# A STUDY OF DIFFERENTIAL AND INTEGRAL CALCULUS OF ARBITRARY ORDER AND NEW GENERALIZED MITTAG-LEFFLER FUNCTION WITH APPLICATIONS

**THESIS**

submitted in fulfilment of the requirements for the degree of

**DOCTOR OF PHILOSOPHY**

**SHEIKH MOHAMMED FAISAL**

**(2011RMA7121)**

Department of Mathematics

**MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY JAIPUR**

**December, 2016**

**DEDICATED**

**TO MY DEAR FATHER**

**LATE MR. MOHAMMED ISHAQ**

# CERTIFICATE

This is to certify that the thesis entitled **"A Study of Differential and Integral Calculus of Arbitrary Order and New Generalized Mittag-Leffler Function with Applications"** submitted by **Sheikh Mohammed Faisal** to the Malaviya National Institute of Technology Jaipur for the award of the degree of **Doctor of Philosophy (Ph.D.)** is a bonafide record of original research work carried out by him under my supervision in conformity with rules and regulations of the institute. The results contained in this thesis have not been submitted, in part or in full, to any University or Institute for the award of any diploma or degree.

**Dr. Sanjay Bhatter**
Assistant Professor
Department of Mathematics
Malaviya National Institute of Technology Jaipur.

Date: December, 2016
Place: Jaipur, India.

# DECLARATION

I hereby declare, the thesis entitled **"A Study of Differential and Integral Calculus of Arbitrary Order and New Generalized Mittag-Leffler Function with Applications"** has been carried out by me during working as a full time research scholar under the supervision of **Dr. Sanjay Bhatter** in the Department of Mathematics, Malaviya National Institute of Technology Jaipur for the degree of **Doctor of Philosophy**.

As per my best knowledge this thesis contains original texts except where otherwise indicated and no material has been published in whole or in part, anywhere in the form of books, monographs or articles, for the award of any academic diploma or degree to any University or Institute. This thesis is originally my own work.

**Sheikh Mohammed Faisal**

Date: December, 2016
Place: Jaipur, India.

## ACKNOWLEDGEMENTS

All praise be to Allah, blessings on prophet Muhammad (saw), on his family and all of his companions.

First of all, I express my sincere thanks to my supervisor, Dr. Sanjay Bhatter, Assistant Professor, Department of Mathematics, Malaviya National Institute of Technology, Jaipur. He has been supportive since I joined as a research scholar under his supervision. I am grateful for his valuable guidance and providing the opportunity and facility to produce and complete this research work with perfection.

I convey lots of thanks to Professor I.K. Bhatt, Director, Malaviya National Institute of Technology Jaipur, for his kind help, support and facilities to produce this research work.

I offer my deep regards and gratitude to Dr. Vatsala Mathur, Head of the Department of Mathematics, for her kind care and help during the entire period of research.

I am indebted to Professor KC Jain, Convenor of DPGC and the former Head of Department of Mathematics for his kind help and also grateful to all DREC faculty members and other faculty members for their encouragement and support during the entire period of research.

Words are inadequate to express my affection and attachment towards my colleagues and friends for their valuable suggestions, motivation and help to complete this research work.

I will be neglecting my duty, if I do not express my sincere regards and thanks to clerical staff and peon staff of the Department of Mathematics, for their constant help and support throughout the period of research work.

Last but not the least I convey my best regards to my beloved mother, brother, sister and wife for their patience, care and encouragement for the entire period of the research work.

**Sheikh Mohammed Faisal**

# ABSTRACT

This thesis contains total 6 chapters along with two appendices including zero chapter which provides an introduction to the topic of study and includes a brief survey of the contribution made by many authors on the earlier matter presented in the thesis. Besides the zero chapter there are five more chapters whose outlines are as follows:

- In **Chapter 1,** we introduce a new Mittag-Leffler (M-L) type function named $E$-function. Then we establish its conditions of convergence and obtain two interesting special cases (generalized sine and cosine function) which are believed to be new and important. Further derive Mellin-Barnes type contour integral representation of $E$-function and finally establish some integral transforms like Mellin transform, Laplace transform, Euler-Beta transform and Whittaker transform of the newly defined function.
- In **Chapter 2,** we prove efficiency and usefulness of the $E$-function, by establishing relations of $E$-function with well-known special functions such as generalized hypergeometric function, Fox's $H$-function, $\overline{H}$-function and Wright function. Further we obtain known M-L type functions as special cases of the $E$-function. Finally we obtain Bessel function, Bessel Maitland function, generalized Bessel Maitland function, Bessel Clifford function, Lommel function, Hurwitz zeta function, Riemann zeta function, Struve function, modified Struve function, Dotsenko function, Rabotnov's function and Mellin-Ross function as particular cases of the $E$-function.
- In **Chapter 3,** we define two fractional integral operators whose kernels involve generalized multivariable polynomial and the $E$-function. We define a pair of multidimensional fractional integral operators $I$ and $J$ and give the conditions of existence. Then under these operators we obtain images of important functions. After this, we prove two theorems connecting the multidimensional generalized Stieltjes transform and the newly introduced integral operators here. Then, we establish Mellin transform, Mellin convolutions and inversion formulae of these

operators. Finally, we study three composition formulae of the multidimensional fractional integral operators and obtain two dimensional analogue of second composition formula.

- In **Chapter 4,** we establish Riemann-Liouville, Erdélyi-Kober and a more generalized fractional integral transformation of the $E$-function and then obtain various special cases. Finally discuss the second form of Mellin-Barnes type contour integral representation of the $E$-function and then obtain various special cases.
- In **Chapter 5,** we discuss essentials of fractional calculus and operate a generalized Saigo-fractional derivative operator upon the $E$-function. Finally establish some important theorems on fractional differentiation of the $E$-function and at the end of the chapter we give a concluding remark.

# Contents

# CHAPTER 0

# INTRODUCTION

The main object of this chapter is to provide an introduction to the topics of study and include a brief survey of the contribution made by many authors on the earlier matter presented in the thesis. A short chapterwise description of the thesis has also been added at the end of the chapter.

Throughout this thesis, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive and non-positive integers, respectively and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

## 0.1 THE GAUSSIAN HYPERGEOMETRIC FUNCTION AND ITS GENERALIZATIONS

The term 'hypergeometric' (from the Greek word $\nu\pi\varepsilon\rho$ for hyper, above or beyond) first used by John Wallis in 1655 in the work Airthmetica Infinitorum, to present any series which was advancement of the ordinary geometric series $1 + x + x^2 + x^3 + \dots$. In particular, he studied the series

$$1 + a + a\,(a+1) + a\,(a+1)\,(a+2) + \dots .$$

A large number of functions of this kind have been defined and studied, but the most common are the hypergeometric functions. In 1812, the well known mathematician C. F. Gauss defined and studied the following infinite series which is an extension of the earlier defined geometric series and called Gauss series or Gauss hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a \cdot (a+1) \cdot b \cdot (b+1)}{1 \cdot 2 \cdot c \cdot (c+1)} z^2 + \ldots \qquad (0.1.1)$$

where

$$(a)_n = \prod_{r=1}^{n} (a + r - 1) = a (a+1) \ldots (a+n-1) \qquad (0.1.2)$$

$a \neq 0, n \in \mathbb{N};\ (a)_0 = 1;\ c \neq 0, -1, -2, \ldots.$

The function $(a)_n$ is called the factorial function (or Pochhammer symbol).

Gauss denoted this series (0.1.1) by ${}_2F_1(a, b; c; z)$, where $a, b, c$ and $z$ may be real or complex. The function reduces to a polynomial, if either of the numbers $a$ or $b$ takes a value as non-positive integer, but if $c$ takes a value as non-positive integer then the function does not remain defined since all but a finite number of terms of the series become infinite.

If we replace $z$ by $z/b$ and let $b \to \infty$ in equation (0.1.1), then we get

$$\frac{(b)_n z^n}{b^n} \to z^n$$

and we obtain the following well known Kummer's series

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n = 1 + \frac{a}{1 \cdot c} z + \frac{a \cdot (a+1)}{1 \cdot 2 \cdot c \cdot (c+1)} z^2 + \ldots \qquad (0.1.3)$$

it is known as confluent hypergeometric function and denoted by ${}_1F_1(a; c; z)$.

Note: If k is a positive integer and n is a non-negative integer, then

$$(\alpha)_{kn} = k^{nk} \left(\frac{\alpha}{k}\right)_n \left(\frac{\alpha+1}{k}\right)_n \cdots \left(\frac{\alpha+k-1}{k}\right)_n. \tag{0.1.4}$$

Generalization of ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$, which is defined by this series

$${}_pF_q\left[\begin{array}{c} a_1, a_2, \ldots, a_p; \\ \\ b_1, b_2, \ldots, b_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{0.1.5}$$

where $z$ is a variable and all the parameters $a_1, a_2, \ldots, a_p$; $b_1, b_2, \ldots, b_q$ are real or complex numbers such that no denominator parameter is negative integer or zero, and $p$ and $q$ are either positive integers or zero, and an empty product is interpreted as unity,

The conditions of convergence of the function ${}_pF_q$ are as follows:

1. When $p \leq q$, then the series on the right hand side of eqution (0.1.5) is convergent for all values of z.

2. When $p = q + 1$, then the series (0.1.5) is convergent if $\mid z \mid < 1$ and divergent when $\mid z \mid > 1$, and on the circle $\mid z \mid = 1$, the series (0.1.5) is

   (a) Absolutely convergent if $\Re(w) > 0$;

   (b) Conditionaly convergent if $-1 < \Re(w) < 0$ for $z \neq 1$;

   (c) Divergent if $\Re(w) \leq -1$,

   where

$$w := \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$$

3. When $p > q+1$, then the series (0.1.5) never converges except when $z = 0$ and the function is defined only when the series terminates.

A detailed description of the functions ${}_2F_1, {}_1F_1$, and ${}_pF_q$ can be found in the works of Exton [39], Luke [112], Rainville[156], and Slater [186] and their applications can be found in Mathai and Saxena [125].

