = u \Rightarrow dt = \frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{\xi}{a}u} f(u)du$$

$$= \frac{1}{a}F\left(\frac{\xi}{a}\right).$$

□

Theorem 2.12 (Initial Value Theorem)**.** *Let $f(t)$ is differentiable on $[0,\infty)$ and exponential order $K \in \mathbb{R}$ such that $Re(P_1 : \xi) > K$ and $Re(P_2 : \xi) > K$, then*

$$\lim_{Re(\xi)\to\infty} \xi F(\xi) = f(0), \quad \xi \in C_2.$$

Proof. We have seen that

$$\xi F(\xi) - f(0) = L\left\{\frac{d}{dt}f(t)\right\} = \int_0^\infty e^{-\xi t}\frac{d}{dt}f(t)dt$$

Taking the limit $\mathrm{Re}(\xi) \to \infty$ on both sides, we have

$$\lim_{\mathrm{Re}(\xi)\to\infty} (\xi F(\xi) - f(0)) = \lim_{\mathrm{Re}(\xi)\to\infty} \int_0^\infty e^{-\xi t}\frac{d}{dt}f(t)dt = 0$$

$$\therefore \lim_{\mathrm{Re}(\xi)\to\infty} \xi F(\xi) = f(0).$$

□

Theorem 2.13 (Final Value Theorem)**.** *Let $F(\xi)$ be the bicomplex Laplace transform of $f(t)$ defined for every $\xi \in C_2$ in a region around zero, then*

$$\lim_{\xi\to 0} \xi F(\xi) = f(\infty).$$

Proof. We have seen that

$$\xi F(\xi) - f(0) = L\left[\frac{d}{dt}f(t);\xi\right] = \int_0^\infty e^{-\xi t}\frac{d}{dt}f(t)dt$$

Taking the limit $\xi \to 0$, we have

$$\begin{aligned}\lim_{\xi\to 0}(\xi F(\xi) - f(0)) &= \lim_{\xi\to 0}\int_0^\infty e^{-\xi t}\frac{d}{dt}f(t)dt \\ &= \int_0^\infty \left(\lim_{\xi\to 0} e^{-\xi t}\right)\frac{d}{dt}f(t)dt = \int_0^\infty \frac{d}{dt}f(t)dt \\ &= f(\infty) - f(0)\end{aligned}$$

$$\therefore \lim_{\xi\to 0}\xi F(\xi) = f(\infty).$$

□

In the following theorem we discuss relationship between the bilateral bicomplex Laplace transform and bicomplex Fourier transform. This relationship is often used to determine the bicomplex frequency spectrum of a signal or dynamical system.

Theorem 2.14. *Let $f(t)$ be a real-valued continuous function with bilateral bicomplex Laplace transform $F(\xi)$ and bicomplex Fourier transform $\hat{f}(w)$ and satisfies the following estimates*

$$|f(t)| \leq C_1 e^{-\alpha t}, \qquad t \geq 0,\ \alpha > 0$$

$$|f(t)| \leq C_1 e^{\beta t}, \qquad t \geq 0,\ \beta > 0$$

Then bilateral bicomplex Laplace transform is equivalent to bicomplex Fourier transform with $w = i_1\xi$, $\xi \in D$, where

$$D = \{\xi \in C_2 : \xi = s_1e_1 + s_2e_2, -\alpha < Re(s_1), Re(s_2) < \beta\}.$$

Proof. Since we know that,

$$
\begin{aligned}
F(\xi) &= \int_{-\infty}^{\infty} e^{-\xi t} f(t)dt = \int_{-\infty}^{\infty} e^{-(s_1e_1+s_2e_2)t} f(t)dt \\
&= \int_{-\infty}^{\infty} e^{-s_1 t} f(t)dt\; e_1 + \int_{-\infty}^{\infty} e^{-s_2 t} f(t)dt\; e_2 \\
&= \int_{-\infty}^{\infty} e^{i_1(i_1 s_1)t} f(t)dt\; e_1 + \int_{-\infty}^{\infty} e^{i_1(i_1 s_2)t} f(t)dt\; e_2 \\
&= \int_{-\infty}^{\infty} e^{i_1(i_1(s_1e_1+s_2e_2)t} f(t)dt \\
&= \int_{-\infty}^{\infty} e^{i_1(i_1\xi)t} f(t)dt \qquad (\because \xi = s_1e_1 + s_2e_2 \in C_2) \\
&= \hat{f}(w)|_{w=i_1\xi}, \qquad w \in C_2.
\end{aligned}
$$

□

2.5 Convolution

The way of combining two signals is known as convolution. It is such a widespread and useful formula that it has its own shorthand notation '$*$'. For any two signals x and y, there will be another signal z obtained by convolving x with y,

$$z(t) = x * y = \int_0^t x(s)y(t-s)ds, \quad t \in \mathbb{R}. \tag{2.21}$$

We derive here the convolution theorem for bicomplex Laplace transform as follows:

Theorem 2.15. *If $L[f(t);\xi] = F(\xi)$ and $L[g(t);\xi] = G(\xi)$, $\quad \xi \in C_2$ with $Re(P_1 : \xi) > K$ and $Re(P_2 : \xi) > K$, where $K = Max(K_1, K_2)$ and $f(t)$ and $g(t)$ are of exponential orders K_1 and K_2 respectively. Then*

$$L\{f * g\} = F(\xi)G(\xi).$$

Proof. By the definition of bicomplex Laplace transform

$$\begin{aligned}
L\{f*g\} &= \int_0^\infty e^{-\xi t}(f*g)(t)dt \\
&= \int_0^\infty e^{-\xi t}\left(\int_0^t f(t-x)g(x)dx\right)dt \\
&= \int_0^\infty \left(\int_0^t f(t-x)g(x)e^{-\xi t}dx\right)dt
\end{aligned}$$

On changing the order of integration, we have

$$\begin{aligned}
&= \int_0^\infty \left(\int_x^\infty f(t-x)g(x)e^{-\xi t}dt\right)dx \\
&= \int_0^\infty \left(\int_0^\infty f(z)e^{-\xi(z+x)}dz\right)g(x)dx, \text{ [On putting } t-x=z] \\
&= \int_0^\infty e^{-\xi z}f(z)dz \int_0^\infty e^{-\xi x}g(x)dx \\
&= F(\xi)G(\xi).
\end{aligned}$$

□

Following is the illustration to find inverse Laplace transform of a bicomplex-valued function using convolution theorem.

Example 2.1. Let bicomplex Laplace transform $F(\xi) = \frac{\xi}{(\xi^2+w^2)^2}$, where $\xi = \xi_1 e_1 + \xi_2 e_2 \in C_2$, $w = w_1 e_1 + w_2 e_2 \in C_2$ with $\text{Re}(\xi_1 + w_1) > 0$ and $\text{Re}(\xi_2 + w_2) > 0$, then find $f(t)$.

Solution. $\because \text{Re}(P_1 : \xi + w) > 0$ and $\text{Re}(P_2 : \xi + w) > 0$

$$\therefore L^{-1}\left\{\frac{w}{\xi^2+w^2}\right\} = \sin wt = h(t) \text{ and } L^{-1}\left\{\frac{\xi}{\xi^2+w^2}\right\} = \cos wt = g(t)$$

Using convolution theorem, we have

$$\begin{aligned}
f(t) = L^{-1}\left\{\frac{\xi}{(\xi^2+w^2)^2}\right\} &= \frac{1}{w}L^{-1}\{g*h\} \\
&= \frac{1}{2w}\int_0^t 2\sin ws\cos(wt-ws)ds \\
&= \frac{1}{2w}\int_0^t (\sin wt + \sin(2ws - wt))\,ds = \frac{t\sin wt}{2w}.
\end{aligned}$$

2.6 Applications of Bicomplex Laplace Transform

(a) Here we find bicomplex solution for Cartesian transverse electric magnetic (TEM) waves in homogeneous space using bicomplex Laplace transform technique. To apply this purely mathematical concept in electromagnetic theory, Maxwell's equations (in a source-free domain) are first written in a form involving the wave number k and the medium intrinsic impedance η, rather than the medium permittivity and permeability. Bicomplex Maxwell's equation is described in *Anastassiu* et al. [5]. i.e.

$$\nabla \times \mathbf{E} = -i_1 k \eta \mathbf{H} \tag{2.22}$$

$$\nabla \times \mathbf{H} = i_1 \frac{k}{\eta} \mathbf{E} \tag{2.23}$$

for the time convention $e^{i_1 wt}$. Vector fields $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic field respectively. The bicomplex vector field $\mathbf{F}$ is defined:

$$\mathbf{F} \equiv \frac{1}{\sqrt{\eta}} \mathbf{E} + i_2 \sqrt{\eta} \mathbf{H} \tag{2.24}$$

with the implication that each directional component of $\mathbf{F}$ is a scalar bicomplex function, combining the corresponding field directional components. Multiplying (2.23) with i_2 and adding the result to (2.22), after some manipulation, the bicomplex Maxwell's equation is derived, i.e.

$$\nabla \times \mathbf{F} = i_2 i_1 k \mathbf{F} \tag{2.25}$$

Assuming a TEM to z wave, i.e., a vanishing z-component, and after introducing $Q_y = i_2 F_y$, in (2.25) is reduced to the following system of bicomplex differential equations

$$\frac{dQ_y}{dz} = i_1 k F_x \tag{2.26}$$

$$\frac{dF_x}{dz} = i_1 k Q_y \tag{2.27}$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 \tag{2.28}$$

with $F_x(0) = A$ and $F_x'(0) = B$, where A and B are bicomplex constants due to (2.28). After solving (2.26) and (2.27), we have

$$\frac{d^2 F_x}{dz^2} + k^2 F_x = 0 \tag{2.29}$$

$$\frac{d^2 Q_y}{dz^2} + k^2 Q_y = 0 \tag{2.30}$$

For the solution, taking the bicomplex Laplace transform of (2.29), we have

$$\xi^2 F_x(\xi) - \xi F_x(0) - F_x'(0) + k^2 F_x(\xi) = 0$$

$$\Rightarrow F_x(\xi) = \frac{\xi F_x(0) + F_x'(0)}{\xi^2 + k^2} \tag{2.31}$$

Taking the bicomplex inverse Laplace transform of (2.31), we get

$$\begin{aligned}
F_x(z) &= \lim_{r_1, r_2 \to \infty} \frac{1}{2\pi i_1} \int_\Gamma e^{\xi z} \frac{A + B\xi}{\xi^2 + k^2} d\xi \\
&= \lim_{r_1 \to \infty} \frac{1}{2\pi i_1} \int_{\Gamma_1} e^{\xi_1 z} \frac{A + B\xi_1}{\xi_1^2 + k^2} d\xi_1 \; e_1 + \lim_{r_2 \to \infty} \frac{1}{2\pi i_1} \int_{\Gamma_2} e^{\xi_2 z} \frac{A + B\xi_2}{\xi_2^2 + k^2} d\xi_2 \; e_2 \\
&= \frac{1}{2\pi i_1} 2\pi i_1 \left[\lim_{\xi_1 \to -i_1 k} (\xi_1 + i_1 k) e^{\xi_1 z} \frac{A + B\xi_1}{\xi_1^2 + k^2} + \lim_{\xi_1 \to i_1 k} (\xi_1 - i_1 k) e^{\xi_1 z} \frac{A + B\xi_1}{\xi_1^2 + k^2} \right] e_1 \\
&\quad + \frac{1}{2\pi i_1} 2\pi i_1 \left[\lim_{\xi_2 \to -i_1 k} (\xi_2 + i_1 k) e^{\xi_2 z} \frac{A + B\xi_2}{\xi_2^2 + k^2} + \lim_{\xi_2 \to i_1 k} (\xi_2 - i_1 k) e^{\xi_2 z} \frac{A + B\xi_2}{\xi_2^2 + k^2} \right] e_2 \\
&= \left[R e^{-i_1 k z} + K e^{i_1 k z} \right] e_1 + \left[R e^{-i_1 k z} + K e^{i_1 k z} \right] e_2 \\
&= \left[R e^{-i_1 k z} + K e^{i_1 k z} \right]
\end{aligned} \tag{2.32}$$

Similarly,

$$Q_y(z) = \left[Le^{-i_1kz} + Se^{i_1kz}\right] \tag{2.33}$$

$$\therefore \mathbf{F} \equiv \frac{1}{\sqrt{\eta}}\mathbf{E} + i_2\sqrt{\eta}\mathbf{H} = \left[Re^{-i_1kz} + Ke^{i_1kz}\right]\hat{x} - i_2\left[Le^{-i_1kz} + Se^{i_1kz}\right]\hat{y} \tag{2.34}$$

where $R = \frac{A-i_1Bk}{-2i_1k} = R_1 + i_2R_2$, $K = \frac{A+i_1Bk}{2i_1k} = K_1 + i_2K_2$, L and S are bicomplex constants and $\hat{x}$ and $\hat{y}$ are the fundamental position unit vectors in the direction of X- axis and Y- axis respectively. Since (2.34) is the solution of bicomplex Maxwell's equation (2.25), therefore it satisfies the Maxwell's equation if $L = -R$ and $S = K$. Hence (2.34) becomes

$$\mathbf{F} \equiv \frac{1}{\sqrt{\eta}}\mathbf{E} + i_2\sqrt{\eta}\mathbf{H} = \left[Re^{-i_1kz} + Ke^{i_1kz}\right]\hat{x} - i_2\left[-Re^{-i_1kz} + Ke^{i_1kz}\right]\hat{y} \tag{2.35}$$

Extracting the bi-real and bi-imaginary parts of the solutions (2.35) yields the electric and magnetic field components

$$\mathbf{E} = \left[R_1e^{-i_1kz} + K_1e^{i_1kz}\right]\hat{x} + \left[-R_2e^{-i_1kz} + K_2e^{i_1kz}\right]\hat{y} \tag{2.36}$$

$$\mathbf{H} = \frac{1}{\eta}\left[R_2e^{-i_1kz} + K_2e^{i_1kz}\right]\hat{x} - \frac{1}{\eta}\left[-R_1e^{-i_1kz} + K_1e^{i_1kz}\right]\hat{y} \tag{2.37}$$

where R_1, R_2, K_1 and K_2 are complex constants and (2.36) *and* (2.37) are the solution of Maxwell's equations (2.22-2.23).

(b) Here we find the solution of the bicomplex *time-dependent Schrödinger Equation* for free particle in one-dimension. For the solution of the time-dependent Schrödinger Equation by Laplace transform method, (refer, *Lin* and *Eyring* [95]). The one-dimensional standard Schrödinger's equation over the bicomplex space functions is given by *Rochon* and *Tremblay* [124, Eq. (4.1)] as

$$i_1\hbar\partial_t\psi(x,t) + \frac{\hbar^2}{2m}\partial_x^2\psi(x,t) - V(x,t)\psi(x,t) = 0 \tag{2.38}$$

where

$$\psi : \mathbb{R}^2 \to C_2 \text{ and } V : \mathbb{R}^2 \to \mathbb{R}.$$

The imaginary unit i_1 has been chosen as it is more appropriate for the decomposition of the bicomplex Schrödinger equation into idempotent components. For free particle $V(x,t) = 0$. Therefore (2.38) becomes

$$i_1 \hbar \partial_t \psi(x,t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x,t) \tag{2.39}$$

For solution, taking the bicomplex Laplace transform of (2.39) we have

$$i_1 \hbar (\xi \Psi(x,\xi) - \psi(x,0)) = -\frac{\hbar^2}{2m} \frac{d^2 \Psi(x,\xi)}{dx^2}, \quad \text{where } \Psi(x,\xi) = L\{\psi(x,t)\}$$

$$\frac{d^2 \Psi(x,\xi)}{dx^2} + \frac{2mi_1}{\hbar} \xi \Psi(x,\xi) = \frac{2m}{\hbar^2} \psi(x,0)$$

$$\therefore \Psi(x,\xi) = Ae^{\left(\cos\frac{3\pi}{4} + i_1 \sin\frac{3\pi}{4}\right)\sqrt{\frac{2m\xi}{\hbar}}x} + Be^{-\left(\cos\frac{3\pi}{4} + i_1 \sin\frac{3\pi}{4}\right)\sqrt{\frac{2m\xi}{\hbar}}x} + \frac{1}{D^2 + \frac{2mi_1}{\hbar}\xi} \psi(x,0).$$

Taking the inverse bicomplex Laplace transform we have

$$\psi(x,t) = \frac{1}{2\pi i_1} \left(A \int_\Gamma e^{\xi t} e^{\left(\cos\frac{3\pi}{4} + i_1 \sin\frac{3\pi}{4}\right)\sqrt{\frac{2m\xi}{\hbar}}x} d\xi \right)$$
$$+ \frac{1}{2\pi i_1} \left(B \int_\Gamma e^{-\left(\cos\frac{3\pi}{4} + i_1 \sin\frac{3\pi}{4}\right)\sqrt{\frac{2m\xi}{\hbar}}x} d\xi + \int_\Gamma e^{\xi t} \frac{1}{D^2 + \frac{2mi_1}{\hbar}\xi} \psi(x,0) d\xi \right)$$

where Γ is the closed contour in bicomplex space. Using the Cauchy's theorem in bicomplex space, (see, e.g. *Rönn* [126, Theorem 5.5]).

$$\int_\Gamma e^{\xi t} e^{\left(\cos\frac{3\pi}{4} + i_1 \sin\frac{3\pi}{4}\right)\sqrt{\frac{2m\xi}{\hbar}}x} d\xi = 0 \text{ and } \int_\Gamma e^{-\left(\cos\frac{3\pi}{4} + i_1 \sin\frac{3\pi}{4}\right)\sqrt{\frac{2m\xi}{\hbar}}x} d\xi = 0$$

$$\therefore \psi(x,t) = \frac{1}{2\pi i_1} \int_\Gamma e^{\xi t} \frac{1}{D^2 + \frac{2mi_1}{\hbar}\xi} \psi(x,0) d\xi \tag{2.40}$$

(2.40) is the solution of the bicomplex Schrödinger equation (2.39) in one dimension.

Generally, $\psi(x,0)$ have one of the forms $\cos\frac{2\pi x}{\lambda}$, $\sin\frac{2\pi x}{\lambda}$ and $e^{\pm i_1\frac{2\pi x}{\lambda}}$. For illustration, let us consider one of the form as $\psi(x,0)=\sin(\frac{2\pi}{\lambda}x)$. Then

$$\begin{aligned}\psi(x,t) &= \frac{1}{2\pi i_1}\int_\Gamma e^{\xi t}\frac{1}{-\frac{4\pi^2}{\lambda^2}+\frac{2mi_1}{\hbar}\xi}\sin\left(\frac{2\pi}{\lambda}x\right)d\xi \\ &= -i_1\frac{\hbar}{2m}\sin\left(\frac{2\pi}{\lambda}x\right)\frac{1}{2\pi i_1}\int_\Gamma e^{\xi t}\frac{1}{\xi+\frac{2\pi^2\hbar}{m\lambda^2}i_1}d\xi \\ &= -i_1\frac{\hbar}{2m}\sin\left(\frac{2\pi}{\lambda}x\right)\frac{1}{2\pi i_1}\left(2\pi i_1\lim_{\xi\to-\frac{2\pi^2\hbar}{m\lambda^2}i_1}e^{\xi t}\right) \\ &= -i_1\frac{\hbar}{2m}\sin\left(\frac{2\pi}{\lambda}x\right)e^{-i_1\frac{2\pi^2\hbar}{m\lambda^2}t} \qquad (2.41)\end{aligned}$$

(2.41) is the solution of bicomplex Schrödinger equation (2.39) for $\psi(x,0)=\sin(\frac{2\pi}{\lambda}x)$.

2.7 Bicomplex Double Laplace Transform

Let $f(x,t)$ be a bicomplex-valued function of two variables $x,t>0$, which is piecewise continuous and has exponential orders K_1 and K_2 w.r.t. x and t respectively. The bicomplex Laplace transform (see, *Kumar* and *Kumar* [88]) w.r.t. x is

$$L_x[f(x,t)]=\int_0^\infty e^{-\xi x}f(x,t)dx=\bar{f}(\xi,t), \qquad \xi\in\Omega_1\subset C_2 \qquad (2.42)$$

where

$$\Omega_1=\{\xi=s_1e_1+s_2e_2\in C_2:\ \mathrm{Re}(P_1:\xi)>K_1 \text{ and } \mathrm{Re}(P_2:\xi)>K_1\} \qquad (2.43)$$

or equivalently,

$$\Omega_1=\{\xi\in C_2\ :\ \mathrm{Re}(\xi)>K_1+|\mathrm{Im}_j(\xi)|\} \qquad (2.44)$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part of ξ w.r.t. j and (2.42) is convergent and analytic in Ω_1. Similarly, bicomplex Laplace transform of $f(x,t)$ w.r.t. t is

$$L_t[f(x,t)] = \int_0^\infty e^{-\eta t} f(x,t)dt = \bar{f}(x,\eta), \qquad \eta \in \Omega_2 \subset C_2 \tag{2.45}$$

where

$$\Omega_2 = \{\eta = p_1e_1 + p_2e_2 \in C_2 : \ \mathrm{Re}(P_1 : \eta) > K_2 \text{ and } \mathrm{Re}(P_2 : \eta) > K_2\} \tag{2.46}$$

or equivalently,

$$\Omega_2 = \{\eta \in C_2 \ : \ \mathrm{Re}(\eta) > K_2 + |\mathrm{Im}_j(\eta)|\} \tag{2.47}$$

where (2.45) is convergent and analytic in Ω_2. Now, taking the bicomplex Laplace transform of (2.42) w.r.t. t and using (2.45), we have

$$\begin{aligned} L_{xt}[f(x,t)] &= L_t[\bar{f}(\xi,t)] = \int_0^\infty e^{-\eta t}\bar{f}(\xi,t)dt \\ &= \int_0^\infty e^{-\eta t}\int_0^\infty e^{-\xi x} f(x,t)dxdt = \bar{\bar{f}}(\xi,\eta), \quad (\xi,\eta) \in \Omega \end{aligned} \tag{2.48}$$

where the integral on right hand side is convergent and analytic in

$$\Omega = \left\{(\xi,\eta) \in C_2^2 \ : \ \xi \in \Omega_1 \text{ and } \eta \in \Omega_2\right\}. \tag{2.49}$$

Now, we define the bicomplex double Laplace transform as follows:

Definition 2.2. Let $f(x,t)$ be a bicomplex-valued function of two variables $x,t > 0$, which is piecewise continuous and has exponential orders K_1 and K_2 w.r.t. x and t respectively. Then bicomplex double Laplace transform is defined as

$$L_{xt}[f(x,t)](\xi,\eta) = \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} f(x,t)dxdt = \bar{\bar{f}}(\xi,\eta), \qquad (\xi,\eta) \in \Omega$$

which exists and is convergent for all $(\xi, \eta) \in \Omega$ as defined as

$$\Omega = \left\{(\xi, \eta) \in C_2^2 \ : \ \xi \in \Omega_1 \text{ and } \eta \in \Omega_2\right\}, \tag{2.50}$$

where

$$\Omega_1 = \{\xi \in C_2 \ : \ \mathrm{Re}(\xi) > K_1 + |\mathrm{Im}_j(\xi)|\} \tag{2.51}$$

$$\Omega_2 = \{\eta \in C_2 \ : \ \mathrm{Re}(\eta) > K_2 + |\mathrm{Im}_j(\eta)|\}. \tag{2.52}$$

2.8 Properties of Bicomplex Double Laplace Transform

In this section, we discuss some properties of bicomplex double Laplace transform viz. linearity property, change of scale property, shifting property etc.

Theorem 2.16 (Linearity Property)**.** *Let $f(x,t)$ and $g(x,t)$ be two bicomplex-valued functions of $x, t > 0$ such that*

$$L_{xt}[f(x,t)] = \bar{\bar{f}}(\xi, \eta), \qquad (\xi, \eta) \in \Omega$$
$$\text{where } \Omega = \left\{(\xi, \eta) \in C_2^2 \ : \ Re(\xi) > K_1 + |Im_j(\xi)| \text{ and } Re(\eta) > K_2 + |Im_j(\eta)|\right\}$$
$$\text{and } L_{xt}[g(x,t)] = \bar{\bar{g}}(\xi, \eta), \qquad (\xi, \eta) \in \Omega$$
$$\text{where } \Omega = \left\{(\xi, \eta) \in C_2^2 \ : \ Re(\xi) > K_3 + |Im_j(\xi)| \text{ and } Re(\eta) > K_4 + |Im_j(\eta)|\right\}.$$

Then,

$$L_{xt}[c_1 f(x,t) + c_2 g(x,t)] = c_1 L_{xt}[f(x,t)] + c_2 L_{xt}[g(x,t)], \qquad (\xi, \eta) \in \Omega$$
$$\text{where } \Omega = \{(\xi, \eta) \in C_2^2 \ : \ Re(\xi) > \max(K_1, K_3) + |Im_j(\xi)|$$
$$\text{and } Re(\eta) > \max(K_2, K_4) + |Im_j(\eta)|\} \text{ and } c_1, c_2 \text{ are constants.}$$

Proof. Applying the definition of bicomplex double Laplace transform,

$$L_{xt}[c_1 f(x,t) + c_2 g(x,t)] = \int_0^\infty \int_0^\infty e^{-\xi x - \eta t}[c_1 f(x,t) + c_2 g(x,t)]dxdt$$
$$= c_1 \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} f(x,t) dxdt + c_2 \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} g(x,t) dxdt$$
$$= c_1 \bar{\bar{f}}(\xi, \eta) + c_2 \bar{\bar{g}}(\xi, \eta).$$

Thus,

$$L_{xt}[c_1 f(x,t) + c_2 g(x,t)] = c_1 L_{xt}[f(x,t)] + c_2 L_{xt}[g(x,t)].$$

□

Theorem 2.17 (Change of Scale Property). *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}[f(\alpha x, \beta t)](\xi,\eta) = \frac{1}{\alpha\beta}\bar{\bar{f}}\left(\frac{\xi}{\alpha},\frac{\eta}{\beta}\right), \qquad (\xi,\eta)\in\Omega \text{ and } \alpha,\beta>0$$

where Ω defined in (2.50).

Proof. From the definition of bicomplex double Laplace transform,

$$\begin{aligned}
L_{xt}[f(\alpha x,\beta t)](\xi,\eta) &= \int_0^\infty \int_0^\infty e^{-\xi x-\eta t} f(\alpha x,\beta t)dxdt \\
&= \int_0^\infty e^{-\eta t}\left(\int_0^\infty e^{-\xi x} f(\alpha x,\beta t)dx\right)dt \\
&= \frac{1}{\alpha}\int_0^\infty e^{-\eta t}\left(\int_0^\infty e^{-\frac{\xi}{\alpha}r} f(r,\beta t)dr\right)dt \quad [\text{Taking } \alpha x = r] \\
&= \frac{1}{\alpha}\int_0^\infty e^{-\eta t}\bar{f}\left(\frac{\xi}{\alpha},\beta t\right)dt \\
&= \frac{1}{\alpha\beta}\int_0^\infty e^{-\frac{\eta}{\beta}s}\bar{f}\left(\frac{\xi}{\alpha},s\right)ds \quad [\text{Taking } \beta t = s] \\
&= \frac{1}{\alpha\beta}\bar{\bar{f}}\left(\frac{\xi}{\alpha},\frac{\eta}{\beta}\right).
\end{aligned}$$

Thus,

$$L_{xt}[f(\alpha x,\beta t)](\xi,\eta) = \frac{1}{\alpha\beta}\bar{\bar{f}}\left(\frac{\xi}{\alpha},\frac{\eta}{\beta}\right).$$

□

Theorem 2.18 (First Shifting Property). *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[e^{ax+bt}f(x,t)\right](\xi,\eta) = \bar{\bar{f}}(\xi-a,\eta-b), \qquad (\xi-a,\eta-b)\in\Omega$$

where Ω defined in (2.50).

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}L_{xt}\left[e^{ax+bt}f(x,t)\right](\xi,\eta) &= \int_0^\infty\int_0^\infty e^{-\xi x-\eta t}e^{ax+bt}f(x,t)dxdt \\ &= \int_0^\infty e^{-(\eta-b)t}\left(\int_0^\infty e^{-(\xi-a)x}f(x,t)dx\right)dt \\ &= \int_0^\infty e^{-(\eta-b)t}\bar{f}(\xi-a,t)dt \\ &= \bar{\bar{f}}(\xi-a,\eta-b).\end{aligned}$$

Thus,

$$L_{xt}\left[e^{ax+bt}f(x,t)\right](\xi,\eta) = \bar{\bar{f}}(\xi-a,\eta-b).$$

□

Theorem 2.19 (Double Laplace Transform of Derivatives)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[f_{xt}(x,t)\right](\xi,\eta) = \xi\eta\bar{\bar{f}}(\xi,\eta) - \xi\bar{f}(\xi,0) - \eta\bar{f}(0,\eta) + f(0,0), \quad (\xi,\eta)\in\Omega$$

where Ω defined in (2.50) and $f_{xt}(x,t) = \frac{\partial^2}{\partial x\partial t}f(x,t)$.

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}L_{xt}\left[f_{xt}(x,t)\right] &= \int_0^\infty e^{-\eta t}\left(\int_0^\infty e^{-\xi x}f_{xt}(x,t)dx\right)dt \\ &= \int_0^\infty e^{-\eta t}\left[\left(e^{-\xi x}f_t(x,t)\right)_{x=0}^{\infty} + \xi\int_0^\infty e^{-\xi x}f_t(x,t)dx\right]dt \\ &= -\int_0^\infty e^{-\eta t}f_t(0,t)dt + \xi\int_0^\infty e^{-\eta t}\int_0^\infty f_t(x,t)dxdt \\ &= f(0,0) - \eta\int_0^\infty e^{-\eta t}f(0,t)dt + \xi\int_0^\infty e^{-\eta t}\left(\int_0^\infty f_t(x,t)dt\right)dx \\ &= f(0,0) - \eta\bar{f}(0,\eta) + \xi\int_0^\infty e^{-\xi x}\left[\left(e^{-\eta t}f(x,t)\right)_{t=0}^{\infty} + \eta\int_0^\infty e^{-\eta t}f(x,t)dt\right]dx \\ &= f(0,0) - \eta\bar{f}(0,\eta) - \xi\int_0^\infty e^{-\xi x}f(x,0)dx + \xi\eta\int_0^\infty\int_0^\infty e^{-\xi x-\eta t}f(x,t)dxdt \\ &= f(0,0) - \eta\bar{f}(0,\eta) - \xi\bar{f}(\xi,0) + \xi\eta\bar{\bar{f}}(\xi,\eta).\end{aligned}$$

Thus,

$$L_{xt}\left[f_{xt}(x,t)\right](\xi,\eta) = \xi\eta\bar{\bar{f}}(\xi,\eta) - \xi\bar{f}(\xi,0) - \eta\bar{f}(0,\eta) + f(0,0).$$

□

Theorem 2.20 (Multiplication by xt)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}[xtf(x,t)](\xi,\eta) = \frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta), \qquad [(\xi,\eta)\in\Omega \text{ as defined in (2.50)}]$$

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}\frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta) &= \left(\frac{\partial^2}{\partial\xi_1\partial\eta_1}\bar{\bar{f}}_{e_1}(\xi_1,\eta_1)\right)e_1 + \left(\frac{\partial^2}{\partial\xi_2\partial\eta_2}\bar{\bar{f}}_{e_2}(\xi_2,\eta_2)\right)e_2\\ &= \left(\frac{\partial^2}{\partial\xi_1\partial\eta_1}\int_0^\infty\int_0^\infty e^{-\xi_1 x-\eta_1 t}f_{e_1}(x,t)dxdt\right)e_1\\ &\quad + \left(\frac{\partial^2}{\partial\xi_2\partial\eta_2}\int_0^\infty\int_0^\infty e^{-\xi_2 x-\eta_2 t}f_{e_2}(x,t)dxdt\right)e_2\end{aligned}$$

$\left[\text{where } \bar{\bar{f}}(\xi,\eta) = \bar{\bar{f}}_{e_1}(\xi_1,\eta_1)e_1 + \bar{\bar{f}}_{e_2}(\xi_2,\eta_2)e_2,\ \xi = \xi_1e_1+\xi_2e_2 \text{ and } \eta = \eta_1e_1+\eta_2e_2\right]$.

Applying Leibniz's rule for complex functions [100, p. 243], we have

$$\begin{aligned}\frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta) &= (-1)^2\left\{\left(\int_0^\infty\int_0^\infty e^{-\xi_1 x-\eta_1 t}xtf_{e_1}(x,t)dxdt\right)e_1\right.\\ &\quad\left. + \left(\int_0^\infty\int_0^\infty e^{-\xi_2 x-\eta_2 t}xtf_{e_2}(x,t)dxdt\right)e_2\right\}\\ &= \int_0^\infty\int_0^\infty e^{-(\xi_1e_1+\xi_2e_2)x-(\eta_1e_1+\eta_2e_2)t}\left(f_{e_1}(x,t)e_1 + f_{e_2}(x,t)e_2\right)dxdt\\ &= \int_0^\infty\int_0^\infty e^{-\xi x-\eta t}xtf(x,t)dxdt.\end{aligned}$$

Thus,

$$L_{xt}[xtf(x,t)](\xi,\eta) = \frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta).$$

In general,

$$L_{xt}\left[x^mt^nf(x,t)\right](\xi,\eta) = (-1)^{m+n}\frac{\partial^{m+n}}{\partial\xi^m\partial\eta^n}\bar{\bar{f}}(\xi,\eta).$$

□

Theorem 2.21 (Division by xt)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[\frac{f(x,t)}{xt}\right](\xi,\eta)=\int_{\xi}^{\infty}\int_{\eta}^{\infty}\bar{\bar{f}}(\xi,\eta)d\xi d\eta,\qquad (\xi,\eta)\in\Omega$$

provided the integral on right hand exists.

Proof. Applying the definition of bicomplex double Laplace transform,

$$\bar{\bar{f}}(\xi,\eta)=\int_0^{\infty}\int_0^{\infty}e^{-\xi x-\eta t}f(x,t)dxdt \tag{2.53}$$

Integrating (2.53) w.r.t. ξ from ξ to ∞ and η from η to ∞, we have

$$\begin{aligned}\int_{\xi}^{\infty}\int_{\eta}^{\infty}\bar{\bar{f}}(\xi,\eta)d\xi d\eta &= \int_{\xi}^{\infty}\int_{\eta}^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\xi x}e^{-\eta t}f(x,t)dxdtd\xi d\eta\\ &=\int_{\eta}^{\infty}\int_0^{\infty}\int_0^{\infty}\left(\frac{e^{-\xi x}}{-x}\right)_{\xi=\xi}^{\infty}e^{-\eta t}f(x,t)dxdtd\eta\\ &=\int_0^{\infty}\int_0^{\infty}\left(0+\frac{e^{-\xi x}}{x}\right)\left(\frac{e^{-\eta t}}{-t}\right)_{\eta=\eta}^{\infty}f(x,t)dxdt\\ &=\int_0^{\infty}\int_0^{\infty}e^{-\xi x}e^{-\eta t}\frac{f(x,t)}{xt}dxdt\\ &=L_{xt}\left[\frac{f(x,t)}{xt}\right](\xi,\eta).\end{aligned}$$

Thus,

$$L_{xt}\left[\frac{f(x,t)}{xt}\right](\xi,\eta)=\int_{\xi}^{\infty}\int_{\eta}^{\infty}\bar{\bar{f}}(\xi,\eta)d\xi d\eta.$$

□

Theorem 2.22 (Double Laplace Transform of Integrals)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[\int_0^x\int_0^t f(u,v)dudv\right]=\frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta},\quad Re(\xi)>|Im_j(\xi)|\,,\ Re(\eta)>|Im_j(\eta)|\,.$$

Proof. Let

$$g(x,t)=\int_0^x\int_0^t f(u,v)dudv.$$

Hence, we have

$$g_{xt}(x,t) = f(x,t) \text{ and } g(0,0) = 0$$
$$\therefore \quad L_{xt}[g_{xt}(x,t)] = L[f(x,t)] = \bar{\bar{f}}(\xi,\eta).$$

Now from the Theorem 2.19 we have

$$L_{xt}\left[g_{xt}(x,t)\right] = \xi\eta\bar{\bar{g}}(\xi,\eta) - \xi\bar{g}(\xi,0) - \eta\bar{g}(0,\eta) + g(0,0)$$
$$\Rightarrow \bar{f}_2(\xi,\eta) = \xi\eta\bar{g}_2(\xi,\eta) - \xi\bar{g}_1(\xi,0) - \eta\bar{g}_1(0,\eta)$$
$$\therefore \quad \bar{\bar{g}}(\xi,\eta) = \frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta} + \frac{\bar{g}(\xi,0)}{\eta} + \frac{\bar{g}(0,\eta)}{\xi}.$$

But $\bar{g}(\xi,0) = 0$ and $\bar{g}(0,\eta) = 0$, therefore

$$\bar{\bar{g}}(\xi,\eta) = \frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}$$
$$\therefore \quad L_{xt}[g(x,t)] = \frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}.$$

Hence,

$$L_{xt}\left[\int_0^x \int_0^t f(u,v)dudv\right] = \frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}.$$

□

Theorem 2.23. *Let $f(x,t)$ be a periodic function of period K and T w.r.t. x and t respectively. Then the bicomplex double Laplace transform is given by*

$$L_{xt}[f(x,t)] = \frac{\int_0^K \int_0^T e^{-\xi x - \eta t} f(x,t)dxdt}{\left(1 - e^{-K\xi}\right)\left(1 - e^{-T\eta}\right)}, \quad Re(\xi) > |Im_j(\xi)| \text{ and } Re(\eta) > |Im_j(\eta)|.$$

Proof. Let $f(x,t)$ be a periodic function with period K w.r.t. x. Then for $\xi \in C_2$ and $\text{Re}(\xi) > |\text{Im}_j(\xi)|$ (see, Theorem 2.10)

$$L_x[f(x,t)] = \frac{\int_0^K e^{-\xi x} f(x,t)dx}{1 - e^{-K\xi}} = \bar{f}(\xi,t). \tag{2.54}$$

Similarly, for $\eta \in C_2$ and $\mathrm{Re}(\eta) > |\mathrm{Im}_j(\eta)|$ taking the bicomplex Laplace transform of (2.54) w.r.t. t, we have

$$\begin{aligned} L_t[\bar{f}(\xi,t)] = \bar{\bar{f}}(\xi,\eta) &= \frac{\int_0^T e^{-\eta t}\bar{f}_1(\xi,t)dt}{1-e^{-T\eta}} \\ &= \frac{1}{1-e^{-T\eta}}\int_0^T e^{-\eta t}\frac{\int_0^K e^{-\xi x}f(x,t)dx}{1-e^{-K\xi}}dt \\ &= \frac{\int_0^K\int_0^T e^{-\xi x-\eta t}f(x,t)dxdt}{(1-e^{-K\xi})(1-e^{-T\eta})}. \end{aligned}$$

Thus,

$$L_{xt}[f(x,t)] = \frac{\int_0^K\int_0^T e^{-\xi x-\eta t}f(x,t)dxdt}{(1-e^{-K\xi})(1-e^{-T\eta})}.$$

□

2.9 Inversion of Bicomplex Double Laplace Transform

In this section, we derive the inversion theorem for bicomplex double Laplace transform.

Theorem 2.24. *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$f(x,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x+\eta t}\bar{\bar{f}}(\xi,\eta)d\xi d\eta, \quad (\xi,\eta)\in\Omega \tag{2.55}$$

where Ω defined in (2.50) and Γ_1 and Γ_2 are Bromwich closed contours in bicomplex space.

Proof. Taking the inverse bicomplex Laplace transform [10] of $\bar{\bar{f}}(\xi,\eta)$ w.r.t. ξ, we have

$$L_\xi^{-1}[\bar{\bar{f}}(\xi,\eta)] = \bar{f}(x,\eta) = \frac{1}{2\pi i_1}\int_{\Gamma_2} e^{\xi x}\bar{\bar{f}}(\xi,\eta)d\xi. \tag{2.56}$$

Similarly, taking inverse bicomplex Laplace transform of (2.56) w.r.t. η, we have

$$\begin{aligned} L_{\eta}^{-1}[\bar{f}(x,\eta)] = f(x,t) &= \frac{1}{2\pi i_1}\int_{\Gamma_1} e^{\eta t}\bar{f}(x,\eta)d\eta \\ &= \frac{1}{(2\pi i_1)^2}\int_{\Gamma_1} e^{\eta t}\int_{\Gamma_2} e^{\xi x}\bar{\bar{f}}(\xi,\eta)d\xi d\eta. \end{aligned}$$

Hence,

$$f(x,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x+\eta t}\bar{\bar{f}}(\xi,\eta)d\xi d\eta.$$

□

2.10 Applications of Bicomplex Double Laplace Transform

In this section, we discuss applications of bicomplex double Laplace transform in finding the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle by two different approaches. In first approach, we find the solution of above equation by taking the bicomplex double Laplace transform under suitable initial and boundary conditions w.r.t. spaces variables x and y and in second approach, w.r.t. space variable x and time variable t.

Rochon and *Tremblay* [124] discussed the extension of time dependent time-dependent complex Schrödinger equation in bicomplex form. In section 2.6, we discussed the solution of one-dimensional time-dependent bicomplex Schrödinger equation for free particle using by bicomplex Laplace transform. In [6], *Arnold* discussed the solution of two-dimensional time-dependent complex Schrödinger equation using by Fourier-Laplace transform. In [35], *Dehghan* et al. discussed the numerical solution of two-dimensional time-dependent Schrödinger equation.

Here, we discuss the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle. We extend the one-dimensional time-dependent bicomplex Schrödinger equation (2.38) in two dimensions as

$$i_1\hbar\partial_t\psi(x,y,t)+\frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t)+\partial_y^2\psi(x,y,t)\right)-V(x,y,t)\psi(x,y,t)=0, \tag{2.57}$$

where

$$\psi:\mathbb{R}^3\to C_2 \text{ and } V:\mathbb{R}^3\to\mathbb{R}.$$

with initial and boundary conditions

$$\begin{aligned}&\psi(x,y,0)=h(x,y),\ \psi(0,y,t)=f_1(y,t),\ \psi(x,0,t)=g_1(x,t),\\&\psi_x(0,y,t)=f_2(y,t),\ \psi_y(x,0,t)=g_2(x,t),\ x>0,\ y>0,\ t>0.\end{aligned} \tag{2.58}$$

For free particle $V(x,y,t)=0$, (2.57) becomes

$$i_1\hbar\partial_t\psi(x,y,t)+\frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t)+\partial_y^2\psi(x,y,t)\right)=0. \tag{2.59}$$

(a) Firstly, we have solved equation (2.59) by using bicomplex double Laplace transform w.r.t. x and y. Taking bicomplex double Laplace transform of (2.59) w.r.t. x and y, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{-\xi x-\eta y}i_1\hbar\partial_t\psi(x,y,t)dxdy\\&+\int_0^\infty\int_0^\infty e^{-\xi x-\eta y}\frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t)+\partial_y^2\psi(x,y,t)\right)dxdy=0\\&\Rightarrow i_1\hbar\frac{d}{dt}\bar{\bar{\psi}}(\xi,\eta,t)+\frac{\hbar^2}{2m}\Big(\left(\xi^2+\eta^2\right)\bar{\bar{\psi}}(\xi,\eta,t)-\xi\bar{\psi}(0,\eta,t)-\eta\bar{\psi}(\xi,0,t)\\&-\bar{\psi}_x(0,\eta,t)-\bar{\psi}_y(\xi,0,t)\Big)=0,\end{aligned}$$

where $\bar{\bar{\psi}}(\xi,\eta,t) = L_{xy}\left[\psi(x,y,t)\right]$ is bicomplex double Laplace transform of $\psi(x,y,t)$.