## 0.2 THE FOX $H$-FUNCTION

To explore the study in the direction of condition $p > q+1$, in the series (0.1.5), C.S. Meijer defined and studied a more generalized function which are now well known in the literature as $G$-function [125]. Although the $G$-function contains many special functions as its particular cases, even though many functions such as Lorenzo Hartley $R$ and $G$-functions [110], reduced Green function [116], Mittag-leffler function [133], Wright generalized hypergeometric function [216], Wright generalized Bessel function [217], and many other functions do not form its particular cases.

In 1961, Charles Fox [43] introduced and studied a more generalized function, named $H$-function, since then it has become well known in literature. Usefulness of this function has been published in many research articles and books during the last five decades and a vast collection of the work on $H$-function can be seen in the literature by Kilbas and Saigo [90], Mathai and Saxena [126], and Srivastava, Gupta and Goyal [196].

The Fox $H$-function is introduced by means of the following Mellin-Barnes type of contour integral as follows:

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_j,\,A_j)_1^p\\ \\ (b_k,\,B_k)_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\Lambda(s)z^s ds,\quad z\neq 0. \tag{0.2.1}$$

Here

$$\Lambda(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_js\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_js\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-A_js\right)}, \tag{0.2.2}$$

where $m, n, p$ and, $q$ are non-negative integers satisfying $0 \leq n \leq p$, $0 \leq m \leq q$ and empty products are taken as unity. Also, $A_j(j = 1, \ldots, p)$ and $B_j(j = 1, \ldots, q)$ are positive real numbers for standardization purpose, $a_j(j = 1, \ldots, p)$ and $b_j(j = 1, \ldots, q)$ are complex numbers satisfying $A_j(b_h+\nu) \neq B_h(a_j-\lambda-1)$ for $\nu, \lambda = 0, 1 \ldots; h = 1, \ldots, m; j = 1, \ldots, n$. The contour $\mathcal{L}$ in $\mathbb{C}$ is such that the poles of $\Gamma(b_j-B_js)(j = 1, \ldots, m)$ are separated from the poles of $\Gamma(1-a_j + A_js)(j = 1, \ldots, n)$ such that the poles of $\Gamma(b_j - B_js)$ lie to the left of $\mathcal{L}$, while the poles of $\Gamma(1-a_j + A_js)$ are to the right of $\mathcal{L}$. The poles of the integrand are assumed to be simple. The $H$-function is an analytic function of $z$ for every $|z| \neq 0$ when $\mu > 0$ and for $0 < |z| < 1/\beta$ when $\mu = 0$, where $\mu$ and $\beta$ are defined as

$$\mu=\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j \tag{0.2.3}$$

and

$$\beta=\prod_{j=1}^{p}A_j^{A_j}\prod_{j=1}^{q}B_j^{-B_j}. \tag{0.2.4}$$

A large number of special functions of one variable are special cases of the $H$-function, so each formula derived for the $H$-function becomes a key formula from which several results involving other simple special functions can be developed by suitably specializing the parameters involved.

## 0.3 THE $\overline{H}$-FUNCTION

A function more general than the Fox $H$-function has been defined in 1987 by Inayat Hussain [79]. A comprehensive account of this function can be found in the work by Buschman and Srivastava [14], Gupta, Jain and Agrawal [67], Rathie [158], Saxena [165], and Saxena et al. [168, 172]. This function is called $\overline{H}$-function and defined as follows:

$$\overline{H}^{m,n}_{p,q}\left[z\left|\begin{array}{l}(a_j,\,A_j;\,\alpha_j)_1^n;(a_j,\,A_j)_{n+1}^p\\ (b_j,\,B_j)_1^m;(b_j,\,B_j;\,\beta_j)_{m+1}^q\end{array}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\chi(s)z^s ds, \qquad (0.3.1)$$

$$z\neq 0; i=\sqrt{(-1)}; \chi(s):=\frac{\prod\limits_{j=1}^{m}\Gamma\left(b_j-B_js\right)\prod\limits_{j=1}^{n}\left\{\Gamma\left(1-a_j+A_js\right)\right\}^{\alpha_j}}{\prod\limits_{j=m+1}^{q}\left\{\Gamma\left(1-b_j+B_js\right)\right\}^{\beta_j}\prod\limits_{j=n+1}^{p}\Gamma\left(a_j-A_js\right)}, \qquad (0.3.2)$$

where $a_i$, $b_j$ are complex parameters and $m, n, p$ and, $q$ are integers satisfying $0 \leq n \leq p$, $0 \leq m \leq q$, it contains *fractional* powers of some of the Gamma functions involved. Here, and in what follows, the parameters

$$A_j \geq 0 \quad (j=1,\ldots,p) \qquad \text{and} \qquad B_j \geq 0 \quad (j=1,\ldots,q),$$

not all zero simultaneously and the exponents

$$\alpha_j \quad (j = 1, \ldots, n) \qquad \text{and} \qquad \beta_j \quad (j = m+1, \ldots, q),$$

can take on noninteger values, and $\mathcal{L} = \mathcal{L}_{(i\tau;\infty)}$ is a Mellin-Barnes type contour starting at the point $\tau - i\infty$ and terminating at the point $\tau + i\infty$ $(\tau \in \mathbb{R})$ with the usual indentations to separate one set of poles from the other set of poles. The sufficient condition for the absolute convergence of the contour integral in (0.3.1) was established by Buschman and Srivastava [14, p. 4708] as follows:

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |\alpha_j A_j| - \sum_{j=m+1}^{q} |\beta_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0, \qquad (0.3.3)$$

which provides the exponential decay of the integrand in (0.3.1), and the region of absolute convergence of the contour integral in (0.3.1) is given by

$$|\arg(z)| < \frac{1}{2}\pi\Omega,$$

where $\Omega$ is defined by (0.3.3).

## 0.4 GENERAL CLASS OF POLYNOMIALS

Jacobi, Laguerre, Hermite, Konhauser polynomials are the classical orthogonal polynomials and extended Jacobi polynomials, Brafman polynomials are hypergeometric polynomials and also several other polynomials play vital role in the study of many branches of mathematical sciences and other sciences. Almost all the above given polynomials can be obtained as particular cases of the following general class of polynomials defined by Srivastava [188]

$$S_V^U[x] = \sum_{k=0}^{\infty} \frac{(-V)_{Uk} A_{V,k}}{k!} x^k, \qquad (V = 0, 1, \ldots), \qquad (0.4.1)$$

where the coefficients $A_{V,k}$ are arbitrary constants (real or complex) and $U$ is an arbitrary positive integer. At the end of this thesis a detail of some of the particular cases of the above given class of polynomials has been given in the Appendix-B.

## 0.5 THE MULTIVARIABLE GENERALIZATIONS OF THE $S_V^U$ POLYNOMIAL

In the present thesis we shall study the following generalization of the $S_V^U$ polynomial (0.4.1) introduced and defined by Srivastava and Garg [195, p. 686, Eq. (1.4)] as follows:

$$S_V^{U_1,\ldots,U_k}[x_1,\ldots,x_k] = \sum_{\substack{R_1,\ldots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \leq V}} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V, R_1, \ldots, R_k) \frac{x_i^{R_i}}{R_i!}, \quad (0.5.1)$$

where V= 0,1,...; $U_1, \ldots, U_k$ are arbitrary positive integers and the coefficients $A(V, R_1, \ldots, R_k)$ are arbitrary constants (real or complex). Several single and general multivariable polynomials can be obtained as special cases of general multivariable polynomial $S_V^{U_1,\ldots,U_k}(x_1, \ldots, x_k)$ by replacing coefficients $A(V, R_1, \ldots, R_k)$ occuring in (0.5.1) with a suitable function. Further details of this polynomial and its special cases can be seen in Appendix B.

## 0.6 FRACTIONAL CALCULUS

In the year 1695, Marquis de L'Hospital asked a question to Gottfried Wilhelm Leibniz regarding a solution of derivative $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$. On September $30^{th}$ of 1695, Leibniz replied to L'Hospital "This is an apparent paradox from which one day, useful consequences will be drawn". It was the begining of a new concept of fractional calculus (calculus of integrals and derivatives of any arbitrary real or complex order). Between 1695 and 1819 several mathematicians such as Euler in 1730, Lagrange in 1772, Laplace in 1812 and S. F. Lacroix in 1819, had studied it. The real journey of progress of fractional calculus started in 1974, when the first article on fractional calculus was published [141].

A detailed description of the development and applications in the field of fractional calculus can be seen in literature by Caputo [18], Gorenflo and Vessella [53], Kiryakova [94], McBride [129], Miller and Ross [132], Nishimoto [135], Podlubny [146] and Samko, Kilbas and Marichev [164].

The fractional calculus is useful in several fields of science and engineering, including the quantitative biology, fluid flow, rhelogy, electromagnetic theory, electro-chemistry, scattering theory, electrical networks, chemical physics, diffusion transport theory and statistical probability theory, potential theory and many more branches of mathematical sciences like integral and differential equations, univalent function theory and operational calculus.

The analysis of fractional calculus is based upon the study of the known fractional integral operator ${}_aD_z^\alpha$ introduced by Lavoie et al. [111] and Ross [159] as follows:

$$ {}_aD_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_a^z (z-y)^{-\alpha-1} f(y)\, dy, \qquad \Re(\alpha) < 0, \tag{0.6.1} $$

$$ = \frac{d^m}{dz^m} {}_aD_z^\alpha f(z), \qquad \Re(\alpha) \geq 0, \tag{0.6.2} $$

where the above involved integral exists and m is the least positive integer greater than $\Re(\alpha)$.

For $a = 0$, the fractional integral operator defined by (0.6.1) becomes the classical Riemann Liouville fractional integral operator of order $(-\alpha)$ and when $a \to \infty$, it can be reduced to the definition of the well known Weyl fractional integral operator of order $(-\alpha)$.

On account of the significance of the fractional calculus operators (FCO) in many problems of mathematical physics and applied mathematics, several generalizations of the FCO defined by Riemann–Liouville and Weyl have been analysed from time to time by many authors like Erdélyi [31, 32], Garg [49], Garg and Purohit [48], Gupta [64], Kalla [83], Kalla and Saxena [85], Koul [100], Kober [98], Manocha [122], Raina and Kiryakova [155], Saigo [160], Sneddon [187].

Details of several fractional integral operators studied by many researchers can be seen in the work of Srivastava and Saxena [202]. In the present work we have defined and studied two unified fractional integral operators whose kernels involve the product of a multivariable polynomial $S_V^{U_1,\ldots,U_k}$ and a newly defined Mittag-Leffler type $E$-function in this thesis.

## 0.7 MITTAG-LEFFLER FUNCTION

The $H$-function [126] is the generalized solution of integer order differential equations. On the other hand, Mittag-Leffler function [133] is recognized as solution of fractional differential and fractional integral equations [132].

The Mittag-Leffler (M-L) function introduced in 1903 due to Gösta Mittag-Leffler is a generalization of the exponential function $e^z$. The first significance of this function was noticed in 1930, when Hille and Tamarkin [76] provided a solution of the Abel-Volterra type integral equation of the 2nd kind in terms of the M-L functions.

Barret [7, (1954)] has proved the most remarkable application of M-L type functions by presenting the general solution of the linear fractional differential equation with constant coefficients in terms of the M-L type functions. Caputo and Mainardi [20, 21, (1971)] have shown that when constitutive equations of linear viscoelastic body involve derivatives of fractional order, then they provide a solution in the form of M-L type functions.

Recently some pioneer work on M-L type functions has been done by Camargo et al. [16], who studied the fractional Langevin equation in terms of the three-parameter M-L function, and also presented the corresponding relaxation function in terms of the convenient M-L functions.

Mittag-Leffler type functions of several parameters have been studied by many authors due to its applications in certain problems such as telegraph equation [17], random walks and anomalous diffusion [131] and kinetic equation [146] in a fractional version and many other problems of mathematics, Physics, Biology and other sciences [105, 106, 116, 119, 219].

The journey of M-L function started as a generalization of exponential function and later its many generalizations were developed and studied by Kiryakova [95], Prabhakar [148], Shukla-Prajapati [180], Srivastava-Tomovski [204], and many other authors and they have proved its importance in many physical phenomena.

### 0.7.1 Journey of Mittag-Leffler Type Functions (1903-2015)

- In 1903, Gösta Mittag-Leffler [133] introduced the function $E_\alpha$(z):

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+1)} z^n , \tag{0.7.1}$$

where $z, \alpha \in \mathbb{C}; \Re(\alpha) \geq 0$ and $|z| < \infty$.

- In 1905, Wiman [215] extended (0.7.1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} z^n , \tag{0.7.2}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0$ and $\Re(\beta) > 0$.

- In 1953, Humbert and Agarwal [78] have studied the properties of a slightly more general function defined by

$$E^*_{\alpha,\beta}(z) = z^{\frac{\beta-1}{\alpha}} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} z^n , \tag{0.7.3}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0$ and $\Re(\beta) > 0$.

- In 1960, Dzrbashjan [29] proposed a generalization of the M-L function in the form

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha_1 n + \beta_1)\,\Gamma(\alpha_2 n + \beta_2)}\, z^n\,, \tag{0.7.4}$$

where $\alpha_1, \alpha_2 > 0; \beta_1, \beta_2 \in \mathbb{R}$ and $z \in \mathbb{C}$.

- In 1971, Prabhakar [148] introduced a generalization of (0.7.1) in terms of series representation as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}\,, \tag{0.7.5}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$ and $\Re(\gamma) > 0$.

- In 1995, Kilbas and Saigo [89] introduced and studied a further generalization of M-L type function in the form

$$E_{\alpha,m,l}(z) = 1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} \frac{\Gamma[\alpha(jm + l) + 1]}{\Gamma[\alpha(jm + l + 1) + 1]}\, z^n\,, \tag{0.7.6}$$

where $z, \alpha \in \mathbb{C}, \Re(\alpha) > 0, m > 0$ and $l \in \mathbb{R}$.

- In 2000, Kiryakova [95] has studied "multiindex M-L functions" defined by

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu_1 + n/\rho_1)\ldots\Gamma(\mu_m + n/\rho_m)}\, z^n\,, \tag{0.7.7}$$

where $m > 1$, is an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ are real numbers.

- In 2002, Kilbas, Saigo and Trujillo [93] considered the following generalized M-L function

$$E_{\rho}\left[(\beta_1,\sigma_1),\ldots,\left(\beta_q,\sigma_q\right);z\right]=\sum_{n=0}^{\infty}\frac{(\rho)_n}{\prod_{j=1}^{q}\Gamma\left(\sigma_j n+\beta_j\right)}\frac{z^n}{n!},\qquad(0.7.8)$$

where $\Re\left(\sigma_j\right)>0,\Re\left(\beta_j\right)>0,j=1,...,q$.

- In 2007, Shukla and Prajapati [180] have studied a generalization of (0.7.5) in the following form

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\qquad(0.7.9)$$

where $z,\alpha,\beta,\gamma\in\mathbb{C};\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0$ and $q\in(0,1)\,U\,\mathbb{N}$.

- In 2009, Srivastava and Tomovski [204] introduced and studied another generalization of M-L function in the form

$$\breve{E}_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\delta}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\qquad(0.7.10)$$

where $z,\alpha,\beta,\gamma,\delta\in\mathbb{C};\Re(\alpha)>\max\{0,\ \Re(\delta)-1\},\Re(\beta)>0,\Re(\gamma)>0$ and

$\Re(\delta)>0$.

- In 2010, Saxena and Nishimoto [171] studied a function as follows:

$$E_{\gamma,\kappa}[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z]=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\kappa}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j\right)}\frac{z^n}{n!},\qquad(0.7.11)$$

where $z,\alpha_j,\beta_j,\gamma\in\mathbb{C},\sum_{j=1}^{m}\Re(\alpha_j)>\Re(\kappa)-1,j=1,\ldots,m$ and $\Re(\kappa)>0$.

- In 2011, Saxena, Kalla and Saxena [169] defined a function as

$$E^{(\gamma_1,\ldots,\gamma_m)}_{(\rho_1,\ldots,\rho_m),\lambda}(z_1,\ldots,z_m)$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{(\gamma_1)_{n_1}\ldots(\gamma_m)_{n_m}}{\Gamma\left[\lambda+\sum_{j=1}^{m}\left(\rho_j n_j\right)\right]} \frac{z_1^{n_1}\ldots z_m^{n_m}}{(n_1)!\ldots(n_m)!}, \tag{0.7.12}$$

where $z_j, \lambda, \gamma_j, \rho_j \in \mathbb{C}$ and $\Re(\rho_j) > 0, j = 1, \ldots, m$.

- In 2012, Kalla, Haidey and Virchenko [84] introduced multiparameter M-L type function in the following form

$$HE^{\lambda_1,\lambda_2,\ldots,\lambda_r}_{\mu_1,\mu_2,\ldots,\mu_r}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\prod_{i=1}^{r}\Gamma(1+\mu_i+\lambda_i n)} \left(\frac{z}{\Lambda}\right)^{\Lambda n+M}, \tag{0.7.13}$$

where $\mu_i \in \mathbb{C}, \lambda_i > 0, i = 1, \ldots, r; \sum_{i=1}^{r}\mu_i = M$ and $\sum_{i=1}^{r}\lambda_i = \Lambda$.

- In 2012, Salim and Faraz [163, see also [162]] defined a function as

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{pn}} z^n, \tag{0.7.14}$$

where $z, \alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$ and

$q \leq \Re(\alpha) + p$.

- In 2013, Khan and Ahmad [88, see also [87]] have defined a function as

$$E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}} z^n, \tag{0.7.15}$$

where $z, \alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}; p, q > 0;\ q \leq \Re(\alpha) + p$ and

$\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\mu), \Re(\nu), \Re(\rho), \Re(\sigma)\} > 0$.

## 0.8 INTEGRAL TRANSFORMS

If $f(x)$ is a function defined on a given interval $[a, b]$ and $K(x, s)$ is a definite function of $x$ on the same interval for each value of parameter s, so the linear integral transform $T\left[f(x); s\right]$ of the function $f(x)$ is defined as follows:

$$T\left[f\left(x\right); s\right] = \int_{a}^{b} K(x, s) f(x) dx, \tag{0.8.1}$$

where the domain of parameter s and the class of functions are so prescribed that the above integral exists. $K(x, s)$ is the kernel of the transform, $T[f(x); s]$ is the image of $f(x)$ and $f(x)$ is the original of $T[f(x); s]$. When an integral equation can be so obtained that

$$f\left(x\right) = \int_{\alpha}^{\beta} \phi\left(s, x\right) T[f(x); s] ds, \tag{0.8.2}$$

then (0.8.2) is called the inversion formula of (0.8.1).

### 0.8.1 Mellin Transform

The Mellin transform [187] of the function $f\left(z\right)$ with respect to $\zeta$ is given by

$$M\left[f\left(z\right); \zeta\right] = \int_{0}^{\infty} z^{\zeta-1} f\left(z\right) dz = f^{*}\left(\zeta\right), \qquad \Re(\zeta) > 0 \tag{0.8.3}$$

and the inverse Mellin transform of $f^{*}\left(\zeta\right)$ with respect to $z$ is given by

$$M^{-1}\left[f^{*}\left(\zeta\right); z\right] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} z^{-\zeta} f^{*}\left(\zeta\right) d\zeta = f\left(z\right), \qquad \gamma \in \mathbb{R}, \tag{0.8.4}$$

provided that both the integrals exist.