Applying the boundary conditions (2.58), we get

$$\begin{aligned} & i_1\hbar\frac{d\bar{\bar{\psi}}}{dt} + \frac{\hbar^2}{2m}\Big(\left(\xi^2+\eta^2\right)\bar{\bar{\psi}} - \xi\bar{f}_1(\eta,t) - \eta\bar{g}_1(\xi,t) - \bar{f}_2(\eta,t) - \bar{g}_2(\xi,t)\Big) = 0 \\ \Rightarrow\ & \frac{d\bar{\bar{\psi}}}{dt} - i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right) = -i_1\frac{\hbar}{2m}\left(\xi\bar{f}_1(\eta,t) + \eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\right). \end{aligned}$$

Rearranging the terms and simplifying, we get

$$\begin{aligned} \bar{\bar{\psi}}(\xi,\eta,t) = & -i_1\frac{\hbar}{2m}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\int \exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\big(\xi\bar{f}_1(\eta,t) \\ & +\eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\big)\,dt + c\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right). \end{aligned} \tag{2.60}$$

Letting $\bar{\bar{\psi}}(\xi,\eta,0) = \bar{\bar{h}}(\xi,\eta)$ in (2.60) we have

$$\begin{aligned} c = & \bar{\bar{h}}(\xi,\eta) \\ & +i_1\frac{\hbar}{2m}\int \exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\left(\xi\bar{f}_1(\eta,t) + \eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\right)dt\Big|_{t=0} \\ = & \bar{\bar{p}}(\xi,\eta) \qquad \text{(say)}. \end{aligned} \tag{2.61}$$

Therefore, (2.60) becomes

$$\begin{aligned} \bar{\bar{\psi}}(\xi,\eta,t) = & -i_1\frac{\hbar}{2m}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\int \exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\big(\xi\bar{f}_1(\eta,t)+ \\ & \eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\big)\,dt + \bar{\bar{p}}(\xi,\eta)\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right). \end{aligned} \tag{2.62}$$

Taking the inverse bicomplex Laplace transform of (2.62), we have

$$\psi(x,y,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x+\eta y}\left[\bar{\bar{p}}(\xi,\eta)\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right) + A(\xi,\eta,t)\right]d\xi d\eta \tag{2.63}$$

where,

$$A(\xi,\eta,t) = -i_1\frac{\hbar}{2m}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\int \exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\left(\xi\bar{f}_1(\eta,t)\right.$$
$$\left.+\eta\bar{g}_1(\xi,t)+\bar{f}_2(\eta,t)+\bar{g}_2(\xi,t)\right)dt$$

and Γ_1 and Γ_2 are closed contours in bicomplex space w.r.t. ξ and η respectively. (2.63) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle.

Illustrative Example:

Let us consider the initial and boundary conditions for equation (2.59) as

$$\psi(x,y,0) = \sin\left(\frac{2\pi}{\lambda}x\right)\cos\left(\frac{2\pi}{\lambda}y\right),\ \psi_x(0,y,t) = \frac{2\pi}{\lambda}\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}y\right),$$
$$\psi(0,y,t)=0,\ \psi(x,0,t)=0,\ \psi_y(x,0,t) = \frac{2\pi}{\lambda}\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right). \tag{2.64}$$

Then (2.63) becomes

$$\begin{aligned}
\psi(x,y,t) &= -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x+\eta y}\frac{2\pi}{\lambda}\frac{\eta\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)}{\left(\xi^2+\left(\frac{2\pi}{\lambda}\right)^2\right)\left(\eta^2+\left(\frac{2\pi}{\lambda}\right)^2\right)}d\xi d\eta \\
&= -\frac{1}{2\pi\lambda}\int_{\Gamma_1}\frac{e^{\xi x}}{\left(\xi^2+\left(\frac{2\pi}{\lambda}\right)^2\right)}2\pi i_1\left(\lim_{\eta\to i_1\frac{2\pi}{\lambda}} e^{\eta y}\frac{\eta\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)}{\left(\eta+i_1\frac{2\pi}{\lambda}\right)}\right. \\
&\quad \left.+\lim_{\eta\to -i_1\frac{2\pi}{\lambda}} e^{\eta y}\frac{\eta\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)}{\left(\eta-i_1\frac{2\pi}{\lambda}\right)}\right)d\xi \\
&= -\frac{i_1}{\lambda}\int_{\Gamma_1}\frac{e^{\xi x}}{\left(\xi^2+\left(\frac{2\pi}{\lambda}\right)^2\right)}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2-\frac{4\pi^2}{\lambda^2}\right)t\right)\cos\left(\frac{2\pi}{\lambda}y\right)d\xi
\end{aligned}$$

$$= -\frac{i_1}{\lambda}\cos\left(\frac{2\pi}{\lambda}y\right) 2\pi i_1 \left(\lim_{\xi \to i_1\frac{2\pi}{\lambda}} \frac{e^{\xi x}}{\left(\xi + i_1\frac{2\pi}{\lambda}\right)} \exp\left(i_1 \frac{\hbar}{2m}\left(\xi^2 - \frac{4\pi^2}{\lambda^2}\right)t\right) \right.$$
$$\left. + \lim_{\xi \to -i_1\frac{2\pi}{\lambda}} \frac{e^{\xi x}}{\left(\xi - i_1\frac{2\pi}{\lambda}\right)} \exp\left(i_1 \frac{\hbar}{2m}\left(\xi^2 - \frac{4\pi^2}{\lambda^2}\right)t\right)\right)$$
$$= \exp\left(-i_1 \frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\cos\left(\frac{2\pi}{\lambda}y\right).$$

Therefore,

$$\psi(x, y, t) = \exp\left(-i_1 \frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\cos\left(\frac{2\pi}{\lambda}y\right). \tag{2.65}$$

Expression in (2.65) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation (2.59) for initial and boundary conditions (2.64).

(b) In this alternative approach, we shall make use of bicomplex double Laplace transform w.r.t. x and t. Consider the two-dimensional time-dependent bicomplex Schrödinger equation (2.59) for free particle with the condition

$$\psi(x, y, t) \text{ is bounded as } |y| \to \infty \text{ and } x > 0,\ t > 0. \tag{2.66}$$

Taking the bicomplex double Laplace transform of (2.59) w.r.t. x and t, we get

$$\int_0^\infty \int_0^\infty e^{-\xi x - \eta t} i_1 \hbar \partial_t \psi(x, y, t) dx dy + \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} \frac{\hbar^2}{2m}\left(\partial_x^2 \psi(x, y, t)\right.$$
$$\left. + \partial_y^2 \psi(x, y, t)\right) dx dy = 0$$
$$\Rightarrow\ i_1 \hbar \left(\eta \bar{\bar{\psi}}(\xi, y, \eta) - \bar{\psi}(\xi, y, 0)\right) + \frac{\hbar^2}{2m}\left(\xi^2 \bar{\bar{\psi}}(\xi, y, \eta) - \xi \bar{\psi}(0, y, \eta) - \bar{\psi}_x(0, y, \eta)\right)$$
$$+ \frac{\hbar^2}{2m}\frac{d^2}{dy^2}\bar{\bar{\psi}}(\xi, y, \eta) = 0$$
$$\Rightarrow\ \frac{d^2 \bar{\bar{\psi}}}{dy^2} + \left(\xi^2 + i_1 \frac{2m\eta}{\hbar}\right)\bar{\bar{\psi}} = i_1 \frac{2m}{\hbar}\bar{\psi}(\xi, y, 0) + \xi \bar{\psi}(0, y, \eta) + \bar{\psi}_x(0, y, \eta).$$

Rearranging the terms and solving, we get

$$\bar{\bar{\psi}}(\xi, y, \eta) = c_1 \exp\left(y\sqrt{-\xi^2 - i_1\frac{2m\eta}{\hbar}}\right) + c_2 \exp\left(-y\sqrt{-\xi^2 - i_1\frac{2m\eta}{\hbar}}\right)$$
$$+ \frac{1}{\frac{d^2}{dy^2} + \left(\xi^2 + i_1\frac{2m\eta}{\hbar}\right)}\left(i_1\frac{2m}{\hbar}\bar{\psi}(\xi, y, 0) + \xi\bar{\psi}(0, y, \eta) + \bar{\psi}_x(0, y, \eta)\right), \tag{2.67}$$

$$\left[\text{where Re}\left(P_1 : \sqrt{-\xi^2 - i_1\frac{2m\eta}{\hbar}}\right) > 0 \text{ and Re}\left(P_2 : \sqrt{-\xi^2 - i_1\frac{2m\eta}{\hbar}}\right) > 0\right].$$

$\because$ $\bar{\bar{\psi}}(\xi, y, \eta)$ is bounded as $|y| \to \infty \Rightarrow c_1 = c_2 = 0$. Then (2.67) becomes

$$\bar{\bar{\psi}}(\xi, y, \eta) = \frac{1}{\frac{d^2}{dy^2} + \left(\xi^2 + i_1\frac{2m\eta}{\hbar}\right)}\left(i_1\frac{2m}{\hbar}\bar{\psi}(\xi, y, 0) + \xi\bar{\psi}(0, y, \eta) + \bar{\psi}_x(0, y, \eta)\right). \tag{2.68}$$

Taking the inverse bicomplex Laplace transform of (2.68), we get

$$\psi(x, y, t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x + \eta t}\bar{\bar{\psi}}(\xi, y, \eta)d\xi d\eta \tag{2.69}$$

where Γ_1 and Γ_2 are closed contours in bicomplex space w.r.t. ξ and η respectively.

(2.69) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation (2.59).

Illustrative Example:

Let us consider the initial and boundary conditions for equation (2.59) as

$$\psi(x, y, 0) = \sin\left(\frac{2\pi}{\lambda}x\right)\sin\left(\frac{2\pi}{\lambda}y\right),\ \psi(0, y, t) = 0$$
$$\psi_x(0, y, t) = \frac{2\pi}{\lambda}\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}y\right). \tag{2.70}$$

Then (2.69) becomes

$$\psi(x, y, t)$$
$$= -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{2\pi}{\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\frac{e^{\xi x + \eta t}}{\xi^2 + i_1\frac{2m\eta}{\hbar} - \frac{4\pi^2}{\lambda^2}}\left(i_1\frac{2m}{\hbar}\frac{1}{\xi^2 + \frac{4\pi^2}{\lambda^2}} + \frac{1}{\eta - i_1\frac{4\hbar\pi^2}{m\lambda^2}}\right)d\xi d\eta$$

$$\begin{aligned}
&= -i_1 \frac{m}{\pi\hbar\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1}\int_{\Gamma_2} \frac{e^{\xi x+\eta t}}{\left(\xi^2 + i_1\frac{2m\eta}{\hbar} - \frac{4\pi^2}{\lambda^2}\right)\left(\xi^2 + \frac{4\pi^2}{\lambda^2}\right)} d\xi d\eta \\
&\quad - \frac{1}{2\pi\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1}\int_{\Gamma_2} \frac{e^{\xi x+\eta t}}{\left(\xi^2 + i_1\frac{2m\eta}{\hbar} - \frac{4\pi^2}{\lambda^2}\right)\left(\eta - i_1\frac{4\hbar\pi^2}{m\lambda^2}\right)} d\xi d\eta \\
&= I_1 + I_2. \quad \text{(say)}
\end{aligned} \tag{2.71}$$

Now,

$$\begin{aligned}
I_1 &= -i_1 \frac{m}{\pi\hbar\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1}\int_{\Gamma_2} \frac{e^{\xi x+\eta t}}{\left(\xi^2 + i_1\frac{2m\eta}{\hbar} - \frac{4\pi^2}{\lambda^2}\right)\left(\xi^2 + \frac{4\pi^2}{\lambda^2}\right)} d\xi d\eta \\
&= -i_1 \frac{m}{\pi\hbar\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1} 2\pi i_1 \frac{\hbar}{2m i_1} e^{\xi x} \lim_{\eta\to -i_1\left(\frac{4\pi^2}{\lambda^2}-\xi^2\right)\frac{\hbar}{2m}} \left(e^{\eta t}\frac{1}{\xi^2 + \frac{4\pi^2}{\lambda^2}}\right) d\xi \\
&= -\frac{i_1}{\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1} \frac{e^{\xi x}}{\xi^2 + \frac{4\pi^2}{\lambda^2}} \exp\left(-i_1\frac{\hbar}{2m}\left(\frac{4\pi^2}{\lambda^2} - \xi^2\right)t\right) d\xi \\
&= -\frac{i_1}{\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) 2\pi i_1 \left[\lim_{\xi\to i_1\frac{2\pi}{\lambda}} \frac{e^{\xi x}}{\xi + i_1\frac{2\pi}{\lambda}} \exp\left(-i_1\frac{\hbar}{2m}\left(\frac{4\pi^2}{\lambda^2} - \xi^2\right)t\right)\right. \\
&\quad \left. + \lim_{\xi\to -i_1\frac{2\pi}{\lambda}} \frac{e^{\xi x}}{\xi - i_1\frac{2\pi}{\lambda}} \exp\left(-i_1\frac{\hbar}{2m}\left(\frac{4\pi^2}{\lambda^2} - \xi^2\right)t\right)\right] \\
&= \exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right) \sin\left(\frac{2\pi}{\lambda}x\right) \sin\left(\frac{2\pi}{\lambda}y\right).
\end{aligned} \tag{2.72}$$

Similarly,

$$\begin{aligned}
I_2 &= -\frac{1}{2\pi\lambda} \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1}\int_{\Gamma_2} \frac{e^{\xi x+\eta t}}{\left(\xi^2 + i_1\frac{2m\eta}{\hbar} - \frac{4\pi^2}{\lambda^2}\right)\left(\eta - i_1\frac{4\hbar\pi^2}{m\lambda^2}\right)} d\xi d\eta \\
&= -i_1\lambda \sin\left(\frac{2\pi}{\lambda}y\right) \int_{\Gamma_1} \frac{\left(\exp\left(i_1\frac{4\pi^2\hbar}{\lambda^2 m}t\right) - \exp\left(i_1\frac{\lambda^2\xi^2\hbar - 4\pi^2\hbar}{2\lambda^2 m}t\right)\right)}{\lambda^2\xi^2 - 12\pi^2} d\xi \\
&= 0, \quad \text{[by using contor integral].}
\end{aligned} \tag{2.73}$$

Using (2.72) and (2.73) in (2.71), we get

$$\psi(x,y,t) = \exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right) \sin\left(\frac{2\pi}{\lambda}x\right) \sin\left(\frac{2\pi}{\lambda}y\right). \tag{2.74}$$

Expression in (2.74) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation (2.59) under the conditions given in (2.66) and (2.70).

2.11 Conclusion

In this chapter, we derived inversion formula, convolution theorem and some properties of bicomplex Laplace transform. Also, we defined double Laplace transform and its inverse in bicomplex space. Applications have been illustrated for the solution of bicomplex Maxwell's equation and bicomplex time-dependent Schrödinger equation for free particle using bicomplex Laplace transform. Also, the solution of two-dimensional time-dependent bicomplex Schrödinger equation has been obtained by using bicomplex double Laplace transform.

Moreover, similar to work of *Rochon* and *Tremblay* [124], under some discrete symmetries time-dependent bicomplex Schrödinger equation can be decomposed into two standard Schrödinger equations. Therefore, solution of two standard Schrödinger equations can be obtained by separating the solution of time-dependent bicomplex Schrödinger equation. The bicomplex analysis has found a great advantage that it separates the electric and magnetic field as complex components. This theory can further be developed to find the solution of the problems in electromagnetic field theory and quantum mechanics also.

3

Sumudu Transform in Bicomplex Space and Applications

The main finding of this chapter has been published as:

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2017), Sumudu transform in bicomplex space and its applications, *Annals of Applied Mathematics*, (In press).

In this chapter, we obtain bicomplex Sumudu transform, its inverse and some of their properties. As applications we find the solution of differential equation of bicomplex-valued function and solution of Cartesian transverse electric magnetic (TEM) waves in homogeneous media.

3.1 Introduction

In literature, various integral transforms are widely used in physics and engineering mathematics. In the sequence of these integral transforms, *Watugala* [146] defined Sumudu transform and applied it for finding the solution of ordinary differential equations in control engineering problems.

Over the set of functions

$$\mathcal{A} = \left\{ f(t) : \exists M,\ \tau_j > 0,\ |f(t)| < Me^{\frac{|t|}{\tau_j}},\ \text{if } t \in (-1)^j[0,\infty),\ j = 1,\, 2 \right\} \tag{3.1}$$

the Sumudu transform is defined by the formula

$$\mathcal{S}[f(t); s] = \frac{1}{s}\int_0^{\infty} e^{-\frac{t}{s}} f(t)dt, \quad s \in (\tau_1, \tau_2). \tag{3.2}$$

Sumudu transform have scale and unit preserving properties. In [17], *Belgacem* et al. discussed fundamental properties of Sumudu transform and shown that Sumudu transform is a theoretical dual of Laplace transform. Also, it used in solving an integral production-depreciation problem. In [16], *Belgacem* et al. have generalized Sumudu differentiations, integrations and convolution theorems existing in literature. Also they have generalized Sumudu shifting theorems and introduced recurrence formulas of the transform.

In [157], *Zhang* developed an algorithm based on Sumudu transform which can be implemented in computer algebra systems like Maple and used in solving differential equations. In [70], *Hussian* et al. obtained the solution of Maxwell's differential equations for transient excitation functions propagating in a lossy conducting medium by using Sumudu transform in time domain.

In [15], *Belgacem* find the electric field solutions of Maxwell's equations, pertaining to transient electromagnetic planner, (TEMP), waves propagating in lossy media, through Sumudu transform. In [42], *Eltayeb* et al. have discussed the Sumudu transform on a space of distributions. In ([80], [81]), *Kilicman* and *Gadain* produced some properties and relationship between double Laplace and double Sumudu transform and also, used the double Sumudu transform for solving wave equation in one dimension having singularity at initial conditions.

In [79], *Kilicman* et al. discussed the existence of double Sumudu transform with convergence conditions and applied it for finding the solution of linear ordinary differential equations with constant coefficients. In [1], *Al-Omari* and *Belgacem* investigated certain class of quaternions and Sumudu transform. Motivated by the work of *Al-Omari* et al., we have made efforts to extend the Sumudu transform to bicomplex variable.

3.2 Bicomplex Sumudu Transform

Let $f(t)$ be bicomplex-valued piecewise continuous function of exponential order K. Then bicomplex Laplace transform (*Kumar* and *Kumar* [88]) of $f(t)$ is

$$L[f(t);\xi] = \int_0^\infty e^{-\xi t} f(t)dt, \quad \xi \in D \tag{3.3}$$

where

$$D = \{\xi = s_1e_1 + s_2e_2 \in C_2 : \text{Re}(s_1) > K \text{ and } \text{Re}(s_2) > K\} \tag{3.4}$$

or, equivalently

$$D = \{\xi \in C_2 : \text{Re}(\xi) > K + |\text{Im}_j(\xi)|\} \tag{3.5}$$

$\text{Im}_j(\xi)$ denotes the imaginary part w.r.t. j. In (3.3) if we replace ξ by $\frac{1}{\xi}$ and multiply the integral obtained by $\frac{1}{\xi}$, we get

$$\frac{1}{\xi}L\left[f(t);\frac{1}{\xi}\right] = \frac{1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} f(t)dt = \mathcal{S}[f(t);\xi] = \bar{f}(\xi), \quad \xi \in \Omega \tag{3.6}$$

where $\mathcal{S}[\cdot]$ denote the Sumudu transform of f and

$$\Omega = \left\{\xi = s_1e_1 + s_2e_2 \in C_2 : \text{Re}\left(\frac{1}{s_1}\right) > K,\ \text{Re}\left(\frac{1}{s_2}\right) > K \text{ and } \xi \notin \mathcal{O}_2\right\} \tag{3.7}$$

or equivalently,

$$\Omega = \left\{\xi \in C_2 : \text{Re}\left(\frac{1}{\xi}\right) > K + \left|\text{Im}_j\left(\frac{1}{\xi}\right)\right| \text{ and } \xi \notin \mathcal{O}_2\right\} \tag{3.8}$$

Further, we show that $\left\|\bar{f}(\xi)\right\| < \infty$. Now, for $\xi = s_1e_1 + s_2e_2$, $s_1 = x_1 + i_1y_1$ and $s_2 = x_2 + i_1y_2$,

$$\begin{aligned}
\left\|\bar{f}(\xi)\right\| &= \left\|\frac{1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} f(t)dt\right\| \\
&\leq \frac{1}{\|\xi\|}\int_0^\infty \left\|e^{-\frac{1}{\xi}t}\right\| \|f(t)\|\, dt, \quad [\xi \neq 0 \text{ i.e. } \xi \notin \mathcal{O}_2] \\
&\leq \frac{1}{\|\xi\|}\int_0^\infty \left\|e^{-\frac{1}{(s_1e_1+s_2e_2)}t}\right\| Me^{Kt}dt \\
&= \frac{M}{\|\xi\|}\int_0^\infty \left\|e^{-\frac{1}{s_1}t}e_1 + e^{-\frac{1}{s_2}t}e_2\right\| e^{Kt}dt, \quad [\text{using (1.15)}] \\
&= \frac{M}{\|\xi\|}\int_0^\infty \frac{1}{\sqrt{2}}\left(\left|e^{-\frac{1}{s_1}t}\right|^2 + \left|e^{-\frac{1}{s_2}t}\right|^2\right)^{\frac{1}{2}} e^{Kt}dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{M}{\sqrt{2}\|\xi\|}\int_0^\infty \left(e^{-\frac{2x_1}{x_1^2+y_1^2}t} + e^{-\frac{2x_2}{x_2^2+y_2^2}t}\right)^{\frac{1}{2}} e^{Kt}dt \\
&\leq \frac{M}{\sqrt{2}\|\xi\|}\left[\int_0^\infty e^{-\frac{x_1}{x_1^2+y_1^2}t}e^{Kt}dt + \int_0^\infty e^{-\frac{x_2}{x_2^2+y_2^2}t}e^{Kt}dt\right] \\
&\quad \left[\because \text{ by Minkowski's inequality } \left(|x|^2+|y|^2\right)^{\frac{1}{2}} \leq |x|+|y|, \quad \forall\, x, y \in \mathbb{R}\right] \\
&= \frac{M}{\sqrt{2}\|\xi\|}\left[\int_0^\infty e^{-\left(\frac{x_1}{x_1^2+y_1^2}-K\right)t}dt + \int_0^\infty e^{-\left(\frac{x_2}{x_2^2+y_2^2}-K\right)t}dt\right] \\
&= \frac{M}{\sqrt{2}\|\xi\|}\left(\frac{1}{\frac{x_1}{x_1^2+y_1^2}-K} + \frac{1}{\frac{x_2}{x_2^2+y_2^2}-K}\right).
\end{aligned}$$

Then the requirement $\left\|\bar{f}(\xi)\right\| < \infty$ only if $\frac{x_1}{x_1^2+y_1^2} > K$ i.e. $\mathrm{Re}\left(\frac{1}{s_1}\right) > K$ and $\frac{x_2}{x_2^2+y_2^2} > K$ i.e. $\mathrm{Re}\left(\frac{1}{s_2}\right) > K$. Therefore, $\bar{f}(\xi)$ is analytic and convergent in the strip Ω, defined by (3.7).

Thus, we can summarize the above discussion to define the bicomplex Sumudu transform as follows:

Definition 3.1. Let $f(t)$ bicomplex-valued piecewise continuous function of exponential order K. Then bicomplex Sumudu transform of $f(t)$ is defined as

$$\mathcal{S}[f(t);\xi] = \frac{1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} f(t)dt = \bar{f}(\xi), \quad \xi \in \Omega \tag{3.9}$$

where Ω is defined as

$$\Omega = \left\{\xi \in C_2 : \text{Re}\left(\frac{1}{\xi}\right) > K + \left|\text{Im}_j\left(\frac{1}{\xi}\right)\right| \text{ and } \xi \notin \mathcal{O}_2\right\}. \tag{3.10}$$

3.3 Properties of Bicomplex Sumudu Transform

In this section, we discuss some properties of bicomplex Sumudu transform viz. linearity property, change of scale property and etc. as follows:

Theorem 3.1 (Linearity Property)**.** *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ and $\bar{g}(\xi) = \mathcal{S}[g(t);\xi]$ be bicomplex Sumudu transforms of bicomplex-valued functions $f(t)$ and $g(t)$ of exponential orders K_1 and K_2, respectively. Then*

$$\mathcal{S}[c_1 f(t) + c_2 g(t)] = c_1\bar{f}(\xi) + c_2\bar{g}(\xi), \quad \xi \in \Omega \tag{3.11}$$

where Ω is defined in (3.10) and $K = \max\{K_1, K_2\}$.

Proof. By applying the definition of bicomplex Sumudu transform, we get

$$\begin{aligned}\mathcal{S}[c_1 f(t) + c_2 g(t)] &= \frac{1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t}[c_1 f(t) + c_2 g(t)]dt \\ &= \frac{c_1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} f(t)dt + \frac{c_2}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} g(t)dt \\ &= c_1\bar{f}(\xi) + c_2\bar{g}(\xi).\end{aligned}$$

□

Theorem 3.2 (Change of scale property)**.** *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

$$\mathcal{S}[f(at);\xi] = \bar{f}(a\xi), \quad a > 0,\ \xi \in \Omega \tag{3.12}$$

where Ω is defined in (3.10).

Proof. By applying the definition of bicomplex Sumudu transform, we get

$$\begin{aligned}\mathcal{S}[f(at);\xi] &= \frac{1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} f(at)dt, \quad [\text{put } at = u] \\ &= \frac{1}{a\xi}\int_0^\infty e^{-\frac{1}{a\xi}u} f(u)du \\ &= \bar{f}(a\xi).\end{aligned}$$

□

Theorem 3.3. *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

(i) $\mathcal{S}[f'(t);\xi] = \frac{1}{\xi}[\bar{f}(\xi) - f(0)], \qquad \xi \in \Omega$

(ii) $\mathcal{S}[f''(t);\xi] = \frac{1}{\xi^2}[\bar{f}(\xi) - f(0) - \xi f'(0)], \quad \xi \in \Omega$

(iii) $\mathcal{S}[f^{(n)}(t);\xi] = \frac{1}{\xi^n}\left[\bar{f}(\xi) - f(0) - \sum_{k=1}^{n-1} \xi^k f^{(k)}(0)\right], \quad \xi \in \Omega$

where Ω is defined in (3.10).

Proof. (*i*) By applying the definition of bicomplex Sumudu transform, we get

$$\begin{aligned}\mathcal{S}[f'(t);\xi] &= \frac{1}{\xi}\int_0^\infty e^{-\frac{1}{\xi}t} f'(t)dt \\ &= \left(\frac{1}{s_1}\int_0^\infty e^{-\frac{1}{s_1}t} f_1'(t)dt\right)e_1 + \left(\frac{1}{s_2}\int_0^\infty e^{-\frac{1}{s_2}t} f_2'(t)dt\right)e_2\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\mathcal{S}[f'(t);\xi] &= \left(-\frac{1}{s_1}f_1(0) + \frac{1}{s_1}\bar{f}_1(s_1)\right)e_1 + \left(-\frac{1}{s_2}f_2(0) + \frac{1}{s_2}\bar{f}_2(s_2)\right)e_2 \\ &= \frac{1}{s_1e_1 + s_2e_2}\left[\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2 - f_1(0)e_1 - f_2(0)e_2\right] \\ &= \frac{1}{\xi}[\bar{f}(\xi) - f(0)]. \qquad (3.13)\end{aligned}$$

(*ii*) If we replace $f'(t) = g(t)$, then

$$\begin{aligned}\mathcal{S}[f''(t);\xi] &= \mathcal{S}[g'(t);\xi] \\ &= \frac{1}{\xi}\left[\bar{g}(\xi) - g(0)\right], \quad \text{[using (3.13)]} \\ &= \frac{1}{\xi}\left[\frac{1}{\xi}\left(\bar{f}(\xi) - f(0)\right) - f'(0)\right] \\ &= \frac{1}{\xi^2}\left[\bar{f}(\xi) - f(0) - \xi f'(0)\right].\end{aligned}$$

(*iii*) By (3.13) the result (*iii*) is true for $n = 1$. Now, let us assume the result is true for $n = m$, i.e.

$$\mathcal{S}[f^{(m)}(t);\xi] = \frac{1}{\xi^m}\left[\bar{f}(\xi) - f(0) - \sum_{k=1}^{m-1}\xi^k f^{(k)}(0)\right] \tag{3.14}$$

Now, for $n = m + 1$

$$\begin{aligned}\mathcal{S}[f^{(m+1)}(t);\xi] &= \frac{1}{\xi}\left[\mathcal{S}[f^{(m)}(t);\xi] - f^{(m)}(0)\right] \\ &= \frac{1}{\xi^{m+1}}\left[\bar{f}(\xi) - f(0) - \sum_{k=1}^{m}\xi^k f^{(k)}(0)\right], \quad \text{[using (3.14)]}\end{aligned}$$

Hence, by the principle of mathematical induction, the result is true for all $n \in \mathbb{N}$. □

Theorem 3.4. *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

$$\mathcal{S}[tf(t);\xi] = \xi^2\frac{d}{d\xi}\bar{f}(\xi) + \xi\bar{f}(\xi), \qquad \xi \in \Omega \tag{3.15}$$

where Ω is defined in (3.10).

Proof. Since,

$$\begin{aligned}\frac{d}{d\xi}\bar{f}(\xi) &= \frac{d}{d\xi}\int_0^\infty \frac{1}{\xi}e^{-\frac{t}{\xi}}f(t)dt \\ &= \left(\frac{d}{ds_1}\int_0^\infty \frac{1}{s_1}e^{-\frac{t}{s_1}}f_1(t)dt\right)e_1 + \left(\frac{d}{ds_2}\int_0^\infty \frac{1}{s_2}e^{-\frac{t}{s_2}}f_2(t)dt\right)e_2\end{aligned}$$

Using Leibniz rule for integration for complex functions, we get

$$\begin{aligned}
\frac{d}{d\xi}\bar{f}(\xi) &= \left(\int_0^\infty \frac{\partial}{\partial s_1}\frac{1}{s_1}e^{-\frac{t}{s_1}}f_1(t)dt\right)e_1 + \left(\int_0^\infty \frac{\partial}{\partial s_2}\frac{1}{s_2}e^{-\frac{t}{s_2}}f_2(t)dt\right)e_2 \\
&= \left(\int_0^\infty \frac{1}{s_1^3}e^{-\frac{t}{s_1}}tf_1(t)dt - \int_0^\infty \frac{1}{s_1^2}e^{-\frac{t}{s_1}}f_1(t)dt\right)e_1 \\
&\quad + \left(\int_0^\infty \frac{1}{s_2^3}e^{-\frac{t}{s_2}}tf_2(t)dt - \int_0^\infty \frac{1}{s_2^2}e^{-\frac{t}{s_2}}f_2(t)dt\right)e_2 \\
&= \left(\frac{1}{s_1^2}\mathcal{S}[tf_1(t);s_1] - \frac{1}{s_1}\mathcal{S}[f_1(t);s_1]\right)e_1 + \left(\frac{1}{s_2^2}\mathcal{S}[tf_2(t);s_2] - \frac{1}{s_2}\mathcal{S}[f_2(t);s_2]\right)e_2 \\
&= \frac{1}{(s_1e_1+s_2e_2)^2}\mathcal{S}[t\left(f_1(t)e_1+f_2(t)e_2\right);s_1e_1+s_2e_2] \\
&\quad - \frac{1}{s_1e_1+s_2e_2}\mathcal{S}[\left(f_1(t)e_1+f_2(t)e_2\right);s_1e_1+s_2e_2] \\
&= \frac{1}{\xi^2}\mathcal{S}[tf(t);\xi] - \frac{1}{\xi}\mathcal{S}[f(t);\xi]
\end{aligned}$$

$$\therefore \quad \mathcal{S}[tf(t);\xi] = \xi^2\frac{d}{d\xi}\bar{f}(\xi) + \xi\bar{f}(\xi).$$

□

Theorem 3.5. *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$, then*

$$\mathcal{S}\left[\int_0^t f(u)du;\xi\right] = \xi\bar{f}(\xi), \quad \xi \in \Omega \tag{3.16}$$

where Ω is defined in (3.10).

Proof. From the definition of bicomplex Sumudu transform, we have

$$\mathcal{S}\left[\int_0^t f(u)du;\xi\right] = \frac{1}{\xi}\int_0^\infty e^{-\frac{t}{\xi}}\int_0^t f(u)dudt$$

by changing the order of integration, we get

$$\begin{aligned}
\mathcal{S}\left[\int_0^t f(u)du;\xi\right] &= \frac{1}{\xi}\int_0^\infty f(u)du\int_u^\infty e^{-\frac{t}{\xi}}dt \\
&= \frac{1}{\xi}\int_0^\infty \xi e^{-\frac{u}{\xi}}f(u)du \\
&= \xi\bar{f}(\xi).
\end{aligned}$$

□

Theorem 3.6. *Let $f(t)$ be the bicomplex-valued function of period $T > 0$, then*

$$\mathcal{S}[f(t);\xi] = \frac{\frac{1}{\xi}\int_0^T e^{-\frac{t}{\xi}} f(t)dt}{1 - e^{-\frac{T}{\xi}}}, \quad \xi \in \Omega \tag{3.17}$$

where Ω is defined in (3.10).

Proof. By definition, we have

$$\begin{aligned}\mathcal{S}[f(t);\xi] &= \frac{1}{\xi}\int_0^\infty e^{-\frac{t}{\xi}} f(t)dt \\ &= \frac{1}{\xi}\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-\frac{t}{\xi}} f(t)dt \end{aligned}\tag{3.18}$$

Taking $t = \tau + nT$ in the n^{th} integral, (3.18) becomes

$$\begin{aligned}\mathcal{S}[f(t);\xi] &= \frac{1}{\xi}\sum_{n=0}^{\infty} e^{-\frac{nT}{\xi}} \int_0^T e^{-\frac{\tau}{\xi}} f(\tau + nT)d\tau \\ &= \frac{1}{\xi}\sum_{n=0}^{\infty} e^{-\frac{nT}{\xi}} \int_0^T e^{-\frac{\tau}{\xi}} f(\tau)d\tau, \quad [\because\ f(t+nT) = f(t)] \\ &= \frac{1}{\xi\left(1 - e^{-\frac{T}{\xi}}\right)}\int_0^T e^{-\frac{t}{\xi}} f(t)dt.\end{aligned}$$

□

3.3.1 Convolution

The convolution of two signals is defined in equation (2.21). We derive here the convolution theorem for bicomplex Sumudu transform as follows:

Theorem 3.7. *Let $\bar{f}(\xi) = \mathcal{S}[f(t);\xi]$ and $\bar{g}(\xi) = \mathcal{S}[g(t);\xi]$ be the bicomplex Sumudu transforms of bicomplex-valued functions $f(t)$ and $g(t)$ of exponential orders K_1 and K_2 respectively, then*

$$\mathcal{S}[(f * g)(t);\xi] = \xi\bar{f}(\xi)\bar{g}(\xi), \quad \xi \in \Omega \tag{3.19}$$

where $K = \max\{K_1, K_2\}$ and Ω is defined in (3.10).

Proof.

$$\mathcal{S}[(f*g)(t);\xi] = \frac{1}{\xi}\int_0^\infty e^{-\frac{t}{\xi}}(f*g)(t)dt$$
$$= \frac{1}{\xi}\int_0^\infty e^{-\frac{t}{\xi}}\int_0^t f(u)g(t-u)dudt$$

Changing the order of integration, we get

$$\mathcal{S}[(f*g)(t);\xi] = \frac{1}{\xi}\int_0^\infty f(u)du\int_u^\infty e^{-\frac{t}{\xi}}g(t-u)dt$$

Put $t-u=z$

$$= \frac{1}{\xi}\int_0^\infty e^{-\frac{u}{\xi}}f(u)du\int_0^\infty e^{-\frac{z}{\xi}}g(z)dz$$
$$= \xi\bar{f}(\xi)\bar{g}(\xi).$$

□

3.4 Inverse Bicomplex Sumudu Transform

In this section, we discuss the inversion theorem for Sumudu transform in bicomplex space as follows:

Theorem 3.8. *Let $\bar{f}(\xi)$ be the bicomplex Sumudu transform of bicomplex-valued function $f(t)$ of exponential order K, analytic in Ω, then*

$$f(t) = \frac{1}{2\pi i_1}\int_\Gamma e^{\frac{t}{\xi}}\xi\bar{f}(\xi)d\xi = \sum \mathrm{Res}\left[e^{\frac{t}{\xi}}\xi\bar{f}(\xi)\right] \tag{3.20}$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ is piecewise continuous differentiable closed contour in Bicomplex space and Ω is defined in (3.10).

Proof. Since we know that bicomplex Laplace transform [88] of bicomplex-valued function $f(t)$ is

$$L\left[f(t);\frac{1}{\xi}\right] = F\left(\frac{1}{\xi}\right) = \int_0^\infty e^{-\frac{t}{\xi}}f(t)dt, \quad \xi\in\Omega$$
$$\Rightarrow \frac{1}{\xi}F\left(\frac{1}{\xi}\right) = \frac{1}{\xi}\int_0^\infty e^{-\frac{t}{\xi}}f(t)dt = \bar{f}(\xi)$$
$$\Rightarrow F\left(\frac{1}{\xi}\right) = \xi\bar{f}(\xi) \tag{3.21}$$

Inverse bicomplex Laplace transform [10, Definition 3.1] of (3.21) is

$$\begin{aligned} f(t) &= \frac{1}{2\pi i_1}\int_\Gamma e^{\frac{t}{\xi}} F\left(\frac{1}{\xi}\right) d\xi \\ &= \frac{1}{2\pi i_1}\int_\Gamma e^{\frac{t}{\xi}} \xi \bar{f}(\xi) d\xi, \quad \text{[Using (3.21)]} \end{aligned}$$

Hence the result (3.20). □

3.5 Applications

In this section, we find the solution of an application of differential equation of bicomplex-valued functions. Also, we find the solution for Cartesian transverse electric magnetic (TEM) waves in homogeneous space using bicomplex Sumudu transform.

(a) Consider the general linear differential equation of order n of bicomplex-valued function $y(t)$

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = f(t) \tag{3.22}$$

with initial conditions $y(0),\ y'(0), \cdots, y^{n-1}(0)$ are given and finite, $a_0,\ a_1, \cdots, a_n$ are bicomplex constants and $f(t)$ be bicomplex-valued function. Taking the bicomplex Sumudu transform of (3.22), we get

$$\begin{aligned} & a_n \frac{1}{\xi^n}\left[\bar{y}(\xi) - y(0) - \sum_{k=1}^{n-1} \xi^k y^{(k)}(0)\right] + a_{n-1}\frac{1}{\xi^{n-1}}\left[\bar{y}(\xi) - y(0) - \sum_{k=1}^{n-2} \xi^k y^{(k)}(0)\right] \\ & \quad + \cdots + a_0 \bar{y}(\xi) = \bar{f}(\xi) \\ \Rightarrow\ & a_n\left[\bar{y}(\xi) - y(0) - \sum_{k=1}^{n-1} \xi^k y^{(k)}(0)\right] + a_{n-1}\xi\left[\bar{y}(\xi) - y(0) - \sum_{k=1}^{n-2} \xi^k y^{(k)}(0)\right] \\ & \quad + \cdots + a_0 \xi^n = \xi^n \bar{f}(\xi) \end{aligned}$$

Therefore,

$$\begin{aligned}\bar{y}(\xi) = & \frac{y(0)\left[a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_1\xi^{n-1}\right]}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \\ & + \frac{y'(0)\xi\left[a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_2\xi^{n-2}\right]}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \\ & + \cdots + \frac{y^{(n-1)}(0)a_n\xi^{n-1}}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \\ & + \frac{\xi^n \bar{f}(\xi)}{a_n + a_{n-1}\xi + a_{n-2}\xi^2 + \cdots + a_0\xi^n} \end{aligned} \tag{3.23}$$

Taking the inverse bicomplex Sumudu transform of (3.23), we get the solution of differential equation (3.22) as

$$y(t) = \frac{1}{2\pi i_1}\int_{\Gamma} e^{\frac{t}{\xi}}\xi\bar{y}(\xi)d\xi \tag{3.24}$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ is piecewise continuous differentiable closed contour in Bicomplex space and Ω and $\bar{y}(\xi)$ are defined in (3.10) and (3.23), respectively.

In particular, consider the differential equation

$$\frac{dy}{dt} + y = e^{at}, \quad a \in C_2 \tag{3.25}$$

with initial condition $y(0) = 0$ and $y(t)$ is bicomplex-valued function.

Taking the bicomplex Sumudu transform of (3.25), we get

$$\begin{aligned} & \frac{\bar{y}(\xi) - y(0)}{\xi} + \bar{y}(\xi) = \sum_{n=0}^{\infty} a^n\xi^n \\ & \bar{y}(\xi) = \frac{\xi}{\xi+1}\sum_{n=0}^{\infty} a^n\xi^n \\ & \bar{y}(\xi) = \xi\left(1 - \xi + \xi^2 - \xi^3 + \cdots\right)\sum_{n=0}^{\infty} a^n\xi^n \end{aligned}$$

Taking the inverse bicomplex Sumudu transform and making use of theorem 3.7, we get

$$y(t) = \int_0^t e^{-u} e^{a(t-u)} du = \frac{1}{1+a} \left(e^{at} - e^{-t} \right)$$

which is the required solution of the given differential equation (3.25).

(b) Here, we shall solve the bicomplex Maxwell's equation using bicomplex Sumudu transform. For details about bicomplex Maxwell's equation see, Anastassiu [5] and section 2.6 of the thesis. Recall, the equation (2.25)

$$\nabla \times \mathbf{F} = i_2 i_1 k \mathbf{F} \tag{3.26}$$

Assuming a TEM to z wave, i.e., a vanishing z-component, and after introducing $Q_y = i_2 F_y$, in (3.26) is reduced to the following system of bicomplex differential equations

$$\frac{dQ_y}{dz} = i_1 k F_x \tag{3.27}$$

$$\frac{dF_x}{dz} = i_1 k Q_y \tag{3.28}$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 \tag{3.29}$$

Differentiating (3.27) and (3.28) and using respectively(3.28) and (3.27) therein, we get

$$\frac{d^2 F_x}{dz^2} + k^2 F_x = 0 \tag{3.30}$$

$$\frac{d^2 Q_y}{dz^2} + k^2 Q_y = 0 \tag{3.31}$$

For the solution, taking the bicomplex Sumudu transform of (3.30), we get

$$\frac{1}{\xi^2}\left[\bar{F}_x(\xi) - F_x(0) - \xi F_x'(0)\right] + k^2\bar{F}_x(\xi) = 0$$
$$\Rightarrow \bar{F}_x(\xi) = \frac{F_x(0) + \xi F_x'(0)}{\xi^2 k^2 + 1} \tag{3.32}$$

Taking the inverse bicomplex Sumudu transform of (3.32), we get

$$\begin{aligned}
F_x(z) &= \frac{1}{2\pi i_1}\int_\Omega e^{\frac{z}{\xi}}\xi \bar{F}_x(\xi)d\xi \\
&= \frac{1}{2\pi i_1}\int_\Omega e^{\frac{z}{\xi}}\xi\frac{F_x(0) + \xi F_x'(0)}{\xi^2 k^2 + 1}d\xi \\
&= \lim_{\xi \to \frac{i_1}{k}}\left(\xi - \frac{i_1}{k}\right)\xi e^{\frac{z}{\xi}}\frac{F_x(0) + \xi F_x'(0)}{\xi^2 k^2 + 1} + \lim_{\xi \to -\frac{i_1}{k}}\left(\xi + \frac{i_1}{k}\right)\xi e^{\frac{z}{\xi}}\frac{F_x(0) + \xi F_x'(0)}{\xi^2 k^2 + 1} \\
&= \frac{kF_x(0) + i_1 F_x'(0)}{k^3}e^{-i_1 kz} + \frac{kF_x(0) - i_1 F_x'(0)}{k^3}e^{i_1 kz} \\
&= Re^{-i_1 kz} + Ke^{i_1 kz}
\end{aligned} \tag{3.33}$$

Similarly

$$F_y(z) = -i_2 Q_y(z) = -i_2\left[Le^{-i_1 kz} + Se^{i_1 kz}\right] \tag{3.34}$$

Therefore,

$$\mathbf{F} \equiv \frac{1}{\sqrt{\eta}}\mathbf{E} + i_2\sqrt{\eta}\mathbf{H} = \left[Re^{-i_1 kz} + Ke^{i_1 kz}\right]\hat{x} - i_2\left[Le^{-i_1 kz} + Se^{i_1 kz}\right]\hat{y} \tag{3.35}$$

$\hat{x}$ and $\hat{y}$ are the fundamental position unit vectors in the direction of X- axis and Y- axis respectively and

$$R = \frac{kF_x(0) + i_1 F_x'(0)}{k^3} = R_1 + i_2 R_2, \quad K = \frac{kF_x(0) - i_1 F_x'(0)}{k^3} = K_1 + i_2 K_2$$
$$L = \frac{kQ_y(0) + i_1 Q_y'(0)}{k^3} = L_1 + i_2 L_2, \quad S = \frac{kQ_y(0) - i_1 Q_y'(0)}{k^3} = S_1 + i_2 S_2.$$

Equation (3.29) implies that L, R, S and K are bicomplex constants. Since (3.35) satisfy bicomplex Maxwell's equation (3.26)

$$L = -R \text{ and } S = K.$$

Hence (3.35) becomes,

$$\mathbf{F} \equiv \frac{1}{\sqrt{\eta}}\mathbf{E} + i_2\sqrt{\eta}\mathbf{H} = \left[Re^{-i_1kz} + Ke^{i_1kz}\right]\hat{x} - i_2\left[-Re^{-i_1kz} + Ke^{i_1kz}\right]\hat{y} \quad (3.36)$$

Extracting the bi-real and bi-imaginary parts of the solutions (3.36) yields the electric and magnetic fields components

$$\mathbf{E} = \sqrt{\eta}\left[R_1e^{-i_1kz} + K_1e^{i_1kz}\right]\hat{x} + \sqrt{\eta}\left[-R_2e^{-i_1kz} + K_2e^{i_1kz}\right]\hat{y} \quad (3.37)$$

$$\mathbf{H} = \frac{1}{\sqrt{\eta}}\left[R_2e^{-i_1kz} + K_2e^{i_1kz}\right]\hat{x} - \frac{1}{\sqrt{\eta}}\left[-R_1e^{-i_1kz} + K_1e^{i_1kz}\right]\hat{y} \quad (3.38)$$

where R_1, R_2, $K_1, \& K_2$ are arbitrary complex constants and (3.37) and (3.38) are the solution of Maxwell's equations (2.22-2.23).