### 0.8.2 Laplace Transform

If the function $f(z) = O(e^{\alpha z})$, z$\rightarrow \infty$ for some $\alpha$ then the Laplace transform [187] of the function $f(z)$ with respect to $s$, is given by

$$L[f(z); s] = \int_0^\infty e^{-sz} f(z)\, dz = F(s), \qquad \Re(s) > \alpha, \tag{0.8.5}$$

it can be obtained by appealing to Euler-integral of the II kind

$$\int_0^\infty e^{-sz} z^{\lambda-1} dz = \frac{\Gamma(\lambda)}{s^\lambda}, \qquad \min\{\Re(\lambda), \Re(s)\} > 0 \tag{0.8.6}$$

and the inverse Laplace transform of $F(s)$ with respect to $z$ is given by

$$L^{-1}[F(s); z] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sz} F(s)\, dz = f(z), \qquad c \in \mathbb{R}, \tag{0.8.7}$$

provided that both the integrals exist.

### 0.8.3 Euler-Beta Transform

The generalized Euler-Beta transform [187] of the function $f(z)$ with respect to $\mu$ and $\nu$ is given by

$$B[f(z); \mu, \nu : a, b] = \int_a^b (z-a)^{\mu-1} (b-z)^{\nu-1} f(z)\, dz, \tag{0.8.8}$$

provided the integral exists and generalized Beta function is defined as

$$\int_a^b (z-a)^{\mu-1} (b-z)^{\nu-1}\, dz = (b-a)^{\mu+\nu-1} \mathbf{B}(\mu, \nu)$$

$$= (b-a)^{\mu+\nu-1} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \tag{0.8.9}$$

where $\Re(\mu) > 0, \Re(\nu) > 0,\ a, b \in \mathbb{R}$.

### 0.8.4 Whittaker Transform

The Whittaker transform [214] of the function $f(z)$ with respect to $\lambda$, $\mu$ and $\nu$ is given by

$$\mathcal{W}[f(z);\lambda,\mu,\nu] = \int_0^\infty e^{-\frac{z}{2}} z^{\nu-1} W_{\lambda,\mu}(z) f(z)\, dz, \tag{0.8.10}$$

provided the integral exists, where $W_{\lambda,\mu}(z)$ is Whittaker's confluent hypergeometric function and associated integral is given in [36, p. 215], the equation (0.8.10) can be solved by appealing the following integral

$$\int_0^\infty e^{-\frac{z}{2}} z^{\nu-1} W_{\lambda,\mu}(z)\, dz = \frac{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}{\Gamma(\nu-\lambda+1)}, \tag{0.8.11}$$

where $\Re(\nu \pm \mu) > -\frac{1}{2}$.

### 0.8.5 Riemann-Liouville Fractional Integral Transform

The Riemann-Liouville fractional integral transform $\left(I_{c+}^{\theta}\Psi\right)(x)$ [164] is defined as

$$\left(I_{c+}^{\theta}\Psi\right)(x) = \frac{1}{\Gamma(\theta)} \int_c^x (x-t)^{\theta-1}\, \Psi(t)\, dt, \tag{0.8.12}$$

where $\theta \in \mathbb{C}$ and $\Re(\theta) > 0$.

### 0.8.6 Erdélyi-Kober Fractional Integral Transform

The Erdélyi-Kober fractional integral transform $\left(\Xi_{0+}^{\eta,\theta} f\right)(x)$ [164] is defined as

$$\left(\Xi_{0+}^{\eta,\theta} f\right)(x) = \frac{x^{-\eta-\theta}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1}\, t^{\theta} f(t)\, dt, \tag{0.8.13}$$

where $\eta, \theta \in \mathbb{C}; \Re(\eta) > 0$ and $\Re(\theta) > 0$.

## 0.9 SOME IMPORTANT SPECIAL FUNCTIONS

### 0.9.1 Ordinary Bessel Function $J_\nu(z)$

The ordinary Bessel function $J_\nu(z)$ of the first kind of order $\nu$ [128] is defined as follows:

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \tag{0.9.1}$$

where $\nu > 0$.

### 0.9.2 Bessel Maitland Function $J_\nu^\mu(z)$

The Bessel Maitland function $J_\nu^\mu(z)$ [128] is defined as follows:

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(n\mu+\nu+1)}, \tag{0.9.2}$$

where $\nu > 0$, $\mu > 0$.

### 0.9.3 Generalized Bessel Maitland Function $J_{\nu,\lambda}^\mu(z)$

The generalized Bessel Maitland function $J_{\nu,\lambda}^\mu(z)$ [128] is defined as follows:

$$J_{\nu,\lambda}^\mu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2\lambda+2n}}{\Gamma(n+\lambda+1)\Gamma(n\mu+\nu+\lambda+1)}, \tag{0.9.3}$$

where $\nu > 0$, $\mu > 0$, $\lambda > 0$.

### 0.9.4 Bessel Clifford Function $C_m(z)$

The Bessel Clifford function $C_m(z)$ [10] is defined as follows:

$$C_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(n+m+1)}, \tag{0.9.4}$$

where $m > 0$.

### 0.9.5 Lommel Function $s_{\mu,\nu}(z)$

The Lommel function $s_{\mu,\nu}(z)$ [112] is defined as follows:

$$s_{\mu,\nu}(z) = \frac{2^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\mu+1}}{\left(\frac{\mu-\nu+3}{2}\right)_n \left(\frac{\mu+\nu+3}{2}\right)_n}, \tag{0.9.5}$$

where $\mu+\nu \neq -1, -2, \ldots$

### 0.9.6 Hurwitz Zeta Function $\zeta(\rho,\nu)$

The Hurwitz zeta function $\zeta(\rho,\nu)$ [86] is defined as follows:

$$\zeta(\rho,\nu) = \sum_{n=0}^{\infty} \frac{1}{(n+\nu)^{\rho}}, \tag{0.9.6}$$

where $\Re(\rho) > 1; \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

### 0.9.7 Riemann Zeta Function $\zeta(\nu)$

The Riemann zeta function $\zeta(\nu)$[86] is defined as follows:

$$\zeta(\nu) = \sum_{n=0}^{\infty} (n+1)^{-\nu}, \tag{0.9.7}$$

where $\Re(\nu) > 1$.

### 0.9.8 Struve Function $H_\alpha(z)$

The Struve function $H_\alpha(z)$[143] is defined as follows:

$$\mathrm{H}_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma(n+\alpha+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+\alpha+1}, \tag{0.9.8}$$

where $\Re(\alpha) > 0$.

### 0.9.9 Modified Struve Function $L_{\alpha}(z)$

The Modified Struve function $L_{\alpha}(z)$ [143] is defined as follows:

$$\mathrm{L}_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n+\frac{3}{2}\right)\Gamma(n+\alpha+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+\alpha+1}, \tag{0.9.9}$$

where $\Re(\alpha) > 0$.

### 0.9.10 Dotsenko Function ${}_2R_1^{\tau}(a, b; c; z)$

The Dotsenko function ${}_2R_1^{\tau}(a, b; c; z)$ [124] is defined as follows:

$${}_2R_1^{\tau}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}, \tag{0.9.10}$$

where $|z| < 1$.

### 0.9.11 Rabotnov's Function $R_{\alpha}(\beta, t)$

The Rabotnov's function $R_{\alpha}(\beta, t)$ [152] is defined as follows:

$$R_{\alpha}(\beta, t) = t^{\alpha} \sum_{n=0}^{\infty} \frac{\beta^n t^{(\alpha+1)n}}{\Gamma\{(1+\alpha)n + (1+\alpha)\}}, \tag{0.9.11}$$

where $\Re(\alpha) > 0$, $\Re(\beta) > 0$.

### 0.9.12 Mellin-Ross Function $E_t(\nu, b)$

The Mellin-Ross function $E_t(\nu, b)$ [124] is defined as follows:

$$E_t(\nu, b) = t^{\nu} \sum_{n=0}^{\infty} \frac{(bt)^n}{\Gamma(\nu+n+1)}, \tag{0.9.12}$$

where $\Re(\nu) > 0$, $\Re(b) > 0$.

# CHAPTER 1

# A FAMILY OF MITTAG-LEFFLER TYPE FUNCTIONS AND THEIR PROPERTIES

**Publications:**

**1. A family of Mittag-Leffler type functions and its properties,** ***Palestine Journal of Mathematics*** **4, No.2(2015) 367-373.**

In this chapter, we first introduce some Mittag-Leffler type functions and then give definition of various integral operators. Next, we define a Mittag-Leffler type function named $E$-function [11] and also we establish its conditions of convergence. Then we define two more functions (generalization of sine and cosine function) as special cases of earlier defined $E$-function which are also believed to be new and important. Further derive Mellin-Barnes type contour integral representation of the $E$-function. Finally establish some integral transforms like Mellin transform, Laplace transform, Euler-Beta transform and Whittaker transform of the newly defined $E$-function.

## 1.1 INTRODUCTION

### 1.1.1 Mittag-Leffler Type Functions

- In 1903, Gösta Mittag-Leffler [133] introduced the function $E_\alpha$(z), defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} z^n , \tag{1.1.1}$$

where $z, \alpha \in \mathbb{C}; \Re(\alpha) \geq 0$ and $|z| < \infty$.

- In 1905, Wiman [215] extended (1.1.1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} z^n , \tag{1.1.2}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0$ and $\Re(\beta) > 0$.

The journey of M-L function started as a generalization of the exponential function $e^z$ and later its many generalizations were developed and studied by Prabhakar [148], Kiryakova [95], Srivastava-Tomovski [204] and many other authors.

### 1.1.2 Integral Transforms

- Mellin transform

The Mellin transform [187] of the function $f(z)$ with respect to $\zeta$ is given by

$$M[f(z);\zeta] = \int_0^\infty z^{\zeta-1} f(z)\, dz = f^*(\zeta), \qquad \Re(\zeta) > 0 \tag{1.1.3}$$

and the inverse Mellin transform of $f^*(\zeta)$ with respect to $z$ is given by

$$M^{-1}\left[f^*(\zeta);z\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} z^{-\zeta}f^*(\zeta)\,d\zeta=f(z)\,,\qquad \gamma\in\mathbb{R},\quad (1.1.4)$$

provided that both the integrals exist.

- Laplace transform

If the function $f(z)=O\left(e^{\alpha z}\right)$, z$\to\infty$ for some $\alpha$, then the Laplace transform [187] of the function $f(z)$ with respect to the parameter $s$, is given as follows:

$$L\left[f(z);s\right]=\int_0^\infty e^{-sz}f(z)\,dz=F(s)\,,\qquad \Re(s)>\alpha,\qquad (1.1.5)$$

it can be obtained by appealing to the Euler-integral of the II kind

$$\int_0^\infty e^{-sz}z^{\lambda-1}dz=\frac{\Gamma(\lambda)}{s^\lambda}\,,\qquad \min\{\Re(\lambda)\,,\Re(s)\}>0\qquad (1.1.6)$$

and the inverse Laplace transform of $F(s)$ with respect to $z$ is given by

$$L^{-1}\left[F(s);z\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz}F(s)\,dz=f(z)\,,\qquad c\in\mathbb{R},\quad (1.1.7)$$

provided that both the integrals exist.

- Euler-Beta transform

The generalized Euler-Beta transform [187] of the function $f(z)$ with respect to $\mu$ and $\nu$ is given by

$$B\left[f\left(z\right);\mu,\nu:a,b\right]=\int_{a}^{b}\left(z-a\right)^{\mu-1}\left(b-z\right)^{\nu-1}f\left(z\right)dz, \tag{1.1.8}$$

where $\Re\left(\mu\right)>0,\Re\left(\nu\right)>0;\ a,b\in\mathbb{R}$,

provided the integral exists and the generalized Beta function is defined as

$$\begin{aligned}\int_{a}^{b}\left(z-a\right)^{\mu-1}\left(b-z\right)^{\nu-1}dz&=\left(b-a\right)^{\mu+\nu-1}\mathbf{B}\left(\mu,\nu\right)\\&=\left(b-a\right)^{\mu+\nu-1}\frac{\Gamma\left(\mu\right)\Gamma\left(\nu\right)}{\Gamma\left(\mu+\nu\right)},\end{aligned} \tag{1.1.9}$$

where $\Re\left(\mu\right)>0,\Re\left(\nu\right)>0;\ a,b\in\mathbb{R}$.

- Whittaker transform

The Whittaker transform [214] of the function $f\left(z\right)$ with respect to $\lambda$, $\mu$ and $\nu$ is given by

$$\mathcal{W}\left[f\left(z\right);\lambda,\mu,\nu\right]=\int_{0}^{\infty}e^{-\frac{z}{2}}z^{\nu-1}W_{\lambda,\mu}\left(z\right)f\left(z\right)dz, \tag{1.1.10}$$

provided the integral exists, where $W_{\lambda,\mu}\left(z\right)$ is the Whittaker's confluent hypergeometric function, and associated integral is given in the literature [60, p. 823, Eq. (11)], Equation (1.1.10) can be solved by appealing to the following integral

$$\int_{0}^{\infty}e^{-\frac{z}{2}}z^{\nu-1}W_{\lambda,\mu}\left(z\right)dz=\frac{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}{\Gamma\left(\nu-\lambda+1\right)}, \tag{1.1.11}$$

where $\Re\left(\nu\pm\mu\right)>-\frac{1}{2}$.

## 1.2 DEFINITION, CONVERGENCE CONDITIONS AND SPECIAL CASES OF THE $E$-FUNCTION

**Definition 1.** The $E$-function [11] is defined as follows:

$$
{}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\ \\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{matrix}\right.\right]={}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\dots,(\gamma_h,q_h,s_h)\\ \\(\alpha,\beta);(\delta_1,p_1,r_1),\dots,(\delta_k,p_k,r_k)\end{matrix}\right.\right]
$$

$$
=\sum_{n=0}^{\infty}\frac{\left[(\gamma_1)_{q_1n}\right]^{s_1}\left[(\gamma_2)_{q_2n}\right]^{s_2}\dots\left[(\gamma_h)_{q_hn}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1n}\right]^{r_1}\left[(\delta_2)_{p_2n}\right]^{r_2}\dots\left[(\delta_k)_{p_kn}\right]^{r_k}\Gamma(\alpha n+\beta)}, \qquad (1.2.1)
$$

where

$$
z,\alpha,\beta,\gamma_i,\delta_j\in\mathbb{C};\Re(\alpha)\geq 0,\Re(\beta)>0,\Re(\gamma_i)>0,\Re(\delta_j)>0,q_i\geq 0,
$$

$$
p_j\geq 0,s_i\geq 0,r_j\geq 0;a,\tau\in\mathbb{R};\rho\in\{0,1\},\left(\sum_{i=1}^{h}q_is_i<\sum_{j=1}^{k}p_jr_j+\Re(\alpha)\right)\text{ or}
$$

$$
\left(\sum_{i=1}^{h}q_is_i=\sum_{j=1}^{k}p_jr_j+\Re(\alpha)\text{ when }\prod_{i=1}^{h}(q_i)^{q_is_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}(p_j)^{p_jr_j}\right]^{-1}|z^a|<1\right).
$$

$$
\text{for } i=1,\dots,h;j=1,\dots,k. \qquad (1.2.2)
$$

### 1.2.1 Domain of Convergence

Equation (1.2.1) can be denoted as

$$
{}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\ \\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{matrix}\right.\right]=\sum_{n=0}^{\infty}c_n, \qquad (1.2.3)
$$

where

$$
c_n=\frac{\left[(\gamma_1)_{q_1n}\right]^{s_1}\left[(\gamma_2)_{q_2n}\right]^{s_2}\dots\left[(\gamma_h)_{q_hn}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1n}\right]^{r_1}\left[(\delta_2)_{p_2n}\right]^{r_2}\dots\left[(\delta_k)_{p_kn}\right]^{r_k}\Gamma(\alpha n+\beta)}. \qquad (1.2.4)
$$

Now applying results due to Olver [142, p. 118-119], Tricomi and Erdélyi [210, p. 133, Eq. (1)] in the ratio $\left|\frac{c_{n+1}}{c_n}\right|$ then after simplification, we get

$$\left|\frac{c_{n+1}}{c_n}\right| = \prod_{i=1}^{h} (q_i n)^{q_i s_i} \left[1 + \frac{q_i (2\gamma_i + q_i - 1)}{2q_i n} + O\left\{\frac{1}{\left|(q_i n)^2\right|}\right\}\right]^{s_i}$$

$$\times \prod_{j=1}^{k} (p_j n)^{-p_j r_j} \left[1 + \frac{-p_j (2\delta_j + p_j - 1)}{2p_j n} + O\left\{\frac{1}{\left|(p_j n)^2\right|}\right\}\right]^{r_j}$$

$$\times (\alpha n)^{-\alpha} \left[1 + \frac{-\alpha (2\beta + \alpha - 1)}{2\alpha n} + O\left\{\frac{1}{\left|(\alpha n)^2\right|}\right\}\right] \left|(-1)^{\rho} z^{a}\right|. \qquad (1.2.5)$$

Now taking the limit $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \prod_{i=1}^{h} (q_i)^{q_i s_i} \prod_{j=1}^{k} (p_j)^{-p_j r_j} (\alpha)^{-\alpha} \left|z^{a}\right| \lim_{n\to\infty} n^{\left(\sum_{i=1}^{h} q_i s_i - \sum_{j=1}^{k} p_j r_j - \alpha\right)}. \qquad (1.2.6)$$

Now applying D'Alembert's simple ratio test, we get

(i)

$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = 0$ provided $\sum_{i=1}^{h} q_i s_i < \sum_{j=1}^{k} p_j r_j + \Re(\alpha)$, then the given series is convergent for all finite values of $\prod_{i=1}^{h} (q_i)^{q_i s_i} \left[\alpha^{\alpha} \prod_{j=1}^{k} (p_j)^{p_j r_j}\right]^{-1} |z^{a}|$.

(ii)

$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$ provided $\sum_{i=1}^{h} q_i s_i = \sum_{j=1}^{k} p_j r_j + \Re(\alpha)$, then the given series is convergent for $\prod_{i=1}^{h} (q_i)^{q_i s_i} \left[\alpha^{\alpha} \prod_{j=1}^{k} (p_j)^{p_j r_j}\right]^{-1} |z^{a}| < 1$.

### 1.2.2 Special Cases

1. Put $h = 1, s_1 = 0$; $k = 1, r_1 = 0$, in (1.2.1) we get **generalized Sine function** as

$$ {}_{\tau}E_1^1 \left[ z \middle| \begin{array}{c} (\rho, a) ; (\gamma_1, q_1, 0) \\ (\alpha, \beta) ; (\delta_1, p_1, 0) \end{array} \right] = \sum_{n=0}^{\infty} (-1)^{\rho n} \frac{z^{an+\tau}}{\Gamma(\alpha n + \beta)} = \sin^*(z). \tag{1.2.7} $$

2. Put $h = 1, s_1 = 0$; $k = 1, r_1 = 0$; $\tau$=0, in (1.2.1) we get **generalized Cosine function** as

$$ {}_{0}E_1^1 \left[ z \middle| \begin{array}{c} (\rho, a) ; (\gamma_1, q_1, 0) \\ (\alpha, \beta) ; (\delta_1, p_1, 0) \end{array} \right] = \sum_{n=0}^{\infty} (-1)^{\rho n} \frac{z^{an}}{\Gamma(\alpha n + \beta)} = \cos^*(z). \tag{1.2.8} $$

where $\sin^*(z)$ and $\cos^*(z)$ are defined as generalization of sine and cosine functions respectively.