3.6 Conclusion

In this chapter, we derived bicomplex Sumudu transform with convergence conditions and some of its basic properties which are useful in finding the solution of differential equations involving bicomplex-valued functions.

4

Stieltjes and Laplace-Stieltjes Transforms in Bicomplex Space and Applications

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The main findings of this chapter have been published as:

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2014), Bicomplex version of Stieltjes transform and applications, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms*, **21**(4-5), 229-246.

2. Agarwal R., Goswami M.P. and Agarwal R.P. (2015), Tauberian theorem and applications of bicomplex Laplace-Stieltjes transform, *Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms*, **22**, 141-153.

In this chapter, motivated by work of *Galue* et al. [51] we investigate bicomplex Stieltjes transform which is generalization of complex Stieltjes transform and also, investigate bicomplex Laplace-Stieltjes transform which is generalization of complex Laplace-Stieltjes transform and its Tauberian theorem. Both transforms are powerful mathematical tool applied in the theory of moments, probability distribution theory, tail probability, Dirichlet series, orthogonal polynomial, signal processing and mathematical physics.

4.1 Introduction

The Stieltjes transform introduced by *Stieltjes* (1856-1894) in his studies on continued fractions. In 1938, *Widder* [147] discussed the Stieltjes transform and its inverse with convergence conditions and relate to probability moments. In 1985, *Sinha* [136] developed two new characterization of the Stieltjes transform for distribution and proved standard theorem as analyticity, uniqueness and invertibility of the Stieltjes transform. In 1987, *Pathak* and *Debnath* [113] discussed recent developments on the Stieltjes transform of generalized functions. In 1989, *Tekale* [141] discussed generalized Stieltjes transform and its inversion in Banach space-valued distributions. In 1995, *Srivastava* and *Tuan* [138] proved a new convolution theorem of the Stieltjes transform and solving a certain class of singular integral equations by using convolution theorem.

In 2003, *Geronimo* and *Hill* [54] discussed the necessary and sufficient condition that point-wise limit of a sequence of Stieltjes transforms of real Borel probability measures is a Stieltjes transform of a Borel probability measure. Also, applications in mathematical physics are studied. In 2005, *Schwarz* [131] formulated generalized Stieltjes transform for all $\rho > 0$, which is iterated Laplace transform and therefore, its inverse can be expressed in the form of iterated inverse Laplace transform. In 2009, *Choi* [30] discussed the inversion formula for the Stieltjes transform of spectral distribution. In 2014, *Yakubivich* and *Martins* [153] established relationship between iterated Stieltjes transform to iterated Hilbert transform on the half axis and proved corresponding convolution and Titchmarsh's theorem.

4.1.1 Complex Stieltjes Transform

Definition 4.1. A real-valued function $F(t)$ of real variable t which satisfies the following three conditions:
(i) non-decreasing,
(ii) right-continuous,
(iii) $\lim_{t\to\infty} F(t) = 1,\ \lim_{t\to-\infty} F(t) = 0$
is called a probability distribution function.

Note. If $F(t)$ be probability distribution function of probability density function $f(t)$. Then $F(t) = \int_{-\infty}^{t} f(x)dx,\ \forall\, t \in \mathbb{R}$. Support of f written supp(f) is the set of points where f is non-zero, i.e.

$$\text{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

Definition 4.2 (Yyh-Shin Huang [67])**.** Let $F(t)$ be a probability distribution function for $t \in \mathbb{R}$. Its Stieltjes transform $S_F(s)$ is defined as

$$S_F(s) = \int_{-\infty}^{\infty} \frac{dF(t)}{t-s}, \tag{4.1}$$

where $s = x + i_1 y \in C_1$, $S_F(s)$ is exist and convergent for $\text{Im}(s) = y > 0$ and C_1 is the complex plane.

Since

$$\left|\int_{-\infty}^{\infty} \frac{dF(t)}{t-s}\right| < \int_{-\infty}^{\infty} \frac{1}{y} dF(t) = \frac{1}{y} < \infty,$$

the existence of the integral is trivial.

We require the following complex inversion formula for Stieltjes transform to define its bicomplex form.

Theorem 4.1 (Choi [30]). *For any $\lambda_1 < \lambda_2$*

$$F(\lambda_2) - F(\lambda_1) = \lim_{y \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} Im(S_F(x + i_1 y))dx, \tag{4.2}$$

where λ_1 and λ_2 are continuity points of distribution function F.

On the other hand , the Laplace-Stieltjes transform is similar to Laplace transform which named for *Pierre-Simon Laplace* and *Thomas Joannes Stieltjes*, it is the Laplace transform of Stieltjes measure, which is defined as

$$F(s) = \int_{0}^{\infty} e^{-st} d\alpha(t), \quad s = x + i_1 y \in C_1 \tag{4.3}$$

where $\alpha(t)$ is function of bounded variation on any interval $[0, X]$, $(0 < X < \infty)$.

In 1960, *Saltz* [128] discussed the inversion theorem for Laplace-Stieltjes transform. In 1966, *Ditzian* and *Jakimovski* [36] proved a general inversion theorem for Laplace-Stieltjes transform which improved the result obtained by *Saltz* [128]. In 1990, *Batty* [14] discussed the Tauberian theorem for Laplace-Stieltjes transform, power series and Dirichlet series. In 1994, *Lin* extended the uniqueness theorem for the moment generating function and for the Laplace-Stieltjes transform.

In 2007, *Nakagawa* [110] discussed that if the abscissa of convergence of the Laplace-Stieltjes transform is negative finite and the real point on the axis of convergence is a pole of the Laplace-Stieltjes transform, then the tail probability decays exponentially. In 2012, *Xu* et al. [152] discussed the convergence of Laplace-Stieltjes transform which represents the proximate order and type function for analytic functions of finite order and investigated growthn of such functions. In [158], *Ziolkowski* discussed methods connected with applications of Laplace-Stieltjes transform and generating functions.

In 2013, *Xu* and *Xuan* [151] discussed the growth and value distribution of Laplace-Stieltjes transformations with infinite order in the right half-plane. In [51], *Galue* et al. established two theorems on Laplace-Stieltjes transform and applications of these theorems to evaluate some integrals. In 2014, *Kong* and *Yang* [87] investigated a type function of Laplace-Stieltjes transforms convergent in the complex plane, which extends some results of Dirichlet series.

4.2 Bicomplex Stieltjes Transform

In the following theorem we find the bicomplex Stieltjes transform with the help of bicomplex Laplace transform already discussed in the chapter 2.

Theorem 4.2. *Let $f(t)$ be any bicomplex-valued function of real variable $t \geq 0$ and $F(\xi)$, $\xi \in C_2$ be bicomplex Laplace transform of $f(t)$. Then the Stieltjes transform of $f(t)$ is*

$$S_f(\xi) = L\{L\{f(t)\}\} = \int_0^\infty \frac{1}{\xi + t} f(t)dt, \tag{4.4}$$

provided that the integral is absolutely convergent.

Proof. The bicomplex Laplace transform (see, section 2.2) is given by

$$F(\xi) = L\{f(t)\} = \int_0^\infty e^{-\xi t} f(t)dt, \text{ where } \xi = s_1 e_1 + s_2 e_2 \in D \subset C_2$$

$\therefore$ Bicomplex Stieltjes transform

$$S_f(\xi) = L\{L\{f(t)\}\} = \int_0^\infty e^{-\xi p} \int_0^\infty e^{-pt} f(t) dt\, dp; \; p = p_1 e_1 + p_2 e_2 \in C_2$$

{where p_1 and p_2 varies from 0 to ∞}

$$= \int_0^\infty \int_0^\infty e^{-(\xi+t)p} f(t) dt\, dp$$

$$= \int_0^\infty \int_0^\infty e^{-(s_1e_1+s_2e_2+t)(p_1e_1+p_2e_2)} f(t) dt\, d(p_1e_1 + p_2e_2)$$

$$= \left(\int_0^\infty \int_0^\infty e^{-(s_1+t)p_1} f(t) dt\, dp_1 \right) e_1 + \left(\int_0^\infty \int_0^\infty e^{-(s_2+t)p_2} f(t) dt\, dp_2 \right) e_2$$

$$= \left(\int_0^\infty \frac{1}{s_1+t} f(t) dt \right) e_1 + \left(\int_0^\infty \frac{1}{s_2+t} f(t) dt \right) e_2$$

$$= \int_0^\infty \frac{1}{(s_1e_1 + s_2e_2) + t} f(t) dt = \int_0^\infty \frac{1}{\xi + t} f(t) dt. \tag{4.5}$$

□

Remark 4.1. The integral (4.5) is analytic in $C_2 \backslash \text{supp}(f)$, where $\text{supp}(f) = \{\xi : \xi = s_1e_1 + s_2e_2;\ \text{Im}(P_1 : \xi) = 0 \text{ and } \text{Im}(P_2 : \xi) = 0\}$, where $\text{Im}(P_1 : \xi) = \text{Im}(s_1)$ and $\text{Im}(P_2 : \xi) = \text{Im}(s_2)$. For convenience and particular applications in probability theory, the bicomplex Stieltjes transform of a probability distribution is defined on the upper bicomplex space excluding support of $f(t)$ and is integrated over $(-\infty, \infty)$ i.e. $\text{Im}(s_1) > 0$ and $\text{Im}(s_2) > 0$.

Let $F(t)$ be probability distribution function defined on $(-\infty, \infty)$ for $t \in \mathbb{R}$. Its Stieltjes transform $S_F(s_1)$ is defined as

$$S_F(s_1) = \int_{-\infty}^\infty \frac{dF(t)}{t - s_1},$$

where $s_1 = x_1 + i_1y_1 \in C_1$, $S_F(s_1)$ exists and is convergent and analytic for $\text{Im}(s_1) = y_1 > 0$ and take another Stieltjes transform for $s_2 \in C_1$ such that

$$S_F(s_2) = \int_{-\infty}^\infty \frac{dF(t)}{t - s_2},$$

where $s_2 = x_2 + i_1 y_2 \in C_1$, $S_F(s_2)$ exists and is convergent and analytic for $\mathrm{Im}(s_2) = y_2 > 0$. Now we have linear combination of $S_F(s_1)$ and $S_F(s_2)$ with e_1 and e_2 such as:

$$\begin{aligned} S_F(s_1)e_1 + S_F(s_2)e_2 &= \int_{-\infty}^{\infty} \frac{dF(t)}{t - s_1} e_1 + \int_{-\infty}^{\infty} \frac{dF(t)}{t - s_2} e_2 = \int_{-\infty}^{\infty} \frac{dF(t)}{t - (s_1 e_1 + s_2 e_2)} \\ &= \int_{-\infty}^{\infty} \frac{dF(t)}{t - \xi} = S_F(\xi) \end{aligned}$$

$S_F(\xi)$ exist for $\mathrm{Im}(s_1) > 0$ and $\mathrm{Im}(s_2) > 0$ that is, $\mathrm{Im}(P_1 : \xi) > 0$ and $\mathrm{Im}(P_2 : \xi) > 0$.

Since $S_F(s_1)$ and $S_F(s_2)$ are complex valued functions which are convergent and analytic for $\mathrm{Im}(s_1) > 0$ and $\mathrm{Im}(s_2) > 0$ respectively, so a bicomplex valued function $S_F(\xi)$=$S_F(s_1)e_1$+$S_F(s_2)e_2$ will be convergent and analytic in the region D defined as:

$$D = \{\xi : \xi = s_1 e_1 + s_2 e_2;\ \mathrm{Im}(P_1 : \xi) > 0 \text{ and } \mathrm{Im}(P_2 : \xi) > 0\}. \qquad (4.6)$$

$\because$ $\mathrm{Im}(s_1) = y_1 > 0$ and $\mathrm{Im}(s_2) = y_2 > 0$, then

$$\begin{aligned} \xi &= (x_1 + i_1 y_1)e_1 + (x_2 + i_1 y_2)e_2 = (x_1 + i_1 y_1)\left(\frac{1 + i_1 i_2}{2}\right) + (x_2 + i_1 y_2)\left(\frac{1 + i_1 i_2}{2}\right) \\ &= \frac{x_1 + x_2}{2} + \left(\frac{y_1 + y_2}{2}\right) i_1 + \left(\frac{y_2 - y_1}{2}\right) i_2 + \left(\frac{x_1 - x_2}{2}\right) i_1 i_2 \end{aligned}$$

Now there are three possible cases:

1. If $y_1 = y_2$ then $\frac{y_2 - y_1}{2} = 0$ and $\frac{y_1 + y_2}{2} = y_1 = y_2 > 0$. Hence if $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2$, then $a_1 > 0$ and $a_2 = 0$.

2. If $y_1 > y_2$ then $\frac{y_2 - y_1}{2} < 0$ and $\frac{y_1 + y_2}{2} > \frac{0 + y_1}{2} > \frac{y_1 - y_2}{2}$. Thus $a_1 > -a_2$ and $a_2 < 0$.

3. If $y_1 < y_2$ then $\frac{y_2-y_1}{2} > 0$ and $\frac{y_1+y_2}{2} > \frac{0+y_2}{2} > \frac{y_2-y_1}{2}$. Thus $a_1 > a_2$ and $a_2 > 0$.

From these three conditions following three sets can be defined:

$$D_1 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ a_1 > 0 \text{ and } a_2 = 0\}$$

$$D_2 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ a_1 > -a_2 \text{ and } a_2 < 0\}$$

$$D_3 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ a_1 > a_2 \text{ and } a_2 > 0\}, \text{ respectively.}$$

Thus, $\mathrm{Im}(P_1 : \xi) > 0$ and $\mathrm{Im}(P_2 : \xi) > 0$ implies $\xi \in D = D_1 \cup D_2 \cup D_3$. Conditions in the set D_1, D_2 and D_3 can be combined and written as $a_1 > |a_2|$ and D defined as:

$$\begin{aligned} D &= \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in C_2 : a_1 > |a_2|\} \\ &= \{\xi \in C_2 : \mathrm{Im}_{i_1}(\xi) > |\mathrm{Im}_{i_2}(\xi)|\} \qquad (4.7) \end{aligned}$$

where $\mathrm{Im}_{i_1}(\xi)$ and $\mathrm{Im}_{i_2}(\xi)$ denote the imaginary part of a bicomplex number w.r.t. i_1 and i_2, respectively.

Conversely, the existence condition of bicomplex Stieltjes transform $F(\xi)$ can be obtained in the following way:

$$\text{If } \xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in \Omega, \ a_1 > |a_2|. \qquad (4.8)$$

Now, in terms of idempotent components, ξ can be expressed as

$$\begin{aligned} \xi &= a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \\ &= [(a_0 + a_3) + i_1(a_1 - a_2)]\, e_1 + [(a_0 - a_3) + i_1(a_1 + a_2)]\, e_2 \\ &= s_1 e_1 + s_2 e_2. \end{aligned}$$

Depending on the value of a_2, there arises three cases:

1. When $a_2 = 0$, from the inequality (4.8) $a_1 > 0$ which trivially leads $a_1 - a_2 > 0$ and $a_1 + a_2 > 0$.

2. When $a_2 > 0$, from the inequality (4.8) $a_1 - a_2 > 0$. This result can be interpreted as $a_1 + a_2 > a_1 - a_2 > 0$.

3. When $a_2 < 0$, from the inequality (4.8) $a_1 + a_2 > 0$. This result can be interpreted as $a_1 - a_2 > a_1 + a_2 > 0$.

Hence the result.

As a consequence of the Theorem 4.2 and above discussion, bicomplex Stieltjes transform can be defined as

Definition 4.3. Let $F(t)$ be probability distribution function defined on $(-\infty, \infty)$ for $t \in \mathbb{R}$. Its bicomplex Stieltjes transform is defined as

$$S_F(\xi) = \int_{-\infty}^{\infty} \frac{dF(t)}{t - \xi}, \quad \xi \in D \subset C_2. \tag{4.9}$$

It is well-defined in D and defined as

$$D = C_2 \backslash \text{supp}(F) = \{\xi \in C_2 : \text{Im}_{i_1}(\xi) > |\text{Im}_{i_2}(\xi)|\}, \tag{4.10}$$

where $\text{supp}(F) = \{\xi : \xi = s_1 e_1 + s_2 e_2;\ \text{Im}(s_1) = 0 \text{ and } \text{Im}(s_2) = 0\}$.

In terms of expected value, the bicomplex-valued Stieltjes transform can be expressed as

$$\begin{aligned} S_F(\xi) &= \int_{-\infty}^{\infty} \frac{dF(t)}{t - \xi} = -\int_{-\infty}^{\infty} \frac{dF(t)}{\xi(1 - t\xi^{-1})} \\ &= -\int_{-\infty}^{\infty} \frac{(1 - t\xi^{-1})^{-1} dF(t)}{\xi} \\ &= -\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{(t\xi^{-1})^n dF(t)}{\xi} \end{aligned}$$

$$= -\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{t^n dF(t)}{\xi^{n+1}} = -\sum_{n=0}^{\infty} \frac{E[t^n]}{\xi^{n+1}}$$

$$\therefore S_F(\xi) = -\sum_{n=0}^{\infty} \frac{E[t^n]}{\xi^{n+1}} \tag{4.11}$$

[where E represent the expected value.]

If instead of probability distribution of real variable, we consider the probability distribution of complex variable, the Stieltjes transform for such distribution can be defined as

Definition 4.4. Let $F(z)$ be probability distribution function of complex variable. Then bicomplex Stieltjes transform is given by

$$S_F(\xi) = \int_{\mathbb{S}} \frac{dF(z)}{z - \xi}, \qquad \xi \in C_2 \tag{4.12}$$

where $\mathbb{S}$ is the support of $F(z)$.

The Stieltjes transform inversion formula for bicomplex function is given in the following theorem:

Theorem 4.3. *For any* $\lambda_1 < \lambda_2$

$$F(\lambda_2) - F(\lambda_1) = \lim_{y_1, y_2 \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} Im_{i_1}(S_F(\xi)) d(Re(\xi)), \;\; \xi = s_1 e_1 + s_2 e_2 \in C_2 \tag{4.13}$$

where λ_1 *and* λ_2 *are continuity points of distribution function* F, $Im_{i_1}(\xi)$ *denotes imaginary part bicomplex number* ξ *w.r.t.* i_1 *component and* $s_1 = x_1 + i_1 y_1,\ s_2 = x_2 + i_1 y_2$ *with* $Im(P_1 : \xi) > 0$, $Im(P_2 : \xi) > 0$.

Proof. We have by definition,

$$\lim_{y_1, y_2 \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}_{i_1}(S_F(\xi)) d(\mathrm{Re}(\xi))$$

$$= \lim_{y_1, y_2 \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}_{i_1}\left(S_F(s_1 e_1 + s_2 e_2)\right) d\left(\mathrm{Re}(s_1 e_1 + s_2 e_2)\right)$$

$$= \left(\lim_{y_1 \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}_{i_1}\left(S_F(s_1)\right) d\left(\mathrm{Re}(s_1)\right)\right) e_1 + \left(\lim_{y_2 \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}_{i_1}\left(S_F(s_2)\right) d\left(\mathrm{Re}(s_2)\right)\right) e_2$$

$$= \left(\lim_{y_1\to 0}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\operatorname{Im}_{i_1}\left(S_F(x_1+i_1y_1)\right)dx_1\right)e_1 + \left(\lim_{y_2\to 0}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\operatorname{Im}_{i_1}\left(S_F(x_2+i_1y_2)\right)dx_2\right)e_2$$
$$= \left(F(\lambda_2)-F(\lambda_1)\right)e_1 + \left(F(\lambda_2)-F(\lambda_1)\right)e_2 \quad \text{[by Theorem 4.1]}$$
$$= F(\lambda_2)-F(\lambda_1).$$

□

4.3 Some Basic Operational Properties of Stieltjes Transform

In this section we derive some basic operational properties of bicomplex Stieltjes transform.

Theorem 4.4. *Let $S_f(\xi)$ be bicomplex Stieltjes transform of bounded variation function $f(t)$, then $S_f(f(t+a)) = S_f(\xi+a)$; where a is constant and $\xi = s_1e_1 + s_2e_2 \in C_2\backslash supp(f)$.*

Proof. We have, by definition

$$S_f(f(t+a)) = \int_{-\infty}^{\infty}\frac{f(t+a)}{t-\xi}dt = \int_{-\infty}^{\infty}\frac{f(t+a)}{t-(s_1e_1+s_2e_2)}dt$$
$$= \int_{-\infty}^{\infty}\frac{f(t+a)}{t-s_1}dt\ e_1 + \int_{-\infty}^{\infty}\frac{f(t+a)}{t-s_2}dt\ e_2$$

Put $t+a = x \Rightarrow dt = dx$

$$= \int_{-\infty}^{\infty}\frac{f(x)}{x-(s_1+a)}dx\ e_1 + \int_{-\infty}^{\infty}\frac{f(x)}{x-(s_2+a)}dx\ e_2$$
$$= S_f(s_1+a)e_1 + S_f(s_2+a)e_2 = S_f((s_1e_1+s_2e_2)+a)$$
$$= S_f(\xi+a).$$

□

Theorem 4.5. *Let $S_f(\xi)$ be bicomplex Stieltjes transform of bounded variation function $f(t)$, then $S_f(f(at)) = S_f(a\xi)$; where $a > 0$ is constant and $\xi = s_1e_1 + s_2e_2 \in C_2\backslash supp(f)$.*

Proof. We have, by definition

$$\begin{aligned} S_f(f(at)) &= \int_{-\infty}^{\infty} \frac{f(at)}{t-\xi} dt = \int_{-\infty}^{\infty} \frac{f(at)}{t-(s_1e_1+s_2e_2)} dt \\ &= \int_{-\infty}^{\infty} \frac{f(at)}{t-s_1} dt\ e_1 + \int_{-\infty}^{\infty} \frac{f(at)}{t-s_2} dt\ e_2 \end{aligned}$$

Put $at = x \Rightarrow dt = \frac{dx}{a}$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{f(x)}{x-as_1} dx\ e_1 + \int_{-\infty}^{\infty} \frac{f(x)}{x-as_2} dx\ e_2 \\ &= S_f(as_1)e_1 + S_f(as_2)e_2 = S_f(a(s_1e_1+s_2e_2)) \\ &= S_f(a\xi). \end{aligned}$$

□

Theorem 4.6. *Let $S_f(\xi)$ be bicomplex Stieltjes transform of bounded variation function $f(t)$, then $S_f(tf(t)) = \xi S_f(\xi) + \int_{-\infty}^{\infty} f(t)dt$; where a is constant and $\xi = s_1e_1 + s_2e_2 \in C_2 \backslash supp(f)$. Provided the integral on the right hand side exists.*

Proof. We have, by definition

$$\begin{aligned} S_f(tf(t)) &= \int_{-\infty}^{\infty} \frac{tf(t)}{t-\xi} dt = \int_{-\infty}^{\infty} \frac{tf(t)}{t-(s_1e_1+s_2e_2)} dt \\ &= \int_{-\infty}^{\infty} \frac{tf(t)}{t-s_1} dt\ e_1 + \int_{-\infty}^{\infty} \frac{tf(t)}{t-s_2} dt\ e_2 \\ &= \int_{-\infty}^{\infty} \frac{(t-s_1+s_1)f(t)}{t-s_1} dt\ e_1 + \int_{-\infty}^{\infty} \frac{(t-s_2+s_2)f(t)}{t-s_2} dt\ e_2 \\ &= \int_{-\infty}^{\infty} f(t)dt\ e_1 + s_1 \int_{-\infty}^{\infty} \frac{f(t)}{t-s_1} dt\ e_1 + \int_{-\infty}^{\infty} f(t)dt\ e_2 + s_2 \int_{-\infty}^{\infty} \frac{f(t)}{t-s_2} dt\ e_2 \\ &= (s_1e_1+s_2e_2) \int_{-\infty}^{\infty} \frac{f(t)}{t-(s_1e_1+s_2e_2)} dt + \int_{-\infty}^{\infty} f(t)dt\ (e_1+e_2) \\ &= \xi \int_{-\infty}^{\infty} \frac{f(t)}{t-\xi} dt + \int_{-\infty}^{\infty} f(t)dt \\ &= \xi S_f(\xi) + \int_{-\infty}^{\infty} f(t)dt. \end{aligned}$$

□

Theorem 4.7. *Let $S_f(\xi)$ be bicomplex Stieltjes transform of bounded variation function $f(t)$, then $S_f\left(\frac{f(t)}{t-a}\right) = \frac{1}{a-\xi}\left(S_f(\xi) - S_f(a)\right)$; where a is constant and $\xi = s_1e_1 + s_2e_2 \in C_2 \backslash supp(f)$.*

Proof. We have, by definition

$$
\begin{aligned}
S_f\left(\frac{f(t)}{t-a}\right) &= \int_{-\infty}^{\infty} \frac{f(t)}{(t-a)(t-\xi)}dt = \int_{-\infty}^{\infty} \frac{f(t)}{(t-a)(t-(s_1e_1+s_2e_2))}dt \\
&= \int_{-\infty}^{\infty} \frac{f(t)}{(t-a)(t-s_1)}dt\, e_1 + \int_{-\infty}^{\infty} \frac{f(t)}{(t-a)(t-s_2)}dt\, e_2 \\
&= \frac{1}{a-s_1}\left(\int_{-\infty}^{\infty}\left(\frac{1}{t-s_1} - \frac{1}{t-a}\right) f(t)dt\right) e_1 \\
&+ \frac{1}{a-s_2}\left(\int_{-\infty}^{\infty}\left(\frac{1}{t-s_2} - \frac{1}{t-a}\right) f(t)dt\right) e_2 \\
&= \frac{1}{a-s_1}\left(S_f(s_1) - S_f(a)\right) e_1 + \frac{1}{a-s_2}\left(S_f(s_2) - S_f(a)\right) e_2 \\
&= \frac{1}{a-(s_1e_1+s_2e_2)}\left(S_f(s_1e_1+s_2e_2) - S_f(a)\right) \\
&= \frac{1}{a-\xi}\left(S_f(\xi) - S_f(a)\right).
\end{aligned}
$$

□

4.4 Convolution Theorem

Motivated by the work on convolution theorem of Stieltjes transform and its applications by *Srivastva* and *Tuan* [138] and also work of *Sinha* [136], we derive the convolution theorem for bicomplex Stieltjes transform in this section.

Theorem 4.8. *Let $S_f(\xi)$ and $S_g(\xi)$ are the bicomplex Stieltjes transforms of bounded variation functions $f(t)$ and $g(t)$ with domains $C_2\backslash supp(f)$ and $C_2\backslash supp(g)$ respectively, then $f * g$ is Stieltjes transformable in Ω, where $\Omega = C_2\backslash supp(f) \cap C_2\backslash supp(f)$ and for every $\xi \in \Omega$*

$$S_{(f*g)}(\xi) = S_f(\xi)S_g(\xi). \tag{4.14}$$

Proof. Since we know that the convolution of two function of bounded variation on the interval $(-\infty, \infty)$ is

$$h(t) = (f * g)(t) = f(t) \int_{-\infty}^{\infty} \frac{g(u)}{u-t} du + g(t) \int_{-\infty}^{\infty} \frac{f(u)}{u-t} du \tag{4.15}$$

Under the hypothesis of the theorem, $(f * g)(t)$ is Stieltjes transformable.

Next, by using the definition of bicomplex Stieltjes transform and (4.15), we have

$$\begin{aligned} S_{(f*g)}(\xi) &= \int_{-\infty}^{\infty} \frac{(f*g)(t)}{t-\xi} dt \\ &= \int_{-\infty}^{\infty} \frac{f(t)}{t-\xi} \left[\int_{-\infty}^{\infty} \frac{g(u)}{u-t} du \right] dt + \int_{-\infty}^{\infty} \frac{g(t)}{t-\xi} \left[\int_{-\infty}^{\infty} \frac{f(u)}{u-t} du \right] dt \end{aligned} \tag{4.16}$$

We now change the order of integration in the second integral on the right-hand side of (4.16),we have

$$S_{(f*g)}(\xi) = \int_{-\infty}^{\infty} \frac{f(t)}{t-\xi} \left[\int_{-\infty}^{\infty} \frac{g(u)}{u-t} du \right] dt + \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} \frac{g(t)}{(t-\xi)(u-t)} dt \right] du \tag{4.17}$$

or, similarly,

$$\begin{aligned} S_{(f*g)}(\xi) &= \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} \frac{g(u)}{(t-\xi)(u-t)} du \right] dt + \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} \frac{g(u)}{(u-\xi)(t-u)} du \right] dt \\ &= \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} \frac{g(u)}{u-t} \left\{ \frac{1}{t-\xi} - \frac{1}{u-\xi} \right\} du \right] dt \end{aligned}$$

Simplifying this last double integral, we have

$$\begin{aligned} S_{(f*g)}(\xi) &= \int_{-\infty}^{\infty} \frac{f(t)}{t-\xi} dt \int_{-\infty}^{\infty} \frac{g(u)}{u-\xi} du \\ &= S_f(\xi) S_g(\xi). \end{aligned}$$

□

4.5 Applications of Bicomplex Stieltjes Transform

The bicomplex Stieltjes transform is highly applicable in theory of moments, probability distribution theory, orthogonal polynomial, signal processing and mathematical physics. In this section discuss the applications of bicomplex Stieltjes transform to check the symmetry of probability distribution function and finding the solution of singular integral equation of bicomplex functions.

Definition 4.5. A probability distribution function F is **symmetric** if $F(t) = 1 - F(t^-)$, $\forall\ t \in \mathbb{R}$. Where $F(t^-) = \lim_{h\to 0^+} F(t-h)$ and $F(t) = 1 - F(t^-)$ is called the conjugate distribution of $F(t)$.

Theorem 4.9 (Huang Yyh-Shin [67])**.** *A probability distribution function F is symmetric if and only if $S_F(z) = -S_F(-z)$, $\forall\ z \in C_1\backslash\mathbb{R}$.*

(a) To check the symmetry of probability distribution function F we may use bicomplex Stieltjes transform as described in the following theorem:

Theorem 4.10. *A probability distribution function F is symmetric if and only if $S_F(\xi) = -S_F(-\xi)$, $\forall\ \xi \in C_2\backslash supp(F)$. Where $supp(F) = \{\xi : \xi = s_1e_1 + s_2e_2;\ Im(s_1) = 0$ and $Im(s_2) = 0\}$.*

Proof. Suppose that F is symmetric, i.e. $F(t) = 1 - F(-t^-)$.

$$\begin{aligned}
S_F(\xi) &= \int_{-\infty}^{\infty} \frac{1}{t-\xi} dF(t) \\
&= \int_{-\infty}^{\infty} \frac{1}{t-(s_1e_1+s_2e_2)} d(1-F(-t^-)) \\
&= -\left(\int_{-\infty}^{\infty} \frac{1}{t-s_1} dF(-t)\right) e_1 - \left(\int_{-\infty}^{\infty} \frac{1}{t-s_2} dF(-t)\right) e_2 \\
&= -S_F(-s_1)e_1 - S_F(-s_2)e_2
\end{aligned}$$

$$
\begin{aligned}
&= -S_F(-s_1e_1 - s_2e_2) \quad \{\because S_F(s_1e_1 + s_2e_2) = S_F(s_1)e_1 = S_F(s_1)e_1\}\\
&= -S_F(-\xi).
\end{aligned}
$$

Suppose that $S_F(\xi) = -S_F(-\xi)$. From the bicomplex inversion formula, if t is a continuity point of F, then

$$F(t) = \lim_{y_1,y_2\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(\xi))d(\mathrm{Re}(\xi)).$$

We then put the discussion into two parts.

(i) Assume that both t and $-t$ are continuity points of F. Then

$$
\begin{aligned}
F(t) &= \lim_{y_1,y_2\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(\xi))d(\mathrm{Re}(\xi))\\
&= \lim_{y_1,y_2\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(s_1e_1+s_2e_2))d(\mathrm{Re}(s_1e_1+s_2e_2))\\
&= \left(\lim_{y_1\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(s_1))d(\mathrm{Re}(s_1))\right)e_1\\
&\quad + \left(\lim_{y_2\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(s_2))d(\mathrm{Re}(s_2))\right)e_2\\
&= \left(\lim_{y_1\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(x_1+i_1y_1))dx_1\right)e_1\\
&\quad + \left(\lim_{y_2\to 0^+} \frac{1}{\pi}\int_{-\infty}^{t} \mathrm{Im}_{i_1}(S_F(x_2+i_1y_2))dx_2\right)e_2\\
&= (1-F(-t))\,e_1 + (1-F(-t))\,e_2 \quad \text{[by Theorem 4.9]}\\
&= 1-F(-t).
\end{aligned}
$$

(ii) Assume that t is any real number. Since F is monotonic, the set of discontinuities of F is countable. Moreover, F is right continuous and the continuity points of F are dense. Hence, we can find a sequence $t_n \to t$ from the right hand side with both $\{t_n\}$ and $\{-t_n\}$ are continuity points of F. Then

$$
\begin{aligned}
F(t) &= \lim_{n\to\infty} F(t_n)\\
&= \lim_{n\to\infty} (1-F(-t_n)) \quad \{\text{by (i)}\}\\
&= 1-\lim_{n\to\infty} F(-t_n)\\
&= 1-F(-t^-).
\end{aligned}
$$

By (i) and (ii), F is symmetric. The proof is complete. □

(b) Here we find the solution by using convolution theorem of bicomplex Stieltjes transform of the following Cauchy's singular integral equation:

$$f(t) + \lambda \int_{-\infty}^{\infty} \frac{f(u)}{u-t} du = g(t) \qquad (\lambda \neq 0), \tag{4.18}$$

where $g(t)$ is known and $f(t)$ is an unknown function to be determined. If $f(t)$ is complex-valued function then solution of (4.18) can be seen in *Chakrabarti* and *Martha* [25, p. 25]. In case, $f(t)$ is bicomplex-valued function, we find solution of (4.18) by using convolution theorem of bicomplex Stieltjes transform.

Then, in view of the well-known integral (*Erdelyi* et al. [43, p. 251]):

$$\int_{-\infty}^{\infty} \frac{e^{i_1 au}}{u-\xi} du = \pi i_1 e^{i_1 a\xi}, \quad a > 0;\ \xi \in D \text{ as defined in (4.10)} \tag{4.19}$$

Multiplying (4.18) by $i_1 \pi e^{i_1 at}$ on both side and putting $\lambda = \frac{-i_1}{\pi}$, we have

$$f(t) \int_{-\infty}^{\infty} \frac{e^{i_1 au}}{u-t} du + e^{i_1 at} \int_{-\infty}^{\infty} \frac{f(u)}{u-t} du = i_1 \pi e^{i_1 at} g(t)$$

or, equivalently,

$$f(t) * e^{i_1 at} = i_1 \pi e^{i_1 at} g(t) \tag{4.20}$$

Applying the convolution theorem of bicomplex Stieltjes transform on (4.20), we have

$$S\{f(t) : \xi\} \int_{-\infty}^{\infty} \frac{e^{i_1 at}}{t-\xi} dt = \pi i_1 S\{e^{i_1 at} g(t) : \xi\} \tag{4.21}$$

By using (4.19), we have

$$S\{f(t) : \xi\} = e^{-i_1 a\xi} S\{e^{i_1 at} g(t) : \xi\} \tag{4.22}$$

$$S\{f(t):s_1e_1+s_2e_2\}=e^{-i_1a(s_1e_1+s_2e_2)}S\{e^{i_1at}g(t):s_1e_1+s_2e_2\}$$

$$S\{f(t):s_1\}e_1+S\{f(t):s_2\}e_2=\left(e^{-i_1as_1}S\{e^{i_1at}g(t):s_1\}\right)e_1 + \left(e^{-i_1as_2}S\{e^{i_1at}g(t):s_2\}\right)e_2 \tag{4.23}$$

Taking the classical inversion Stieltjes transform (*Widder* [149]) of (4.23) on both sides , we have

$$f(t)= \left(\lim_{\epsilon_1\to 0^+}\frac{1}{2\pi i_1}\left(e^{i_1a(-t+i_1\epsilon_1)}S\{e^{i_1at}g(t):-t+i_1\epsilon_1\}-e^{i_1a(-t-i_1\epsilon_1)}S\{e^{i_1at}g(t):-t-i_1\epsilon_1\}\right)\right)e_1 + \left(\lim_{\epsilon_2\to 0^+}\frac{1}{2\pi i_1}\left(e^{i_1a(-t+i_1\epsilon_2)}S\{e^{i_1at}g(t):-t+i_1\epsilon_2\}-e^{i_1a(-t-i_1\epsilon_2)}S\{e^{i_1at}g(t):-t-i_1\epsilon_2\}\right)\right)e_2$$

$$\therefore f(t)= \lim_{\epsilon_1,\epsilon_2\to 0^+}\frac{1}{2\pi i_1}\left(e^{i_1a(-t+(\epsilon_1+\epsilon_2)i_1-(\epsilon_1-\epsilon_2)\frac{i_2}{2})}S\left\{e^{i_1at}g(t):(-t+(\epsilon_1+\epsilon_2)i_1-(\epsilon_1-\epsilon_2)\frac{i_2}{2}\right\} - e^{i_1a(-t-(\epsilon_1+\epsilon_2)i_1+(\epsilon_1-\epsilon_2)\frac{i_2}{2})}S\left\{e^{i_1at}g(t):(-t-(\epsilon_1+\epsilon_2)i_1+(\epsilon_1-\epsilon_2)\frac{i_2}{2}\right\}\right). \tag{4.24}$$

equation (4.24) is the solution of the singular integral equation (4.18).

(c) Here we discuss the application of bicomplex Stieltjes transform in spectral analysis of large dimensional random matrices, *Bai* et al. [9] as follows:

Similar to bicomplex Fourier transform in classical probability theory and signal processing which allows to perform simpler analysis in the bicomplexified frequency domain than the time domain, the bicomplex Stieltjes transform often used for spectral analysis of large dimensional random matrices.

For $X \in C_1^{N\times N}$ matrix, (see, for details, *Couillet* and *Debbah* [32]) with real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ and eigenvalue distribution F^X, we use Stieltjes transform for $s \in C_1 \setminus \{\lambda_1, \lambda_2, ..., \lambda_N\}$,

$$S_{F^X}(s) = \int_{\mathbb{S}} \frac{dF^X(t)}{t-s} = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{\lambda_k - s} = \frac{1}{N} tr(X - sI_N)^{-1}.$$

If $X \in C_2^{N\times N}$ matrix with bicomplex entries, (*Alpay* et al. [4]) with real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ and eigenvalue distribution F^X, we use bicomplex Stieltjes transform for $\xi \in C_2 \setminus \{\lambda_1, \lambda_2, ..., \lambda_N\}$,

$$S_{F^X}(\xi) = \int_{\mathbb{S}} \frac{dF^X(t)}{t-\xi} = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{\lambda_k - \xi} = \frac{1}{N} tr(X - \xi I_N)^{-1}.$$

In case, if $X \in C_2^{N\times N}$ have complex eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ and eigenvalue distribution F^X, we use bicomplex Stieltjes transform for $\xi \in C_2 \setminus \{\lambda_1, \lambda_2, ..., \lambda_N\}$,

$$S_{F^X}(\xi) = \int_{\mathbb{S}} \frac{dF^X(z)}{z-\xi} = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{\lambda_k - \xi} = \frac{1}{N} tr(X - \xi I_N)^{-1}$$

where $\mathbb{S}$ is the support of the distribution F^X.

If the bicomplex Stieltjes transform of random matrix $X \in C_2^{N\times N}$ is known, then its eigenvalue distribution can be obtained, which is often more difficult to obtain in the spectral domain than in the domain of bicomplex Stieltjes transform of F^X.

Let the wireless Single Input Multiple Output (SIMO) system is given by

$$y = \sqrt{\gamma} Hx + n \tag{4.25}$$

where $H \in C_2^{R\times T}$, $x \in \mathbb{R}^T$, $y \in C_2^R$, $n \in C_2^R$, and γ are the channel matrix, channel input, channel output, additive white Gaussian noise, and the signal-to-noise ratio respectively. Moreover, entries of the channel $H \in C_2^{R\times T}$ represent the fading coefficient between each transmission path from a transmit antenna to receive antenna. For the spectral analysis of random matrix $H \in C_2^{R\times T}$, we use the bicomplex Stieltjes transform.

The bicomplex Stieltjes transform is advantageous than Stieltjes transform in complex form because in bicomplex Stieltjes transform the frequency domain is of large class than Stieltjes transform in complex form. So it gives simpler analysis than Stieltjes transform in complex form. It is also advantageous than quaternionic Stieltjes transform, (*Müller* and *Cakmak* [109] and *Cakmak* [22]) due to commutativity property for multiplication of two bicomplex numbers. The quaternions are inconvenient to deal with since multiplication of quaternions does not commute, in general.

For this we give an illustration of bicomplex Stiletjes transform as follows:

Example 4.1. Let H be semicircle element, (*Wigner* [150]) then find the bicomplex limiting spectrum of H i.e. $S_{F^H}(\xi)$.

Solution. Since,

$$\int_{-\infty}^{\infty} \frac{dF_H(t)}{t-\xi} = -\sum_{n=0}^{\infty} \frac{C_n}{\xi^{2n+1}} \tag{4.26}$$

Since odd moments of a even distribution vanishes and C_n is n^{th} catalan number, (*Nica* and *Speicher* [111]) we have

$$\begin{aligned} S_{F^H}(\xi) &= -\frac{1}{\xi} - \sum_{n=1}^{\infty} \frac{1}{\xi^{2n+1}} \left(\sum_{m=1}^{n} C_{m-1} C_{n-m} \right) \\ &= -\frac{1}{\xi} - \frac{1}{\xi} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{C_{m-1}}{\xi^{2m+1}} \frac{C_{n-m}}{\xi^{2(n-m)+1}} \\ &= -\frac{1}{\xi} - \frac{1}{\xi} \sum_{m=1}^{\infty} \frac{C_{m-1}}{\xi^{2m+1}} \left(\sum_{n=m}^{\infty} \frac{C_{n-m}}{\xi^{2(n-m)+1}} \right) \end{aligned}$$

$$= -\frac{1}{\xi} - \frac{1}{\xi}\sum_{m=1}^{\infty}\frac{C_{m-1}}{\xi^{2m+1}}S_{F^H}(\xi)$$
$$= -\frac{1}{\xi} - \frac{1}{\xi}S^2_{F^H}(\xi)$$
$$S^2_{F^H}(\xi) + \xi S_{F^H}(\xi) + 1 = 0$$
$$\therefore S_{F^H}(\xi) = \frac{-\xi \pm \sqrt{\xi^2 - 4}}{2} \tag{4.27}$$

(**Note:** Since $\xi \in D$ defined in (4.6) and (4.10) and by definition $\lim_{\|\xi\|\to\infty} \xi S_F(\xi) = -1$. Therefore $+$ sign has to be chosen in $\pm$ of (4.27))

$$\therefore S_{F^H}(\xi) = \frac{-\xi + \sqrt{\xi^2 - 4}}{2}.$$

4.6 Bicomplex Laplace-Stieltjes Transform

Let X be a non-negative random variable with distribution function $F(t)$ defined on $[0, \infty)$ with $F(0) = 0$. Its Laplace-Stieltjes transform defined as

$$(L_S F)(s_1) = \int_0^\infty e^{-s_1 t} dF(t), \qquad s_1 \in C_1$$

is absolutely convergent and analytic for $\text{Re}(s_1) > K$, where K is the exponential order of $F(t)$. Take another Laplace-Stieltjes transform for $s_2 \in C_1$ such that

$$(L_S F)(s_2) = \int_0^\infty e^{-s_2 t} dF(t), \qquad s_2 \in C_1$$

is absolutely convergent and analytic for $\text{Re}(s_2) > K$. The linear combination of $(L_S F)(s_1)$ and $(L_S F)(s_2)$ with e_1 and e_2 is

$$(L_S F)(s_1)e_1 + (L_S F)(s_2)e_2 = \int_0^\infty e^{-s_1 t} dF(t)e_1 + \int_0^\infty e^{-s_2 t} dF(t)e_2$$
$$= \int_0^\infty e^{-(s_1 e_1 + s_2 e_2)t} dF(t)$$
$$= \int_0^\infty e^{-\xi t} dF(t) = (L_S F)(\xi).$$

$(L_S F)(\xi)$ exists for $\text{Re}(s_1) > K$ and $\text{Re}(s_2) > K$ or $\text{Re}(P_1 : \xi) > K$ and $\text{Re}(P_2 : \xi) > K$.

Since $(L_S F)(s_1)$ and $(L_S F)(s_2)$ are complex-valued functions which are absolutely convergent and analytic for $\text{Re}(s_1) > K$ and $\text{Re}(s_2) > K$ respectively, so bicomplex-valued function $(L_S F)(\xi) = (L_S F)(s_1)e_1 + (L_S F)(s_2)e_2$ will be absolutely convergent and analytic in the region D defined as:

$$D = \{\xi : \xi = s_1 e_1 + s_2 e_2;\ \text{Re}(P_1 : \xi) > K \text{ and } \text{Re}(P_2 : \xi) > K\}. \tag{4.28}$$

Let $s_1 = x_1 + i_1 y_1$ and $s_2 = x_2 + i_1 y_2$. Thus $\text{Re}(s_1) = x_1 > K$ and $\text{Re}(s_2) = x_2 > K$, then

$$\begin{aligned}\xi &= (x_1 + i_1 y_1)e_1 + (x_2 + i_1 y_2)e_2 = (x_1 + i_1 y_1)\left(\frac{1 + i_1 i_2}{2}\right) + (x_2 + i_1 y_2)\left(\frac{1 + i_1 i_2}{2}\right)\\ &= \frac{x_1 + x_2}{2} + \left(\frac{y_1 + y_2}{2}\right) i_1 + \left(\frac{y_2 - y_1}{2}\right) i_2 + \left(\frac{x_1 - x_2}{2}\right) i_1 i_2\end{aligned}$$

Now there are three possible cases:

1. If $x_1 = x_2$ then $\frac{x_1 - x_2}{2} = 0$ and $\frac{x_1 + x_2}{2} = x_1 = x_2 > K$. Hence if $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2$, then $a_0 > K$ and $a_3 = 0$.

2. If $x_1 > x_2$ then $\frac{x_1 - x_2}{2} > 0$ and $\frac{x_1 + x_2}{2} > \frac{K + x_1}{2} > \frac{K + x_1}{2} + \frac{K - x_2}{2} = K + \frac{x_1 - x_2}{2}$. Thus $a_0 > K + a_3$ and $a_3 > 0$.

3. If $x_1 < x_2$ then $\frac{x_1 - x_2}{2} < 0$ and $\frac{x_1 + x_2}{2} > \frac{K + x_2}{2} > \frac{K + x_1}{2} + \frac{K - x_2}{2} = K - \frac{x_2 - x_1}{2}$. Thus $a_0 > K - a_3$ and $a_3 < 0$.

These three conditions will make following three sets:

$$D_1 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ a_0 > K \text{ and } a_3 = 0\}$$

$$D_2 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ a_0 > K + a_3 \text{ and } a_3 > 0\}$$

$$D_3 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ a_0 > K - a_3 \text{ and } a_3 < 0\}, \text{ respectively.}$$

Thus, $\mathrm{Re}(P_1 : \xi) > 0$ and $\mathrm{Re}(P_2 : \xi) > 0$ implies $\xi \in D = D_1 \cup D_2 \cup D_3$. Conditions in the set D_1, D_2 and D_3 can be combined and written as $a_0 > K + |a_3|$. Thus (4.28) can also be written in an equivalent form as

$$\begin{aligned} D &= \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in C_2 : a_0 > K + |a_3|\} \\ &= \{\xi \in C_2 : \mathrm{Re}(\xi) > K + |\mathrm{Im}_j(\xi)|\} \end{aligned} \tag{4.29}$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part of a bicomplex number w.r.t. j.

Conversely, the existence condition of bicomplex Laplace-Stieltjes transform $F(\xi)$ can be obtained in the following way:

$$\text{If } \xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in \Omega, \ a_0 > K + |a_3|. \tag{4.30}$$

Now, in terms of idempotent components, ξ can be expressed as

$$\begin{aligned} \xi &= a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \\ &= [(a_0 + a_3) + i_1(a_1 - a_2)]\, e_1 + [(a_0 - a_3) + i_1(a_1 + a_2)]\, e_2 \\ &= s_1 e_1 + s_2 e_2. \end{aligned}$$

Depending on the value of a_3, there arises three cases:

1. When $a_3 = 0$, from the inequality (4.30) $a_0 > K$, which trivially leads $a_0 - a_3 > K$ and $a_0 + a_3 > K$.

2. When $a_3 > 0$, from the inequality (4.30) $a_0 > K + a_3$, we get $a_0 - a_3 > K$. This result can be interpreted as $a_0 + a_3 > a_0 - a_3 > K$.

3. When $a_3 < 0$, from the inequality (4.30) $a_0 > K - a_3$, we get $a_0 + a_3 > K$. This result can be interpreted as $a_0 - a_3 > a_0 + a_3 > K$.

Hence the result.

Thus, the bicomplex Laplace-Stieltjes transform can be defined as

Definition 4.6. Let X be a non-negative random variable with distribution function $F(t)$ defined on $[0, \infty)$ with $F(0) = 0$. Its bicomplex Laplace-Stieltjes transform defined as

$$(L_S F)(\xi) = \int_0^\infty e^{-\xi t} dF(t), \qquad \xi \in D \subset C_2 \tag{4.31}$$

Or in general, if $\alpha : \mathbb{R} \to C_2$ is bounded variation function on $[0, \infty)$. Then bicomplex Laplace-Stieltjes transform defined as

$$(L_S \alpha)(\xi) = \int_0^\infty e^{-\xi t} d\alpha(t), \qquad \xi \in D \subset C_2 \tag{4.32}$$

are absolutely convergent and analytic in D is defined as

$$D = \{\xi \in C_2 : \mathrm{Re}(\xi) > K + |\mathrm{Im}_j(\xi)|\} \tag{4.33}$$

where K is exponential order of $F(t)$ and $\mathrm{Im}_j(\xi)$ denotes the imaginary part of a bicomplex number w.r.t. j.

4.7 Some Useful Properties of Bicomplex Laplace-Stieltjes Transform

Motivated by work of *Lin* [94], *Ziolkowski* [158] and *Debnath* and *Bhatta* [34] we are deriving some useful properties of Bicomplex Laplace-Stieltjes transform.

Theorem 4.11 (Linearity Property). *Let probability distribution functions $F(t)$ and $G(t)$ have bicomplex Laplace-Stieltjes transforms $(L_SF)(\xi)$ and $(L_SG)(\xi)$ with $Re(P_1 : \xi) > K_1,\ Re(P_2 : \xi) > K_1$ and $Re(P_1 : \xi) > K_2,\ Re(P_2 : \xi) > K_2$, respectively. Then for $Re(P_1 : \xi) > K,\ Re(P_2 : \xi) > K$, where $K = max(K_1, K_2)$*

$$L_S\{c_1F(t) + c_2G(t)\} = c_1(L_SF)(\xi) + c_2(L_SG)(\xi) \tag{4.34}$$

for arbitrary constants c_1, c_2.

Proof. Applying the definition of bicomplex Laplace-Stieltjes transform,

$$L_S\{c_1F(t) + c_2G(t)\} = \int_0^\infty e^{-\xi t} d\left(c_1F(t) + c_2G(t)\right)$$

$$= \int_0^\infty e^{-(s_1e_1+s_2e_2)t}\left(c_1dF(t) + c_2dG(t)\right), \qquad [\text{where } \xi = s_1e_1 + s_2e_2]$$

$$= c_1\left(\int_0^\infty e^{-s_1t}dF(t)\right)e_1 + c_1\left(\int_0^\infty e^{-s_2t}dF(t)\right)e_2 + c_2\left(\int_0^\infty e^{-s_1x}dF(t)\right)e_1 + c_2\left(\int_0^\infty e^{-s_2t}dF(t)\right)e_2$$

$$= c_1(L_SF)(s_1)e_1 + c_1(L_SF)(s_2)e_2 + c_2(L_SG)(s_1)e_1 + c_2(L_SG)(s_2)e_2$$

$$= c_1(L_SF)(s_1e_1 + s_2e_2) + c_2(L_SG)(s_1e_1 + s_2e_2)$$

$$= c_1(L_SF)(\xi) + c_2(L_SG)(\xi).$$

□

Theorem 4.12 (Change of Scale Property). *Let $L_SF(\xi)$ be the bicomplex Laplace-Stieltjes transform of probability distribution function $F(t)$ for $Re(P_1 : \xi) > K,\ Re(P_2 : \xi) > K$, then for $a > 0$*

$$L_S\{F(at)\} = \frac{1}{a}(L_SF)\left(\frac{\xi}{a}\right). \tag{4.35}$$

Proof. Applying the definition of bicomplex Laplace-Stieltjes transform,

$$L_S\{F(at)\} = \int_0^\infty e^{-\xi t} dF(at)$$

$$\text{put } at = u \text{ then } dF(at) = \frac{dF(u)}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{\xi}{a}u} dF(u)$$

$$= \frac{1}{a}(L_S F)\left(\frac{\xi}{a}\right).$$

□

Theorem 4.13. *Let $X_1, X_2, \cdots, X_n$ denote the sequence of independent non-negative random variables and $(L_SF_1)(\xi), (L_SF_2)(\xi), \cdots, (L_SF_n)(\xi)$ denote the sequence of bicomplex Laplace-Stieltjes transform of these random variables respectively. Also, let $X = X_1 + X_2 + \cdots + X_n$ be the sum of random variables $X_1, X_2, \cdots, X_n$. Let $(L_SF)(\xi)$ denote the bicomplex Laplace-Stieltjes transform of random variable X. Then*

$$(L_SF)(\xi) = \prod_{i=1}^{n}(L_SF_i)(\xi), \quad \xi \in D \text{ as defined in (4.33).}$$

Proof. Applying the definition of bicomplex Laplace-Stieltjes transform, $E(X)$ being expected value of X

$$(L_SF)(\xi) = \int_0^\infty e^{-\xi t} dF(t) = E\left(e^{-\xi X}\right)$$

$$\therefore\ (L_SF)(\xi) = E\left(e^{-\xi X}\right) = E\left(e^{\xi \sum_{i=1}^{n} X_i}\right) = E\left(\prod_{i=1}^{n} e^{-\xi X_i}\right)$$

by mean value property of independent random variables product, we have

$$(L_SF)(\xi) = \prod_{i=1}^{n} E\left(e^{-\xi X_i}\right) = \prod_{i=1}^{n}(L_SF_i)(\xi).$$

□

Theorem 4.14. *Let X be a non-negative random variable and $F(t)$ and $(L_SF)(\xi)$ are distribution function and bicomplex Laplace-Stieltjes transform of random* *variable X respectively. Then*

$$(L_S F)(\xi) = \int_0^\infty e^{-\xi t} dF(t) = \xi \int_0^\infty e^{-\xi t} F(t) dt, \qquad \xi \in D \tag{4.36}$$

where D is defined in (4.33).

Proof. Applying the definition of bicomplex Laplace-Stieltjes transform,

$$\begin{aligned}
(L_S F)(\xi) &= \int_0^\infty e^{-\xi t} dF(t) = \int_0^\infty e^{-(s_1 e_1 + s_2 e_2)t} dF(t) \\
&= \left(\int_0^\infty e^{-s_1 t} dF(t) \right) e_1 + \left(\int_0^\infty e^{-s_2 t} dF(t) \right) e_2 \\
&= \left(s_1 \int_0^\infty e^{-s_1 t} F(t) dt \right) e_1 + \left(s_2 \int_0^\infty e^{-s_2 t} F(t) dt \right) e_2 \\
&\qquad\qquad\qquad\qquad \text{[using bi-part integration]} \\
&= (s_1 e_1 + s_2 e_2) \int_0^\infty e^{-(s_1 e_1 + s_2 e_2)t} F(t) dt \\
&= \xi \int_0^\infty e^{-\xi t} F(t) dt.
\end{aligned}$$

□

4.8 Tauberian Theorem for Bicomplex Laplace-Stieltjes Transform

Tauberian Theorem used to study asymptotic behaviour of $F(t)$ from asymptotic behaviour of $(L_S F)(\xi)$. This has found applications in communications, networking, high-SNR analysis of performance over fading channels and proving prime number theorem (*Widder* [148]). In this section, we have made efforts to extend the following *Ikehara's* Tauberian theorem (*Ikehara* [71], see also *Widder* [148, p. 233, Theorem 17]) for bicomplex variable.

Theorem 4.15. *Let $F(t)$ be a non-negative and non-decreasing function defined in $0 \leq t < \infty$, and let the integral*

$$(L_S F)(s) = \int_0^\infty e^{-st} dF(t), \qquad s = x + i_1 y \in C_1$$

converges for $Re(s) > 1$. If for some constant A and some function $g(y)$

$$\lim_{Re(s)\to 1^+} \left((L_S F)(s) - \frac{A}{s-1} \right) = g(y)$$

uniformly in every finite interval of y, then we have

$$\lim_{t\to\infty} e^{-t} F(t) = A.$$

Motivated by their work, we find the Tauberian theorem of bicomplex Laplace-Stieltjes transform as follows:

Theorem 4.16 (Bicomplex Tauberian Theorem)**.** *Let $F(t)$ be a probability distribution function defined in $0 \leq t < \infty$, and let the integral*

$$(L_S F)(\xi) = \int_0^\infty e^{-\xi t} dF(t), \qquad \xi = s_1 e_1 + s_2 e_2 \in C_2$$

converges for $Re(P_1 : \xi) > 1$ and $Re(P_2 : \xi) > 1$. If for some constant A and some function $g(y)$

$$\lim_{Re(\xi)\to 1^+,\, Im_j(\xi)\to 0} \left((L_S F)(\xi) - \frac{A}{\xi - 1} \right) = g(y),$$

$$(\textit{where } y = y_1 e_1 + y_2 e_2 \textit{ and } A = A_1 e_1 + A_2 e_2)$$

uniformly in every finite interval of y_1 and y_2, then we have

$$\lim_{t\to\infty} e^{-t} F(t) = A,$$

where $Im_j(\xi)$ denotes the imaginary part of bicomplex number ξ w.r.t. j component.

Proof. Applying the definition of bicomplex Laplace-Stieltjes transform,

$$\begin{aligned}(L_SF)(\xi) &= \int_0^\infty e^{-\xi t} dF(t), \qquad \xi = s_1e_1 + s_2e_2 \in C_2 \\ &= \int_0^\infty e^{-(s_1e_1+s_2e_2)t} dF(t) \\ &= \left(\int_0^\infty e^{-s_1t} dF(t)\right) e_1 + \left(\int_0^\infty e^{-s_2t} dF(t)\right) e_2 \\ &= (L_SF)(s_1)e_1 + (L_SF)(s_2)e_2 \end{aligned} \tag{4.37}$$

Now,

$$(L_SF)(s_1) = \int_0^\infty e^{-s_1t} dF(t), \qquad s_1 = x_1 + i_1y_1 \in C_1 \tag{4.38}$$

The integral in (4.38) converges for $\mathrm{Re}(s_1) > 1$. If for some constant A_1 and some function $g(y_1)$

$$\lim_{\mathrm{Re}(s_1)\to 1^+} \left\{(L_SF)(s_1) - \frac{A_1}{s_1 - 1}\right\} = g(y_1) \tag{4.39}$$

uniformly in every finite interval of y_1, then we have

$$\lim_{t\to\infty} e^{-t}F(t) = A_1. \tag{4.40}$$

Similarly,

$$(L_SF)(s_2) = \int_0^\infty e^{-s_2t} dF(t), \qquad s_2 = x_1 + i_1y_1 \in C_1$$

converges for $\mathrm{Re}(s_2) > 1$. If for some constant A_2 and some function

$$\lim_{\mathrm{Re}(s_2)\to 1^+} \left\{(L_SF)(s_2) - \frac{A_2}{s_2 - 1}\right\} = g(y_2) \tag{4.41}$$

uniformly in every finite interval of y_2, then we have

$$\lim_{t\to\infty} e^{-t}F(t) = A_2. \tag{4.42}$$

Since $g(y_1)$ and $g(y_2)$ are analytic functions, therefore by taking linear combination of (4.39) and (4.41) with e_1 and e_2 respectively, we have

$$\lim_{\mathrm{Re}(s_1)\to 1^+}\left\{(L_SF)(s_1)-\frac{A_1}{s_1-1}\right\}e_1+\lim_{\mathrm{Re}(s_2)\to 1^+}\left\{(L_SF)(s_2)-\frac{A_2}{s_2-1}\right\}e_2=g(y_1)e_1+g(y_2)e_2$$

$$\lim_{\mathrm{Re}(\xi)\to 1^+,\ \mathrm{Im}_j(\xi)\to 0}\left\{(L_SF)(s_1e_1+s_2e_2)-\frac{A_1e_1+A_2e_2}{(s_1e_1+s_2e_2)-1}\right\}=g(y_1e_1+y_2e_2)$$

$$\lim_{\mathrm{Re}(\xi)\to 1^+,\ \mathrm{Im}_j(\xi)\to 0}\left\{(L_SF)(\xi)-\frac{A}{\xi-1}\right\}=g(y) \tag{4.43}$$

(where $\xi=s_1e_1+s_2e_2,\ y=y_1e_1+y_2e_2$ and $A=A_1e_1+A_2e_2$)

uniformly in every finite interval of y_1 and y_2.

Therefore, by taking the linear combination of (4.40) and (4.42) with e_1 and e_2 respectively, we have

$$\left(\lim_{t\to\infty}e^{-t}F(t)\right)e_1+\left(\lim_{t\to\infty}e^{-t}F(t)\right)e_2=A_1e_1+A_2e_2$$

$$\lim_{t\to\infty}e^{-t}F(t)=A.$$

□

4.9 Applications of Bicomplex Laplace-Stieltjes Transform

The bicomplex Laplace-Stieltjes transform has found applications in applied physics, moments, probability distribution theory, signal processing and other related problem. In this section, we discuss applications of bicomplex Laplace-Stieltjes transform to tail probability and bicomplex Dirichlet series.

Theorem 4.17 (Nakagawa [110])**.** *Let X be a non-negative random variable and $F(t)=P(X\le t)$ be the probability distribution function of X. Let*

$$(L_SF)(s)=\int_0^\infty e^{-st}dF(t),\qquad s\in C_1$$

be the Laplace-Stieltjes transform of $F(t)$ *and* σ_0 *be the abscissa of convergence of* $(L_S F)(s)$*. We assume* $-\infty < \sigma_0 < 0$*. If* $s = \sigma_0$ *is a pole of* $(L_S F)(s)$*, then*

$$\lim_{t\to\infty} \frac{1}{t} \log P(X > t) = \sigma_0.$$

(a) To find the exponential decay tail probability in bicomplex space we may use the the bicomplex Laplace-Stieltjes transform as described in the following theorem:

Theorem 4.18. *Let* X *be a non-negative random variable and* $F(t) = P(X \leq t)$ *be the bicomplex-valued probability distribution function of* X *for* $t \in \mathbb{R}$*. Let*

$$(L_S F)(\xi) = \int_0^\infty e^{-\xi t} dF(t), \qquad \xi \in D \text{ [where } D \text{ defined in (4.28) and (4.33)]}$$

be the bicomplex Laplace-Stieltjes transform of $F(t)$ *and* $(L_S F)(\xi)$ *converges for* $Re(P_1 : (\xi) > \sigma_0$ *and* $Re(P_2 : (\xi) > \sigma_0$*. We assume* $-\infty < \sigma_0 < 0$*. If* $\xi = \sigma_0$ *is a pole of* $(L_S F)(\xi)$*, then*

$$\lim_{t\to\infty} \frac{1}{t} \log P(X > t) = \sigma_0.$$

Proof. Applying the definition of bicomplex Laplace-Stieltjes transform,

$$\begin{aligned}
(L_S F)(\xi) &= \int_0^\infty e^{-\xi t} dF(t) \\
&= \int_0^\infty e^{-(s_1 e_1 + s_2 e_2)t} dF(t) \\
&= \left(\int_0^\infty e^{-s_1 t} dF(t) \right) e_1 + \left(\int_0^\infty e^{-s_2 t} dF(t) \right) e_2 \\
&= (L_S F)(s_1) e_1 + (L_S F)(s_2) e_2
\end{aligned}$$

Now,
$(L_S F)(s_1) = \int_0^\infty e^{-s_1 t} dF(t)$ is converges for $\text{Re}(s_1) > \sigma_0, \ -\infty < \sigma_0 < 0$. Therefore by Theorem 4.17, we have

$$\lim_{t\to\infty} \frac{1}{t} \log P(X > t) = \sigma_0 \tag{4.44}$$

where $s_1 = \sigma_0$ be the pole of $(L_SF)(s_1)$.
Similarly,
$(L_SF)(s_2) = \int_0^\infty e^{-s_2t}dF(t)$ is converges for $\text{Re}(s_2) > \sigma_0,\ -\infty < \sigma_0 < 0$. Therefore, by Theorem 4.17, we have

$$\lim_{t\to\infty} \frac{1}{t} \log P(X > t) = \sigma_0 \tag{4.45}$$

where $s_2 = \sigma_0$ be the pole of $(L_SF)(s_2)$.
Now taking the linear combination of (4.44) and (4.45) with e_1 and e_2 respectively, we have

$$\left(\lim_{t\to\infty} \frac{1}{t} \log P(X > t)\right) e_1 + \left(\lim_{t\to\infty} \frac{1}{t} \log P(X > t)\right) e_2 = \sigma_0 e_1 + \sigma_0 e_2$$
$$\therefore \lim_{t\to\infty} \frac{1}{t} \log P(X > t) = \sigma_0$$

where $\xi = \sigma_0$ is a pole of $(L_SF)(\xi)$. Therefore, bicomplex-valued tail probability exponentially decay in bicomplex space. □

(b) Here we discuss the relation between bicomplex Laplace-Stieltjes transform and bicomplex Dirichlet series (*Price* [119, p. 61]) as follows:

The complex Dirichlet series is given by

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{4.46}$$
$$(s = x + iy,\ 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty,\ x, y \in \mathbb{R})$$

where $a_n,\ n \in N$ is complex-valued coefficient of Dirichlet series.

Jiarong [73] and *Knopp* [83] have shown in their papers that complex Dirichlet series is a particular case of complex Laplace-Stieltjes transform. Therefore, all the properties of Dirichlet series can be discussed by using complex Laplace-Stieltjes transform. Some results of complex Dirichlet series are obtained by complex Laplace-Stieltjes transform in *Yinying* and *Daochun* [154] and *Kong* and

Yang [87].

In a similar way, consider bicomplex Dirichlet series given by

$$f(\xi)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n \xi} \tag{4.47}$$
$$(\xi = z_1+i_2z_2 \in C_2,\ 0=\lambda_1<\lambda_2<\cdots<\lambda_n \to +\infty,\ z_1,z_2\in C_1)$$

where $a_n,\ n\in N$ is bicomplex-valued coefficient of bicomplex Dirichlet series.

Let us consider bicomplex Laplace-Stieltjes transform

$$(L_S\alpha)(\xi)=\int_0^{\infty} e^{-\xi t} d\alpha(t) \qquad (\xi = z_1+i_2z_2\in D,\ z_1,z_2\in C_1), \tag{4.48}$$

where $\alpha:\mathbb{R}\to C_2$ is a function of bounded variation on any interval $[0,X]$, $(0< X<\infty)$ and D defined in (4.28) and (4.33). If we construct

$$\alpha(t)=\begin{cases}0, & \text{if } 0\le t<\lambda_1;\\ \sum_{m=1}^{n} a_m, & \text{if } \lambda_n\le t\le\lambda_{n+1}.\end{cases} \tag{4.49}$$

Then, by using Theorem 4.14, equation (4.48) becomes

$$\begin{aligned}(L_S\alpha)(\xi) &= \xi\int_0^{\infty} e^{-\xi t}\alpha(t)dt\\ &= \xi\int_0^{\lambda_1} e^{-\xi t}\alpha(t)dt+\xi\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} e^{-\xi t}\alpha(t)dt\\ &= \xi\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}}\left(e^{-\xi t}\sum_{m=1}^{n} a_m\right)dt \qquad \text{[using by (4.49)]}\\ &= \xi\sum_{n=1}^{\infty}\left(a_n\int_{\lambda_n}^{\infty} e^{-\xi t}dt\right)\\ &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n\xi}.\end{aligned}$$

Therefore, bicomplex Dirichlet series is a particular case of the bicomplex Laplace-Stieltjes transform. Therefore, all the properties of bicomplex Dirichlet series can be discussed by bicomplex Laplace-Stieltjes transform. In particularly, the convergence conditions of bicomplex Dirichlet series are difficult to find out. So by using Bicomplex Laplace-Stieltjes transform we can easily find out the convergence conditions.

4.10 Conclusion

In this chapter, we introduced the Stieltjes and Laplace-Stieltjes transforms with Tauberian theorem in bicomplex space which are the generalization of Stieltjes and Laplace-Stieltjes transforms from complexified frequency domain to bicomplexified frequency domain, respectively. The applications of bicomplex Stieltjes transform have been illustrated to check the symmetry of probability distribution function, find the solution of singular integral equation of bicomplex-valued function and spectral analysis of large dimensional random matrices. In quaternion-valued Stieltjes transform to define the inversion formula is challenging in practice. This problem can be solved easily in bicomplex Stieltjes transform due to commutative property of bicomplex numbers.

Also, the applications of bicomplex Laplace-Stieltjes transform have been illustration to find the exponential decay of tail probability of the bicomplex-vauled distribution and analysis of bicomplex Dirichlet series. Since bicomplex Dirichlet series is a particular case of bicomplex Laplace-Stieltjes transform, properties of the former can be discussed later. The work can further be applied to obtain the properties of bicomplex Dirichlet series using by bicomplex Laplace-Stiletjes transform.

5

Bochner Theorem of Fourier-Stieltjes Transform and Applications of Fourier Transform in Bicomplex Space

The main findings of this chapter have been published as:

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Bochner theorem and applications of bicomplex Fourier-Stieltjes transform, *Advanced Studies in Contemporary Mathematics,* **26**(2), 355-369.

2. Agarwal R., Goswami M.P., Agarwal R.P., Venkataratnam K.K. and Baleanu D. (2017), Solution of Maxwell's wave equations in bicomplex space, *Romanian Journal of Physics,* **62**(5-6), Article no. 115, 1-11.

5. BOCHNER THEOREM OF FOURIER-STIELTJES TRANSFORM AND APPLICATIONS OF FOURIER TRANSFORM IN BICOMPLEX SPACE

In this chapter, we derive the bicomplex Fourier-Stieltjes transform and related Bochner theorem with convergence conditions that can be capable of transferring signals from real-valued (t) domain to bicomplexified frequency (ξ) domain. Bicomplex Fourier-Stieltjes transform is highly applicable in applied physics, moments, probability distribution theory, signal processing, image processing and other related problems.

Applications of bicomplex Fourier transform in finding the solution of initial value heat equation in bicomplex algebra, algebraic reduction of complicated bicomplex linear time-invariant systems and solution of Maxwell's equations in vacuum have been discussed. Illustrations have been given to find the solution of bicomplex heat equation and check the unboundedness condition of non-homogeneous bicomplex-valued wave equation.

5.1 Introduction

The Fourier-Stieltjes transform is similar to Fourier transform which named for *Joseph Fourier* and *Thomas Joannes Stieltjes*. The Fourier-Stieltjes transform is the generalization of the standard Fourier transform and it has certain applications in the area of theoretical and applied probability and stochastic process. In 1939, *Cameron* and *Wiener* [23] discussed convergence conditions of the Fourier-Stieltjes transform. In 1967, *Rosenthal* [127] discussed some theorems of bounded measurable function defined on Lebesgue measurable subset of $\mathbb{R}$ with restrictions of Fourier-Stieltjes transform. In 1977,*Blei* [19] proved a theorem on continuous measures of Fourier-Stieltjes transform.

In 1989, *Assiamoua* and *Olubummo* [7] discussed the Fourier-Stieltjes transform of a Banach algebra valued measure on a compact group which is a collection of some continuous sesquilinear mappings. In 2012, *Gulhane* [61] studied different versions of inversion formula for the conventional Fourier-Stieltjes transform. In 2013, *Mensah* [104] studied the modern technique of tensor product to keep the interpretation of the Fourier-Stieltjes transform of a vector measure which is a collection of operators.

On the other hand, the classical Bochner theorem on Fourier integral transform, can be seen in *Bochner* ([20], [21]). In Bochner theorem, Fourier-Stieltjes transform of non-decreasing bounded functions can be easily seen to result in continuous function of positive type. In 2013, *Georgier* and *Morais* [52] extended the classical Bochner theorem in the framework of quaternion analysis.

We shall need the following definitions and results for our work.

Definition 5.1 (Alpay et al. [4])**.** Let $\Omega \subset C_2$ be some set. A function $f : \Omega \to C_2$ is said to positive definite in Ω if,

$$\sum_{p=1}^{n}\sum_{k=1}^{n} c_p c_k^* f(\xi_p - \xi_k^*) \in \mathbb{D}^+, \qquad \forall\, \xi_1, \xi_2, \cdots, \xi_n \in \Omega \tag{5.1}$$

for every choice of $c_1, c_2, \cdots, c_n \in C_2$.

The relation between the Fourier transform and the positive definiteness of a continuous function is given by *Bochner* [20] in the following theorem:

Theorem 5.1 (Bochner)**.** *A continuous function $f : \mathbb{R} \to C_1$ is positive definite iff it is the Fourier transform of a finite positive measure μ on $\mathbb{R}$, i.e.*

$$f(x) = \int_{-\infty}^{\infty} e^{itx} d\mu(t).$$

The bicomplex Fourier transform defied by *Banerjee* et al. [11] in the following way:

Definition 5.2. Let $f(t)$ be a real-valued continuous function in $(-\infty, \infty)$ that satisfies the following estimates

$$\begin{aligned} |f(t)| &\leq c_1 e^{-\alpha t}, \qquad t \geq 0,\ \alpha > 0 \\ |f(t)| &\leq c_2 e^{\beta t}, \qquad t < 0,\ \beta > 0. \end{aligned} \tag{5.2}$$

The bicomplex Fourier transform defined as

$$\mathcal{F}[f(t);\xi] = \int_{-\infty}^{\infty} e^{i_1 \xi t} f(t) dt = \bar{f}(\xi), \quad \xi \in \Omega \subset C_2 \tag{5.3}$$

(5.3) exists and is analytic for all $\xi \in \Omega$, defined as

$$\begin{aligned} \Omega = \{\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in C_2 : -\infty < a_0, a_3 < \infty; \\ -\alpha + |a_2| < a_1 < \beta - |a_2|; 0 \leq |a_2| < (\alpha + \beta)/2\}. \end{aligned} \tag{5.4}$$

5.2 Bicomplex Fourier-Stieltjes Transform

In this section, we discuss the convergence conditions and define bicomplex Fourier-Stieltjes transform.

Let $f(t)$ be a bicomplex-valued continuous function for $-\infty < t < \infty$ and satisfies the estimates

$$\begin{aligned} \|f(t)\| &\leq c_1 e^{-\alpha t}, \qquad t \geq 0,\ \alpha > 0 \\ \|f(t)\| &\leq c_2 e^{\beta t}, \qquad t < 0,\ \beta > 0 \end{aligned} \tag{5.5}$$

which guarantees that f is absolutely integrable on whole real line. For bicomplex-valued function $f(t)$, bicomplex Fourier transform can be defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i_1 \xi t} f(t) dt, \qquad \xi \in C_2 \tag{5.6}$$

together with the requirement of $\|\hat{f}(\xi)\| < \infty$. Now for $\xi = s_1e_1 + s_2e_2$, $s_1 = x_1 + i_1y_1$ and $s_2 = x_2 + i_1y_2$,

$$
\begin{aligned}
&\|\hat{f}(\xi)\| = \left\|\int_{-\infty}^{\infty} e^{i_1\xi t} f(t)dt\right\| \\
&\leq \int_{-\infty}^{\infty} \left\|e^{i_1\xi t}\right\| \|f(t)\|dt \\
&\leq \int_{-\infty}^{0} \left\|e^{i_1\xi t}\right\| c_2e^{\beta t}dt + \int_{0}^{\infty} \left\|e^{i_1\xi t}\right\| c_1e^{-\alpha t}dt \\
&= c_2\int_{-\infty}^{0} \left\|e^{i_1s_1t}e_1 + e^{i_1s_2t}e_2\right\| e^{\beta t}dt + c_1\int_{0}^{\infty} \left\|e^{i_1s_1t}e_1 + e^{i_1s_2t}e_2\right\| e^{-\alpha t}dt, \\
&\qquad\qquad [\because\ \xi = s_1e_1 + s_2e_2] \\
&= c_2\int_{-\infty}^{0} \frac{1}{\sqrt{2}}\left(\left|e^{i_1s_1t}\right|^2 + \left|e^{i_1s_2t}\right|^2\right)^{\frac{1}{2}} e^{\beta t}dt \\
&\quad + c_1\int_{0}^{\infty} \frac{1}{\sqrt{2}}\left(\left|e^{i_1s_1t}\right|^2 + \left|e^{i_1s_2t}\right|^2\right)^{\frac{1}{2}} e^{-\alpha t}dt, \quad [\text{using } (1.15)] \\
&= \frac{c_2}{\sqrt{2}}\int_{-\infty}^{0} \left(e^{-2y_1t} + e^{-2y_2t}\right)^{\frac{1}{2}} e^{\beta t}dt + \frac{c_1}{\sqrt{2}}\int_{0}^{\infty} \left(e^{-2y_1t} + e^{-2y_2t}\right)^{\frac{1}{2}} e^{-\alpha t}dt, \\
&\qquad\qquad \left[\because\ \left|e^{i_1xt}\right| = 1\right] \\
&\leq \frac{c_2}{\sqrt{2}}\int_{-\infty}^{0} e^{-y_1t}e^{\beta t}dt + \frac{c_2}{\sqrt{2}}\int_{-\infty}^{0} e^{-y_2t}e^{\beta t}dt + \frac{c_1}{\sqrt{2}}\int_{0}^{\infty} e^{-y_1t}e^{-\alpha t}dt \\
&\quad + \frac{c_1}{\sqrt{2}}\int_{0}^{\infty} e^{-y_2t}e^{-\alpha t}dt \\
&\quad \left[\because\ \left(|x|^2 + |y|^2\right)^{\frac{1}{2}} \leq |x| + |y|, \quad \forall\, x, y \in \mathbb{R} \text{ i.e. Minkowski's inequality}\right] \\
&= \frac{c_2}{\sqrt{2}}\int_{-\infty}^{0} e^{(\beta-y_1)t}dt + \frac{c_2}{\sqrt{2}}\int_{-\infty}^{0} e^{(\beta-y_2)t}dt + \frac{c_1}{\sqrt{2}}\int_{0}^{\infty} e^{-(\alpha+y_1)t}dt \\
&\quad + \frac{c_1}{\sqrt{2}}\int_{0}^{\infty} e^{-(\alpha+y_2)t}dt \\
&= \frac{c_2}{\sqrt{2}}\left(\frac{1}{\beta - y_1} + \frac{1}{\beta - y_2}\right) + \frac{c_1}{\sqrt{2}}\left(\frac{1}{\alpha + y_1} + \frac{1}{\alpha + y_2}\right).
\end{aligned}
$$

Then, the requirement $\|\hat{f}(\xi)\| < \infty$ is fulfilled only if $-\alpha < y_1 < \beta$ and $-\alpha < y_2 < \beta$.

Therefore, $\hat{f}(\xi)$ is analytic and convergent in the strip

$$D = \{\xi = s_1e_1 + s_2e_2 \in C_2 : -\alpha < \mathrm{Im}(P_1 : \xi) < \beta \text{ and } -\alpha < \mathrm{Im}(P_2 : \xi) < \beta\}. \tag{5.7}$$

Now, let $\mu : \mathbb{R} \to C_2$ be the Stieltjes measure which satisfies the estimate (5.5). Equation (5.6) can be written as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i_1\xi t} d\mu(t), \qquad \xi \in D \tag{5.8}$$

where D is defined in (5.7). Then (5.8) is known as bicomplex Fourier-Stieltjes transform, which is analytic and convergent in D. For better geometrical representation of the region of convergence of bicomplex Fourier-Stieltjes transform, it will be advantageous to use the four dimensional representation of bicomplex number. Let $s_1 = x_1 + i_1y_1$ and $s_1 = x_2 + i_1y_2$, then in accordance with (5.7),

$$\begin{aligned}\xi = s_1e_1 + s_2e_2 &= \frac{x_1 + x_2}{2} + i_1\frac{y_1 + y_2}{2} + i_2\frac{y_1 - y_2}{2} + i_1i_2\frac{x_1 - x_2}{2} \\ &= a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3, \quad \text{where } a_0, a_1, a_2, a_3 \in \mathbb{R}. \end{aligned} \tag{5.9}$$

There are three possibilities for the equivalent form (5.9)

1. If $y_1 = y_2$, then $-\alpha < a_1 < \beta$ and $a_2 = 0$.

2. If $y_1 > y_2$, then we may infer $-\alpha - a_2 < a_1 < \beta + a_2$ and $-\frac{\alpha+\beta}{2} < a_2 < 0$.

3. If $y_1 < y_2$, then we have $-\alpha + a_2 < a_1 < \beta - a_2$ and $0 < a_2 < \frac{\alpha+\beta}{2}$.

By combining all three cases we have

$$-\alpha + |a_2| < a_1 < \beta - |a_2|, \;\; 0 \leq |a_2| < \frac{\alpha + \beta}{2}$$

and hence the region of convergence D of $\hat{f}(\xi)$ is given by

$$D = \{\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in C_2 : -\alpha + |a_2| < a_1 < \beta - |a_2|, \\ 0 \leq |a_2| < (\alpha+\beta)/2\} \tag{5.10}$$

or, equivalently

$$D = \left\{\xi \in C_2 : -\alpha + |\mathrm{Im}_{i_2}(\xi)| < \mathrm{Im}_{i_1}(\xi) < \beta - |\mathrm{Im}_{i_2}(\xi)|,\ 0 \leq |\mathrm{Im}_{i_2}(\xi)| < \frac{\alpha+\beta}{2}\right\} \tag{5.11}$$

where $\mathrm{Im}_{i_1}(\xi)$ and $\mathrm{Im}_{i_2}(\xi)$ denotes the imaginary part of a bicomplex number w.r.t. i_1 and i_2, respectively. Therefore, we the above discussion can be summarized in the following proposition:

Proposition 5.1. *Let $\mu(t)$ be bicomplex-valued Stieltjes measurable function satisfying the estimates*

$$\|\mu(t)\| \leq c_1 e^{-\alpha t}, \qquad t \geq 0,\ \alpha > 0$$
$$\|\mu(t)\| \leq c_2 e^{\beta t}, \qquad t < 0,\ \beta > 0.$$

Then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i_1 \xi t} d\mu(t),$$

exists and analytic in the strip

$$D = \{\xi = s_1 e_1 + s_2 e_2 \in C_2 : -\alpha < Im(P_1 : \xi) < \beta \text{ and } -\alpha < Im(P_2 : \xi) < \beta\}.$$

Now, we define the bicomplex Fourier-Stieltjes transform as follows:

Definition 5.3. Let $\mu(t)$ be bicomplex-valued Stieltjes measurable function in $(-\infty, \infty)$ for $t \in \mathbb{R}$ that satisfies the estimates

$$\begin{aligned} \|\mu(t)\| &\leq c_1 e^{-\alpha t}, \qquad t \geq 0,\ \alpha > 0 \\ \|\mu(t)\| &\leq c_2 e^{\beta t}, \qquad t < 0,\ \beta > 0. \end{aligned} \tag{5.12}$$

Then the bicomplex Fourier-Stieltjes transform of $\mu(t)$ is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i_1 \xi t} d\mu(t), \qquad \xi \in D \subset C_2. \tag{5.13}$$

The bicomplex Fourier-Stieltjes transform $\hat{f}(\xi)$ exists and analytic for all $\xi \in D$, where D defined as

$$D = \left\{ \xi \in C_2 : -\alpha + |\mathrm{Im}_{i_2}(\xi)| < \mathrm{Im}_{i_1}(\xi) < \beta - |\mathrm{Im}_{i_2}(\xi)|,\ 0 \leq |\mathrm{Im}_{i_2}(\xi)| < \frac{\alpha + \beta}{2} \right\} \tag{5.14}$$

where $\mathrm{Im}_{i_1}(\xi)$ and $\mathrm{Im}_{i_2}(\xi)$ denotes the imaginary part of a bicomplex number w.r.t. i_1 and i_2, respectively.

Definition 5.4. The class of functions $\hat{f}(\xi),\ \xi \in D$ represented by (5.13) is called the class of functions and it is denoted as $\mathfrak{B}$. $\mathfrak{B}$ will be referred as the *bicomplex Bochner set*.

Since bicomplex Fourier-Stieltjes transform $\hat{f}(\xi)$ analytic for all $\xi \in D$. It follows that all members of $\mathfrak{B}$ are analytic functions of bicomplex variable ξ in D.

5.3 Properties of the Class of Bicomplex Bochner Functions

In this section, we discuss some basic properties of the class $\mathfrak{B}$ of bicomplex Bochner functions.