## 1.3 MELLIN-BARNES TYPE CONTOUR INTEGRAL REPRESENTATION OF $E$-FUNCTION

**Theorem 1.** *If convergence conditions (1.2.2) are satisfied then the $E$-function ${}_{\tau}E_k^h[z]$ can be represented as the Mellin-Barnes type integral as follows:*

$$ {}_{\tau}E_k^h \left[ z \middle| \begin{array}{c} (\rho, a) ; (\gamma_1, q_1, s_1), \ldots, (\gamma_h, q_h, s_h) \\ (\alpha, \beta) ; (\delta_1, p_1, r_1), \ldots, (\delta_k, p_k, r_k) \end{array} \right] = \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} $$

$$\times\frac{z^{\tau}}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(\zeta)\Gamma(1-\zeta)\prod_{i=1}^{h}\left[\Gamma(\gamma_{i}-q_{i}\zeta)\right]^{s_{i}}}{\Gamma(\beta-\alpha\zeta)\prod_{j=1}^{k}\left[\Gamma\left(\delta_{j}-p_{j}\zeta\right)\right]^{r_{j}}}\left[(-1)^{\rho}(-z^{a})\right]^{-\zeta}d\zeta, \quad (1.3.1)$$

*where $\mathcal{L}$ is a suitable contour of integration that runs from $c-i\infty$ to $c+i\infty, c\in\mathbb{R}$ and intended to separate the poles of the integrand at $\zeta=-n$ for all $n\in\mathbb{N}_0\,(to\,the\,left)$ from those at $\zeta=n+1$ and at $\zeta=\frac{\gamma_i+n}{q_i}, i=1,\ldots h;$ for all $n\in\mathbb{N}_0\,(to\,the\,right)$.*

**Proof.** Rewriting the definition (1.2.1) in the form

$$ {}_{\tau}E_{k}^{h}\left[z\left|\begin{array}{c}(\rho,a);(\gamma_{1},q_{1},s_{1}),\ldots,(\gamma_{h},q_{h},s_{h})\\(\alpha,\beta);(\delta_{1},p_{1},r_{1}),\ldots,(\delta_{k},p_{k},r_{k})\end{array}\right.\right]$$

$$=\frac{z^{\tau}\prod_{v=1}^{k}\left[\Gamma(\delta_{v})\right]^{r_{v}}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_{u})\right]^{s_{u}}}\sum_{n=0}^{\infty}\frac{(-1)^{\rho n}\prod_{i=1}^{h}\left[\Gamma(\gamma_{i}+q_{i}n)\right]^{s_{i}}}{\Gamma(\alpha n+\beta)\prod_{j=1}^{k}\left[\Gamma\left(\delta_{j}+p_{j}n\right)\right]^{r_{j}}}z^{an} \quad (1.3.2)$$

$$=\frac{z^{\tau}\prod_{v=1}^{k}\left[\Gamma(\delta_{v})\right]^{r_{v}}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_{u})\right]^{s_{u}}}\sum_{n=0}^{\infty}\frac{(-1)^{n}\prod_{i=1}^{h}\left[\Gamma(\gamma_{i}+q_{i}n)\right]^{s_{i}}}{\Gamma(\alpha n+\beta)\prod_{j=1}^{k}\left[\Gamma\left(\delta_{j}+p_{j}n\right)\right]^{r_{j}}}\left[(-1)^{\rho}(-z^{a})\right]^{n} \quad (1.3.3)$$

$$=\frac{z^{\tau}\prod_{v=1}^{k}\left[\Gamma(\delta_{v})\right]^{r_{v}}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_{u})\right]^{s_{u}}}\sum_{n=0}^{\infty}\lim_{\zeta\to-n}\Gamma(\zeta)\Gamma(1-\zeta)(\zeta+n)\,g(\zeta)\left[(-1)^{\rho}(-z^{a})\right]^{-\zeta} \quad (1.3.4)$$

where

$$g\left(\zeta\right)=\frac{\displaystyle\prod_{i=1}^{h}\left[\Gamma\left(\gamma_{i}-q_{i}\zeta\right)\right]^{s_i}}{\Gamma\left(\beta-\alpha\zeta\right)\displaystyle\prod_{j=1}^{k}\left[\Gamma\left(\delta_{j}-p_{j}\zeta\right)\right]^{r_j}}. \tag{1.3.5}$$

Then

$${}_{\tau}E_{k}^{h}\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\ (\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right]=\frac{\displaystyle\prod_{v=1}^{k}\left[\Gamma\left(\delta_{v}\right)\right]^{r_v}}{\displaystyle\prod_{u=1}^{h}\left[\Gamma\left(\gamma_{u}\right)\right]^{s_u}}$$

$$\times\frac{z^{\tau}}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma\left(\zeta\right)\Gamma\left(1-\zeta\right)\displaystyle\prod_{i=1}^{h}\left[\Gamma\left(\gamma_{i}-q_{i}\zeta\right)\right]^{s_i}}{\Gamma\left(\beta-\alpha\zeta\right)\displaystyle\prod_{j=1}^{k}\left[\Gamma\left(\delta_{j}-p_{j}\zeta\right)\right]^{r_j}}\left[(-1)^{\rho}\left(-z^{a}\right)\right]^{-\zeta}d\zeta. \tag{1.3.6}$$

This completes the proof. □

## 1.4 SOME INTEGRAL TRANSFORMS

**Theorem 2.** (Mellin transform) *Let conditions associated with Mellin-Barnes type contour integral representation (1.3.1) of the E-function are satisfied and* $\Re\left(\zeta\right)>0$, *then the Mellin transform of the E-function is*

$$M\left[\frac{1}{\left\{(-1)^{\rho}\left(-z\right)\right\}^{\frac{\tau}{a}}}\,{}_{\tau}E_{k}^{h}\left(\left\{(-1)^{\rho}\left(-z\right)\right\}^{\frac{1}{a}}\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\ (\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right);\zeta\right]$$

$$= \frac{\Gamma(\zeta)\Gamma(1-\zeta)}{\Gamma(\beta-\alpha\zeta)} \frac{\prod_{i=1}^{h}\left[\frac{\Gamma(\gamma_i-q_i\zeta)}{\Gamma(\gamma_i)}\right]^{s_i}}{\prod_{j=1}^{k}\left[\frac{\Gamma(\delta_j-p_j\zeta)}{\Gamma(\delta_j)}\right]^{r_j}}, \tag{1.4.1}$$

*provided that the parameters are adjusted in such a way that the right-hand side is meaningful.*

**Proof.** According to Theorem 1, the $E$-function can be written as follows:

$$\frac{1}{\{(-1)^{\rho}(-z)\}^{\frac{\tau}{a}}}\,{}_{\tau}E_k^h\left(\{(-1)^{\rho}(-z)\}^{\frac{1}{a}}\,\middle|\,\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} g(\zeta)(z)^{-\zeta}\,d\zeta, \tag{1.4.2}$$

where

$$g(\zeta) = \frac{\Gamma(\zeta)\Gamma(1-\zeta)}{\Gamma(\beta-\alpha\zeta)} \frac{\prod_{i=1}^{h}\left[\frac{\Gamma(\gamma_i-q_i\zeta)}{\Gamma(\gamma_i)}\right]^{s_i}}{\prod_{j=1}^{k}\left[\frac{\Gamma(\delta_j-p_j\zeta)}{\Gamma(\delta_j)}\right]^{r_j}}. \tag{1.4.3}$$

Then by using definition of the Mellin transform in (1.4.2), we have

$$L.H.S. = M^{-1}[g(\zeta);z], \tag{1.4.4}$$

or

$$M\left[\frac{1}{\{(-1)^{\rho}(-z)\}^{\frac{\tau}{a}}}\,{}_{\tau}E_k^h\left(\{(-1)^{\rho}(-z)\}^{\frac{1}{a}}\,\middle|\,\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right);\zeta\right]$$

$$= \frac{\Gamma[\zeta]\,\Gamma[1-\zeta]}{\Gamma(\beta-\alpha\zeta)} \frac{\prod_{i=1}^{h}\left[\frac{\Gamma(\gamma_i - q_i\zeta)}{\Gamma(\gamma_i)}\right]^{s_i}}{\prod_{j=1}^{k}\left[\frac{\Gamma(\delta_j - p_j\zeta)}{\Gamma(\delta_j)}\right]^{r_j}}. \tag{1.4.5}$$

This completes the proof. □

**Theorem 3.** (Laplace transform) *If conditions associated with Mellin-Barnes type contour integral representation (1.3.1) of E-function are satisfied then the Laplace transform of the E-function is*

$$L\left[z^{\mu-1}{}_{\tau}E_k^h\left(xz^{\sigma}\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right);\nu\right]$$

$$= \frac{1}{\nu^{\mu}}\left(\frac{x}{\nu^{\sigma}}\right)^{\tau} \frac{\prod_{v=1}^{k}[\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h}[\Gamma(\gamma_u)]^{s_u}}$$

$$\times \overline{H}_{h+2,k+2}^{1,h+2}\left[(-1)^{\rho}\left\{-\left(\frac{x}{\nu^{\sigma}}\right)^{a}\right\}\left|\begin{matrix}(0,1;1),(1-\mu-\sigma\tau,\sigma a;1),(1-\gamma_i,q_i;s_i)_1^h;\text{---}\\(0,1);(1-\beta,\alpha;1),\left(1-\delta_j,p_j;r_j\right)_1^k\end{matrix}\right.\right], \tag{1.4.6}$$

*provided that the function on the right-hand side is convergent and has a meaning.*

**Proof.** We obtain the Laplace transform of the $E$-function as follows:

$$L\left[z^{\mu-1}{}_{\tau}E_k^h\left(xz^{\sigma}\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right);\nu\right]$$

$$= \int_0^\infty z^{\mu-1} e^{-\nu z} {}_{\tau}E_k^h \left( xz^\sigma \middle| \begin{matrix} (\rho, a); (\gamma_1, q_1, s_1), \ldots, (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), \ldots, (\delta_k, p_k, r_k) \end{matrix} \right) dz, \ \Re(\nu) > 0. \tag{1.4.7}$$

Now using (1.3.1) and interchanging the order of integrations, which is permissible under suitable convergence conditions, we have

$$L.H.S. = \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \frac{x^\tau}{2\pi i} \int_{\mathcal{L}} g(\zeta) [(-1)^\rho \{-(x^a)\}]^{-\zeta} \left\{ \int_0^\infty e^{-\nu z} z^{\mu+\sigma\tau-\sigma a\zeta-1} dz \right\} d\zeta, \tag{1.4.8}$$

where $g(\zeta)$ can be written as

$$g(\zeta) = \frac{\Gamma(0+\zeta)\Gamma(1-0-\zeta)\prod_{i=1}^{h}[\Gamma\{1-(1-\gamma_i)-q_i\zeta\}]^{s_i}}{\Gamma[1-(1-\beta)-\alpha\zeta]\prod_{j=1}^{k}\left[\Gamma\left\{1-\left(1-\delta_j\right)-p_j\zeta\right\}\right]^{r_j}}. \tag{1.4.9}$$