Theorem 5.2. *$\mathfrak{B}$ is a linear space.*

Proof. Let $\hat{f}$, $\hat{g} \in \mathfrak{B}$ and $a, b \in C_2$, then for $\xi \in D$ (as defined in (5.14))

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i_1 \xi t} d\mu_1(t)$$
$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{i_1 \xi t} d\mu_2(t).$$

Therefore,

$$\begin{aligned} a\hat{f}(\xi) + b\hat{g}(\xi) &= a\int_{-\infty}^{\infty} e^{i_1\xi t} d\mu_1(t) + b\int_{-\infty}^{\infty} e^{i_1\xi t} d\mu_2(t) \\ &= \int_{-\infty}^{\infty} e^{i_1\xi t} d(a\mu_1(t) + b\mu_1(t)) \\ &= \int_{-\infty}^{\infty} e^{i_1\xi t} d\mu(t) \in \mathfrak{B}, \qquad \text{(where } \mu(t) = a\mu_1(t) + b\mu_1(t)\text{)}. \end{aligned}$$

Hence, $\mathfrak{B}$ is a linear space. □

Theorem 5.3. *Every element of $\mathfrak{B}$ is a continuous bounded function.*

Proof. Let $\hat{f}$ be any element of $\mathfrak{B}$, then

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i_1\xi t} d\alpha(t), \qquad \xi \in D$$

where D is defined in (5.14) and (5.10). Since, $\hat{f}(\xi)$ is analytic in D, it is differentiable and hence continuous in D. Therefore, every element of $\mathfrak{B}$ is continuous function. Now,

$$\begin{aligned} &\left\|\hat{f}(\xi)\right\| = \left\|\int_{-\infty}^{\infty} e^{i_1\xi t} d\alpha(t)\right\| \\ &\leq \int_{-\infty}^{\infty} \left\|e^{i_1\xi t}\right\| \|d\alpha(t)\| \\ &= \int_{-\infty}^{\infty} \left\|e^{i_1 s_1 t} e_1 + e^{i_1 s_2 t} e_2\right\| \|d\alpha(t)\| \qquad (\because\ \xi = s_1 e_1 + s_2 e_2) \\ &= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \left(\left|e^{i_1 s_1 t}\right|^2 + \left|e^{i_1 s_2 t}\right|^2\right)^{\frac{1}{2}} \|d\alpha(t)\|, \left[\because\ \|s_1 e_1 + s_2 e_2\| = \left(\frac{|s_1|^2 + |s_2|^2}{2}\right)^{\frac{1}{2}}\right] \\ &= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \left(e^{-2y_1 t} + e^{-2y_2 t}\right)^{\frac{1}{2}} \|d\alpha(t)\|, \ (\because\ s_1 = x_1 + i_1 y_1 \text{ and } s_1 = x_2 + i_1 y_2) \\ &\leq \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \left(e^{-y_1 t} + e^{-y_2 t}\right) \|d\alpha(t)\|, \left(\because\ \left(|x|^2 + |y|^2\right)^{\frac{1}{2}} \leq |x| + |y|, \quad \forall\, x, y \in \mathbb{R}\right) \\ &= \frac{1}{\sqrt{2}} \left(\int_{-\infty}^{0} e^{-y_1 t} \|d\alpha(t)\| + \int_{0}^{\infty} e^{-y_1 t} \|d\alpha(t)\|\right. \\ &\quad \left. + \int_{-\infty}^{0} e^{-y_2 t} \|d\alpha(t)\| + \int_{0}^{\infty} e^{-y_2 t} \|d\alpha(t)\|\right) \end{aligned}$$

By (5.5) and using integration by parts, we have

$$\leq \sqrt{2}c_2 + \frac{c_2 y_1}{\sqrt{2}(\beta - y_1)} + \sqrt{2}c_1 + \frac{c_1 y_1}{\sqrt{2}(\alpha + y_1)} + \frac{c_2 y_2}{\sqrt{2}(\beta - y_2)} + \frac{c_1 y_2}{\sqrt{2}(\alpha + y_2)}$$

$$= M(\text{say}) < \infty.$$

Therefore, $\hat{f}(\xi)$ is bounded function. Hence, every element of $\mathfrak{B}$ is a continuous bounded function. □

5.4 Bicomplex Bochner Theorem

In this section, we discuss the Bochner Theorem of bicomplex Fourier-Stieltjes transform, which is the generalization of the complex Bochner Theorem. Also, this is advantageous than quaternionic Bochner's Theorem (*Georgiev* and *Morais* [52]) due to *commutative property* of bicomplex numbers.

Motivated by the work of *Georgiev* and *Morais* [52], we generalize the Bochner's theorem in the framework of bicomplex analysis.

Theorem 5.4. *A continuous function $f : C_2 \to C_2$ is positive definite if it is the Fourier transform of a finite positive measure μ on C_2 satisfying the estimates (5.5), i.e.*

$$f(\xi) = \int_{-\infty}^{\infty} e^{i_1 \xi t} d\mu(t).$$

Proof. $f : C_2 \to C_2$ will be positive definite if it satisfies the Definition (5.1).

For any $\xi_1, \xi_2, \cdots, \xi_n \in C_2$, $c_1, c_2, \cdots, c_n \in C_2$ and $\mu : \mathbb{R} \to C_2$, straightforward computations show that

$$\sum_{p=1}^{n}\sum_{k=1}^{n} c_p c_k^* f(\xi_p - \xi_k^*) = \sum_{p=1}^{n}\sum_{k=1}^{n} c_p c_k^* \int_{-\infty}^{\infty} e^{i_1(\xi_p - \xi_k^*)t} d\mu(t)$$

$$= \sum_{p=1}^{n}\sum_{k=1}^{n} c_p c_k^* \int_{-\infty}^{\infty} e^{i_1 \xi_p t} e^{-i_1 \xi_k^* t} d\mu(t)$$

$$= \sum_{p=1}^{n}\sum_{k=1}^{n} c_p c_k^* \int_{-\infty}^{\infty} e^{i_1\xi_p t}\left(e^{i_1\xi_k t}\right)^* d\mu(t)$$

$$= \int_{-\infty}^{\infty} \sum_{p=1}^{n} c_p e^{i_1\xi_p t} \sum_{k=1}^{n} \left(c_k e^{i_1\xi_k t}\right)^* d\mu(t)$$

$$= \left(\int_{-\infty}^{\infty} \sum_{p=1}^{n} (c_1)_p e^{i_1(s_1)_p t} \sum_{k=1}^{n} \overline{(c_1)_k e^{i_1(s_1)_k t}} d\mu_1(t)\right) e_1$$

$$+ \left(\int_{-\infty}^{\infty} \sum_{p=1}^{n} (c_2)_p e^{i_1(s_2)_p t} \sum_{k=1}^{n} \overline{(c_2)_k e^{i_1(s_2)_k t}} d\mu_2(t)\right) e_2$$

(where $\xi = s_1e_1 + s_2e_2,\ c = c_1e_1 + c_2e_2$ and $\mu(t) = \mu_1(t)e_1 + \mu_2(t)e_2$)

$$= \left(\int_{-\infty}^{\infty} \sum_{p=1}^{n} \left|(c_1)_p e^{i_1(s_1)_p t}\right|^2 d\mu_1(t)\right) e_1$$

$$+ \left(\int_{-\infty}^{\infty} \sum_{p=1}^{n} \left|(c_2)_p e^{i_1(s_2)_p t}\right|^2 d\mu_2(t)\right) e_2$$

$$= \alpha e_1 + \beta e_2 \in \mathbb{D}^+,$$

where,

$$\alpha = \int_{-\infty}^{\infty} \sum_{p=1}^{n} \left|(c_1)_p e^{i_1(s_1)_p t}\right|^2 d\mu_1(t) \geq 0$$

$$\beta = \int_{-\infty}^{\infty} \sum_{p=1}^{n} \left|(c_2)_p e^{i_1(s_2)_p t}\right|^2 d\mu_2(t) \geq 0.$$

This proves that f is positive definite. □

5.5 Applications

In [11], *Banerjee* et al. proposed to use bicomplex functions in order to define a bicomplex Fourier transform applicable to signal processing, image processing, solving differential equations, quantum mechanics and other related fields. In this section, we discuss applications of bicomplex Fourier transform to solve partial differential equations in bicomplex algebra and reduction of one-dimensional bicomplex-valued linear time-invariant systems. The details of the solution of parabolic initial value problem in quaternion algebra by using quaternion Fourier transform can be seen in *Bahri* et al. [8].

(a) Consider the parabolic initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{5.15}$$

with

$$u(x,0) = f(x), \tag{5.16}$$

where $u : \mathbb{R} \times \mathbb{R} \to C_2$ and $f : \mathbb{R} \to C_2$ are bicomplex-valued functions satisfying the estimates defined in (5.5). To find the solution of above bicomplex heat equation take the bicomplex Fourier transform of (5.15). We get

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i_1 \xi x} \frac{\partial u}{\partial t} dx &= \int_{-\infty}^{\infty} e^{i_1 \xi x} \frac{\partial^2 u}{\partial x^2} dx \\ \Rightarrow \frac{d\bar{u}(\xi,t)}{dt} &= (i_1 \xi)^2 \bar{u}(\xi,t), \end{aligned} \tag{5.17}$$

where $\bar{u}(\xi,t)$ is the bicomplex Fourier transform of $u(x,t)$. The general solution of (5.17) is given by

$$\bar{u}(\xi,t) = ce^{-\xi^2 t}, \tag{5.18}$$

where c is a bicomplex constant. By taking bicomlex Fourier transform of initial condition (5.16), we have

$$\bar{u}(\xi,0) = \bar{f}(\xi), \ \left(\text{where } \bar{f}(\xi) \text{ is the bicomplex Fourier transform of } f(x)\right). \tag{5.19}$$

Using (5.19) in (5.18), we have

$$\bar{u}(\xi,t) = \bar{f}(\xi)e^{-\xi^2 t} \tag{5.20}$$

Taking the bicomplex inverse Fourier transform of (5.20) and making use of the following result

$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{i_1 \xi x} e^{-\frac{x^2}{4t}}\,dx = e^{-\xi^2 t} \tag{5.21}$$

and convolution theorem therein, we obtain

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u) e^{-\frac{(x-u)^2}{4t}}\,du, \tag{5.22}$$

which is the solution of the heat equation (5.15).

(b) In [39], *Ell* discussed the use of the quaternion Fourier transform for quaternion linear time-invariant systems analysis and reduction in easy form of complicated two-dimensional quaternion systems. The work in [39] is suitable for the case where impulse response of the quaternion linear time-invariant systems is a pure real function. Further, *Pei* et al. [115] developed the relationship between quaternion convolution and quaternion Fourier transform. With these relations, quaternion Fourier transform analyzes the quaternion linear time-invariant systems easily. For the analysis of commutative linear time-invariant systems, quaternion Fourier transform is difficult to use due to non-commutative property of quaternions.

In this section, we use bicomplex Fourier transform to analyze the bicomplex linear time-invariant systems and reduction to easy form from complicated *one-dimensional linear time-invariant systems composed* in series and in parallel connections of one-dimensional linear time-invariant subsystems. Bicomplex linear time-invariant systems in one variable can be defined in terms of a convolution operator as

$$y(t) = \int_{-\infty}^{\infty} h(t-u)x(u)du$$

where $x(\cdot)$ is the input, $y(\cdot)$ is the output and $h(\cdot)$ impulse response of the system. The pictorial representation of this system is given in Fig. 5.1 below

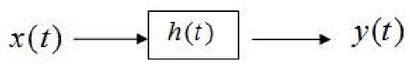

Figure 5.1: Block diagram for bicomplex linear time-invariant systems

Convolution theorem (*Banerjee* et al. [11, Theorem 4]) for the bicomplex Fourier transform is as follows:

Theorem 5.5. *The Fourier transform of two functions $f(t)$ and $g(t)$, $-\infty < t < \infty$ is the product of their Fourier transforms, respectively $\bar{f}(\xi)$ and $\bar{g}(\xi)$ i.e.*

$$F\{f(t) *_b g(t)\} = F\left\{\int_{-\infty}^{\infty} f(u)g(t-u)du\right\} = \bar{f}(\xi)\bar{g}(\xi).$$

When we combine the bicomplex linear time-invariant systems in parallel (as in Fig. 5.2), the relation between the input $f(x)$ and output $g(x)$ can be expressed as

$$g(x) = f(x) *_b h_t(x) \tag{5.23}$$

$$\text{where} \qquad h_t(x) = \sum_{n=1}^{p} h_n(x) \tag{5.24}$$

and $*_b$ is the bicomplex convolution. In the frequency domain, bicomplex Fourier transforms of (5.23) and (5.24) are as follows

$$\bar{g}(\xi) = \bar{f}(\xi)\bar{h}_t(\xi) \tag{5.25}$$

$$\text{where} \qquad \bar{h}_t(\xi) = \sum_{n=1}^{p} h_n(\xi). \tag{5.26}$$

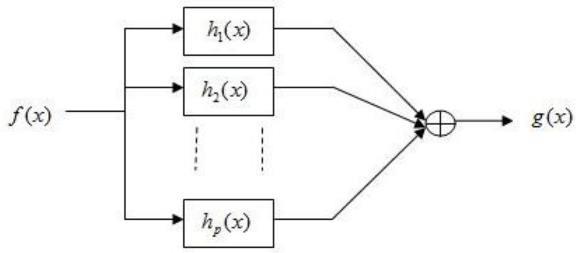

Figure 5.2: Combination of bicomplex linear time-invariant systems in parallel

When we combine the bicomplex linear time-invariant systems in series (as in Fig. 5.3), the relation between input $f(x)$ and output $g(x)$ can be expressed as

$$g(x) = f(x) *_b h_t(x) \tag{5.27}$$

$$\text{where} \qquad h_t(x) = h_1(x) *_b h_2(x) *_b \cdots *_b h_p(x). \tag{5.28}$$

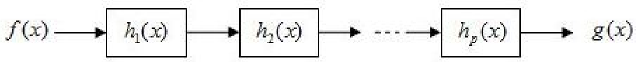

Figure 5.3: Combination of bicomplex linear time-invariant systems in series

In the frequency domain, bicomplex Fourier transforms of (5.27) and (5.28) are as follows

$$\bar{g}(\xi) = \bar{f}(\xi)\bar{h}_t(\xi) \tag{5.29}$$

$$\text{where} \qquad \bar{h}_t(\xi) = \prod_{n=1}^{p} h_n(\xi). \tag{5.30}$$

For the analysis of these kind of bicomplex linear time-invariant systems, we need large class of frequency domain. So bicomplex Fourier transform permits easy analysis of the complicated one-dimensional bicomplex-valued linear time-invariant systems by composing in parallel and in series conversion.

Following are the illustrations to find the solution of bicomplex heat equation with initial condition generated by bicomplex linear time-invariant systems and check the unbounded condition of non-homogeneous bicomplex-valued wave equation.

Example 5.1. Find the solution of the bicomplex heat equation described by the following differential system

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{with initial condition } u(x,0) = i_1\delta(x) \tag{5.31}$$

where $\delta(x)$ is the Dirac-delta function.

Solution. The solution of (5.31) can be obtained using (5.22) as

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} i_1\delta(u) e^{-\frac{(x-u)^2}{4t}} du \\ &= \frac{i_1}{2\sqrt{\pi t}}\, e^{-\frac{(x-u)^2}{4t}} \Big|_{u=0} \\ &= \frac{i_1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \end{aligned}$$

$\therefore \quad u(x,t) = \frac{i_1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$

Example 5.2. Consider the non-homogeneous bicomplex-valued wave equation with zero initial conditions described by the following differential system

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = p(x,t), \quad u(x,0) = 0, \ \frac{\partial u(x,0)}{\partial t} = 0, \tag{5.32}$$

where $u : \mathbb{R} \times \mathbb{R} \to C_2$. Show that the above system becomes unbounded if the forcing function i.e. bicomplex input is of the form

$$p(x,t) = \cos(\xi x)\cos(\eta t), \qquad \xi,\ \eta \in C_2.$$

Solution. Taking the bicomplex Fourier transform of (5.32) w.r.t. x, we have

$$\int_{-\infty}^{\infty} e^{i_1 \xi x} \frac{\partial^2 u(x,t)}{\partial t^2} dx - c^2 \int_{-\infty}^{\infty} e^{i_1 \xi x} \frac{\partial^2 u(x,t)}{\partial x^2} dx = \int_{-\infty}^{\infty} e^{i_1 \xi x} p(x,t) dx$$

$$\frac{d^2 \bar{u}(\xi,t)}{dt^2} + c^2 \xi^2 \bar{u}(\xi,t) = \bar{p}(\xi,t). \tag{5.33}$$

Again taking bicomplex Fourier transform of (5.33) w.r.t. t, we have

$$\int_{-\infty}^{\infty} e^{i_1 \eta t} \frac{d^2 \bar{u}(\xi,t)}{dt^2} dt + c^2 \xi^2 \int_{-\infty}^{\infty} e^{i_1 \eta t} \bar{u}(\xi,t) dt = \int_{-\infty}^{\infty} e^{i_1 \eta t} \bar{p}(\xi,t) dt$$

$$\Rightarrow\ -\eta^2 \bar{\bar{u}}(\xi,\eta) + c^2 \xi^2 \bar{\bar{u}} = \bar{\bar{p}}(\xi,\eta)$$

$$\Rightarrow \bar{\bar{u}}(\xi,\eta) = \frac{\bar{\bar{p}}(\xi,\eta)}{(c^2 \xi^2 - \eta^2)}.$$

This is singular whenever

$$\eta^2 = c^2 \xi^2. \tag{5.34}$$

Now, the d'Alembert's solution of the wave equation is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2c} \int_0^t \int_{x-c(t-\bar{t})}^{x+c(t-\bar{t})} p(\bar{x},\bar{t}) d\bar{x} d\bar{t} \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-\bar{t})}^{x+c(t-\bar{t})} \cos(\xi \bar{t}) \cos(\eta \bar{x}) d\bar{x} d\bar{t} \\ &= \frac{(\cos(\eta t) - \cos(ct\xi)) \cos(\eta x)}{-\eta^2 + c^2 \xi^2} \end{aligned} \tag{5.35}$$

which is unbounded under the condition mentioned in (5.34).

Let $\xi = \xi_1 e_1 + \xi_2 e_2$ and $\eta = \eta_1 e_1 + \eta_2 e_2$, where $\xi_1,\ \xi_2,\ \eta_1,\ \eta_2 \in C_1$. Then (5.35) can be written as

$$u(x,t) = \frac{(\cos(\eta_1 t) - \cos(\xi_1 ct))\cos(\eta_1 x)}{c^2\xi_1^2 - \eta_1^2} e_1 + \frac{(\cos(\eta_2 t) - \cos(\xi_2 ct))\cos(\eta_2 x)}{c^2\xi_2^2 - \eta_2^2} e_2.$$

Alternatively, using the results by *Luna-Elizarraras* et. al [97, p. 75] in (5.35), we can write

$$\begin{aligned} u(x,t) =& \frac{1}{A + i_2 B}\left[\cos\left(\frac{\eta_1+\eta_2}{2}x\right)\cosh\left(i_1\frac{\eta_1-\eta_2}{2}x\right)\left\{\cos\left(\frac{\eta_1+\eta_2}{2}t\right)\right.\right. \\ & \cosh\left(i_1\frac{\eta_1-\eta_2}{2}t\right) - \cos\left(\frac{\xi_1+\xi_2}{2}ct\right)\cosh\left(i_1\frac{\xi_1-\xi_2}{2}ct\right)\bigg\} - \\ & \sin\left(\frac{\eta_1+\eta_2}{2}x\right)\sinh\left(i_1\frac{\eta_1-\eta_2}{2}x\right)\bigg\{\sin(\frac{\eta_1+\eta_2}{2}t)\sinh\left(i_1\frac{\eta_1-\eta_2}{2}t\right) \\ & -\sin\left(\frac{\xi_1+\xi_2}{2}ct\right)\sinh\left(i_1\frac{\xi_1-\xi_2}{2}ct\right)\bigg\} - i_2\left(\sin\left(\frac{\eta_1+\eta_2}{2}x\right)\right. \\ & \sinh\left(i_1\frac{\eta_1-\eta_2}{2}x\right)\bigg\{\cos\left(\frac{\eta_1+\eta_2}{2}t\right)\cosh\left(i_1\frac{\eta_1-\eta_2}{2}t\right) - \cos\left(\frac{\xi_1+\xi_2}{2}ct\right) \\ & \cosh\left(i_1\frac{\xi_1-\xi_2}{2}ct\right)\bigg\} + \cos\left(\frac{\eta_1+\eta_2}{2}x\right)\cosh\left(i_1\frac{\eta_1-\eta_2}{2}x\right)\bigg\{\sin\left(\frac{\eta_1+\eta_2}{2}t\right) \\ & \left.\left.\sinh\left(i_1\frac{\eta_1-\eta_2}{2}t\right) - \sin\left(\frac{\xi_1+\xi_2}{2}ct\right)\sinh\left(i_1\frac{\xi_1-\xi_2}{2}ct\right)\bigg\}\right)\right] \end{aligned}$$

where,

$$\begin{aligned} A &= \frac{1}{2}\left(c^2\xi_1^2 + c^2\xi_2^2 - \eta_1^2 - \eta_2^2\right) \\ B &= \frac{i_1}{2}\left(c^2\xi_1^2 - c^2\xi_2^2 - \eta_1^2 + \eta_2^2\right). \end{aligned}$$

James Maxwell published his first paper in 1853 after obtaining his graduate degree. In this paper, he published *Faraday's* concept *lines of force*. He gave the mathematical explanation of *Faraday's* work (see, e.g. *Guilmette* [60]). Maxwell's equations describe how electric and magnetic fields are generated and influenced by each other and by charges and currents. These equations are named after the mathematician and physicist *Maxwell,* who published these equations between 1861 and 1862.

In 1864, *Maxwell* [102] discussed that an electromagnetic disturbance travels in free space with the velocity of light. In 1873, *Maxwell* [103] records the transformation of Maxwells complete theory of electromagnetism. *Hertz* discussed electromagnetic waves in the year 1888 [65, Chapter 7, p. 107-123], in which *Hertz* confirmed Maxwell's prediction and helped in the acceptance of Maxwell's electromagnetic theory. By the efforts of *Hertz, George Francis Fitzgerald* (1851-1901), *Oliver Lodge* (1851-1940) and *Oliver Heaviside* (1850-1925) (see, e.g. *Sengupta* and *Sarkar* [133]) Maxwell's ideas and equations made understandable. These developments are well documented in [69] and [64].

Motivated by the work of *Anastassiu* et al. [5] for finding solution of Maxwell's equations in source free domain for electric and magnetic fields using quaternions, we have made efforts to solve the Maxwell's equations in vacuum using bicomplex analysis. The method discussed here has the advantage of dealing both the vector fields (electric and magnetic) together as a single vector field in bicomplex space.

5.6 Bicomplex Solution for Electromagnetic Wave Equation in Vacuum

The Maxwell's equations in vacuum for electromagnetic field are (see, e.g. *Longren* and *Savov* [96, Chapter 7])

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{5.36}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{5.37}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{5.38}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{5.39}$$

where electric field $\mathbf{E}$ and magnetic field intensity $\mathbf{H}$ are complex-valued vector, μ_0 is permeability and ϵ_0 is the permittivity of free space. Let us define bicomplex vector field $\mathbf{F}$ as

$$\mathbf{F} \equiv \sqrt{\epsilon_0}\mathbf{E} + i_2\sqrt{\mu_0}\mathbf{H} \tag{5.40}$$

with the intimation that each directional component of $\mathbf{F}$ is a scalar bicomplex function, obtaining by combining the corresponding field directional components. Now, taking curl of (5.40) on both sides,

$$\begin{aligned}
\nabla \times \mathbf{F} &= \sqrt{\epsilon_0}\nabla \times \mathbf{E} + i_2\sqrt{\mu_0}\nabla \times \mathbf{H} \\
&= -\mu_0\sqrt{\epsilon_0}\frac{\partial \mathbf{H}}{\partial t} + i_2\sqrt{\mu_0}\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \\
&= i_2\sqrt{\mu_0\epsilon_0}\frac{\partial}{\partial t}\left(\sqrt{\epsilon_0}\mathbf{E} + i_2\sqrt{\mu_0}\mathbf{H}\right) \\
&= i_2\sqrt{\mu_0\epsilon_0}\frac{\partial \mathbf{F}}{\partial t}
\end{aligned}$$

Therefore, we obtain the bicomplex Maxwell's vector equations as,

$$\nabla \times \mathbf{F} = i_2 \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t}, \quad \left[\text{where } c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}\right] \tag{5.41}$$

$$\nabla \cdot \mathbf{F} = 0 \tag{5.42}$$

Assuming that the wave is travelling in x-direction, i.e., a vanishing x- component, then (5.41) and (5.42) are reduced to the following system of bicomplex differential equations,

$$-\frac{\partial F_z}{\partial x} = i_2 \frac{1}{c}\frac{\partial F_y}{\partial t} \tag{5.43}$$

$$\frac{\partial F_y}{\partial x} = i_2 \frac{1}{c}\frac{\partial F_z}{\partial t} \tag{5.44}$$

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0 \tag{5.45}$$

$$\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 \tag{5.46}$$

Put $Q_z = i_2 F_z$. The equations (5.43) and (5.44) become

$$\frac{\partial Q_z}{\partial x} = \frac{1}{c}\frac{\partial F_y}{\partial t}, \tag{5.47}$$

$$\text{and} \quad \frac{\partial F_y}{\partial x} = \frac{1}{c}\frac{\partial Q_z}{\partial t}, \text{ repectively.} \tag{5.48}$$

Differentiating (5.47) and (5.48) and using respectively (5.48) and (5.47) therein, we get

$$\frac{\partial^2}{\partial x^2} F_y(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} F_y(x,t) \tag{5.49}$$

$$\frac{\partial^2}{\partial x^2} Q_z(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} Q_z(x,t) \tag{5.50}$$

Due to (5.45) and (5.46) initial conditions of F_y and Q_z are only functions of the variable x only. Let initial conditions $F_y(x,0) = Af_1(x)$, $\frac{\partial}{\partial t}F_y(x,0) =$

$Bg_1(x)$, $Q_z(x,0) = Df_1(x)$ and $\frac{\partial}{\partial t}Q_z(x,0) = Gg_1(x)$, where $f_1(x)$, $g_1(x)$ are bicomplex-valued functions and A, B, D, G are bicomplex constants.

Taking the bicomplex Fourier transform (for details, refer [11]) of (5.49) w.r.t. x, we get

$$\frac{d^2}{dt^2}\bar{F}_y(\xi,t) + c^2\xi^2\bar{F}_y(\xi,t) = 0 \tag{5.51}$$

Solving (5.51) and applying initial conditions therein, we get

$$\bar{F}_y(\xi,t) = \frac{A}{2}\bar{f}_1(\xi)\left(e^{i_1c\xi t} + e^{-i_1c\xi t}\right) - i_1B\frac{\bar{g}_1(\xi)}{2c\xi}\left(e^{i_1c\xi t} - e^{-i_1c\xi t}\right). \tag{5.52}$$

[where $\bar{f}_1(\xi) = \mathcal{F}[f_1(x)](\xi)$ and $\bar{g}_1(\xi) = \mathcal{F}[g_1(x)](\xi)$]

Remark 5.1. There is no major reason to prefer i_1 instead of i_2, however i_1 is more appropriate than i_2 for the decomposition of bicomplex form in idempotent components.

Taking the inverse bicomplex Fourier transform [12, Eq. 11] of (5.52) w.r.t. ξ, we have

$$F_y(x,t) = \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi x}\bar{F}_y(\xi,t)d\xi$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ is closed contour in bicomplex space, where Γ_1 and Γ_2 are closed contours in complex space along the horizontal lines $\{-\alpha < \mathrm{Im}(P_1 : \xi) <$

$\beta\}$ and $\{-\alpha < \mathrm{Im}(P_2 : \xi) < \beta\}$, respectively.

$$\begin{aligned}
F_y(x,t) &= \frac{A}{2}\left[\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x-ct)}\bar{f}_1(\xi)d\xi + \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x+ct)}\bar{f}_1(\xi)d\xi\right] \\
&+ \frac{B}{2c}\left\{\frac{1}{2\pi}\int_\Gamma \frac{\bar{g}_1(\xi)}{i_1\xi}\left(e^{-i_1\xi(x-ct)} - e^{-i_1\xi(x+ct)}\right)d\xi\right\} \\
&= \frac{A}{2}\left[\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x-ct)}\bar{f}_1(\xi)d\xi + \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x+ct)}\bar{f}_1(\xi)d\xi\right] \\
&+ \frac{B}{2c}\left\{\frac{1}{2\pi}\int_\Gamma \bar{g}_1(\xi)\int_{x-ct}^{x+ct} e^{-i_1\xi p}dpd\xi\right\} \\
&= \frac{A}{2}\left[\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x-ct)}\bar{f}_1(\xi)d\xi + \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi(x+ct)}\bar{f}_1(\xi)d\xi\right] \\
&+ \frac{B}{2c}\left\{\int_{x-ct}^{x+ct} dp\left(\frac{1}{2\pi}\int_\Gamma e^{-i_1\xi p}\bar{g}_1(\xi)d\xi\right)\right\}. \qquad (5.53)
\end{aligned}$$

By simplifying (5.53), we get

$$F_y(x,t) = \frac{A}{2}\left[f_1(x-ct) + f_1(x+ct)\right] + \frac{B}{2c}\int_{x-ct}^{x+ct} g_1(p)dp. \qquad (5.54)$$

Similarly,

$$F_z(x,t) = -i_2Q_z(x,t) = -i_2\frac{D}{2}\left[f_1(x-ct) + f_1(x+ct)\right] - i_2\frac{G}{2c}\int_{x-ct}^{x+ct} g_1(p)dp. \qquad (5.55)$$

Therefore, wave travelling in x-direction with vector field $\mathbf{F_x}$ is

$$\mathbf{F_x} = F_y(x,t)\hat{y} + F_z(x,t)\hat{z} \qquad (5.56)$$

Since (5.56) is the solution of bicomplex Maxwell's equations, it satisfies the equations (5.41-5.42). So we obtain the values of D and G in terms of A and B.

Similarly, the wave travelling in y-direction with vector field $\mathbf{F_y}$ with initial conditions $F_x(y,0) = Rf_2(y)$, $Q_z(y,0) = Mf_2(y)$, $\frac{\partial}{\partial t}F_x(y,0) = Sg_2(y)$ and $\frac{\partial}{\partial t}Q_z(y,0) = Ng_2(y)$ are

$$\mathbf{F_y} = F_x(y,t)\hat{x} + F_z(y,t)\hat{z} \qquad (5.57)$$

where,

$$F_x(y,t) = \frac{R}{2}\left[f_2(y-ct) + f_2(y+ct)\right] + \frac{S}{2c}\int_{y-ct}^{y+ct} g_2(p)dp,$$

and

$$F_z(y,t) = -i_2\frac{M}{2}\left[f_2(y-ct) + f_2(y+ct)\right] - i_2\frac{N}{2c}\int_{y-ct}^{y+ct} g_2(p)dp.$$

Again, since (5.57) is the solution of bicomplex Maxwell's equations, it satisfies the equations (5.41-5.42). So we obtain the values of M and N in terms of R and S.

Also, wave travelling in z-direction with vector field $\mathbf{F_z}$ and initial conditions $F_x(z,0) = Lf_3(z)$, $Q_y(z,0) = If_3(z)$, $\frac{\partial}{\partial t}Q_y(z,0) = Jg_3(z)$ and $\frac{\partial}{\partial t}F_x(z,0) = Gg_3(z)$ are

$$\mathbf{F_z} = F_x(z,t)\hat{x} + F_y(z,t)\hat{y} \tag{5.58}$$

where,

$$F_x(z,t) = \frac{L}{2}\left[f_3(z-ct) + f_3(z+ct)\right] + \frac{G}{2c}\int_{z-ct}^{z+ct} g_3(p)dp,$$

and

$$F_y(z,t) = -i_2\frac{I}{2}\left[f_3(z-ct) + f_3(z+ct)\right] - i_2\frac{J}{2c}\int_{z-ct}^{z+ct} g_3(p)dp.$$

Again, since (5.58) is the solution of bicomplex Maxwell's equations, it satisfies the equations (5.41-5.42). So we obtain the values of I and J in terms of L and G.

Now, by applying the superposition principle on equations (5.56), (5.57) and (5.58), we obtain the solution of equations (5.41) and (5.42)as

$$\mathbf{F} = [F_x(y,t) + F_x(z,t)]\,\hat{x} + [F_y(x,t) + F_y(z,t)]\,\hat{y} + [F_z(x,t) + F_z(y,t)]\,\hat{z}. \quad (5.59)$$

By separating bi-real and bi-imaginary part we obtain the electric and magnetic fields in all three dimensions which satisfy the Maxwell's equations.

5.7 Bicomplex Gaussian Pulse Wave

In this section, we find the complete solution of the bicomplex Gaussian pulse travelling electromagnetic wave equation. For Gaussian pulse wave function is a solution of Gaussian pulse travelling wave (see, e.g. *Lonngren* and *Savov* [96, p. 345-346]). A two-pulse synthesis model presented by *Goswami* et al. [55] successfully reconstructed digital volume pulse waveforms using Rayleigh functions with small Mean Square Error. In [144], *Wang Lu* et al. presented a multi-Gaussian model to fit real pulse waveforms using an adaptive number of Gaussian waves.

Consider the bicomplex Gaussian pulse travelling electromagnetic wave equations are

$$\nabla \times \mathbf{F} = i_2 \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t}$$

$$\nabla \cdot \mathbf{F} = 0$$

For the bicomplex Gaussian pulse wave travelling in x- direction, the initial conditions (see, e.g. *Lonngren* and *Savov* [96, p. 345-346]) are of the form

$$F_y(x,0) = Ae^{-x^2}, \quad \frac{\partial}{\partial t}F_y(x,0) = Bxe^{-x^2}, \quad A,\, B \in C_2 \tag{5.60}$$

$$Q_z(x,0) = De^{-x^2}, \quad \frac{\partial}{\partial t}Q_z(x,0) = Gxe^{-x^2}, \quad D,\, G \in C_2 \tag{5.61}$$

Since (5.56) satisfies bicomplex Maxwell's equations and using (5.60) and (5.61) in (5.54) and (5.55), respectively. We get

$$F_y(x,t) = \left(\frac{A}{2} + \frac{B}{4c}\right) e^{-(x-ct)^2} + \left(\frac{A}{2} - \frac{B}{4c}\right) e^{-(x+ct)^2}$$

and

$$F_z(x,t) = i_2 \left(\frac{A}{2} + \frac{B}{4c}\right) e^{-(x-ct)^2} - i_2 \left(\frac{A}{2} - \frac{B}{4c}\right) e^{-(x+ct)^2}$$

Let $\frac{A}{2} + \frac{B}{4c} = \alpha \in C_2$ and $\frac{A}{2} - \frac{B}{4c} = \beta \in C_2$, then (5.56) becomes

$$\mathbf{F_x} = \left[\alpha e^{-(x-ct)^2} + \beta e^{-(x+ct)^2}\right] \hat{y} + i_2 \left[\alpha e^{-(x-ct)^2} - \beta e^{-(x+ct)^2}\right] \hat{z}. \tag{5.62}$$

Similarly, for the bicomplex Gaussian pulse wave travelling in y- direction with initial conditions

$$F_x(y,0) = Re^{-y^2}, \quad \frac{\partial}{\partial t}F_x(y,0) = Sye^{-y^2}, \quad R,\, S \in C_2 \tag{5.63}$$

$$Q_z(y,0) = Me^{-y^2}, \quad \frac{\partial}{\partial t}Q_z(y,0) = Nye^{-y^2}, \quad M,\, N \in C_2 \tag{5.64}$$

is

$$\mathbf{F_y} = \left[\delta e^{-(y-ct)^2} + \gamma e^{-(y+ct)^2}\right] \hat{x} - i_2 \left[\delta e^{-(y-ct)^2} - \gamma e^{-(y+ct)^2}\right] \hat{z}, \tag{5.65}$$

where $\delta = \frac{R}{2} + \frac{S}{4c}$ and $\gamma = \frac{R}{2} - \frac{S}{4c}$. And the bicomplex Gaussian pulse wave travelling in z- direction with initial conditions

$$F_x(z,0) = Le^{-z^2}, \quad \frac{\partial}{\partial t}F_x(z,0) = Gze^{-y^2}, \quad L,\, G \in C_2, \tag{5.66}$$

$$Q_y(z,0) = Ie^{-z^2}, \quad \frac{\partial}{\partial t}Q_y(z,0) = Jze^{-z^2}, \quad I,\, J \in C_2 \tag{5.67}$$

is

$$\mathbf{F_z} = \left[\phi e^{-(z-ct)^2} + \psi e^{-(z+ct)^2}\right]\hat{x} + i_2\left[\phi e^{-(z-ct)^2} - \psi e^{-(z+ct)^2}\right]\hat{y}, \tag{5.68}$$

where $\phi = \frac{L}{2} + \frac{G}{4c}$, $\psi = \frac{L}{2} - \frac{G}{4c}$ and α, β, ϕ, ψ, δ and γ are bicomplex constants. Now, by applying the superposition principle on equations (5.62), (5.65) and (5.68), we get vector field as

$$\mathbf{F} = \left[\delta e^{-(y-ct)^2} + \gamma e^{-(y+ct)^2} + \phi e^{-(z-ct)^2} + \psi e^{-(z+ct)^2}\right]\hat{x} + \left[\alpha e^{-(x-ct)^2} + \beta e^{-(x+ct)^2} + \right.$$
$$\left. i_2\phi e^{-(z-ct)^2} - i_2\psi e^{-(z+ct)^2}\right]\hat{y} + i_2\left[\alpha e^{-(x-ct)^2} - \beta e^{-(x+ct)^2} - \delta e^{-(y-ct)^2} + \gamma e^{-(y+ct)^2}\right]\hat{z}.$$

Therefore,

$$\begin{aligned}
\mathbf{F} \equiv \sqrt{\epsilon_0}\mathbf{E} + i_2\sqrt{\mu_0}\mathbf{H} = {} & \left[\delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2} + \phi_1 e^{-(z-ct)^2} + \psi_1 e^{-(z+ct)^2}\right]\hat{x} \\
& + \left[\alpha_1 e^{-(x-ct)^2} + \beta_1 e^{-(x+ct)^2} - \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{y} \\
& + \left[-\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \delta_2 e^{-(y-ct)^2} - \gamma_2 e^{-(y+ct)^2}\right]\hat{z} \\
& + i_2\left\{\left[\delta_2 e^{-(y-ct)^2} + \gamma_2 e^{-(y+ct)^2} + \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{x}\right. \\
& + \left[\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \phi_1 e^{-(z-ct)^2} - \psi_1 e^{-(z+ct)^2}\right]\hat{y} \\
& \left. + \left[\alpha_1 e^{-(x-ct)^2} - \beta_1 e^{-(x+ct)^2} - \delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2}\right]\hat{z}\right\}
\end{aligned} \tag{5.69}$$

where $\delta_1, \delta_2, \gamma_1, \gamma_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \phi_1, \phi_2, \psi_1, \psi_2 \in C_1$. By separating bi-real and bi-imaginary part of (5.69), we get

$$\begin{aligned}
\mathbf{E} = \frac{1}{\sqrt{\epsilon_0}} \Big\{ & \Big[\delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2} + \phi_1 e^{-(z-ct)^2} + \psi_1 e^{-(z+ct)^2}\Big] \hat{x} \\
& + \Big[\alpha_1 e^{-(x-ct)^2} + \beta_1 e^{-(x+ct)^2} - \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\Big] \hat{y} \\
& + \Big[-\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \delta_2 e^{-(y-ct)^2} - \gamma_2 e^{-(y+ct)^2}\Big] \hat{z} \Big\} \qquad (5.70)
\end{aligned}$$

$$\begin{aligned}
\mathbf{H} = \frac{1}{\sqrt{\mu_0}} \Big\{ & \Big[\delta_2 e^{-(y-ct)^2} + \gamma_2 e^{-(y+ct)^2} + \phi_2 e^{-(z-ct)^2} + \psi_2 e^{-(z+ct)^2}\Big] \hat{x} \\
& + \Big[\alpha_2 e^{-(x-ct)^2} + \beta_2 e^{-(x+ct)^2} + \phi_1 e^{-(z-ct)^2} - \psi_1 e^{-(z+ct)^2}\Big] \hat{y} \\
& + \Big[\alpha_1 e^{-(x-ct)^2} - \beta_1 e^{-(x+ct)^2} - \delta_1 e^{-(y-ct)^2} + \gamma_1 e^{-(y+ct)^2}\Big] \hat{z} \Big\}. \qquad (5.71)
\end{aligned}$$

Therefore, electric field **E** and and magnetic field **H** satisfies the Maxwell's equations (5.36-5.39). Therefore, electric and magnetic fields of Gaussian pulse wave propagating in positive direction are

$$\begin{aligned}
\mathbf{E} = \frac{1}{\sqrt{\epsilon_0}} \Big\{ & \Big[\delta_1 e^{-(y-ct)^2} + \phi_1 e^{-(z-ct)^2}\Big] \hat{x} + \Big[\alpha_1 e^{-(x-ct)^2} - \phi_2 e^{-(z-ct)^2}\Big] \hat{y} \\
& + \Big[-\alpha_2 e^{-(x-ct)^2} + \delta_2 e^{-(y-ct)^2}\Big] \hat{z} \Big\} \qquad (5.72)
\end{aligned}$$

$$\begin{aligned}
\mathbf{H} = \frac{1}{\sqrt{\mu_0}} \Big\{ & \Big[\delta_2 e^{-(y-ct)^2} + \phi_2 e^{-(z-ct)^2}\Big] \hat{x} + \Big[\alpha_2 e^{-(x-ct)^2} + \phi_1 e^{-(z-ct)^2}\Big] \hat{y} \\
& + \Big[\alpha_1 e^{-(x-ct)^2} - \delta_1 e^{-(y-ct)^2}\Big] \hat{z} \Big\}. \qquad (5.73)
\end{aligned}$$

Similarly, electric and magnetic fields of Gaussian pulse wave propagating in negative direction are

$$\begin{aligned}\mathbf{E} &= \frac{1}{\sqrt{\epsilon_0}}\left\{\left[\gamma_1 e^{-(y+ct)^2} + \psi_1 e^{-(z+ct)^2}\right]\hat{x} + \left[\beta_1 e^{-(x+ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{y}\right.\\ &\quad \left. + \left[\beta_2 e^{-(x+ct)^2} - \gamma_2 e^{-(y+ct)^2}\right]\hat{z}\right\} \qquad (5.74)\\ \mathbf{H} &= \frac{1}{\sqrt{\mu_0}}\left\{\left[\gamma_2 e^{-(y+ct)^2} + \psi_2 e^{-(z+ct)^2}\right]\hat{x} + \left[\beta_2 e^{-(x+ct)^2} - \psi_1 e^{-(z+ct)^2}\right]\hat{y}\right.\\ &\quad \left. + \left[-\beta_1 e^{-(x+ct)^2} + \gamma_1 e^{-(y+ct)^2}\right]\hat{z}\right\}. \qquad (5.75)\end{aligned}$$

Also, electric and magnetic fields of Gaussian pulse wave in equations (5.72-5.75) satisfies the Maxwell's equations (5.36-5.39). Bicomplex approach is advantageous than quaternionic approach due to the commutativity property of bicomplex numbers. The authors have discussed application of bicomplex Mellin transform in chapter 7 in RLC circuit also.

5.8 Conclusion

In this chapter, we define bicomplex Fourier-Stieltjes transform which is the generalization of complex Fourier-Stieltjes transform. We have discussed the positive-definiteness of bicomplex Fourier-Stieltjes transform through Bochner's theorem, which is advantageous than Bochner's theorem for quaternion Fourier-Stieltjes transform due to commutativity property of bicomplex numbers. In case of quaternions, three types of Bochner's theorem are required to be discussed. Also, we find the solution of bicomplex electromagnetic Maxwell's equations by defining bicomplex vector field.

5. BOCHNER THEOREM OF FOURIER-STIELTJES TRANSFORM AND APPLICATIONS OF FOURIER TRANSFORM IN BICOMPLEX SPACE

Bicomplex Fourier transform has many applications in image processing, signal processing, solving differential equations in quantum mechanics and other related problems in which large class of frequency domain required. The applications have been illustrated to find the solution of parabolic initial value problem in bicomplex algebra and algebraic reduction of complicated one-dimensional bicomplex-valued linear time-invariant systems.

The concept of bicomplex numbers has been applied for finding the solution of Maxwell's equations. We conclude that the bicomplex analysis has great advantage that both the vector fields (electric and magnetic) toghter as a single vector field in bicomplex space. This approach is also advantageous than quaternion approach due to the commutativity property of bicomplex numbers.

6

Hankel Transform in Bicomplex Space and Applications

The main finding of this chapter has been published as:

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Hankel transform in bicomplex sapce with applications, *Transylvanian Journal of Mathematics and Mechanics*, **8**(1), 1-14.

In this chapter, we investigate Hankel transform and its properties in bicomplex space which is generalization of complex Hankel transform which is given by *Koh* and *Zemanian* [85]. The application of Hankel transform in bicomplex space has been illustrated by solving bicomplex Cauchy problem. Bicomplex Hankel transform is highly applicable in solving partial differential equation of bicomplex-valued function, signal processing, optics and other related problems.

6.1 Introduction

In 1966, *Zemanian* [155] extended the classical Hankel transformation which introduced by Germen mathematician *Hermann Hankel* (1839-1873), to generalized functions of slow growth and in 1968, *Koh* and *Zemanian* [85] generalized the Hankel transform in complex variable. In 1985, *Singh* and *Pathak* [134] obtained various representations of finite Hankel transforms of generalized functions with inversion theorem, which gives a Fourier-Bessel series representation of generalized functions. In 1991, *Betancor* [18] proved characterization theorem for the elements of H'_μ space of generalized functions defined by *Zemanian*.

In 1994, *Koh* and *Li* [84] extended the complex Hankel transform defined by *Zemanian* in a large space of generalized functions. In 1997, *Tuan* [142] extended the range of the Hankel transform. In 2008, *Molina* and *Trione* extended n- dimensional Hankel transform to arbitrary values of $\mu \in \mathbb{R}^n$. In 2012, *Taywade* et al. [139], [140] derived fractional Hankel transform and its inversion theorem in the *Zemanian* space.

Complex Hankel transform of a complex-valued function is defined by *Koh* and *Zemanian* [85, Eq. 11] as

Definition 6.1. Let $\mu \in \mathbb{R}$ be restricted to $\mu \geq -\frac{1}{2}$. If $a > b > 0$, then $\mathfrak{J}_{\mu,b} \subset \mathfrak{J}_{\mu,a}$. This follows immediately from the inequality $\tau_k^{\mu,a}(\phi) \leq \tau_k^{\mu,b}(\phi)$ for $\phi \in \mathfrak{J}_{\mu,a}$. Hence, the restriction of $f \in \mathfrak{J}'_{\mu,a}$ to $\mathfrak{J}_{\mu,b}$ is in $\mathfrak{J}'_{\mu,b}$ and convergence in $\mathfrak{J}'_{\mu,a}$ implies convergence in $\mathfrak{J}'_{\mu,b}$. For every $f \in \mathfrak{J}'_{\mu,a}$, $\exists$ a unique real number σ such that

$$f \in \mathfrak{J}'_{\mu,b} \text{ if } b < \sigma$$
$$f \notin \mathfrak{J}'_{\mu,b} \text{ if } b > \sigma.$$

Therefore, $f \in \mathfrak{J}'_{\mu}(\sigma)$. The Hankel transform $F(s)$ of f of order μ defined as

$$F(s) = H_\mu\{f(x)\} = \left\langle f(x), \sqrt{xs}J_\mu(xs) \right\rangle, \tag{6.1}$$

where

$$s \in \Omega_f = \{s : |\text{Im}(s)| < \sigma,\ s \notin (-\infty, 0]\}, \tag{6.2}$$

where $\mathfrak{J}_{\mu,a}$ is a space of complex-valued testing function $\phi(x)$, which are defined and smooth on $0 < x < \infty$ and for which

$$\tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} \left| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right| < \infty,$$
$$k = 0, 1, 2, \cdots,\ D \equiv \frac{d}{dx}$$

and $\mathfrak{J}'_{\mu,a}$ denotes the dual space of $\mathfrak{J}_{\mu,a}$.

Analyticity of complex function $F(s)$ is given by the following theorem:

Theorem 6.1 (Koh and Zemanian [85])**.** *$F(s)$, as defined in (6.1), is an analytic function of s in the region Ω_f defined in (6.2), and*

$$D_s F(s) = \left\langle f(x), \frac{\partial}{\partial s}\sqrt{xs}J_\mu(xs) \right\rangle, \qquad s \in \Omega_f.$$

6.2 The Testing Spaces $\mathfrak{J}_{\mu,a}$ and $\mathfrak{J}_{\mu}(\sigma)$ and their Duals

In this section, we define the space of bicomplex-valued testing functions extending the space defined by *Koh* and *Zemanian* [85]. Let a denote a positive real number and μ any bicomplex number. Then for each pair of a and μ we define $\mathfrak{J}_{\mu,a}$ as the space of testing functions ϕ which are bicomplex-valued, which are defined and smooth on $0 < x < \infty$ and for which

$$\tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} \left\| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty,$$
$$k = 0, 1, 2, \cdots, \; D \equiv \frac{d}{dx},$$

where $\| \cdot \|$ is as defined in (1.16). The space of testing function satisfy the inclusion relation as discussed in the following theorem.

Theorem 6.2. *Let $a > b > 0$, then $\mathfrak{J}_{\mu,b} \subset \mathfrak{J}_{\mu,a}$.*

Proof. Let $\phi \in \mathfrak{J}_{\mu,b}$, then

$$\tau_k^{\mu,b}(\phi) = \sup_{0<x<\infty} \left\| e^{-bx} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty.$$

Since $a > b > 0$, therefore

$$\begin{aligned} & \sup_{0<x<\infty} \left\| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| \\ & \le \sup_{0<x<\infty} \left\| e^{-bx} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty \\ & \Rightarrow \quad \tau_k^{\mu,a}(\phi) \le \tau_k^{\mu,b}(\phi) < \infty. \end{aligned}$$

Therefore,

$$\tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} \left\| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty$$
$$\Rightarrow \phi \in \mathfrak{J}_{\mu,a}$$
$$\therefore \quad \mathfrak{J}_{\mu,b} \subset \mathfrak{J}_{\mu,a}.$$

□

$\mathfrak{J}_{\mu,a}$ is the linear space over the field of complex numbers as c_1, $c_2 \in C_1$ and ϕ, $\psi \in \mathfrak{J}_{\mu,a} \Rightarrow c_1\phi + c_2\psi \in \mathfrak{J}_{\mu,a}$. Let $\{a_\nu\}_{\nu=1}^{\infty}$ be a monotonically increasing sequence of positive numbers tending to σ. By the Theorem 6.2, if $a_1 > b_1 > 0$, then $\mathfrak{J}_{\mu_1,b_1} \subset \mathfrak{J}_{\mu_1,a_1}$.

This follows that $\{\mathfrak{J}_{\mu,a_\nu}\}_{\nu=1}^{\infty}$ is a sequence such that $\mathfrak{J}_{\mu,a_1} \subset \mathfrak{J}_{\mu,a_2} \subset \mathfrak{J}_{\mu,a_3} \cdots$. Let $\mathfrak{J}_\mu(\sigma) = \bigcup_{\nu=1}^{\infty} \mathfrak{J}_{\mu,a_\nu}$ denote the countable-union space generated by the above sequence of spaces. The dual of $\mathfrak{J}_{\mu,a}$ and $\mathfrak{J}_\mu(\sigma)$ are denoted by $\mathfrak{J}'_{\mu,a}$ and $\mathfrak{J}'_\mu(\sigma)$ respectively.

Let bicomplex-valued function $f(x)$ be locally integrable on $0 < x < \infty$ and such that $\int_0^\infty \left\| f(x) e^{ax} x^{\mu+\frac{1}{2}} \right\| dx < \infty$. Then $f(x)$ generates a regular generalized function in $\mathfrak{J}'_{\mu,a}$ defined by

$$\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)dx, \qquad \phi \in \mathfrak{J}_{\mu,a}.$$

6.3 Bicomplex Hankel Transform

Let $\mu_1 \in C_1$ be restricted to $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$. If $a_1 > b_1 > 0$, then $\mathfrak{J}_{\mu_1,b_1} \subset \mathfrak{J}_{\mu_1,a_1}$. This follows immediately from the inequality $\tau_k^{\mu_1,a_1}(\phi) \leq \tau_k^{\mu_1,b_1}(\phi)$ for $\phi \in \mathfrak{J}_{\mu_1,a_1}$. Hence, the restriction of $f_1 \in \mathfrak{J}'_{\mu_1,a_1}$ to $\mathfrak{J}_{\mu_1,b_1}$ is in $\mathfrak{J}'_{\mu_1,b_1}$, and convergence in $\mathfrak{J}'_{\mu_1,a_1}$

implies convergence in $\mathcal{J}'_{\mu_1,b_1}$. For every $f_1 \in \mathcal{J}'_{\mu_1,a_1}$, $\exists$ a unique real number σ_1 such that

$$f_1 \in \mathcal{J}'_{\mu_1,b_1} \text{ if } b_1 < \sigma_1$$

$$f_1 \notin \mathcal{J}'_{\mu_1,b_1} \text{ if } b_1 > \sigma_1.$$

Therefore, $f_1 \in \mathcal{J}'_{\mu_1}(\sigma_1)$. The Hankel transform $F(s_1)$ of f_1 of order μ_1 is defined as

$$F(s_1) = H_{\mu_1}\{f_1(x)\} = \langle f_1(x), \sqrt{xs_1} J_{\mu_1}(xs_1)\rangle , \tag{6.3}$$

where

$$s_1 \in \Omega_1 = \{s_1 : |\mathrm{Im}(s_1)| < \sigma_1,\ s_1 \notin (-\infty, 0]\} . \tag{6.4}$$

Similarly, for every $f_2 \in \mathcal{J}'_{\mu_2}(\sigma_2)$, another Hankel transform $F(s_2)$ of f_2 of order μ_2 is defined as

$$F(s_2) = H_{\mu_2}\{f_2(x)\} = \langle f_2(x), \sqrt{xs_2} J_{\mu_2}(xs_2)\rangle , \tag{6.5}$$

where

$$s_2 \in \Omega_2 = \{s_2 : |\mathrm{Im}(s_2)| < \sigma_2,\ s_2 \notin (-\infty, 0]\} . \tag{6.6}$$

Since $F(s_1)$ and $F(s_2)$ are analytic and convergent in Ω_1 and Ω_2 respectively for $\sigma = \min(\sigma_1, \sigma_2)$, taking the linear combination with idempotent components e_1 and e_2 as:

$$\begin{aligned}
&F(s_1)e_1 + F(s_2)e_2 \\
&= \langle f_1(x), \sqrt{xs_1} J_{\mu_1}(xs_1)\rangle\, e_1 + \langle f_2(x), \sqrt{xs_2} J_{\mu_2}(xs_2)\rangle\, e_2 \\
&= \left\langle f_1(x)e_1 + f_2(x)e_2, \sqrt{x(s_1e_1 + s_2e_2)} J_{\mu_1 e_1 + \mu_2 e_2}(x(s_1e_1 + s_2e_2))\right\rangle
\end{aligned}$$

$$= \left\langle f(x), \sqrt{x\xi} J_\mu(x\xi) \right\rangle = F(\xi) \tag{6.7}$$

(where $f(x) = f_1(x)e_1 + f_2(x)e_2,\ \mu = \mu_1 e_1 + \mu_2 e_2$ and $\xi = s_1 e_1 + s_2 e_2$).

Since $F(s_1)$ and $F(s_2)$ are complex functions which are convergent and analytic in Ω_1 and Ω_2 respectively, so a bicomplex function $F(\xi) = F(s_1)e_1 + F(s_2)e_2$ will be convergent and analytic in the region Ω defined as:

$$\Omega = \{\xi : \xi = s_1 e_1 + s_2 e_2;\ |\mathrm{Im}(s_1)| < \sigma,\ |\mathrm{Im}(s_2)| < \sigma \text{ and } s_1, s_2 \notin (-\infty, 0]\} \tag{6.8}$$

For better geometrical understanding of the region of convergence of bicomplex Hankel transform it will be advantageous to use the general four dimensional representation of bicomplex numbers. For this we take conventional representation of $s_1, s_2 \in C_1$ as

$$s_1 = x_1 + i_1 y_1, \quad s_2 = x_2 + i_1 y_2; \qquad x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

Then by (6.8), $|y_1| < \sigma,\ |y_2| < \sigma$ and if $y_1 = y_2 = 0$ then $x_1, x_2 \notin (-\infty, 0]$. Now,

$$\begin{aligned}
\xi &= s_1 e_1 + s_2 e_2 = (x_1 + i_1 y_1)e_1 + (x_2 + i_1 y_2)e_2 \\
&= (x_1 + i_1 y_1)\left(\frac{1 + i_1 i_2}{2}\right) + (x_2 + i_1 y_2)\left(\frac{1 + i_1 i_2}{2}\right) \\
&= \frac{x_1 + x_2}{2} + \left(\frac{y_1 + y_2}{2}\right) i_1 + \left(\frac{y_2 - y_1}{2}\right) i_2 + \left(\frac{x_1 - x_2}{2}\right) i_1 i_2. \\
&= a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \qquad \text{(say)}
\end{aligned}$$

On the basis of restriction on y_1 and y_2, three possible cases occur:

1. If $y_1 = y_2$ then $\frac{y_2 - y_1}{2} = 0$ and $\frac{y_1 + y_2}{2} = y_1 = y_2$. Hence $a_2 = 0$ and $|a_1| < \sigma$. In particular, if $y_1 = y_2 = 0$ and $x_1, x_2 \notin (-\infty, 0]$. Clearly if $a_1, a_2 = 0$ then $|a_3| < a_0$.

2. If $y_1 > y_2$ then $-\sigma < \frac{y_2-y_1}{2} < 0$, $\frac{y_1+y_2}{2} < \frac{y_2+\sigma}{2} < \frac{y_2+\sigma}{2} + \frac{\sigma-y_1}{2} = \sigma + \frac{y_2-y_1}{2}$ and $\frac{y_1+y_2}{2} > \frac{y_1-\sigma}{2} > \frac{y_1-\sigma}{2} - \frac{\sigma+y_2}{2} = -\sigma - \frac{y_2-y_1}{2}$. Hence $-\sigma < a_2 < 0$ and $-\sigma - a_2 < a_1 < \sigma + a_2$.

3. If $y_1 < y_2$ then $0 < \frac{y_2-y_1}{2} < \sigma$, $\frac{y_1+y_2}{2} < \frac{y_1+\sigma}{2} < \frac{y_1+\sigma}{2} + \frac{\sigma-y_1}{2} = \sigma - \frac{y_2-y_1}{2}$ and $\frac{y_1+y_2}{2} > \frac{y_2-\sigma}{2} > \frac{y_2-\sigma}{2} + \frac{-\sigma-y_1}{2} = -\sigma + \frac{y_2-y_1}{2}$. Hence $0 < a_2 < \sigma$ and $-\sigma + a_2 < a_1 < \sigma - a_2$.

Considering all of these results we conclude that the region of convergence of $F(\xi)$ as

$$\begin{aligned}\Omega = \{\xi : \xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 \in C_2,\ -\sigma + |a_2| < a_1 < \sigma - |a_2|, \\ 0 < |a_2| < \sigma \text{ and if } a_1 = a_2 = 0 \text{ then } |a_3| < a_0\} \qquad (6.9)\end{aligned}$$

or, equivalently

$$\begin{aligned}\Omega = \{\xi \in C_2 : -\sigma + |\mathrm{Im}_{i_2}(\xi)| < \mathrm{Im}_{i_1}(\xi) < \sigma - |\mathrm{Im}_{i_2}(\xi)|,\ 0 < |\mathrm{Im}_{i_2}(\xi)| < \sigma \\ \text{and if } \mathrm{Im}_{i_1}(\xi) = \mathrm{Im}_{i_2}(\xi) = 0 \text{ then } |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi)\}, \qquad (6.10)\end{aligned}$$

where $\mathrm{Re}(\xi)$, $\mathrm{Im}_{i_1}(\xi)$, $\mathrm{Im}_{i_2}(\xi)$ and $\mathrm{Im}_j(\xi)$ are real part, imaginary partof ξ w.r.t. i_1, i_2 and j, respectively.

Conversely, the existence condition of bicomplex Hankel transform $F(\xi)$ can be obtained in the following way:

If $\xi = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2 \in \Omega$,

$$-\sigma + |a_2| < a_1 < \sigma - |a_2|,\ 0 < |a_2| < \sigma \text{ and if } a_1 = a_2 = 0 \text{ then } |a_3| < a_0 \qquad (6.11)$$

Now, in terms of idempotent components, ξ can be expressed as

$$\begin{aligned}\xi &= a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \\ &= \left[(a_0 + a_3) + i_1(a_1 - a_2)\right] e_1 + \left[(a_0 - a_3) + i_1(a_1 + a_2)\right] e_2 \\ &= s_1 e_1 + s_2 e_2.\end{aligned}$$

Depending on the value of a_3, there arises three cases:

1. When $a_2 = 0$, from inequality (6.11) $-\sigma < a_1 < \sigma$ which trivially leads $-\sigma < a_1 + a_2 < \sigma$ and $-\sigma < a_1 - a_2 < \sigma$. If $a_1 = a_2 = 0$ then $a_0 - a_3 > 0$, $a_0 + a_3 > 0$ i.e. $a_0 - a_3, a_0 + a_3 \notin (-\infty, 0]$.

2. When $a_2 > 0$, from the inequality (6.11) $-\sigma + a_2 < a_1 < \sigma - a_2$, we get $-\sigma < a_1 - a_2$ and $a_1 + a_2 < \sigma$. This result can be interpreted as $-\sigma < a_1 - a_2 < a_1 + a_2 < \sigma$.

3. When $a_2 < 0$, from the inequality (6.11) $-\sigma - a_2 < a_1 < \sigma + a_2$, we get $-\sigma < a_1 + a_2$ and $a_1 - a_2 < \sigma$. This result can be interpreted as $-\sigma < a_1 + a_2 < a_1 - a_2 < \sigma$.

Hence the result.

Similarly, combining $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$ and $\mathrm{Re}(\mu_2) \geq -\frac{1}{2}$ using idempotent components, we get

$$\mathrm{Re}(\mu) \geq -\frac{1}{2} + \left|\mathrm{Im}_j(\mu)\right|,$$

where $\mu = \mu_1 e_1 + \mu_2 e_2 \in C_2$ and Im_j denotes imaginary part w.r.t. j. Let bicomplex-valued function $f(x)$ be locally integrable on $0 < x < \infty$ and such that

$$\int_0^\infty \left\| f(x) e^{ax} x^{\mu+\frac{1}{2}} \right\| dx < \infty, \qquad \forall\, a < \sigma. \tag{6.12}$$

Therefore, (6.7) can be written as

$$F(\xi) = \left\langle f(x), \sqrt{x\xi} J_\mu(x\xi) \right\rangle = \int_0^\infty f(x) \sqrt{x\xi} J_\mu(x\xi) dx$$

$$\therefore\ F(\xi) = \int_0^\infty f(x) \sqrt{x\xi} J_\mu(x\xi) dx, \qquad \forall\, \xi \in \Omega. \tag{6.13}$$

After all the above discussion, now we are in the position to define bicomplex Hankel transform as follows:

Definition 6.2. Let $\mu \in C_2$ be restricted to $\mathrm{Re}(\mu) \geq -\frac{1}{2} + |\mathrm{Im}_j(\mu)|$. If $a > b > 0$, then $\mathfrak{J}_{\mu,b} \subset \mathfrak{J}_{\mu,a}$. This follows immediately from the inequality $\tau_k^{\mu,a}(\phi) \leq \tau_k^{\mu,b}(\phi)$ for $\phi \in \mathfrak{J}_{\mu,a}$. Hence, the restriction of $f \in \mathfrak{J}'_{\mu,a}$ to $\mathfrak{J}_{\mu,b}$ is in $\mathfrak{J}'_{\mu,b}$, and convergence in $\mathfrak{J}'_{\mu,a}$ implies convergence in $\mathfrak{J}'_{\mu,b}$. For every bicomplex-valued function $f \in \mathfrak{J}'_{\mu,a}$, $\exists$ a unique real number σ such that $f \in \mathfrak{J}'_{\mu,b}$ if $b < \sigma$ and $f \notin \mathfrak{J}'_{\mu,b}$ if $b > \sigma$. Therefore, $f \in \mathfrak{J}'_\mu(\sigma)$. The μ^{th} order bicomplex Hankel transform $F(\xi)$ of f is defined as

$$F(\xi) = H_\mu\{f(x)\} = \left\langle f(x), \sqrt{x\xi} J_\mu(x\xi) \right\rangle, \qquad \forall\, \xi \in \Omega$$

where

$$\begin{aligned} \Omega = \{\xi \in C_2 : -\sigma + |\mathrm{Im}_{i_2}(\xi)| < \mathrm{Im}_{i_1}(\xi) < \sigma - |\mathrm{Im}_{i_2}(\xi)|,\ 0 < |\mathrm{Im}_{i_2}(\xi)| < \sigma \\ \text{and if } \mathrm{Im}_{i_1}(\xi) = \mathrm{Im}_{i_2}(\xi) = 0 \text{ then } |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi)\} \end{aligned} \tag{6.14}$$

where $\mathrm{Re}(\xi)$, $\mathrm{Im}_{i_1}(\xi)$, $\mathrm{Im}_{i_2}(\xi)$ and $\mathrm{Im}_j(\xi)$ are real part, imaginary parts of ξ w.r.t. i_1, i_2 and j, respectively. If bicomplex-valued function $f(x)$ is locally integrable on $0 < x < \infty$ and satisfies the condition

$$\int_0^\infty \left\| f(x) e^{ax} x^{\mu+\frac{1}{2}} \right\| dx < \infty, \qquad \forall\, a < \sigma. \tag{6.15}$$

Then bicomplex Hankel transform of $f(x)$ is defined as

$$F(\xi) = \int_0^\infty f(x)\sqrt{x\xi}J_\mu(x\xi)dx, \qquad \forall\, \xi \in \Omega.$$

6.4 Properties of Bicomplex Hankel Transform

In this section, some properties of bicomplex Hankel transform viz. linearity property, change of scale property, analyticity of $F(\xi)$, relationship with bicomplex Laplace transform and others have been discussed.

Theorem 6.3 (Linearity Property)**.** *Let $F(\xi)$ and $G(\xi)$ be the bicomplex Hankel transforms of order μ of bicomplex-valued functions $f(x)$ and $g(x)$ respectively, then*

$$H_\mu\{f(x) + g(x)\} = F(\xi) + G(\xi), \qquad \xi \in \Omega \tag{6.16}$$

where Ω defined in (6.14).

Proof. By applying the definition of bicomplex Hankel transform,

$$\begin{aligned}
&H_\mu\{f(x) + g(x)\} \\
=&\left\langle f(x) + g(x), \sqrt{x\xi}J_\mu(x\xi)\right\rangle \\
=&\left\langle f_1(x)e_1 + f_2(x)e_2 + g_1(x)e_1 + g_2(x)e_2, \sqrt{x(s_1e_1 + s_2e_2)}J_\mu(x(s_1e_1 + s_2e_2))\right\rangle \\
=&\left\langle f_1(x) + g_1(x), \sqrt{xs_1}J_\mu(xs_1)\right\rangle e_1 + \left\langle f_2(x) + g_2(x), \sqrt{xs_2}J_\mu(xs_2)\right\rangle e_2 \\
=&\left\langle f_1(x), \sqrt{xs_1}J_\mu(xs_1)\right\rangle e_1 + \left\langle g_1(x), \sqrt{xs_1}J_\mu(xs_1)\right\rangle e_1 \\
&+ \left\langle f_2(x), \sqrt{xs_2}J_\mu(xs_2)\right\rangle e_2 + \left\langle g_2(x), \sqrt{xs_2}J_\mu(xs_2)\right\rangle e_2 \\
=&\left\langle f_1(x)e_1 + f_2(x)e_2, \sqrt{x(s_1e_1 + s_2e_2)}J_\mu(x(s_1e_1 + s_2e_2))\right\rangle \\
&+ \left\langle g_1(x)e_1 + g_2(x)e_2, \sqrt{x(s_1e_1 + s_2e_2)}J_\mu(x(s_1e_1 + s_2e_2))\right\rangle \\
=&\left\langle f(x), \sqrt{x\xi}J_\mu(x\xi)\right\rangle + \left\langle g(x), \sqrt{x\xi}J_\mu(x\xi)\right\rangle \\
=&F(\xi) + G(\xi).
\end{aligned}$$

□

Theorem 6.4 (Change of Scale Property). *Let $F(\xi)$ be the bicomplex Hankel transform of order μ of bicomplex-valued function $f(x)$ and satisfy the condition (6.15). Then*

$$H_\mu\{f(ax)\} = \frac{1}{a}F\left(\frac{\xi}{a}\right), \quad a \neq 0 \in \mathbb{R},\ \xi \in \Omega \tag{6.17}$$

where Ω defined in (6.14).

Proof. By applying the definition of bicomplex Hankel transform

$$\begin{aligned} H_\mu\{f(ax)\} &= \int_0^\infty f(ax)\sqrt{x\xi}J_\mu(x\xi)dx \\ &\text{Put } ax = t \\ &= \frac{1}{a}\int_0^\infty f(t)\sqrt{t\frac{\xi}{a}}J_\mu\left(t\frac{\xi}{a}\right)dt \\ &= \frac{1}{a}F\left(\frac{\xi}{a}\right). \end{aligned}$$

□

Theorem 6.5. *Let $H_\mu\{f(x)\}$ be the bicomplex Hankel transform of order μ of bicomplex-valued locally integrable function $f(x)$ and satisfy the condition (6.15). Then*

$$H_\mu\left\{\frac{df}{dx}\right\} = \frac{\xi}{2}\left(H_{\mu+1}\{f(x)\} - H_{\mu-1}\{f(x)\}\right) - \frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}, \quad \xi \in \Omega \tag{6.18}$$

where Ω defined in (6.14).

Proof. If $F(\xi)$ be the bicomplex Hankel transform of order μ of $f(x)$ i.e.

$$H_\mu\{f(x)\} = \int_0^\infty f(x)\sqrt{x\xi}J_\mu(x\xi)dx,$$

then the bicomplex Hankel transform of $\frac{df}{dx}$ is

$$H_\mu\left\{\frac{df}{dx}\right\} = \int_0^\infty \frac{df}{dx}\sqrt{x\xi}J_\mu(x\xi)dx$$

on integrating by parts and assuming that $\sqrt{x}f(x) \to 0$ as $x \to 0,\ x \to \infty$, we get

$$\begin{aligned}
H_\mu\left\{\frac{df}{dx}\right\} =& -\int_0^\infty f(x)\frac{d}{dx}\left(\sqrt{x\xi}J_\mu(x\xi)\right)dx \\
=& -\int_0^\infty f(x)\left(\frac{\sqrt{\xi}}{2\sqrt{x}}J_\mu(x\xi)+\frac{\xi}{2}\sqrt{x\xi}J_{\mu-1}(x\xi)-\frac{\xi}{2}\sqrt{x\xi}J_{\mu+1}(x\xi)\right)dx \\
=& -\frac{1}{2}\int_0^\infty \frac{f(x)}{x}\sqrt{x\xi}J_\mu(x\xi)dx \\
& -\frac{\xi}{2}\int_0^\infty f(x)\sqrt{x\xi}J_{\mu-1}(x\xi)dx+\frac{\xi}{2}\int_0^\infty f(x)\sqrt{x\xi}J_{\mu+1}(x\xi)dx \\
=& -\frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}-\frac{\xi}{2}H_{\mu-1}(\xi)+\frac{\xi}{2}H_{\mu+1}(\xi)
\end{aligned}$$

$$\therefore\ H_\mu\left\{\frac{df}{dx}\right\} =\frac{\xi}{2}\left(H_{\mu+1}\left\{f(x)\right\}-H_{\mu-1}\left\{f(x)\right\}\right)-\frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}.$$

□

Theorem 6.6. *Let $H_\mu\{f(x)\}$ be the bicomplex Hankel transform of order μ of bicomplex-valued locally integrable function $f(x)$ and satisfy the condition (6.15). Then*

$$\begin{aligned}
H_\mu\left\{\frac{d^2f}{dx^2}\right\} =& \frac{\xi^2}{4}\left(H_{\mu+2}\{f(x)\}-2H_\mu\{f(x)\}\right) \\
& -\frac{\xi}{4}\left(H_{\mu-1}\left\{\frac{f}{x}\right\}-2H_{\mu+1}\left\{\frac{f}{x}\right\}\right)-\frac{1}{4}H_\mu\left\{\frac{f}{x^2}\right\},\quad \xi\in\Omega
\end{aligned}\tag{6.19}$$

where Ω defined in (6.14).

Proof. By Theorem 6.5 we have,

$$H_\mu\left\{\frac{df}{dx}\right\} = \frac{\xi}{2}\left(H_{\mu+1}\{f(x)\}-H_{\mu-1}\{f(x)\}\right)-\frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}\tag{6.20}$$

By inserting $\frac{df}{dx}$ in place of f in (6.20) we have

$$\begin{aligned}
H_\mu\left\{\frac{d^2f}{dx^2}\right\} =& \frac{\xi}{2}\left(H_{\mu+1}\left\{\frac{df}{dx}\right\}-H_{\mu-1}\left\{\frac{df}{dx}\right\}\right)-\frac{1}{2}H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\} \\
=& \frac{\xi}{2}\left(\xi H_{\mu+2}\{f(x)\}-2\xi H_\mu\{f(x)\}+\xi H_{\mu-2}\{f(x)\}-H_{\mu+1}\left\{\frac{f}{x}\right\}\right. \\
& \left.+H_{\mu-1}\left\{\frac{f}{x}\right\}\right)-\frac{1}{2}H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\}
\end{aligned}\tag{6.21}$$

Now,

$$H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\} = \int_0^\infty f(x)\frac{1}{x}\sqrt{x\xi}J_\mu(x\xi)dx$$

On integrating by parts and assuming that $\frac{f(x)}{\sqrt{x}} \to 0$ as $x \to 0,\ x \to \infty$ we have

$$\begin{aligned}
&H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\}\\
&= -\int_0^\infty f(x)\frac{d}{dx}\left(\sqrt{\frac{\xi}{x}}J_\mu(x\xi)\right)dx\\
&= -\int_0^\infty \sqrt{\xi}f(x)\left(-\frac{1}{2x^{3/2}}J_\mu(x\xi) + \frac{\xi}{\sqrt{x}}\left(\frac{1}{2}\left(J_{\mu-1}(x\xi) - J_{\mu+1}(x\xi)\right)\right)\right)dx\\
&=\frac{1}{2}\int_0^\infty \frac{f(x)}{x^2}\sqrt{x\xi}J_\mu(x\xi)dx - \frac{\xi}{2}\int_0^\infty \frac{f(x)}{x}\sqrt{x\xi}J_{\mu-1}(x\xi)dx\\
&\quad + \frac{\xi}{2}\int_0^\infty \frac{f(x)}{x}\sqrt{x\xi}J_{\mu+1}(x\xi)dx\\
&=\frac{1}{2}H_\mu\left\{\frac{f}{x^2}\right\} - \frac{\xi}{2}H_{\mu-1}\left\{\frac{f}{x}\right\} + \frac{\xi}{2}H_{\mu+1}\left\{\frac{f}{x}\right\}
\end{aligned}$$

By putting the value in (6.21) and after simplification we have

$$\begin{aligned}
H_\mu\left\{\frac{d^2f}{dx^2}\right\} =&\frac{\xi^2}{4}\left(H_{\mu+2}\{f(x)\} - 2H_\mu\{f(x)\}\right)\\
&- \frac{\xi}{4}\left(H_{\mu-1}\left\{\frac{f}{x}\right\} - 2H_{\mu+1}\left\{\frac{f}{x}\right\}\right) - \frac{1}{4}H_\mu\left\{\frac{f}{x^2}\right\}.
\end{aligned}$$

□

Theorem 6.7 (Relationship between bicomplex Hankel transform and bicomplex Laplace transform)**.** *Let $H_\mu\{f(x);\ \xi\}$ and $L\{f(x);\ \eta\}$ be bicomplex Hankel transform and bicomplex Laplace transform respectively. Then*

$$H_\mu\left\{e^{-\eta x}f(x);\ \xi\right\} = L\left\{\sqrt{x\xi}J_\mu(x\xi)f(x);\ \eta\right\}, \quad \xi \in \Omega \tag{6.22}$$

where $Re(P_1 : \eta) > 0,\ Re(P_2 : \eta) > 0$ and Ω defined in (6.14).

Proof. By the definition of bicomplex Hankel transform, we have

$$\begin{aligned} H_\mu\left\{e^{-\eta x} f(x);\ \xi\right\} &= \int_0^\infty e^{-\eta x} f(x)\sqrt{x\xi}J_\mu(x\xi)dx \\ &= \int_0^\infty e^{-\eta x}\left(f(x)\sqrt{x\xi}J_\mu(x\xi)\right)dx \\ &= L\left\{\sqrt{x\xi}J_\mu(x\xi)f(x);\ \eta\right\}. \end{aligned}$$

□

Theorem 6.8 (Analyticity of $F(\xi)$)**.** *$F(\xi)$, as defined in (6.7), is an analytic function of ξ in the region Ω defined in (6.14), and*

$$D_\xi F(\xi) = \left\langle f(x), \frac{\partial}{\partial \xi}\sqrt{x\xi}J_\mu(x\xi)\right\rangle, \qquad \xi = s_1e_1 + s_2e_2 \in \Omega. \tag{6.23}$$

Proof. Clearly, the bicomplex function $F(\xi)$ is analytic in Ω. By Theorem 6.1, we have

$$D_{s_1} F(s_1) = \left\langle f_1(x), \frac{\partial}{\partial s_1}\sqrt{xs_1}J_{\mu_1}(xs_1)\right\rangle, \qquad s_1 \in \Omega_1. \tag{6.24}$$

Similarly,

$$D_{s_2} F(s_2) = \left\langle f_2(x), \frac{\partial}{\partial s_2}\sqrt{xs_2}J_{\mu_2}(xs_2)\right\rangle, \qquad s_2 \in \Omega_2. \tag{6.25}$$

Since (6.24) and (6.25) are analytic in Ω_1 and Ω_2 respectively. Therefore, taking linear combination of (6.24) and (6.25) with e_1 and e_2 respectively.

$$\begin{aligned} &D_{s_1}F(s_1)e_1 + D_{s_2}F(s_2)e_2 \\ &= \left\langle f_1(x), \frac{\partial}{\partial s_1}\sqrt{xs_1}J_{\mu_1}(xs_1)\right\rangle e_1 + \left\langle f_2(x), \frac{\partial}{\partial s_2}\sqrt{xs_2}J_{\mu_2}(xs_2)\right\rangle e_2 \end{aligned}$$

or

$$\begin{aligned} &D_{(s_1e_1+s_2e_2)}F(s_1e_1+s_2e_2) \\ &= \left\langle f_1(x)e_1 + f_2(x)e_2, \frac{\partial}{\partial(s_1e_1+s_2e_2)}\sqrt{x(s_1e_1+s_2e_2)}J_{\mu_1e_1+\mu_2e_2}(x(s_1e_1+s_2e_2))\right\rangle \\ &\therefore\ \ D_\xi F(\xi) = \left\langle f(x), \frac{\partial}{\partial\xi}\sqrt{x\xi}J_\mu(x\xi)\right\rangle \end{aligned}$$

(where $\xi = s_1e_1 + s_2e_2$, $f(x) = f_1(x)e_1 + f_2(x)e_2$ and $\mu = \mu_1e_1 + \mu_2e_2$). □

6.5 Inversion of Bicomplex Hankel Transform

In this section, we discuss inversion formula for bicomplex Hankel transform. We require the following theorem by *Koh* and *Zemanian* [85, Theorem 4] for inverse Hankel transform to define its bicomplex form.

Theorem 6.9. *Let $F(s) = H_\mu\{f(x)\}$, $f \in \mathcal{J}'_\mu(\sigma)$ be complex Hankel transform of $f(x)$ where s is restricted to the real positive axis. Let $Re(\mu) \geq -\frac{1}{2}$. Then, in the sense of convergence in $\mathcal{D}'(I)$,*

$$f(x) = \lim_{r\to\infty} \int_0^r H(s)\sqrt{xs}J_\mu(xs)ds. \tag{6.26}$$

$\mathcal{D}(I)$ denotes the space of smooth functions that have compact support on I and $\mathcal{D}'(I)$ is dual of space $\mathcal{D}(I)$. Now, we shall find the inversion formula for bicomplex Hankel transform with the help of Theorem 6.9.

Theorem 6.10. *Let $F(\xi) = H_\mu\{f(x)\}$, $f \in \mathcal{J}'_\mu(\sigma)$ as in (6.7) where ξ restricted to the real positive axis and $f(x)$ and $F(\xi)$ are bicomplex-valued functions. Then for $Re(\mu) \geq -\frac{1}{2} + |Im_j(\mu)|$, in the sense of convergence in $\mathcal{D}'(I)$,*

$$f(x) = \lim_{r\to\infty} \int_0^r F(\xi)\sqrt{x\xi}J_\mu(x\xi)d\xi. \tag{6.27}$$

Proof. Let $F_1(\xi) = H_{\mu_1}\{f_1(x)\}$, $f_1 \in J'_{\mu_1}(\sigma)$, $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$ be complex-valued Hankel transform of complex-valued function $f(x)$, where ξ restricted to the real positive axis. Then, from Theorem 6.9 we have

$$f_1(x) = \lim_{r\to\infty} \int_0^r F_1(\xi)\sqrt{x\xi}J_{\mu_1}(x\xi)d\xi \tag{6.28}$$

Similarly, let $F_2(\xi) = H_{\mu_2}\{f_2(x)\}$, $f_2 \in J'_{\mu_2}(\sigma)$, $\mathrm{Re}(\mu_2) \geq -\frac{1}{2}$. Then

$$f_2(x) = \lim_{r\to\infty} \int_0^r F_2(\xi)\sqrt{x\xi}J_{\mu_2}(x\xi)d\xi \tag{6.29}$$

Taking the linear combination of (6.28) and (6.29) with e_1 and e_2 respectively, we have

$$
\begin{aligned}
f_1(x)e_1 + f_2(x)e_2 = \lim_{r\to\infty} \Bigg[\left(\int_0^r F_1(\xi)\sqrt{x\xi}J_{\mu_1}(x\xi)d\xi \right) e_1 \\
+ \left(\int_0^r F_2(\xi)\sqrt{x\xi}J_{\mu_2}(x\xi)d\xi \right) e_2 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \lim_{r\to\infty} \int_0^r \{F_1(\xi)e_1 + F_2(\xi)e_2\} \sqrt{x\xi} J_{(\mu_1 e_1 + \mu_2 e_2)}(x\xi)d\xi \\
f(x) &= \lim_{r\to\infty} \int_0^r F(\xi)\sqrt{x\xi}J_\mu(x\xi)d\xi
\end{aligned}
$$

where $f(x) = f_1(x)e_1 + f_2(x)e_2,\ \mu = \mu_1 e_1 + \mu_2 e_2$ and $F(\xi) = F_1(\xi)e_1 + F_2(\xi)e_2$. Combining the conditions $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$, $\mathrm{Re}(\mu_2) \geq -\frac{1}{2}$ using idempotent components, we get

$$
\mathrm{Re}(\mu) \geq -\frac{1}{2} + \left|\mathrm{Im}_j(\mu)\right|.
$$

This completes our proof. □

Following is the illustration to find bicomplex Hankel transform of a bicomplex-valued function.

Example 6.1. If $F(\xi) = H_\mu\{f(x);\ \xi\},\ \xi \in \Omega$ be the bicomplex Hankel transform, then show that

$$
H_\mu \left\{ x^{\mu-\frac{1}{2}} e^{-\eta x};\ \xi \right\} = \frac{2^\mu \Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{\pi}(\xi^2 + \eta^2)^{\mu+\frac{1}{2}}},
$$

where $\mathrm{Re}(P_1 : \eta) > 0$, $\mathrm{Re}(P_2 : \eta) > 0$, $\mathrm{Re}(\mu) > -\frac{1}{2} + |\mathrm{Im}_j(\mu)|$ and Ω defined in (6.14).

Solution. By the definition of bicomplex Hankel transform, we have

$$
\begin{aligned}
H_\mu \left\{ x^{\mu-\frac{1}{2}} e^{-\eta x};\ \xi \right\} &= \int_0^\infty x^{\mu-\frac{1}{2}} e^{-\eta x} \sqrt{x\xi} J_\mu(x\xi)dx \\
&= \sqrt{\xi} \int_0^\infty e^{-\eta x} x^\mu J_\mu(x\xi)dx
\end{aligned}
$$

Table 6.1: Bicomplex Hankel transform of some functions

S.No.	$f(x)$	Order μ	Bicomplex Hankel Transform $F(\xi)$	Region of Convergence
1.	$x^{-\frac{1}{2}}$	$\mu = 0$	$\xi^{-\frac{1}{2}}$	$\xi \in \Omega$
2.	$\begin{cases} x^{\mu+\frac{1}{2}}, & 0 < x < a \\ 0, & x > a \end{cases}$	$\text{Re}(\mu) \geq -\frac{1}{2} + \lvert\text{Im}_j(\mu)\rvert$	$\frac{a^{\mu+1}}{\sqrt{\xi}} J_{\mu+1}(a\xi)$	$\xi \in \Omega$
3.	$x^{\frac{1}{2}}(a^2 + x^2)^{-\frac{1}{2}}$	$\mu = 0$	$\xi^{-\frac{1}{2}} e^{-a\xi}$	$\xi \in \Omega$, $\text{Re}(P_1 : a) > 0$, $\text{Re}(P_2 : a) > 0$
4.	$x^{\frac{1}{2}}(x^2 + a^2)$	$\mu = 0$	$\frac{\xi^{\frac{1}{2}}}{a} e^{-a\xi}$	$\xi \in \Omega$, $\text{Re}(P_1 : a) > 0$, $\text{Re}(P_2 : a) > 0$
5.	$x^{-\frac{1}{2}} e^{-ax}$	$\mu = 0$	$\xi^{\frac{1}{2}}(\xi^2 + a^2)^{-\frac{1}{2}}$	$\xi \in \Omega$, $\text{Re}(P_1 : a) > 0$, $\text{Re}(P_2 : a) > 0$

$$=\sqrt{\xi} \int_0^\infty e^{-\eta x} x^\mu \sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\mu + r + 1)} \left(\frac{x\xi}{2}\right)^{\mu+2r} dx$$

$$=\sqrt{\xi} \sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\mu + r + 1)} \left(\frac{\xi}{2}\right)^{\mu+2r} \int_0^\infty e^{-\eta x} x^{2\mu+2r} dx$$

$$=\frac{\xi^{\mu+\frac{1}{2}}}{2^\mu} \sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\mu + r + 1)} \left(\frac{\xi}{2}\right)^{2r} \frac{\Gamma(2\mu + 2r + 1)}{\eta^{2\mu+2r+1}}$$

Applying Duplication formula for Gamma function *Rainville* [120, p. 24]

$$=\frac{\xi^{\mu+\frac{1}{2}}}{2^\mu \eta^{2\mu+1}} \sum_{r=0}^\infty \frac{(-1)^r}{r!} \left(\frac{\xi}{2\eta}\right)^{2r} \frac{\Gamma(\mu + r + \frac{1}{2})}{\sqrt{\pi}} 2^{2\mu+2r}$$

$$=\frac{2^\mu \Gamma(\mu + \frac{1}{2}) \xi^{\mu+\frac{1}{2}}}{\sqrt{\pi} \eta^{2\mu+1}} \sum_{r=0}^\infty \frac{(-1)^r}{r!} \left(\mu + \frac{1}{2}\right)_r \left(\frac{\xi^2}{a^2}\right)^r$$

$$=\frac{2^\mu \Gamma(\mu + \frac{1}{2}) \xi^{\mu+\frac{1}{2}}}{\sqrt{\pi} \eta^{2\mu+1}} \left(1 + \frac{\xi^2}{\eta^2}\right)^{-\mu-\frac{1}{2}}$$

$$=\frac{2^\mu \Gamma(\mu + \frac{1}{2}) \xi^{\mu+\frac{1}{2}}}{\sqrt{\pi} (\xi^2 + \eta^2)^{\mu+\frac{1}{2}}}.$$

6.6 An Operational Calculus

$y = x^{\frac{1}{2}} J_\mu(x)$ satisfies the following differential equation (see, *Watson* [145, p. 158])

$$x^2 \frac{d^2 y}{dx^2} + \left[x^2 - \left(\mu^2 - \frac{1}{4} \right) \right] y = 0. \tag{6.30}$$

Let $\Delta_\mu \equiv x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\frac{1}{2}-\mu}$, $D \equiv \frac{d}{dx}$, Then

$$\begin{aligned}
\Delta_\mu(f) =& x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\frac{1}{2}-\mu} f(x) \\
=& x^{-\mu-\frac{1}{2}} \left\{ (2\mu+1) x^{2\mu+1} D x^{-\mu-\frac{1}{2}} f(x) + x^{2\mu+1} D^2 \left[x^{-\mu-\frac{1}{2}} f(x) \right] \right\} \\
=& x^{-\mu-\frac{1}{2}} \left\{ (2\mu+1) x^{2\mu+1} \left(-\mu - \frac{1}{2} \right) x^{-\mu-\frac{3}{2}} f(x) + (2\mu+1) x^{2\mu+1} x^{-\mu-\frac{1}{2}} f'(x) \right. \\
&+ x^{2\mu+1} \left[\left(-\mu - \frac{1}{2} \right) \left(-\mu - \frac{3}{2} \right) x^{-\mu-\frac{5}{2}} f(x) + 2 \left(-\mu - \frac{1}{2} \right) x^{-\mu-\frac{3}{2}} f'(x) \right. \\
&\left. \left. + x^{-\mu-\frac{1}{2}} f''(x) \right] \right\} \\
=& f''(x) + x^{-2} \left(\frac{1}{4} - \mu^2 \right) f(x).
\end{aligned}$$

Therefore,

$$\Delta_\mu \equiv x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\frac{1}{2}-\mu} \equiv D^2 + x^{-2} \left(\frac{1}{4} - \mu^2 \right). \tag{6.31}$$

The operator satisfies (see, *Koh* and *Zemanian* [85, p. 951])

$$\Delta_\mu^k \left[\sqrt{x s_1} J_\mu(x s_1) \right] = (-1)^k s_1^{2k} J_\mu(x s_1), \quad s_1 \in C_1 \tag{6.32}$$

Similarly,

$$\Delta_\mu^k \left[\sqrt{x s_2} J_\mu(x s_2) \right] = (-1)^k s_2^{2k} J_\mu(x s_2), \quad s_2 \in C_1 \tag{6.33}$$

By taking linear combination of (6.32) and (6.33) w.r.t. e_1 and e_2 respectively, we have

$$\begin{aligned}\Delta_\mu^k\left[\sqrt{xs_1}J_\mu(xs_1)\right]e_1+\Delta_\mu^k\left[\sqrt{xs_2}J_\mu(xs_2)\right]e_2 &= \left((-1)^k s_1^{2k}\sqrt{xs_1}J_\mu(xs_1)\right)e_1\\ &\quad+\left((-1)^k s_2^{2k}\sqrt{xs_2}J_\mu(xs_2)\right)e_2\\ \Delta_\mu^k\left[\sqrt{x\xi}J_\mu(x\xi)\right] &= (-1)^k\xi^{2k}\sqrt{x\xi}J_\mu(x\xi),\ (\text{where } \xi=s_1e_1+s_2e_2\in C_2). \quad (6.34)\end{aligned}$$

Now we define the operator

$$\Delta_\mu : \mathcal{J}_\mu(\sigma)\to\mathcal{J}_\mu(\sigma)$$

such that

$$\langle \Delta_\mu f(x),\phi(x)\rangle = \langle f(x),\Delta_\mu\phi(x)\rangle, \quad \forall\ f\in\mathcal{J}'_\mu(\sigma),\ \phi\in\mathcal{J}_\mu(\sigma). \quad (6.35)$$

From (6.34) and (6.35), we get

$$\left\langle \Delta_\mu^k f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle = (-1)^k\xi^{2k}\left\langle f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle. \quad (6.36)$$

Therefore, bicomplex Hankel transform of $\Delta_\mu^k f(x)$ is

$$\begin{aligned}H_\mu\left\{\Delta_\mu^k f(x)\right\} &= \left\langle \Delta_\mu^k f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle\\ &= (-1)^k\xi^{2k}\left\langle f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle, \qquad [\text{Using (6.36)}]\\ &= (-1)^k\xi^{2k}H_\mu\{f(x)\}\\ &= (-1)^k\xi^{2k}F(\xi). \quad (6.37)\end{aligned}$$

6.7 Applications

Hankel transform is an important transform and has found many applications in the fields of science and engineering. In [86], *Kong* discussed the application of Hankel transform in the dipole antenna radiation in conductive medium. In [62], *Gupta* et al. discussed that computation of electromagnetic fields, for one-dimensional layered earth model, requires evaluation of Hankel transform of the electromagnetic kernel function. Bicomplex Hankel transform is advantageous than complex Hankel transform as the former can deal with the large class of frequency domain. In [99], *Malgonde* et al. used generalized Hankel transform to solve the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\mu + 1}{x}\frac{\partial u}{\partial x} - \frac{\nu^2 - \mu^2}{x^2}u = \lambda\frac{\partial u}{\partial t}, \tag{6.38}$$

with initial condition

$$u(x,t) \to f(x) \text{ in } D'(I), \text{ where } f \in \mathbb{H}'_{\mu,\nu}(\sigma) \text{ for some } \sigma > 0 \text{ as } t \to 0^+.$$

In the above equation notations and terminologies are as defined in *Zemanian* [156] and I denotes the open interval $(0, \infty)$.