Now applying gamma integral (1.1.6) and replacing $\zeta$ by $-\zeta$ and contour $\mathcal{L}$ by other suitable contour then comparing it with the definition of Inayat-Hussain $\overline{H}$-function (0.3.1), we get

$$L.H.S. = \frac{1}{\nu^\mu} \left(\frac{x}{\nu^\sigma}\right)^\tau \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}}$$

$$\times \overline{H}^{1,h+2}_{h+2,k+2} \left[ (-1)^\rho \left\{ -\left(\frac{x}{\nu^\sigma}\right)^a \right\} \middle| \begin{matrix} (0,1;1), (1-\mu-\sigma\tau, \sigma a; 1), (1-\gamma_i, q_i; s_i)_1^h; \text{——} \\ (0,1); (1-\beta, \alpha; 1), \left(1-\delta_j, p_j; r_j\right)_1^k \end{matrix} \right]. \tag{1.4.10}$$

This completes the proof. □

**Theorem 4.** (Euler-Beta transform) *If conditions associated with Mellin-Barnes type contour integral representation (1.3.1) of E-function are satisfied then the Euler-Beta transform of the E-function is*

$$B\left[{}_{\tau}E_{k}^{h}\left(xz^{\sigma}\left|\begin{array}{c}(\rho,a);(\gamma_{i},q_{i},s_{i})_{1,h}\\(\alpha,\beta);(\delta_{j},p_{j},r_{j})_{1,k}\end{array}\right.\right);\mu,\nu:0,1\right]=\Gamma(\nu)(x)^{\tau}\frac{\prod\limits_{v=1}^{k}[\Gamma(\delta_{v})]^{r_{v}}}{\prod\limits_{u=1}^{h}[\Gamma(\gamma_{u})]^{s_{u}}}$$

$$\times\overline{H}_{h+2,k+3}^{1,h+2}\left[\begin{array}{c}(-1)^{\rho}\\(-x^{a})\end{array}\left|\begin{array}{c}(0,1;1),(1-\mu-\sigma\tau,\sigma a;1),(1-\gamma_{i},q_{i};s_{i})_{1}^{h};---\\(0,1);(1-\beta,\alpha;1),(1-\mu-\nu-\sigma\tau,\sigma a;1),(1-\delta_{j},p_{j};r_{j})_{1}^{k}\end{array}\right.\right],\tag{1.4.11}$$

*provided that the function on the right-hand side is convergent and has a meaning.*

**Proof.** Using definition (1.1.9), we obtain the Euler-Beta transform of the $E$-function as follows:

$$B\left[{}_{\tau}E_{k}^{h}\left(xz^{\sigma}\left|\begin{array}{c}(\rho,a);(\gamma_{1},q_{1},s_{1}),\ldots,(\gamma_{h},q_{h},s_{h})\\(\alpha,\beta);(\delta_{1},p_{1},r_{1}),\ldots,(\delta_{k},p_{k},r_{k})\end{array}\right.\right);\mu,\nu:0,1\right]$$

$$=\int_{0}^{\infty}z^{\mu-1}(1-z)^{\nu-1}\,{}_{\tau}E_{k}^{h}\left(xz^{\sigma}\left|\begin{array}{c}(\rho,a);(\gamma_{1},q_{1},s_{1}),\ldots,(\gamma_{h},q_{h},s_{h})\\(\alpha,\beta);(\delta_{1},p_{1},r_{1}),\ldots,(\delta_{k},p_{k},r_{k})\end{array}\right.\right)dz.\tag{1.4.12}$$

Now using (1.3.1) and interchanging the order of integrations, which is permissible under suitable convergence conditions, we have

$$= \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \frac{1}{2\pi i} \int_{\mathcal{L}} g(\zeta)\, x^{\tau} \left[(-1)^{\rho}(-x^{a})\right]^{-\zeta} \left[\int_0^1 z^{\mu+\sigma\tau-\sigma a\zeta-1}(1-z)^{\nu-1}\, dz\right] d\zeta, \tag{1.4.13}$$

where $g(\zeta)$ can be written as

$$g(\zeta) = \frac{\Gamma(0+\zeta)\,\Gamma(1-0-\zeta) \prod_{i=1}^{h} [\Gamma\{1-(1-\gamma_i)-q_i\zeta\}]^{s_i}}{\Gamma[1-(1-\beta)-\alpha\zeta] \prod_{j=1}^{k} \left[\Gamma\left(1-\left(1-\delta_j\right)-p_j\zeta\right)\right]^{r_j}}. \tag{1.4.14}$$

Applying Beta integral (1.1.9), we get

$$B\left[{}_{\tau}E_k^h\left(xz^{\sigma}\left|\begin{array}{c}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{array}\right.\right);\mu,\nu:0,1\right]$$

$$= \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \frac{x^{\tau}}{2\pi i} \int_{\mathcal{L}} g(\zeta) \left(\frac{\Gamma(\mu+\sigma\tau-\sigma a\zeta)\,\Gamma(\nu)}{\Gamma(\mu+\sigma\tau+\nu-\sigma a\zeta)}\right) \left[(-1)^{\rho}(-x^{a})\right]^{-\zeta} d\zeta. \tag{1.4.15}$$

Now by replacing $\zeta$ by $-\zeta$ and contour $\mathcal{L}$ by other suitable contour then comparing it with the definition of Inayat-Hussain $\overline{H}$-function of one variable (0.3.1), we get

$$= \Gamma(\nu)\,(x)^{\tau} \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}}$$

$$\times \overline{H}^{1,h+2}_{h+2,k+3}\left[\begin{array}{c|c} (-1)^{\rho} & (0,1;1),(1-\mu-\sigma\tau,\sigma a;1),(1-\gamma_i,q_i;s_i)_1^h;\text{——} \\ (-x^a) & (0,1);(1-\beta,\alpha;1),(1-\mu-\nu-\sigma\tau,\sigma a;1),\left(1-\delta_j,p_j;r_j\right)_1^k \end{array}\right]. \tag{1.4.16}$$

This completes the proof. □

**Theorem 5.** (Whittaker transform) *If conditions associated with Mellin-Barnes type contour integral representation (1.3.1) of the E-function are satisfied then the Whittaker transform of the E-function is*

$$\mathcal{W}\left[{}_{\tau}E_k^h\left(xz^{\sigma}\left|\begin{array}{c}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{array}\right.\right);\lambda,\mu,\nu\right]=x^{\tau}\frac{\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}}$$

$$\times \overline{H}^{1,h+3}_{h+3,k+3}\left[\begin{array}{c|c} (-1)^{\rho}(-x^a) & (0,1;1),\left(\tfrac{1}{2}\pm\mu-\nu-\sigma\tau,\sigma a;1\right),(1-\gamma_i,q_i;s_i)_1^h;\text{——} \\ & (0,1);(1-\beta,\alpha;1),(\lambda-\nu-\sigma\tau,\sigma a;1),\left(1-\delta_j,p_j;r_j\right)_1^k \end{array}\right], \tag{1.4.17}$$

*provided that the function on the right-hand side is convergent and has a meaning.*

**Proof.** The proof can be done on the lines similar to that of Theorem 3. □

# CHAPTER 2

# MITTAG-LEFFLER TYPE $E$-FUNCTION AND ASSOCIATED SPECIAL FUNCTIONS

**Publications:**

**1. A family of Mittag-Leffler type functions and its relation with basic special functions,** ***International Journal of Pure and Applied Mathematics*** **101, No. 3(2015), 369-379.**

**2. Mittag-Leffler type $E$-function and related functions,** ***International Journal of Mathematical Sciences and Enginnering Applications*** **8, No. 6(2014), 69-79.**

In this chapter, we prove efficiency and usefulness of the $E$-function [12]. For this we establish relations of the $E$-function with well known special functions such as generalized hypergeometric function, Fox's $H$-function, $\overline{H}$-function and Wright function. Further we obtain known M-L type functions as special cases of the $E$-function. Finally, we obtain Bessel function $J_\nu(z)$, Bessel Maitland function $J_\nu^\mu(z)$, generalized Bessel Maitland function

$J^{\mu}_{\nu,\lambda}(z)$, Bessel Clifford function $C_m(z)$, Lommel function $s_{\mu,\nu}(z)$, Hurwitz zeta function $\zeta(\rho,\nu)$, Riemann zeta function $\zeta(\nu)$, Struve function $H_\nu(z)$, modified Struve function $L_\nu(z)$, Rabotnov's function $R_\nu(\zeta,t)$, Dotsenko function ${}_2R_1^{\frac{\omega}{\mu}}(\nu,\sigma;\theta,\omega;\mu;z)$ and Mellin-Ross function $E_t(\nu,b)$ as particular cases of the $E$-function defined in this thesis.