In the similar manner, consider Cauchy problem for bicomplex-valued function $u : \mathbb{R} \to C_2$ is in the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{\mu^2 - \frac{1}{4}}{x^2}u = \lambda\frac{\partial u}{\partial t}, \tag{6.39}$$

with initial condition

$$u(x,t) \to f(x) \text{ in } \mathcal{D}'(I), \text{ where } f \in \mathfrak{J}'_{\mu,\nu}(\sigma) \text{ for some } \sigma > 0 \text{ as } t \to 0^+.$$

For the solution of (6.39), we shall make use of bicomplex Hankel transform. By taking bicomplex Hankel transform of (6.39) and by using (6.37) we have

$$\begin{aligned} & -\xi^2 U(\xi,t) = \lambda \frac{\partial}{\partial t} U(\xi,t), \quad [\text{where } U(\xi,t) = H_\mu \{u(x,t)\}] \\ \Rightarrow \quad & \frac{d}{dt} U(\xi,t) + \frac{\xi^2}{\lambda} U(\xi,t) = 0. \end{aligned}$$

This is first order differential equation w.r.t. variable t. Solving and making use of initial condition, we get

$$U(\xi,t) = F(\xi) e^{-\frac{\xi^2}{\lambda}t}, \quad [\text{where } F(\xi) = H_\mu \{f(x)\}]. \tag{6.40}$$

Further, taking the bicomplex inverse Hankel transform of (6.40), we get

$$u(x,t) = \lim_{r \to \infty} \int_0^r F(\xi) e^{-\frac{\xi^2}{\lambda}t} \sqrt{x\xi} J_\mu(x\xi) d\xi \tag{6.41}$$

which is the solution of equation (6.39).

6.8 Conclusion

In this chapter, we define bicomplex Hankel transform and its properties which is a natural extension of the complex Hankel transform *Koh* and *Zemanian* [85]. It is applicable in signal processing, solving partial differential equation of bicomplex-valued functions, optics, electromagnetic field theory and other related problems due to large class. Bicomplex numbers being basically four dimensional hyper-complex numbers, provide large class of frequency domain.

7

Mellin Transform in Bicomplex Space, Fractional Calculus and Applications

The main findings of this chapter have been published as:

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Mellin transform in bicomplex Space and its application, *Studia Universitatis Babes-Bolyai Mathematica*, (in press).

2. Agarwal R., Goswami M.P. and Agarwal R.P. (2017), A study of Mellin transform of fractional operators in bicomplex space and application, *Journal of Fractional Calculus and Applications*, **8**(2), 211-226.

In this chapter, we extend the Mellin transform of complex-valued function in complex variable to Mellin transform of bicomplex-valued function in bicomplex variable. Also, we obtain bicomplex Mellin transform of Riemann-Liouville integral, differential and Caputo fractional derivative of order $\alpha \geq 0$ of certain functions and some of their properties.

7.1 Introduction

Hjalmar Mellin (1854-1933, see, e.g. [118]) gave his name to the Mellin transform that associates to a complex-valued function $f(t)$ defined over the interval $(0, \infty)$, the function of complex variable s, as

$$\bar{f}(s) = \int_0^\infty t^{s-1} f(t) dt.$$

The change of variables $t = e^{-x}$ shows that the Mellin transform is closely related to the Laplace transform. General properties of the Mellin transform are usually treated in detail in books on integral transforms, like those of *Poularikas* [118] and *Davies* [33]. In 1959, *Francis* [53] discussed the application of complex Mellin transform to networks with time-varying parameters. In 1992, *Pilipovic* and *Stojanovic* [116] discussed the modified Mellin transform, its inverse, convolution and properties over the investigated space. Also, applied modified Mellin convolution in solving an integro-differential equation. In 1995, *Flajolet* et al. [47] used Mellin transform for the asymptotic analysis of harmonic sums.

In 2007, *Fitouhi* and *Bettaibi* [46] discussed the applications of *q*- Mellin transform in quantum calculus and derived the asymptotic expansion of some functions. In 2009, *Erfani* and *Bayan* [44] studied the application of two-dimensional Laplace,Hankel and Mellin transforms in linear time-varying networks systems. In 2015, *Patil* and *Patil* [114] discussed some properties of Mellin transform to obtained electrical analogous with the use of force-voltage analogy of the given mechanical system. In [3], *Alotta* et al. proposed a wavelet transform of an arbitrary function $f(t)$ which can quickly computed by Mellin transform expression. In 2016, *Bardaro* et al. [13] established the Paley-Wiener theorem of Fourier analysis in the frame of Mellin transform.

For defining bicomplex Mellin transform, we shall need the definitions of bicomplex gamma and beta functions. In [58], *Goyal* et al. defined bicomplex gamma and beta function and discussed its various properties.

Definition 7.1. (Bicomplex Gamma function [58, p. 137]). Let $\xi \in C_2$, $p = p_1e_1 + p_2e_2 \in C_2$, p_1, $p_2 \in (0, \infty)$, then

$$\Gamma(\xi) = \int_H e^{-p} p^{\xi-1} dp \tag{7.1}$$

where $H = (\gamma_1, \gamma_2)$, $\gamma_1 \equiv \gamma_1(p_1)$ and $\gamma_2 \equiv \gamma_2(p_2)$. $\Gamma(\xi)$ exists provided the integral exists.

Definition 7.2. (Bicomplex Beta function [58, p. 137]). Let $\xi = u_1 + i_2u_2$, $\eta = v_1 + i_2v_2 \in C_2$, $p = p_1e_1 + p_2e_2 \in C_2$, p_1, $p_2 \in [0, 1]$ with $\text{Re}(u_1) > |\text{Im}(u_2)|$ and $\text{Re}(v_1) > |\text{Im}(v_2)|$ then

$$B(\xi, \eta) = \int_H p^{\xi-1}(1-p)^{\eta-1} dp \tag{7.2}$$

where $H = (\gamma_1, \gamma_2)$, $\gamma_1 \equiv \gamma_1(p_1)$ and $\gamma_2 \equiv \gamma_2(p_2)$.

Mellin convolution of two bicomplex-valued functions can be defined as:

$$f(t) * g(t) = \int_0^\infty \frac{1}{x} f(x) g\left(\frac{t}{x}\right) dx \tag{7.3}$$

$$f(t) \circ g(t) = \int_0^\infty f(xt) g(x) dx. \tag{7.4}$$

7.1.1 Basics of Fractional Calculus

Fractional calculus is a generalization of the classical calculus and it has been used in various fields of science and engineering. The fractional calculus is a powerful mathematical tool for the physical description systems that have long-term memory and long term spatial interactions (see, for details, *Podlubny* [117], *Miller* and *Ross* [106], *Hilfer* [66], *Kilbas* et al. [77] and *Samko* et al. [129]).

In [82], *Klimek* and *Dziembowski* applied Mellin transform to find the solution of fractional differential equations of complex-valued function. In [48], *Francisco* et al. proposed a fractional differential equation for the electrical RC and LC circuit in terms of the fractional time derivative of the Caputo type. In [93], *Liang* and *Liu* deduced a fractional-order model based on skin effect for frequency dependent transmission line model. In their paper voltage and currents at any location in transmission line can be calculated by the proposed fractional partial differential equations.

In this section, we give the definitions of Riemann-Liouville and Caputo fractional operators along the main properties.

Definition 7.3. (see, e.g. Miller and Ross [106, p. 45]). Let $\alpha > 0$ and f be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$.

Then for $t > 0$

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx $$

the Riemann-Liouville fractional integral of f of order α.

Definition 7.4. (*see, e.g. Miller and Ross* [106, p. 82]). Let f be a function of class $\mathbf{C}$ and let $\alpha > 0$. Let n be the smallest integer that exceeds α. Then the fractional derivative of f of order α is defined as

$$ {}_0D_t^{\alpha} f(t) = {}_0D_t^{n} \left[{}_0D_t^{-\beta} f(t) \right], \quad \alpha > 0,\ t > 0 $$

where $\beta = n - \alpha > 0$.

Some properties of Riemann-Liouville fractional operator are as follows:

Theorem 7.1. *(see, e.g. Miller and Ross [106, Eq. 5.25, 6.1]). Let α, β are two positive real number, then*

(a) ${}_0D_t^{\alpha} \left({}_0D_t^{-\beta} f(t) \right) = {}_0D_t^{\alpha-\beta} f(t)$,

(b) ${}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t) = {}_0D_t^{-\alpha-\beta} f(t)$,

(c) ${}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t) = {}_0D_t^{-\beta}\, {}_0D_t^{-\alpha} f(t)$.

For Riemann-Liouville operator ${}_0D_t^{\alpha}$ and $\alpha, n > 0$ the fractional derivative of the power function t^n (see, e.g. *Miller* and *Ross* [106, p. 36]) is given by

$$ {}_0D_t^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}. \tag{7.5} $$

Definition 7.5. (Caputo [24] and see, e.g. Podlubny [117, Eq. (2.138)]). The Caputo fractional derivative of f for $\alpha > 0$ is defined as

$$ {}_0^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx, \quad n-1 < \alpha \leq n \tag{7.6} $$

$$ {}_0^CD_t^{\alpha} f(t) = {}_0D_t^{-(n-\alpha)} g(t), \quad g(t) = f^{(n)}(t),\ n-1 < \alpha \leq n \tag{7.7} $$

provided the integral exists.

Some properties of Caputo fractional derivative are as follows:

Theorem 7.2. *(see, e.g. Kilbas et al. [77, p. 95, 96]). If* $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ *and function* f *s.t. integral (7.6) exist, then*

(a) ${}_0^C D_t^{\alpha}\left({}_0D_t^{-\alpha} f(t)\right) = f(t)$,

(b) ${}_0D_t^{-\alpha}\left({}_0^C D_t^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\left(\frac{x^k}{k!}\right)$.

7.2 Bicomplex Mellin Transform

Let $f_1(t)$ be a complex-valued continuous function on the interval $(0, \infty)$ with $f_1(t) = O\left(t^{-\alpha_1}\right)$ as $t \to 0^+$ and $f_1(t) = O\left(t^{-\beta_1}\right)$ as $t \to \infty$, where $\alpha_1 < \beta_1$. Then Mellin transform of $f_1(t)$ is

$$\mathfrak{M}\left[f_1(t); s_1\right] = \int_0^{\infty} t^{s_1-1} f_1(t) dt = \bar{f}_1(s_1), \quad s_1 \in C_1 \tag{7.8}$$

where $\bar{f}_1(s_1)$ is analytic and convergent in the vertical strip

$$\Omega_1 = \left\{s_1 \in C_1 : \alpha_1 < \mathrm{Re}(s_1) < \beta_1\right\}. \tag{7.9}$$

Similarly, $f_2(t)$ be a complex-valued continuous function on the interval $(0, \infty)$ with $f_2(t) = O\left(t^{-\alpha_2}\right)$ as $t \to 0^+$ and $f_2(t) = O\left(t^{-\beta_2}\right)$ as $t \to \infty$, where $\alpha_2 < \beta_2$. Then Mellin transform of $f_2(t)$ is

$$\mathfrak{M}\left[f_2(t); s_2\right] = \int_0^{\infty} t^{s_2-1} f_2(t) dt = \bar{f}_2(s_2), \quad s_2 \in C_1 \tag{7.10}$$

where $\bar{f}_2(s_2)$ is analytic and convergent in the vertical strip

$$\Omega_2 = \left\{s_2 \in C_1 : \alpha_1 < \mathrm{Re}(s_2) < \beta_1\right\}. \tag{7.11}$$

Since $\bar{f}_1(s_1)$ and $\bar{f}_2(s_2)$ are complex functions which are analytic and convergent in the strips Ω_1 and Ω_2 respectively. Now, we take linear combination of $\bar{f}_1(s_1)$ and $\bar{f}_2(s_2)$ w.r.t. e_1 and e_2 respectively, denote by $\bar{f}(\xi)$, $\xi = s_1e_1 + s_2e_2$

$$\begin{aligned}
\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2 &= \left(\int_0^\infty t^{s_1-1} f_1(t)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1} f_2(t)dt\right) e_2 \\
\bar{f}(\xi) &= \int_0^\infty t^{(s_1e_1+s_2e_2)-1} \left(f_1(t)e_1 + f_2(t)e_2\right) dt \\
\bar{f}(\xi) &= \int_0^\infty t^{\xi-1} f(t)dt
\end{aligned} \tag{7.12}$$

where $\xi = s_1e_1 + s_2e_2$ and $\bar{f}(\xi)$ is analytic and convergent in the strip

$$\begin{aligned}
\Omega = \{\xi : \xi = s_1e_1 + s_2e_2 \in C_2; \alpha < \mathrm{Re}(P_1 : \xi) < \beta; \alpha < \mathrm{Re}(P_2 : \xi) < \beta; \\
\alpha = \max(\alpha_1, \alpha_2) \text{ and } \beta = \min(\beta_1, \beta_2)\}.
\end{aligned} \tag{7.13}$$

$\because\ \alpha < \mathrm{Re}(s_1) = x_1 < \beta$ and $\alpha < \mathrm{Re}(s_2) = x_2 < \beta$, we have

$$\begin{aligned}
\xi &= (x_1 + i_1y_1)e_1 + (x_2 + i_1y_2)e_2 \\
&= (x_1 + i_1y_1)\left(\frac{1 + i_1i_2}{2}\right) + (x_2 + i_1y_2)\left(\frac{1 - i_1i_2}{2}\right) \\
&= \frac{x_1 + x_2}{2} + \left(\frac{y_1 + y_2}{2}\right) i_1 + \left(\frac{y_2 - y_1}{2}\right) i_2 + \left(\frac{x_1 - x_2}{2}\right) i_1i_2.
\end{aligned}$$

Now, there are three possible cases:

1. If $x_1 = x_2 = a_0$ (say) then $\frac{x_1-x_2}{2} = 0$ and $\frac{x_1+x_2}{2} = a_0$.

   Hence, if $\xi = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2$, then $\alpha < a_0 < \beta$ and $a_3 = 0$.

2. If $x_1 > x_2$, then $\frac{x_1-x_2}{2} > 0$,

   $\frac{x_1+x_2}{2} < \frac{\beta+x_2}{2} < \frac{\beta+x_2}{2} + \frac{\beta-x_1}{2} = \beta - \frac{x_1-x_2}{2}$

   and $\frac{x_1+x_2}{2} > \frac{\alpha+x_1}{2} > \frac{\alpha+x_1}{2} + \frac{\alpha-x_2}{2} = \alpha + \frac{x_1-x_2}{2}$.

   Thus, $\alpha + a_3 < a_0 < \beta - a_3$ and $a_3 > 0$.

3. If $x_1 < x_2$, then $\frac{x_1-x_2}{2} < 0$,

   $\frac{x_1+x_2}{2} < \frac{\beta+x_1}{2} < \frac{\beta+x_1}{2} + \frac{\beta-x_2}{2} = \beta + \frac{x_1-x_2}{2}$

   and $\frac{x_1+x_2}{2} > \frac{\alpha+x_2}{2} > \frac{\alpha+x_2}{2} + \frac{\alpha-x_1}{2} = \alpha - \frac{x_1-x_2}{2}$.

   Thus, $\alpha - a_3 < a_0 < \beta + a_3$ and $a_3 < 0$.

These three conditions can be written in the following set builder form

$$\Omega_1 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ \alpha < a_0 < \beta \text{ and } a_3 = 0\},$$

$$\Omega_2 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ \alpha + a_3 < a_0 < \beta - a_3 \text{ and } a_3 > 0\},$$

$$\Omega_3 = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \ : \ \alpha - a_3 < a_0 < \beta + a_3 \text{ and } a_3 < 0\}.$$

Thus, $\alpha < \mathrm{Re}(P_1 : \xi) < \beta$ and $\alpha < \mathrm{Re}(P_2 : \xi) < \beta$ implies $\xi \in \Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$ which can be defined as:

$$\Omega = \{\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in C_2 : \alpha + |a_3| < a_0 < \beta - |a_3|\} \qquad (7.14)$$

or equivalently,

$$\Omega = \{\xi \in C_2 : \alpha + |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|\}$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part w.r.t. j unit of a bicomplex number.

Conversely, the existence condition of bicomplex Mellin transform $\bar{f}(\xi)$ can be obtained in the following way:

If $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in \Omega$,

$$\alpha + |a_3| < a_0 < \beta - |a_3|. \qquad (7.15)$$

Now, in terms of idempotent components, ξ can be expressed as

$$\begin{aligned}\xi &= a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \\ &= \left[(a_0 + a_3) + i_1(a_1 - a_2)\right] e_1 + \left[(a_0 - a_3) + i_1(a_1 + a_2)\right] e_2 \\ &= s_1 e_1 + s_2 e_2.\end{aligned}$$

Depending on the value of a_3, there arises three cases:

1. When $a_3 = 0$ and $\alpha < a_0 < \beta$ which trivially leads $\alpha < a_0 + a_3 < \beta$ and $\alpha < a_0 - a_3 < \beta$.

2. When $a_3 > 0$, from the inequality (7.15) $\alpha + a_3 < a_0 < \beta - a_3$, we get $\alpha < a_0 - a_3$ and $a_0 + a_3 < \beta$. This result can be interpreted as $\alpha < a_0 - a_3 < a_0 + a_3 < \beta$.

3. When $a_3 < 0$, from the inequality (7.15) $\alpha - a_3 < a_0 < \beta + a_3$, we get $\alpha < a_0 + a_3$ and $a_0 - a_3 < \beta$. This result can be interpreted as $\alpha < a_0 + a_3 < a_0 - a_3 < \beta$.

Hence the result.

Now, we define the Mellin transform in the bicomplex space as follows:

Definition 7.6. Let $f(t)$ be a bicomplex-valued continuous function on the interval $(0, \infty)$ with $f(t) = O\left(t^{-\alpha}\right)$ as $t \to 0^+$ and $f(t) = O\left(t^{-\beta}\right)$ as $t \to \infty$, where $\alpha < \beta$. Then bicomplex Mellin transform of $f(t)$ defined as

$$\mathfrak{M}[f(t); \xi] = \int_0^\infty t^{\xi - 1} f(t) dt = \bar{f}(\xi), \quad \xi \in \Omega$$

where $\bar{f}(\xi)$ is analytic and convergent in Ω defined as

$$\Omega = \{\xi \in C_2 : \alpha + |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|\} \tag{7.16}$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part w.r.t. j unit of a bicomplex number.

Following is the illustration to explain the process of finding the bicomplex Mellin transform of a bicomplex valued function.

Example 7.1. Let $f(t) = t^a U(t - t_0)$, where $U(t - t_0)$ is unit-step function, then

$$\mathfrak{M}[f(t);\xi] = -\frac{t_0^{\xi+a}}{\xi + a}, \quad \mathrm{Re}(\xi + a) < -\left|\mathrm{Im}_j(\xi + a)\right|.$$

Solution. By applying the definition of bicomplex Mellin transform

$$\begin{aligned}\mathfrak{M}[f(t);\xi] &= \int_0^\infty t^{\xi-1} t^a U(t - t_0) dt \\ &= \int_{t_0}^\infty t^{\xi+a-1} dt \\ &= -\frac{t_0^{\xi+a}}{\xi + a}.\end{aligned}$$

Example 7.2. Let $g(t) = \begin{cases} (1-t)^{\alpha-1}, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$ Then

$$\mathfrak{M}[g(t);\xi] = \frac{\Gamma(\alpha)\Gamma(\xi)}{\Gamma(\xi + \alpha)}, \ \mathrm{Re}(\xi_1) > |\mathrm{Im}(\xi_2)| \ \text{and} \ \mathrm{Re}(\alpha_1) > |\mathrm{Im}(\alpha_2)| \tag{7.17}$$

where $\xi = \xi_1 + i_2\xi_2$ and $\alpha = \alpha_1 + i_2\alpha_2$.

Solution. By applying the definition of bicomplex Mellin transform

$$\begin{aligned}\mathfrak{M}[g(t);\xi] &= \int_0^\infty t^{\xi-1} g(t) dt \\ &= \int_0^1 t^{\xi-1}(1-t)^{\alpha-1} dt\end{aligned}$$

where $\xi = \xi_1 + i_2\xi_2$ and $\alpha = \alpha_1 + i_2\alpha_2$ with $\mathrm{Re}(\xi_1) > |\mathrm{Im}(\xi_2)|$ and $\mathrm{Re}(\alpha_1) > |\mathrm{Im}(\alpha_2)|$, then

$$\mathfrak{M}[g(t);\xi] = B(\xi, \alpha) = \frac{\Gamma(\alpha)\Gamma(\xi)}{\Gamma(\xi + \alpha)}. \tag{7.18}$$

Table 7.1: Bicomplex Mellin transform of some basic functions

S.No.	$f(t)$	Bicomplex Hankel Transform $F(\xi)$	Region of Convergence
1.	$(1+t)^{-a}$	$\frac{\Gamma(\xi)\Gamma(a-\xi)}{\Gamma(a)}$	$\lvert \mathrm{Im}_j(a-\xi)\rvert < \mathrm{Re}(a-\xi)$
2.	$(1+t)^{-1}$	$\frac{\pi}{\sin(\pi\xi)}$	$\lvert \mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < 1 - \lvert \mathrm{Im}_j(\xi)\rvert$
3.	$e^{nt},\ n>0$	$\frac{\Gamma(\xi)}{n^{\xi}}$	$\mathrm{Re}(\xi) > \lvert \mathrm{Im}_j(\xi)\rvert$
4.	$\sin(at),\ a>0$	$\frac{\Gamma(\xi)\sin\left(\frac{\pi\xi}{2}\right)}{a^{\xi}}$	$-1+\lvert \mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < 1-\lvert \mathrm{Im}_j(\xi)\rvert$
5.	$\cos(at),\ a>0$	$\frac{\Gamma(\xi)\cos\left(\frac{\pi\xi}{2}\right)}{a^{\xi}}$	$\lvert \mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < 1 - \lvert \mathrm{Im}_j(\xi)\rvert$
6.	$\log(1+t)$	$\frac{\pi}{\xi\sin(\pi\xi)}$	$-1 + \lvert \mathrm{Im}_j(\xi)\rvert < \mathrm{Re}(\xi) < -\lvert \mathrm{Im}_j(\xi)\rvert$
7.	t^{-a}	$-\frac{1}{\xi-a}$	$\mathrm{Re}(\xi-a) < -\lvert \mathrm{Im}_j(\xi-a)\rvert$

7.3 Properties of Bicomplex Mellin Transform

In this section, we discuss the basic properties of bicomplex Mellin transform viz. linearity property, change of scale property, shifting property, Mellin transform of derivatives and operators, relation with bicomplex Laplace transform and some other properties. Also, we discuss bicomplex Mellin transform of convolution of functions, Riemann-Liouville fractional integral and Caputo derivative of order $\alpha \geq 0$ of certain functions and some of their properties.

Theorem 7.3. *Let $f(t)$ and $g(t)$ are bicomplex-valued functions with $f(t) = O\left(t^{-\alpha_1}\right)$, $g(t) = O\left(t^{-\alpha_2}\right)$ as $t \to 0^+$ and $f(t) = O\left(t^{-\beta_1}\right)$, $g(t) = O\left(t^{-\beta_2}\right)$ as $t \to \infty$, with $\max(\alpha_1, \alpha_2) + |Im_j(\xi)| < Re(\xi) < \min(\beta_1, \beta_2) - |Im_j(\xi)|$, then*

$$\mathfrak{M}[c_1 f(t) + c_2 g(t); \xi] = c_1 \mathfrak{M}[f(t); \xi] + c_2 \mathfrak{M}[g(t); \xi]$$

where c_1 and c_2 are arbitrary constants.

Proof. By applying the definition of bicomplex Mellin transform

$$\begin{aligned}\mathfrak{M}[c_1 f(t) + c_2 g(t); \xi] &= \int_0^\infty t^{\xi-1}[c_1 f(t) + c_2 g(t)]dt \\ &= c_1 \int_0^\infty t^{\xi-1} f(t)dt + c_2 \int_0^\infty t^{\xi-1} g(t)dt \\ &= c_1 \mathfrak{M}[f(t); \xi] + c_2 \mathfrak{M}[g(t); \xi].\end{aligned}$$

□

Theorem 7.4 (Change of scale property)**.** *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$\mathfrak{M}[f(at); \xi] = a^{-\xi} \bar{f}(\xi), \quad \xi \in \Omega,\ a > 0 \tag{7.19}$$

where Ω is defined in (7.16).

Proof. By applying the definition of bicomplex Mellin transform

$$\begin{aligned}\mathfrak{M}[f(at); \xi] &= \int_0^\infty t^{\xi-1} f(at)dt, \qquad\qquad [\text{where } \xi = s_1 e_1 + s_2 e_2] \\ &= \left(\int_0^\infty t^{s_1-1} f_1(at)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1} f_2(at)dt\right) e_2\end{aligned}$$

Put $at = u$, to obtain

$$\begin{aligned}&= \frac{1}{a^{s_1}} \left(\int_0^\infty t^{s_1-1} f_1(u)dt\right) e_1 + \frac{1}{a^{s_2}} \left(\int_0^\infty t^{s_2-1} f_2(u)dt\right) e_2 \\ &= \frac{1}{a^{s_1 e_1 + s_2 e_2}} \int_0^\infty t^{s_1 e_1 + s_2 e_2 - 1} \left(f_1(u) e_1 + f_2(u) e_2\right) dt \\ &= \frac{1}{a^\xi} \int_0^\infty t^{\xi-1} f(u)dt \\ &= \frac{\bar{f}(\xi)}{a^\xi}.\end{aligned}$$

□

Theorem 7.5 (Bicomplex Mellin Transform of Derivatives)**.** *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$\mathfrak{M}\left[f^{(n)}(t); \xi\right] = (-1)^n \frac{\Gamma(\xi)}{\Gamma(\xi - n)} \bar{f}(\xi - n), \quad (\xi - n) \in \Omega \tag{7.20}$$

where Ω is defined in (7.16) and provided $t^{\xi-r-1}f^{(r)}(t)$ vanishes as $t \to 0$ and as $t \to \infty$ for $r = 0, 1, 2, \cdots, (n-1)$.

Proof. For $n = 1$, according to the definition of bicomplex Mellin transform,

$$\mathfrak{M}\left[f'(t); \xi\right] = \int_0^\infty t^{\xi-1} f'(t) dt$$

which on integration by parts, gives

$$\begin{aligned}\mathfrak{M}\left[f'(t); \xi\right] &= t^{\xi-1} f(t)|_0^\infty - (\xi - 1) \int_0^\infty t^{\xi-2} f(t) dt \\ &= -(\xi - 1)\bar{f}(\xi - 1).\end{aligned}$$

Therefore, the result is true for $n = 1$. Let the the above result is true for $n = m$

$$\mathfrak{M}\left[f^{(m)}(t); \xi\right] = (-1)^m \frac{\Gamma(\xi)}{\Gamma(\xi - m)} \bar{f}(\xi - m). \tag{7.21}$$

Now, for $n = m + 1$

$$\mathfrak{M}\left[f^{(m+1)}(t); \xi\right] = \int_0^\infty t^{\xi-1} f^{(m+1)}(t) dt$$

Integrating by parts, we get

$$\begin{aligned}\mathfrak{M}\left[f^{(m+1)}(t); \xi\right] &= t^{\xi-1} f^{(m)}(t)|_0^\infty - (\xi - 1) \int_0^\infty t^{\xi-2} f^{(m)}(t) dt \\ &= -(\xi - 1)(-1)^m \frac{\Gamma(\xi - 1)}{\Gamma(\xi - m - 1)} \bar{f}(\xi - m - 1), \text{ [using (7.21)]} \\ &= (-1)^{m+1} \frac{\Gamma(\xi)}{\Gamma(\xi - m - 1)} \bar{f}(\xi - m - 1).\end{aligned}$$

Therefore, the result is true for $n = m+1$. Hence, by the principal of mathematical induction the result is true for all $n = 1, 2, \cdots$. Therefore,

$$\mathfrak{M}\left[f^{(n)}(t); \xi\right] = (-1)^n \frac{\Gamma(\xi)}{\Gamma(\xi - n)} \bar{f}(\xi - n).$$

□

Theorem 7.6 (Shifting Property). *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[t^a f(t); \xi\right] = \bar{f}(\xi + a), \quad (\xi + a) \in \Omega, \ a \in C_2 \tag{7.22}$$

where Ω is defined in (7.16).

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}\mathfrak{M}\left[t^a f(t);\xi\right] &= \int_0^\infty t^{\xi-1} t^a f(t)dt \\ &= \int_0^\infty t^{\xi+a-1} f(t)dt \\ &= \bar{f}(\xi+a).\end{aligned}$$

□

Theorem 7.7. *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[f(t^a);\xi\right] = \frac{1}{a}\bar{f}\left(\frac{\xi}{a}\right), \quad \frac{\xi}{a} \in \Omega,\ 0 \neq a \in \mathbb{R} \tag{7.23}$$

where Ω is defined in (7.16).

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}\mathfrak{M}\left[f(t^a);\xi\right] &= \int_0^\infty t^{\xi-1} f(t^a)dt \\ &= \frac{1}{a}\int_0^\infty u^{\frac{\xi}{a}-1} f(u)du \qquad [\text{substituting } t^a = u] \\ &= \frac{1}{a}\bar{f}\left(\frac{\xi}{a}\right).\end{aligned}$$

□

Theorem 7.8. *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[t^n f^{(n)}(t);\xi\right] = (-1)^n \frac{\Gamma(\xi+n)}{\Gamma(\xi)}\bar{f}(\xi), \quad \xi \in \Omega \tag{7.24}$$

where Ω is defined in (7.16) and provided $t^{\xi-r} f^{(r)}(\xi)$ vanishes as $t \to 0$ and as $t \to \infty$ for $r = 0, 1, 2, \cdots, (n-1)$.

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}
\mathfrak{M}\left[t^n f^{(n)}(t);\xi\right] &= \int_0^\infty t^{\xi-1} t^n f^{(n)}(t)dt, \qquad [\text{where } \xi = s_1e_1 + s_2e_2] \\
&= \left(\int_0^\infty t^{s_1-1} t^n f_1^{(n)}(t)dt\right) e_1 + \left(\int_0^\infty t^{s_2-1} t^n f_2^{(n)}(t)dt\right) e_2 \\
&= (-1)^n \frac{\Gamma(s_1+n)}{\Gamma(s_1)} \bar{f}_1(s_1)e_1 + (-1)^n \frac{\Gamma(s_2+n)}{\Gamma(s_2)} \bar{f}_2(s_2)e_2, \\
&\qquad\qquad [\text{using [34, Equation (8.3.12)]}] \\
&= (-1)^n \frac{\Gamma(s_1e_1+s_2e_2+n)}{\Gamma(s_1e_1+s_2e_2)} \left(\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2\right) \\
&= (-1)^n \frac{\Gamma(\xi+n)}{\Gamma(\xi)} \bar{f}(\xi).
\end{aligned}$$

□

Theorem 7.9 (Bicomplex Mellin Transform of Differential Operators)**.** *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[\left(t\frac{d}{dt}\right)^2 f(t);\xi\right] = \mathfrak{M}\left[t^2 f''(t) + tf'(t);\xi\right] = (-1)^2\xi^2\bar{f}(\xi),\ \xi\in\Omega \tag{7.25}$$

where Ω is defined in (7.16).

Proof. By applying the definition of bicomplex Mellin transform,

$$\begin{aligned}
\mathfrak{M}\left[\left(t\frac{d}{dt}\right)^2 f(t);\xi\right] &= \mathfrak{M}\left[t^2 f''(t) + tf'(t);\xi\right] \\
&= \mathfrak{M}\left[t^2 f''(t);\xi\right] + \mathfrak{M}\left[tf'(t);\xi\right] \\
&= \xi(\xi+1)\bar{f}(\xi) - \xi\bar{f}(\xi) \\
&= (-1)^2\xi^2\bar{f}(\xi).
\end{aligned}$$

In general,

$$\mathfrak{M}\left[\left(t\frac{d}{dt}\right)^n f(t);\xi\right] = (-1)^n\xi^n\bar{f}(\xi).$$

□

Theorem 7.10 (Bicomplex Mellin Transform of Integrals)**.** *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}\left[\int_0^t f(x)dx;\xi\right] = -\frac{1}{\xi}\bar{f}(\xi+1), \quad (\xi+1)\in\Omega \tag{7.26}$$

where Ω is defined in (7.16).

Proof. We write

$$g(t) = \int_0^t f(x)dx$$

so that $g'(t) = f(t)$ with $g(0) = 0$. Taking the bicomplex Mellin transform of $g'(t)$ and using Theorem 7.5 therein, we get

$$\begin{aligned}\mathfrak{M}\left[g'(t);\xi\right] &= -(\xi-1)\mathfrak{M}[g(t);\xi-1] \\ &= -(\xi-1)\mathfrak{M}\left[\int_0^t f(x)dx;\xi-1\right]\end{aligned}$$

Replacing ξ by $\xi+1$, we get the desired result (7.26). □

7.3.1 Relation with Bicomplex Laplace Transform

The bicomplex Laplace transform and its properties are already discussed in section 2.2 of chapter 2. Therefore, the usual right-sided bicomplex Laplace transform is analytic in half-plane $\mathrm{Re}(\xi) > \alpha + |\mathrm{Im}_j(\xi)|$. In the same way, left-sided bicomplex Laplace transform is analytic in the region $\mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|$. If the two half-planes overlap, the region of analyticity of the two-sided bicomplex Laplace transform is thus the strip

$$D = \left\{\xi \in C_2 : \alpha + |\mathrm{Im}_j(\xi)| < \mathrm{Re}(\xi) < \beta - |\mathrm{Im}_j(\xi)|\right\}.$$

Hence, D is equivalent to Ω defined in (7.16).

Theorem 7.11. *Let $\bar{f}(\xi)$ be bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then*

$$\mathfrak{M}[f(t);\xi] = \int_{-\infty}^{\infty} e^{\xi x} f(e^{-x})dx = L\left[f(e^{-x});\xi\right], \quad \xi \in \Omega \tag{7.27}$$

where Ω is defined in (7.16).

Proof. Taking $t = e^{-x}$ in the definition of bicomplex Mellin transform

$$\mathfrak{M}[f(t);\xi] = \int_{0}^{\infty} t^{\xi-1} f(t)dt,$$

we get

$$\mathfrak{M}[f(t);\xi] = \int_{-\infty}^{\infty} e^{\xi x} f(e^{-x})dx = L\left[f(e^{-x});\xi\right].$$

□

Theorem 7.12. *Let $\bar{f}(\xi)$ and $\bar{g}(\xi)$ are bicomplex Mellin transforms of bicomplex-valued functions $f(t)$ and $g(t)$ respectively. Then*

$$\mathfrak{M}[f(t) * g(t);\xi] = \mathfrak{M}\left[\int_{0}^{\infty} \frac{1}{x} f(x) g\left(\frac{t}{x}\right) dx;\xi\right] = \bar{f}(\xi)\bar{g}(\xi), \quad \xi \in \Omega \tag{7.28}$$

and

$$\mathfrak{M}[f(t) \circ g(t);\xi] = \mathfrak{M}\left[\int_{0}^{\infty} f(xt) g(x) dx;\xi\right] = \bar{f}(\xi)\bar{g}(1-\xi), \quad \xi \in \Omega \tag{7.29}$$

where Ω is defined in (7.16).

Proof. We have, by definition,

$$\begin{aligned}\mathfrak{M}[f(t) * g(t);\xi] &= \mathfrak{M}\left[\int_0^\infty \frac{1}{x} f(x) g\left(\frac{t}{x}\right) dx;\xi\right] \\ &= \int_0^\infty t^{\xi-1} dt \int_0^\infty f(x) g\left(\frac{t}{x}\right) \frac{dx}{x}\end{aligned}$$

By changing the order of integration

$$\begin{aligned}&= \int_0^\infty f(x) \frac{dx}{x} \int_0^\infty t^{\xi-1} g\left(\frac{t}{x}\right) dt \\ &= \int_0^\infty f(x) dx \int_0^\infty (xy)^{\xi-1} g(y) dy, \qquad \left[y = \frac{t}{x}\right] \\ &= \int_0^\infty x^{\xi-1} f(x) dx \int_0^\infty y^{\xi-1} g(y) dy \\ &= \bar{f}(\xi)\bar{g}(\xi).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathfrak{M}[f(t) \circ g(t);\xi] &= \mathfrak{M}\left[\int_0^\infty f(xt) g(x) dx;\xi\right] \\ &= \int_0^\infty t^{\xi-1} dt \int_0^\infty f(xt) g(x) dx\end{aligned}$$

By changing the order of integration

$$\begin{aligned}&= \int_0^\infty g(x) dx \int_0^\infty t^{\xi-1} f(xt) dt \\ &= \int_0^\infty g(x) dx \int_0^\infty y^{\xi-1} x^{1-\xi} f(y) \frac{dy}{x}, \quad [y = xt] \\ &= \int_0^\infty x^{1-\xi-1} g(x) dx \int_0^\infty y^{\xi-1} f(y) dy \\ &= \bar{f}(\xi)\bar{g}(1-\xi).\end{aligned}$$

□

In the following theorem, we make efforts to find the bicomplex Mellin transform of the Riemann-Liouville fractional integrals.

Theorem 7.13. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$. Then for $\alpha > 0$*

$$\mathfrak{M}\left[{}_0D_t^{-\alpha} f(t);\xi\right] = \frac{\Gamma(1-\xi-\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi+\alpha), \quad \xi+\alpha \in \Omega \tag{7.30}$$

where Ω is defined in (7.16).

Proof. Since we know that

$$\begin{aligned} {}_0D_t^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \\ &= \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-x)^{\alpha-1} f(tx)dx, \quad \left[x = \frac{\tau}{t}\right] \\ &= \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 f(tx)g(x)dx \end{aligned} \tag{7.31}$$

where

$$g(t) = \begin{cases} (1-t)^{\alpha-1}, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases} \tag{7.32}$$

Then using equations (7.17), (7.29), (7.31) and (7.32), we get

$$\begin{aligned} \mathfrak{M}\left[{}_0D_t^{-\alpha} f(t);\xi\right] &= \frac{1}{\Gamma(\alpha)}\bar{f}(\xi+\alpha)B(\alpha, 1-\xi-\alpha) \\ &= \frac{\Gamma(1-\xi-\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi+\alpha). \end{aligned}$$

□

In the following theorem, we make efforts to find the bicomplex Mellin transform of the Riemann-Liouville fractional derivative.

Theorem 7.14. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of the bicomplex-valued function $f(t)$. Then for $0 \le n-1 < \alpha < n$*

$$\mathfrak{M}\left[{}_0D_t^{\alpha} f(t);\xi\right] = \sum_{k=0}^{n-1}\frac{\Gamma(1-\xi+k)}{\Gamma(1-\xi)}\left[{}_0D_t^{\alpha-k-1} f(t)t^{\xi-k-1}\right]_0^\infty + \frac{\Gamma(1-\xi+\alpha)}{1-\xi}\bar{f}(\xi-\alpha),$$
$$\xi-\alpha \in \Omega \tag{7.33}$$

where Ω defined in (7.16).

Proof. By taking the bicomplex Mellin transform, we get

$$\begin{aligned}
\mathfrak{M}\left[{}_0D_t^{\alpha} f(t);\xi\right] &= \int_0^{\infty} t^{\xi-1}\,{}_0D_t^{\alpha} f(t)dt \\
&= \left(\int_0^{\infty} t^{s_1-1}\,{}_0D_t^{\alpha} f_1(t)dt\right)e_1 + \left(\int_0^{\infty} t^{s_2-1}\,{}_0D_t^{\alpha} f_2(t)dt\right)e_2 \\
&\qquad [\text{where } \xi = s_1e_1 + s_2e_2 \text{ and } f(t) = f_1(t)e_1 + f_2(t)e_2] \\
&= \left(\sum_{k=0}^{n-1} \frac{\Gamma(1-s_1+k)}{\Gamma(1-s_1)}\left[{}_0D_t^{\alpha} f_1(t)t^{s_1-k-1}\right]_0^{\infty} + \frac{\Gamma(1-s_1+\alpha)}{\Gamma(1-s_1)}\bar{f}_1(s_1-\alpha)\right)e_1 \\
&\quad + \left(\sum_{k=0}^{n-1} \frac{\Gamma(1-s_2+k)}{\Gamma(1-s_1)}\left[{}_0D_t^{\alpha} f_2(t)t^{s_2-k-1}\right]_0^{\infty} + \frac{\Gamma(1-s_2+\alpha)}{\Gamma(1-s_2)}\bar{f}_2(s_2-\alpha)\right)e_2 \\
&\qquad (\text{using } [117, Eq.(2.287)]) \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(1-s_1e_1-s_2e_2+k)}{\Gamma(1-s_1e_1-s_2e_2)}\left[{}_0D_t^{\alpha}(f_1(t)e_1+f_2(t)e_2)t^{s_1e_1+s_2e_2-k-1}\right]_0^{\infty} \\
&\quad + \frac{\Gamma(1-s_1e_1-s_2e_2+\alpha)}{\Gamma(1-s_1e_1-s_2e_2)}(\bar{f}_1(s_1-\alpha)e_1+\bar{f}_2(s_2-\alpha)e_2) \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(1-\xi+k)}{\Gamma(1-\xi)}\left[{}_0D_t^{\alpha} f(t)t^{\xi-k-1}\right]_0^{\infty} + \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha). \qquad (7.34)
\end{aligned}$$

□

Remark 7.1. In its particular case, if $0 < \alpha < 1$, then (7.34) becomes

$$\mathfrak{M}\left[{}_0D_t^{\alpha} f(t);\xi\right] = \left[{}_0D_t^{\alpha} f(t)t^{\xi-k-1}\right]_0^{\infty} + \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha). \qquad (7.35)$$

If the function $f(t)$, $\text{Re}(s_1)$ and $\text{Re}(s_2)$, where $\xi = s_1e_1 + s_2e_2$ are such that the substitutions of the limit $t = 0$ and $t = \infty$ make the first term of (7.35) zero, then (7.35) reduces to the

$$\mathfrak{M}\left[{}_0D_t^{\alpha} f(t);\xi\right] = \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha). \qquad (7.36)$$

In the following theorem, we have found the bicomplex Mellin transform of the Caputo fractional derivative.

Theorem 7.15. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \leq n-1 \leq \alpha < n$, $n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0^C D_t^\alpha f(t);\xi\right] = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+k-\xi)}{\Gamma(1-\xi)} \left[f^{(k)}(t)t^{\xi-\alpha+k}\right]_0^\infty + \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha),\ \xi-\alpha \in \Omega \tag{7.37}$$

where Ω defined in (7.16).

Proof. By taking the bicomplex Mellin transform, we get

$$\mathfrak{M}\left[{}_0^C D_t^\alpha f(t);\xi\right] = \int_0^\infty t^{\xi-1}\,{}_0^C D_t^\alpha f(t)dt$$
$$= \left(\int_0^\infty t^{s_1-1}\,{}_0^C D_t^\alpha f_1(t)dt\right)e_1 + \left(\int_0^\infty t^{s_2-1}\,{}_0^C D_t^\alpha f_2(t)dt\right)e_2$$
$$[\text{where } \xi = s_1e_1 + s_2e_2 \text{ and } f(t) = f_1(t)e_1 + f_2(t)e_2$$
$$= \left(\sum_{k=0}^{n-1} \frac{\Gamma(\alpha+k-s_1)}{\Gamma(1-s_1)} \left[f_1^{(k)}(t)t^{s_1-\alpha+k}\right]_0^\infty + \frac{\Gamma(1-s_1+\alpha)}{\Gamma(1-s_1)}\bar{f}_1(s_1-\alpha)\right)e_1$$
$$+ \left(\sum_{k=0}^{n-1} \frac{\Gamma(\alpha+k-s_2)}{\Gamma(1-s_2)} \left[f_2^{(k)}(t)t^{s_2-\alpha+k}\right]_0^\infty + \frac{\Gamma(1-s_2+\alpha)}{\Gamma(1-s_2)}\bar{f}_2(s_2-\alpha)\right)e_2$$
$$(\text{using } [117, Eq.(2.291)])$$
$$= \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+k-s_1e_1-s_2e_2)}{\Gamma(1-s_1e_1-s_2e_2)} \left[\left(f_1^{(k)}(t)e_1 + f_2^{(k)}(t)e_2\right)t^{s_1e_1+s_2e_2-\alpha+k}\right]_0^\infty$$
$$+ \frac{\Gamma(1-s_1e_1-s_2e_2+\alpha)}{\Gamma(1-s_1e_1-s_2e_2)}\left(\bar{f}_1(s_1-\alpha)e_1 + \bar{f}_2(s_2-\alpha)e_2\right)$$
$$= \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+k-\xi)}{\Gamma(1-\xi)} \left[f^{(k)}(t)t^{\xi-\alpha+k}\right]_0^\infty + \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha). \tag{7.38}$$

□

Remark 7.2. In its particular case, if $0 < \alpha < 1$, then (7.38) becomes

$$\mathfrak{M}\left[{}_0^C D_t^\alpha f(t);\xi\right] = \frac{\Gamma(\alpha-\xi)}{\Gamma(1-\xi)} \left[f(t)t^{\xi-\alpha}\right]_0^\infty + \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha) \tag{7.39}$$

If the function $f(t)$, $\text{Re}(s_1)$ and $\text{Re}(s_2)$, where $\xi = s_1e_1 + s_2e_2$ are such that the substitutions of the limit $t = 0$ and $t = \infty$ make the first term of (7.38) zero, then (7.38) reduces to the

$$\mathfrak{M}\left[{}_0^C D_t^\alpha f(t);\xi\right] = \frac{\Gamma(1-\xi+\alpha)}{\Gamma(1-\xi)}\bar{f}(\xi-\alpha). \tag{7.40}$$

Theorem 7.16. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \leq n-1 < \alpha < n$, $n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0^C D_t^\alpha \, {}_0D_t^{-\alpha} f(t);\xi\right] = \bar{f}(\xi), \quad \xi \in \Omega \tag{7.41}$$

where Ω defined in (7.16).

Proof. Since we know that

$${}_0^C D_t^\alpha \left[{}_0D_t^{-\alpha} f(t)\right] = f(t).$$

By taking the bicomplex Mellin transform on both side, we have

$$\mathfrak{M}\left[{}_0^C D_t^\alpha \, {}_0D_t^{-\alpha} f(t);\xi\right] = \bar{f}(\xi).$$

□

Deduction 7.1. *If we take $f(t) = t^n U(t-t_0)$, then*

$${}_0D_t^{-\alpha} t^n U(t-t_0) = \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \sum_{r=0}^{n} (-1)^r \, {}^nC_r t^{n-r} u^r (t-t_0)^r du. \tag{7.42}$$

where $U(t-t_0)$ is unit step function and hence

$$\mathfrak{M}\left[\frac{1}{\Gamma(\alpha)} {}_0^C D_t^\alpha \, {}_0D_t^{-\alpha} f(t);\xi\right] = -\frac{t_0^{\xi+n}}{\xi+n}, \quad Re(\xi+n) < -\left|Im_j(\xi+n)\right|. \tag{7.43}$$

Proof. By applying the definition of Riemann-Liouville integral operator on $t^n U(t-t_0)$

$$\begin{aligned}
{}_0D_t^{-\alpha} t^n U(t-t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} x^n U(x-t_0) dx \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-x)^{\alpha-1} x^n dx \\
&= \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \left[t-u(t-t_0)\right]^n du, \quad \left[\text{put } u = \frac{t-x}{t-t_0}\right] \\
&= \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} \sum_{r=0}^{n} (-1)^r \, {}^nC_r t^{n-r} u^r (t-t_0)^r du.
\end{aligned}$$

Changing the order of integration and summation which is valid under the conditions of convergence, we get

$$\begin{aligned}{}_0D_t^{-\alpha} t^n U(t-t_0) &= \frac{(t-t_0)^\alpha}{\Gamma(\alpha)} \sum_{r=0}^{n} (-1)^r\, {}^nC_r t^{n-r} (t-t_0)^r \int_0^1 u^{\alpha+r-1} du \\ &= \frac{1}{\Gamma(\alpha)} \sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{\alpha+r} t^{n-r} (t-t_0)^{\alpha+r}. \qquad (7.44)\end{aligned}$$

Therefore, making use of (7.41) and (7.44), we get the desired result (7.43). □

Theorem 7.17. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \le n-1 < \alpha < n$, $n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}\, {}_0^C D_t^\alpha f(t); \xi\right] = \bar{f}(\xi) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!(k+\xi)}, \; |Im_j(k+\xi)| < Re(k+\xi) \qquad (7.45)$$

where Ω defined in (7.16).

Proof. Since we know that

$${}_0D_t^{-\alpha}\left[{}_0^C D_t^\alpha f(t)\right] = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \left(\frac{t^k}{k!}\right).$$

By taking the bicomplex Mellin transform on both side, we have

$$\begin{aligned}\mathfrak{M}\left[{}_0D_t^{-\alpha}\, {}_0^C D_t^\alpha f(t); \xi\right] &= \mathfrak{M}\left[f(t);\xi\right] - \mathfrak{M}\left[\sum_{k=0}^{m-1} f^{(k)}(0) \left(\frac{t^k}{k!}\right); \xi\right] \\ &= \bar{f}(\xi) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} \int_0^\infty t^{\xi+k-1} dt \\ &= \bar{f}(\xi) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!(k+\xi)}.\end{aligned}$$

□

Theorem 7.18. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, where $0 \le n-1 < \alpha,\ \beta < n$, $n \in \mathbb{N}$, then*

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t); \xi\right] = \frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)} \bar{f}(\alpha+\beta+\xi), \qquad (7.46)$$

$$Re(\alpha+\beta+\xi) < 1 - |Im_j(\alpha+\beta+\xi)|,\ \alpha+\beta+\xi \in \Omega$$

where Ω defined in (7.16).

Proof. Since we know that

$$ {}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t) = {}_0D_t^{-\alpha-\beta} f(t). $$

By taking the bicomplex Mellin transform on both side, we have

$$ \begin{aligned} \mathfrak{M}\left[{}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t);\xi\right] &= \mathfrak{M}\left[{}_0D_t^{-\alpha-\beta} f(t);\xi\right] \\ &= \int_0^\infty t^{\xi-1}\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-x)^{\alpha+\beta-1} f(x)dxdt \end{aligned} $$

By changing the order of integration

$$ = \frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty f(x)dx \int_x^\infty t^{\xi-1}(t-x)^{\alpha+\beta-1}dt $$

Put $t = \frac{x}{u}$, then

$$ \int_x^\infty t^{\xi-1}(t-x)^{\alpha+\beta-1}dt = x^{\alpha+\beta+\xi-1}\int_0^1 u^{-\alpha-\beta-\xi}(1-u)^{\alpha+\beta-1}du $$

Therefore,

$$ \mathfrak{M}\left[{}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t);\xi\right] = \frac{1}{\Gamma(\alpha+\beta)}\int_0^\infty x^{\alpha+\beta+\xi-1} f(x)dx \int_0^1 u^{-\alpha-\beta-\xi}(1-u)^{\alpha+\beta-1}du $$

where $\alpha+\beta > 0,$ and $\mathrm{Re}(\alpha+\beta+\xi) < 1 - |\mathrm{Im}_j(\alpha+\beta+\xi)|$

After using beta function (7.2), we have

$$ \begin{aligned} \mathfrak{M}\left[{}_0D_t^{-\alpha}\, {}_0D_t^{-\beta} f(t);\xi\right] &= \frac{B(1-\alpha-\beta-\xi,\alpha+\beta)}{\Gamma(\alpha+\beta)}\int_0^\infty x^{\alpha+\beta+\xi-1} f(x)dx \\ &= \frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\bar{f}(\alpha+\beta+\xi). \end{aligned} $$

□

Deduction 7.2. *For* $0 \le n-1 < \alpha,\ \beta < n,\ n \in \mathbb{N}$

$$ \mathfrak{M}\left[\frac{1}{\Gamma(\beta)}\, {}_0D_t^{-\alpha}\left\{\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{\beta+r}t^{n-r}(t-t_0)^{\beta+r}\right\};\xi\right] $$

$$ = -\frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\frac{t_0^{\xi+\alpha+\beta+n}}{\xi+\alpha+\beta+n}, \tag{7.47} $$

$Re(\xi+\alpha+\beta) < 1 - |Im_j(\xi+\alpha+\beta)|,\ Re(\xi+\alpha+\beta+n) < -|Im_j(\xi+\alpha+\beta+n)|.$

Proof. In the similar manner of equation (7.44) and using result (7.46), we get

$$\mathfrak{M}\left[{}_0D_t^{-\alpha}\,{}_0D_t^{-\beta}t^nU(t-t_0)\right] = -\frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\frac{t_0^{\xi+\alpha+\beta+n}}{\xi+\alpha+\beta+n}$$

$$\mathfrak{M}\left[\frac{1}{\Gamma(\beta)}\,{}_0D_t^{-\alpha}\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{\beta+r}t^{n-r}(t-t_0)^{\beta+r};\xi\right] = -\frac{\Gamma(1-\alpha-\beta-\xi)}{\Gamma(1-\xi)}\frac{t_0^{\xi+\alpha+\beta+n}}{\xi+\alpha+\beta+n}.$$

□

Theorem 7.19. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$(a)\quad \mathfrak{M}\left[{}_t^CD_\infty^{\frac{1}{2}}f(t);\xi\right] = \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{1}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{1}{2}\right],\qquad \xi-\frac{1}{2}\in\Omega \tag{7.48}$$

$$(b)\quad \mathfrak{M}\left[{}_t^CD_\infty^{\frac{3}{2}}f(t);\xi\right] = \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{3}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{3}{2}\right],\qquad \xi-\frac{3}{2}\in\Omega \tag{7.49}$$

where Ω defined in (7.16) and ${}_0D_t^{\alpha+j-n}t^{s_i-1}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to 0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$, $f(t)=f_1(t)e_1+f_2(t)e_2$ and $\alpha=\frac{1}{2},\frac{3}{2}$.

Proof. (a) By applying the definition of bicomplex Mellin transform, we get

$$\begin{aligned}\mathfrak{M}\left[{}_t^CD_\infty^{\frac{1}{2}}f(t);\xi\right] &= \int_0^\infty t^{\xi-1}\,{}_t^CD_\infty^{\frac{1}{2}}f(t)dt\\ &= \left(\int_0^\infty t^{s_1-1}\,{}_t^CD_\infty^{\frac{1}{2}}f_1(t)dt\right)e_1+\left(\int_0^\infty t^{s_2-1}\,{}_t^CD_\infty^{\frac{1}{2}}f_2(t)dt\right)e_2\end{aligned}$$

[where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$].

We know the result of fractional integration by parts (see, e.g. *Almeida and Torres* [2]) as

$$\int_a^b g(t)\,{}_t^CD_b^\alpha f(t)dt = \int_a^b f(t)\,{}_aD_t^\alpha g(t)dt+\sum_{j=0}^{n-1}\left[(-1)^{n+j}\,{}_aD_t^{\alpha+j-n}g(t)\,{}_aD_t^{n-1-j}f(t)\right]_a^b. \tag{7.50}$$

By using (7.50) and using given conditions, we obtain

$$\begin{aligned}
\mathfrak{M}\left[{}_tD_{\infty}^{\frac{1}{2}}f(t);\xi\right] &= \left(\int_0^{\infty} f_1(t)\,{}_0D_t^{\frac{1}{2}}t^{s_1-1}dt\right)e_1 + \left(\int_0^{\infty} f_2(t)\,{}_0D_t^{\frac{1}{2}}t^{s_2-1}dt\right)e_2 \\
&= \frac{\Gamma(s_1)}{\Gamma\left(s_1-\frac{1}{2}\right)}\mathfrak{M}\left[f_1(t);s_1-\frac{1}{2}\right]e_1 + \frac{\Gamma(s_2)}{\Gamma\left(s_2-\frac{1}{2}\right)}\mathfrak{M}\left[f_2(t);s_2-\frac{1}{2}\right]e_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{using } (7.5)] \\
&= \frac{\Gamma(s_1e_1+s_2e_2)}{\Gamma\left(s_1e_1+s_2e_2-\frac{1}{2}\right)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2-\frac{1}{2}\right] \\
&= \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{1}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{1}{2}\right].
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
\mathfrak{M}\left[{}_t^CD_{\infty}^{\frac{3}{2}}f(t);\xi\right] &= \left(\int_0^{\infty} f_1(t)\,{}_0D_t^{\frac{3}{2}}t^{s_1-1}dt\right)e_1 + \left(\int_0^{\infty} f_2(t)\,{}_0D_t^{\frac{3}{2}}t^{s_2-1}dt\right)e_2 \\
&= \frac{\Gamma(s_1)}{\Gamma\left(s_1-\frac{3}{2}\right)}\mathfrak{M}\left[f_1(t);s_1-\frac{3}{2}\right]e_1 + \frac{\Gamma(s_2)}{\Gamma\left(s_2-\frac{3}{2}\right)}\mathfrak{M}\left[f_2(t);s_2-\frac{3}{2}\right]e_2 \\
&= \frac{\Gamma(s_1e_1+s_2e_2)}{\Gamma\left(s_1e_1+s_2e_2-\frac{3}{2}\right)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2-\frac{3}{2}\right] \\
&= \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{3}{2}\right)}\mathfrak{M}\left[f(t);\xi-\frac{3}{2}\right].
\end{aligned}$$

□

Continuing by the induction, the results in Theorem 7.19 can be extended further to fractional derivatives as in the following theorem:

Theorem 7.20. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$ for all $n-1<\alpha<n$, $n\in\mathbb{N}$, then*

$$\mathfrak{M}\left[{}_t^CD_{\infty}^{\alpha}f(t);\xi\right] = \frac{\Gamma(\xi)}{\Gamma(\xi-\alpha)}\mathfrak{M}\left[f(t);\xi-\alpha\right], \qquad \xi-\alpha\in\Omega \tag{7.51}$$

where Ω defined in (7.16) and ${}_0D_t^{\alpha+j-n}t^{s_i-1}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to 0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$.

Theorem 7.21. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$, then*

$$(a)\quad \mathfrak{M}\left[t^{\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f(t);\xi\right] = \frac{\Gamma\left(\xi+\frac{1}{2}\right)}{\Gamma(\xi)}\mathfrak{M}\left[f(t);\xi\right], \qquad \xi+\frac{1}{2}\in\Omega \tag{7.52}$$

$$(b)\quad \mathfrak{M}\left[t^{\frac{3}{2}}\,{}_t^C D_\infty^{\frac{3}{2}} f(t);\xi\right] = \frac{\Gamma\left(\xi+\frac{3}{2}\right)}{\Gamma(\xi)}\mathfrak{M}\left[f(t);\xi\right], \qquad \xi+\frac{3}{2}\in\Omega \tag{7.53}$$

where Ω defined in (7.16) and ${}_0D_t^{\alpha+j-n}t^{s_i-\alpha}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to 0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$, $f(t)=f_1(t)e_1+f_2(t)e_2$ and $\alpha=\frac{1}{2},\frac{3}{2}$.

Proof. (a) By applying the definition of bicomplex Mellin transform, we get

$$\mathfrak{M}\left[t^{\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f(t);\xi\right] = \int_0^\infty t^{\xi-\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f(t)dt$$
$$= \left(\int_0^\infty t^{s_1-\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f_1(t)dt\right)e_1 + \left(\int_0^\infty t^{s_2-\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f_2(t)dt\right)e_2$$

[where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$].

By using (7.50) and using given conditions, we obtain

$$\mathfrak{M}\left[t^{\frac{1}{2}}\,{}_t^C D_\infty^{\frac{1}{2}} f(t);\xi\right] = \left(\int_0^\infty f_1(t)\,{}_0D_t^{\frac{1}{2}}t^{s_1-\frac{1}{2}}dt\right)e_1 + \left(\int_0^\infty f_2(t)\,{}_0D_t^{\frac{1}{2}}t^{s_2-\frac{1}{2}}dt\right)e_2$$
$$= \frac{\Gamma(s_1+\frac{1}{2})}{\Gamma(s_1)}\left(\int_0^\infty t^{s_1-1}f_1(t)dt\right)e_1 + \frac{\Gamma(s_2+\frac{1}{2})}{\Gamma(s_2)}\left(\int_0^\infty t^{s_2-1}f_2(t)dt\right)e_2$$

[using (7.5)]

$$= \frac{\Gamma\left(s_1e_1+s_2e_2+\frac{1}{2}\right)}{\Gamma\left(s_1e_1+s_2e_2\right)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2\right]$$
$$= \frac{\Gamma\left(\xi+\frac{1}{2}\right)}{\Gamma(\xi)}\mathfrak{M}\left[f(t);\xi\right].$$

(b) Similarly,

$$\begin{aligned}
\mathfrak{M}\left[t^{\frac{3}{2}}\,{}_t^CD_\infty^{\frac{3}{2}}f(t);\xi\right] &= \left(\int_0^\infty f_1(t)\,{}_0D_t^{\frac{3}{2}}t^{s_1+\frac{1}{2}}dt\right)e_1 + \left(\int_0^\infty f_2(t)\,{}_0D_t^{\frac{3}{2}}t^{s_2+\frac{1}{2}}dt\right)e_2 \\
&= \frac{\Gamma\left(s_1+\frac{3}{2}\right)}{\Gamma(s_1)}\mathfrak{M}\left[f_1(t);s_1\right]e_1 + \frac{\Gamma\left(s_2+\frac{3}{2}\right)}{\Gamma(s_2)}\mathfrak{M}\left[f_2(t);s_2\right]e_2 \\
&= \frac{\Gamma\left(s_1e_1+s_2e_2+\frac{1}{2}\right)}{\Gamma(s_1e_1+s_2e_2)}\mathfrak{M}\left[f_1(t)e_1+f_2(t)e_2;s_1e_1+s_2e_2\right] \\
&= \frac{\Gamma(\xi)}{\Gamma\left(\xi-\frac{3}{2}\right)}\mathfrak{M}\left[f(t);\xi\right].
\end{aligned}$$

□

Following the similar technique as in the above theorem, follows Theorem 7.22.

Theorem 7.22. *Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued function $f(t)$ for all $n-1<\alpha<n$, $n\in\mathbb{N}$, then*

$$\mathfrak{M}\left[t^\alpha\,{}_t^CD_\infty^\alpha f(t);\xi\right] = \frac{\Gamma(\xi+\alpha)}{\Gamma(\xi)}\mathfrak{M}\left[f(t);\xi\right], \qquad \xi+\alpha\in\Omega \tag{7.54}$$

where Ω defined in (7.16) and ${}_0D_t^{\alpha+j-n}t^{s_i-\alpha}\,{}_0D_t^{n-1-j}f_i(t)$ vanishes as $t\to 0$ and $t\to\infty$ for $j=0,1,\cdots,n-1$ and $i=1,2$. Where $\xi=s_1e_1+s_2e_2$ and $f(t)=f_1(t)e_1+f_2(t)e_2$.

7.4 Inversion of Bicomplex Mellin Transform

In this section, we discuss the inversion of bicomplex Mellin transform. Let $\bar{f}(\xi)$ be the bicomplex Mellin transform of bicomplex-valued continuous function $f(t)$. Then $\bar{f}(\xi)=\bar{f}_1(s_1)e_1+\bar{f}_2(s_2)e_2$ is analytic in the strip Ω, which is defined in (7.13). The inverse formula for complex mellin transform (see, e.g. *Poularikas*

[118, chapter 11] and *Davies* [33, p. 195-210]) is

$$\begin{aligned} f_1(t) &= \frac{1}{2\pi i_1}\int_{c_1-i_1\infty}^{c_1+i_1\infty} t^{-s_1}\bar{f}_1(s_1)ds_1, \quad \alpha_1 < c_1 < \beta_1 \\ &= \frac{1}{2\pi i_1}\int_{\Omega_1} t^{-s_1}\bar{f}_1(s_1)ds_1 \end{aligned} \tag{7.55}$$

where, Ω_1 is defined in (7.9). Similarly, another inverse formula for complex Mellin transform is

$$\begin{aligned} f_2(t) &= \frac{1}{2\pi i_1}\int_{c_2-i_1\infty}^{c_2+i_1\infty} t^{-s_2}\bar{f}_2(s_2)ds_2, \quad \alpha_2 < c_2 < \beta_2 \\ &= \frac{1}{2\pi i_1}\int_{\Omega_2} t^{-s_2}\bar{f}_2(s_2)ds_2 \end{aligned} \tag{7.56}$$

where, Ω_2 is defined in (7.10).

Now, using complex inversions (7.55) and (7.56), we obtain the bicomplex-valued function as

$$\begin{aligned} f(t) &= f_1(t)e_1 + f_2(t)e_2 \\ &= \left(\frac{1}{2\pi i_1}\int_{\Omega_1} t^{-s_1}\bar{f}_1(s_1)ds_1\right)e_1 + \left(\frac{1}{2\pi i_1}\int_{\Omega_2} t^{-s_2}\bar{f}_2(s_2)ds_2\right)e_2 \\ &= \frac{1}{2\pi i_1}\left(\int_{(\Omega_1,\Omega_2)} t^{-(s_1e_1+s_2e_2)}\left(\bar{f}_1(s_1)e_1 + \bar{f}_2(s_2)e_2\right)d(s_1e_1+s_2e_2)\right) \\ &= \frac{1}{2\pi i_1}\int_{\Omega} t^{-\xi}\bar{f}(\xi)d\xi \end{aligned} \tag{7.57}$$

where, Ω is defined in (7.16).

Consider the problem of asymptotically expanding $f(t)$ as $t \to 0^+$, when $\bar{f}(\xi)$ is known to be continuable in $-M + |\mathrm{Im}_j(\xi)| \leq \mathrm{Re}(\xi) \leq \alpha - |\mathrm{Im}_j(\xi)|$ for some $M > 0$. We also postulate that $\bar{f}(\xi)$ has finitely many poles λ_k such that $\mathrm{Re}(\lambda_k) > -M + |\mathrm{Im}_j(\lambda_k)|$. Then

$$f(t) = \sum_{\lambda_k \in \mathcal{K}} \mathrm{Res}\left[t^{-\xi}\bar{f}(\xi),\ \xi = \lambda_k\right] + O\left(t^M\right), \quad \text{as } t \to 0^+$$

where $\mathcal{K}$ is the set of singularities and M is as large as we want. Similarly, for problem of asymptotically expanding $f(t)$ as $t \to \infty$. Then contour taken in right and side of the fundamental strip, we have

$$f(t) = -\sum_{\lambda_k \in \mathcal{K}} \text{Res}\left[t^{-\xi}\bar{f}(\xi),\ \xi = \lambda_k\right] + O\left(t^{-M}\right), \quad \text{as } t \to \infty.$$

Following is the illustration to explain the process of finding the inverse bicomplex Mellin transform.

Example 7.3. Let $\bar{f}(\xi) = \frac{1}{(\xi-a)(\xi-b)}$, for $\text{Re}(\xi-a) < -|\text{Im}_j(\xi - a)|$ and $\text{Re}(a-b) < -|\text{Im}_j(a - b)|$. Then find the inverse bicomplex Mellin transform $f(t)$ of $\bar{f}(\xi)$.

Solution. By applying the inverse bicomplex Mellin transform on $\bar{f}(\xi)$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i_1}\int_\Omega t^{-\xi}\bar{f}(\xi)d\xi \\ &= -\left[\text{Res}\left(t^{-\xi}\frac{1}{(\xi-a)(\xi-b)},\ \xi = a\right) + \text{Res}\left(t^{-\xi}\frac{1}{(\xi-a)(\xi-b)},\ \xi = b\right)\right] \\ &= \frac{1}{b-a}\left(t^{-a} - t^{-b}\right). \end{aligned}$$

7.5 Applications of Bicomplex Mellin Transform

Here, we are interested in determining the extent to which the output voltage V and current I using by bicomplex concept differs from their input values as the length of the transmission line tends to a very small value. In this section, we also discuss the application of bicomplex Mellin transform in solving Caputo fractional equation of bicomplex-valued function.

(a) Now, let us define bicomplex scalar field as

$$F \equiv V + i_2 I \tag{7.58}$$

where voltage V and current I are complex scalar fields. Now, we consider an equivalent circuit of a transmission line of small length Δx containing resistance $R\Delta x$, capacitance $C\Delta x$, and inductance $L\Delta x$ as shown in Figure 7.1.

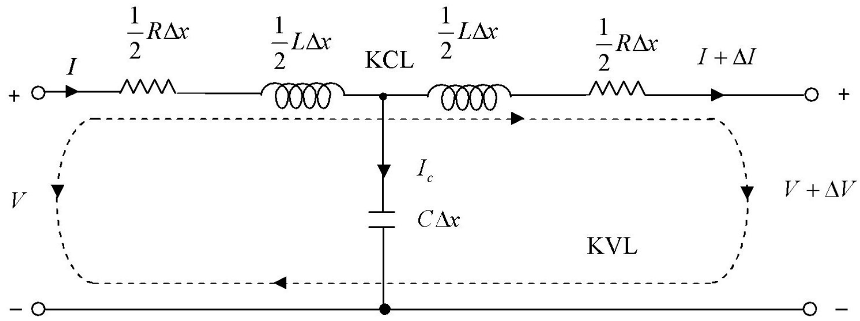

Figure 7.1: Equivalent circuit of a transmission line

The above figure is a symmetrical network. By using the Kirchhoff's voltage law (KVL), we have

$$V = \frac{1}{2}RI\Delta x + \frac{1}{2}L\frac{\partial I}{\partial t}\Delta x + \frac{1}{2}L\frac{\partial}{\partial t}(I+\Delta I)\Delta x + \frac{1}{2}R(I+\Delta I)\Delta x + V + \Delta V. \tag{7.59}$$

Dividing (7.59) by Δx and simplifying, we get

$$\frac{\Delta V}{\Delta x} = -\left[RI + L\frac{\partial I}{\partial t} + \left(\frac{L}{2}\frac{\partial}{\partial t}\frac{\Delta I}{\Delta x} + \frac{R}{2}\frac{\Delta I}{\Delta x}\right)\Delta x\right]. \tag{7.60}$$

Taking limit as $\Delta x \to 0$, we get

$$\frac{\partial V}{\partial x} = -\left[RI + L\frac{\partial I}{\partial t}\right]. \tag{7.61}$$

By applying Kirchhoff's current law (KCL) on the equivalent circuit of the transmission line, we get

$$\begin{aligned} I &= I_c + I + \Delta I \\ &= C\frac{\partial}{\partial t}\left(V + \frac{\Delta V}{2}\right)\Delta x + I + \Delta I. \end{aligned} \tag{7.62}$$

Dividing (7.62) by Δx and simplifying, we get

$$\frac{\Delta I}{\Delta x} = -\left[C\frac{\partial V}{\partial t} + \frac{C}{2}\frac{\partial}{\partial t}\left(\frac{\Delta V}{\Delta x}\right)\Delta x\right]. \tag{7.63}$$

Taking limit as $\Delta x \to 0$, we get

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \tag{7.64}$$

The differential equations in (7.61) and (7.64) describes the evaluation of current and voltage in a lossy transmission line. Differentiating (7.61) w.r.t. x and simplifying using (7.64), we get

$$\frac{\partial^2 V}{\partial x^2} = CL\frac{\partial^2 V}{\partial t^2} + CR\frac{\partial V}{\partial t}. \tag{7.65}$$

Similarly, differentiating (7.64) w.r.t. x and simplifying using (7.61), we get

$$\frac{\partial^2 I}{\partial x^2} = CL\frac{\partial^2 I}{\partial t^2} + CR\frac{\partial I}{\partial t}. \tag{7.66}$$

Equations (7.65) and (7.66) are hyperbolic partial differential equations which describes the voltage and current along power transmission lines.

Combining equation (7.65) and (7.66) with the help of bicomplex unit i_2 as

$$\begin{aligned}
&\frac{\partial^2 V}{\partial x^2} + i_2\frac{\partial^2 I}{\partial x^2} = CL\left(\frac{\partial^2 V}{\partial t^2} + i_2\frac{\partial^2 I}{\partial t^2}\right) + CR\left(\frac{\partial V}{\partial t} + i_2\frac{\partial I}{\partial t}\right)\\
\Rightarrow\quad &\frac{\partial^2}{\partial x^2}(V + i_2 I) = CL\frac{\partial^2}{\partial t^2}(V + i_2 I) + CR\frac{\partial}{\partial t}(V + i_2 I)\\
\Rightarrow\quad &\frac{\partial^2}{\partial x^2}F(x,t) = CL\frac{\partial^2}{\partial t^2}F(x,t) + CR\frac{\partial}{\partial t}F(x,t)
\end{aligned} \tag{7.67}$$

where $F(x,t)$ is bicomplex-valued function defined by (7.58).

In particular, a circuit which has resistance $R = \frac{1}{t}$, capacitance $C = t^2$ and inductance $L = 1$. The differential equation (7.67) of bicomplex-valued function becomes

$$\frac{\partial^2}{\partial x^2}F(x,t) = t^2\frac{\partial^2}{\partial t^2}F(x,t) + t\frac{\partial}{\partial t}F(x,t). \tag{7.68}$$

For finding the solution of partial differential equation (7.68), we assume boundary conditions as

$$F(0,t) = 0 \quad \text{and} \quad F(1,t) = A\left(\frac{1}{t^a} + \frac{1}{t^b}\right) \tag{7.69}$$

where $A \in C_2,\ \text{Re}(b-a) > |\text{Im}_j(b-a)|$. By taking the bicomplex Mellin transform of (7.68) w.r.t. t and making use of Theorem 7.9, we get

$$\frac{d^2}{dx^2}\bar{F}(x,\xi) = \xi^2\bar{F}(x,\xi). \tag{7.70}$$

Therefore, by taking the bicomplex Mellin transform of (7.69) and using in solution of (7.70), we get

$$\bar{F}(x,\xi) = A\left[\frac{(-2\xi + a + b)\left(e^{\xi x} - e^{-\xi x}\right)}{(\xi - a)(\xi - b)\left(e^{\xi} - e^{-\xi}\right)}\right]. \tag{7.71}$$

By taking the inverse bicomplex Mellin transform (7.71), we get

$$F(x,t) = \frac{1}{2\pi i_1}\int_{\Omega} t^{-\xi}\bar{F}(x,\xi)d\xi \tag{7.72}$$

where $\bar{F}(x,\xi)$ is analytic in $\text{Re}(\xi - a) > |\text{Im}_j(\xi - a)|$. Then taking a semi-circle on the right-hand side of a large radius and using by residue theorem, we have

$$\begin{aligned} F(x,t) &= A\left[\frac{\sinh(ax)}{\sinh(a)}t^{-a} + \frac{\sinh(bx)}{\sinh(b)}t^{-b}\right] \\ &= A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]e_1 \\ &\quad + A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]e_2 \end{aligned}$$

where $A = A_1e_1 + A_2e_2$, $a = a_1e_1 + a_2e_2$ and $b = b_1e_1 + b_2e_2$. Therefore,

$$\begin{aligned}F(x,t) &\equiv V + i_2 I\\ &= \frac{1}{2}\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right.\\ &\left.+A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\}\\ &+ i_2\frac{i_1}{2}\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right.\\ &\left.-A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\}. \end{aligned} \tag{7.73}$$

Separating the bi-real and bi-imaginary parts of (7.73), we obtain the voltage and current of above model as

$$\begin{aligned}V(x,t) = \frac{1}{2}&\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right.\\ &\left.+A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\}\end{aligned}$$

and

$$\begin{aligned}I(x,t) = \frac{i_1}{2}&\left\{A_1\left[\frac{\sinh(a_1x)}{\sinh(a_1)}t^{-a_1} + \frac{\sinh(b_1x)}{\sinh(b_1)}t^{-b_1}\right]\right.\\ &\left.-A_2\left[\frac{\sinh(a_2x)}{\sinh(a_2)}t^{-a_2} + \frac{\sinh(b_2x)}{\sinh(b_2)}t^{-b_2}\right]\right\}.\end{aligned}$$

(b) In similar manner to equation (7.68), we can write a Caputo fractional differential equation of bicomplex-valued function of a circuit of transmission line as follows:

$$t^{\alpha}\,{}^{C}_{t}D^{\alpha}_{\infty}F(x,t) + t^{\beta}\,{}^{C}_{t}D^{\beta}_{\infty}F(x,t) = A\delta(t-a)\delta(x-a), \quad A \in C_2 \tag{7.74}$$

Taking the bicomplex Mellin transform of (7.74) w.r.t. t, we get

$$\frac{\Gamma(\xi+\alpha)}{\Gamma(\xi)}\bar{F}(x,\xi)+\frac{\Gamma(\xi+\beta)}{\Gamma(\xi)}\bar{F}(x,\xi)=A\delta(x-a)a^{\xi-1}$$
$$\therefore\ \bar{F}(x,\xi)=A\delta(x-a)\frac{\Gamma(\xi)a^{\xi-1}}{\Gamma(\xi+\alpha)+\Gamma(\xi+\beta)}.$$

Taking the inverse bicomplex Mellin transform of $\bar{F}(x,\xi)$, we get

$$F(x,t)=V+i_2I=\frac{A}{2\pi i_1}\delta(x-a)\int_{\Omega}t^{-\xi}\frac{\Gamma(\xi)a^{\xi-1}}{\Gamma(\xi+\alpha)+\Gamma(\xi+\beta)}d\xi \tag{7.75}$$

where Ω defined in (7.16). By separating the bi-real and bi-imaginary part of (7.75), we obtain the voltage and current of the given circuit of transmission line.

7.6 Conclusion

The concept of bicomplex numbers has been applied for finding the solution of differential equations of bicomplex-valued function generated by network diagram. In this chapter, we define Mellin transform and its inverse in bicomplex space which is the generalization of complex Mellin transform. Also, we find bicomplex Mellin transform of some useful properties of fractional operators, which are useful for finding the solution of fractional differential equation of bicomplex-valued function.

The applications have been illustrated to find the solution of partial differential equation of bicomplex-valued function generated by a network and for finding the solution of transmission line equations in fractional form. The bicomplex analysis has great advantage that it separates the voltage and current as complex components.

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Bio-Data

Mahesh Puri Goswami

Academic Qualifications

July, 2013 – Till Date	:	Research Scholar, Department of Mathematics Malaviya National Institute of Technology, Jaipur.
2012	:	RPSC State Eligibility Test Qualified.
2013	:	GATE Qualified.
July, 2010 – June, 2012	:	Master of Science Central University of Rajasthan, Ajmer
July, 2006 – June, 2009	:	Bachelor of Scienece Maharaja's College, University of Rajasthan, Jaipur.

Teaching Experience

Assistant Professor	:	July, 2012 – January 2013, Poornima Group of Institutions, Jaipur.
Lecturer	:	February, 2013 – July 2013, Vivekananda Institute of Technology, Jaipur.

7. BIO-DATA

Research Publications

1. Agarwal R., Goswami M.P. and Agarwal R.P. (2017), Sumudu Transform in Bicomplex Space and Its Applications, *Annals of Applied Mathematics*, (In Press).

2. Agarwal R., Goswami M.P., Agarwal R.P., Venkataratnam K.K. and Baleanu D. (2017), Solution of Maxwell's wave equations in bicomplex space, *Romanian Journal of Physics,* **62**(5-6), Article no. 115, 1-11.

3. Agarwal R., Goswami M.P. and Agarwal R.P. (2017), A study of Mellin transform of fractional operators in bicomplex space and application, *Journal of Fractional Calculus and Applications,* **8**(2), 211-226.

4. Agarwal R., Goswami M.P. and Agarwal R.P. (2017), Mellin transform in bicomplex Space and its application, *Studia Universitatis Babes-Bolyai Mathematica*, **62**(2), 217-232.

5. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Hankel Transform in Bicomplex Sapce with Application, *Transylvanian Journal of Mathematics and Mechanics*, **8**(1), 1-14.

6. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Bochner Theorem and Applications of Bicomplex Fourier-Stieltjes Transform, *Advanced Studies in Contemporary Mathematics,* **26**(2), 355-369.

7. Agarwal R., Goswami M.P. and Agarwal R.P. (2016), Double Laplace Transform in Bicomplex Space with Applications, *CUBO: A Mathematical Journal*, **18**(2), (In press).

8. Agarwal R., Goswami M.P. and Agarwal R.P. (2015), Tauberian Theorem and Applications of Bicomplex Laplace-Stieltjes Transform, *Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms*, **22**, 141-153.

9. Agarwal R., Goswami M.P. and Agarwal R.P. (2014), Bicomplex Version of Stieltjes Transform and Applications, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms*, **21**(4-5), 229-246.

10. Agarwal R., Goswami M.P. and Agarwal R.P. (2014), Convolution Theorem and Applications of Bicomplex Laplace Transform, *Advances in Mathematical Sciences and Applications*, **24**(1), 113-127.

Participation in the events

A. Conferences

1. Presented paper titled *"Mellin Transform in Bicomplex Space and Its Application"* in the **International conference on Finite Infinite Dimensional Complex Analysis and Applications (24'ICFIDCA 2016)** held at Department of Mathematics, Anand International Engineering college, Jaipur during August 22-26, 2016.

2. Presented paper titled as *"A Study of Mellin Transform of Fractional operators in Bicomplex Space and Application"* in the **International conference on Recent Advances in Mathematics and Their Applications (ICRAMTA 2016)** held at Department of Mathematics, University of Rajasthan, Jaipur during July 10-12, 2016.

3. Presented a paper titled as *"Hankel Transform in Bicomplex Space"* in the **International conference on Analysis and Its Applications (ICAA 2015)** held at Department of Mathematics, Aligarh Muslim University, Aligarh during December 19-21, 2015.

4. Presented paper titled as *"Bicomplex Double Laplace Transform and Applications"* in the **International conference on Special Functions** & **Their Applications (ICSFA 2015)** held at Department of Mathematics Amity Institute of Applied Sciences, Amity university, Noida during September 10-12, 2015.

5. Presented paper titled as *"Tauberian Theorem and Applications of Bicomplex Laplace-Stieltjes Transform"* in **National conference on Mathematical Analysis and Computation (NCMAC 2015)** held at Department of Mathematics and Department of Computer Science and Engineering, Malaviya National Institute of Technology, Jaipur during February 20-21, 2015.

6. Presented paper titled as *"Bochner's Theorem of Bicomplex Fourier-Stieltjes Transform"* in the **National conference on Computational and Mathematical Sciences (COMPUTATIA-IV 14)** held at Vivekananda Institute of Technology Jaipur during November 25-26, 2014.

7. Presented paper titled as *"Bicomplex version of Laplace-Stieltjes Transform and Applications"* in the **National conference on Science Engineering (NCSE 14)** held at JK Laxmipat University Jaipur during July 27-28, 2014.

8. Presented paper titled as *"Bicomplex version of Stieltjes Transform"* in the **National on conference Mathematical Techniques in Engineering Applications** held at Vivekananda Global University Jaipur during April 4-5, 2014.

9. Presented paper titled as *"Bicomplex version of Inverse Laplace Transform"* in the **National conference on Complex Analysis in Honour of Late prof. K.S. Padmanabhan** held at Department of Mathematics, Central University of Rajasthan Kishangarh, Ajmer during March 8-9, 2014.

B. Short-Term courses

1. Worked as tutorial assistant in the workshop on Latex for Research held at Malaviya National Institute of Technology, Jaipur during July 23-24, 2016.

2. Participated in the National Centre for Mathematics Instructional School for Teachers on Algebra organized at the department of Mathematics, Malaviya National Institute of Technology, Jaipur form December 07-19, 2015.

3. Participated in the Five days Short Term Course on Mathematical Modeling, MATLAB Programming and their Applications in Engineering and Sciences organized by the department of Mathematics, Malaviya National Institute of Technology, Jaipur form January 19-23, 2015.

4. Participated in the Five days Short Term Course on Analysis and Applications (STCAA 2014) organized by the department of Mathematics, Malaviya National Institute of Technology, Jaipur form November 10-14, 2014.

5. Participated in the Two day workshop on Professional Communication organized by the department of Humanities & Social Sciences, Malaviya National Institute of Technology, Jaipur form July 10-11, 2014.