## 2.1 DEFINITIONS

### 2.1.1 The $H$-Function

The Fox's $H$-function [126, p. 1] is defined by means of the following Mellin-Barnes type of contour integral

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\,A_j)_1^p\\ \\ (b_k,\,B_k)_1^q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\Lambda(s)z^s ds,\quad z\neq 0. \tag{2.1.1}$$

Here

$$\Lambda(s)=\frac{\prod\limits_{j=1}^{m}\Gamma\left(b_j-B_js\right)\prod\limits_{j=1}^{n}\Gamma\left(1-a_j+A_js\right)}{\prod\limits_{j=m+1}^{q}\Gamma\left(1-b_j+B_js\right)\prod\limits_{j=n+1}^{p}\Gamma\left(a_j-A_js\right)}, \tag{2.1.2}$$

where $m,n,p$ and, $q$ are non-negative integers satisfying $0\le n\le p$, $0\le m\le q$ and empty products are taken as unity. Also, $A_j(j=1,\ldots,p)$ and $B_j(j=1,\ldots,q)$ are positive real numbers for standardization purpose, $a_j(j=1,\ldots,p)$ and $b_j(j=1,\ldots,q)$ are complex numbers satisfying $A_j(b_h+\nu)\neq B_h(a_j-\lambda-1)$ for $\nu,\lambda=0,1,\ldots;h=1,\ldots,m;j=1,\ldots,n$. The contour $\mathcal{L}$ in $\mathbb{C}$ is such that the poles of $\Gamma(b_j-B_js)(j=1,\ldots,m)$ are separated from the poles of $\Gamma(1-a_j + A_j s)(j = 1, \ldots, n)$ such that the poles of $\Gamma(b_j - B_j s)$ lie to the left of $\mathcal{L}$, while the poles of $\Gamma(1-a_j + A_j s)$ are to the right of $\mathcal{L}$. The poles of the integrand are assumed to be simple. The $H$-function is an analytic function of $z$ for every $|z| \neq 0$ when $\mu > 0$ and for $0 < |z| < 1/\beta$ when $\mu = 0$, where $\mu$ and $\beta$ are defined as

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \tag{2.1.3}$$

and

$$\beta = \prod_{j=1}^{p} A_j^{A_j} \prod_{j=1}^{q} B_j^{-B_j}. \tag{2.1.4}$$

### 2.1.2 The $\overline{H}$-Function

Inayat Hussain defined a more general function named $\overline{H}$-function [79] in following manner:

$$\overline{H}^{m,n}_{p,q}\left[z\left| \begin{matrix} (a_j, A_j; \alpha_j)_1^n; (a_j, A_j)_{n+1}^p \\ (b_j, B_j)_1^m; (b_j, B_j; \beta_j)_{m+1}^q \end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \chi(s) z^s ds, \tag{2.1.5}$$

where

$$z \neq 0; i = \sqrt{(-1)}; \chi(s) := \frac{\prod_{j=1}^{m}\Gamma\left(b_j - B_j s\right)\prod_{j=1}^{n}\left\{\Gamma\left(1 - a_j + A_j s\right)\right\}^{\alpha_j}}{\prod_{j=m+1}^{q}\left\{\Gamma\left(1 - b_j + B_j s\right)\right\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s\right)}, \tag{2.1.6}$$

where $a_i$, $b_j$ are complex parameters and $m, n, p$ and, $q$ are integers satisfying $0 \leq n \leq p$, $0 \leq m \leq q$, it contains *fractional* powers of some of the Gamma functions involved. Here, and in what follows, the parameters

$$A_j \geq 0 \quad (j = 1, \ldots, p) \qquad \text{and} \qquad B_j \geq 0 \quad (j = 1, \ldots, q),$$

not all zero simultaneously and the exponents

$$\alpha_j \quad (j = 1, \ldots, n) \qquad \text{and} \qquad \beta_j \quad (j = m+1, \ldots, q),$$

can take on noninteger values, and $\mathcal{L} = \mathcal{L}_{(i\tau;\infty)}$ is a Mellin-Barnes type contour starting at the point $\tau - i\infty$ and terminating at the point $\tau + i\infty$ ($\tau \in \mathbb{R}$) with the usual indentations to separate one set of poles from the other set of poles. The sufficient condition for the absolute convergence of the contour integral in (2.1.5) was established by Buschman and Srivastava [14, p. 4708] as follows:

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |\alpha_j A_j| - \sum_{j=m+1}^{q} |\beta_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0, \tag{2.1.7}$$

which provides the exponential decay of the integrand in (2.1.5), and the region of absolute convergence of the contour integral in (2.1.5) is given by

$$|\arg(z)| < \frac{1}{2}\pi\Omega,$$

where $\Omega$ is defined by (2.1.7).

A comprehensive account of this function can be found in the work by Buschman and Srivastava [14], Gupta, Jain and Agrawal [67], Rathie[158], Saxena [165], and Saxena et al. [168, 172].

## 2.2 RELATION WITH BASIC SPECIAL FUNCTIONS

**Theorem 1.** (Generalized hypergeometric function) *Let condition* (1.2.2) *is satisfied with restriction* $s_i, r_j \in \mathbb{N}_0,\ i = 1,\ldots,h; j = 1,\ldots,k$ *then the* $E$*-function can be written as follows:*

$$
{}_\tau E_k^h \left[ z \middle| \begin{matrix} (\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h) \\ (\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k) \end{matrix} \right]
$$

$$
= \frac{z^\tau}{\Gamma(\beta)} {}_{q*}F_{p*} \left[ \begin{matrix} [\Delta(q_i,\gamma_i)^{s_i}]_{1,h}, 1; \\ \Delta(\alpha,\beta), \left[\Delta(p_j,\delta_j)^{r_j}\right]_{1,k}; \end{matrix} \ \frac{z^a(-1)^\rho \prod_{i=1}^{h}(q_i)^{q_i s_i}}{(\alpha)^\alpha \prod_{j=1}^{k}(p_j)^{p_j r_j}} \right], \quad (2.2.1)
$$

*where*

$$
q^* = \sum_{i=1}^{h} q_i s_i + 1, p^* = \sum_{j=1}^{k} r_j p_j + \alpha; \Delta(\alpha,\beta) = \frac{\beta}{\alpha}, \frac{\beta+1}{\alpha}, \frac{\beta+2}{\alpha}, \ldots, \frac{\beta+\alpha-1}{\alpha};
$$

$$
[\Delta(q_i,\gamma_i)^{s_i}]_{1,h} = \overbrace{\Delta(q_1,\gamma_1),\ldots,\Delta(q_1,\gamma_1)}^{s_1\, times},\ldots\ldots,\overbrace{\Delta(q_h,\gamma_h),\ldots,\Delta(q_h,\gamma_h)}^{s_h\, times}
$$

*and*

$$
\left[\Delta(p_j,\delta_j)^{r_j}\right]_{1,k} = \underbrace{\Delta(p_1,\delta_1),\ldots,\Delta(p_1,\delta_1)}_{r_1\, times},\ldots\ldots,\underbrace{\Delta(p_k,\delta_k),\ldots,\Delta(p_k,\delta_k)}_{r_k\, times}. \quad (2.2.2)
$$

**Proof.** The $E$-function is defined by (1.2.1) as follows

$$
{}_\tau E_k^h \left[ z \middle| \begin{matrix} (\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h) \\ (\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k) \end{matrix} \right]
$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1} \left[(\gamma_2)_{q_2 n}\right]^{s_2} \cdots \left[(\gamma_h)_{q_h n}\right]^{s_h} (-1)^{\rho n} z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1} \left[(\delta_2)_{p_2 n}\right]^{r_2} \cdots \left[(\delta_k)_{p_k n}\right]^{r_k} \Gamma(\alpha n + \beta)} . \tag{2.2.3}$$

Now applying (0.1.4), then by comparing result with definition of generalized hypergeometric function (0.1.5), we get

$$= \frac{z^{\tau}}{\Gamma(\beta)} {}_{q*}F_{p*} \left[ \begin{matrix} [\Delta(q_i, \gamma_i)^{s_i}]_{1,h}, 1; \\ \Delta(\alpha, \beta), [\Delta(p_j, \delta_j)^{r_j}]_{1,k}; \end{matrix} \; \frac{z^a (-1)^{\rho} \prod_{i=1}^{h} (q_i)^{q_i s_i}}{(\alpha)^{\alpha} \prod_{j=1}^{k} (p_j)^{p_j r_j}} \right], \tag{2.2.4}$$

where

$$q^* = \sum_{i=1}^{h} q_i s_i + 1, p^* = \sum_{j=1}^{k} r_j p_j + \alpha; \Delta(\alpha, \beta) = \frac{\beta}{\alpha}, \frac{\beta+1}{\alpha}, \frac{\beta+2}{\alpha}, \ldots, \frac{\beta+\alpha-1}{\alpha};$$

$$[\Delta(q_i, \gamma_i)^{s_i}]_{1,h} = \overbrace{\Delta(q_1, \gamma_1), \ldots, \Delta(q_1, \gamma_1)}^{s_1 \, times}, \ldots\ldots, \overbrace{\Delta(q_h, \gamma_h), \ldots, \Delta(q_h, \gamma_h)}^{s_h \, times}$$

and

$$\left[\Delta(p_j, \delta_j)^{r_j}\right]_{1,k} = \underbrace{\Delta(p_1, \delta_1), \ldots, \Delta(p_1, \delta_1)}_{r_1 \, times}, \ldots\ldots, \underbrace{\Delta(p_k, \delta_k), \ldots, \Delta(p_k, \delta_k)}_{r_k \, times}. \tag{2.2.5}$$

**Theorem 2.** (Fox's $H$-function and $\overline{H}$-function) *Let condition* (1.2.2) *is satisfied with restriction* $s_i, r_j \in \mathbb{N}_0$, $i = 1, \ldots, h; j = 1, \ldots, k$ *then the* $E$*-function can be written as follows:*

$${}_{\tau}E_k^h \left[ z \middle| \begin{matrix} (\rho, a); (\gamma_1, q_1, s_1), \ldots, (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), \ldots, (\delta_k, p_k, r_k) \end{matrix} \right] = z^{\tau} \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}}$$

$$\times H^{1,n^*}_{n^*,q^*}\left[(-1)^{\rho}(-z^{a})\left|\begin{matrix}(0,1),(A,B)\\(0,1),(1-\beta,\alpha),(C,D)\end{matrix}\right.\right], \tag{2.2.6}$$

*where*

$$(A,B)=\overbrace{(1-\gamma_{1},q_{1}),\ldots,(1-\gamma_{1},q_{1})}^{s_1\,times},\ldots\ldots,\overbrace{(1-\gamma_{h},q_{h}),\ldots,(1-\gamma_{h},q_{h})}^{s_h\,times};$$

$$(C,D)=\underbrace{(1-\delta_{1},p_{1}),\ldots,(1-\delta_{1},p_{1})}_{r_1\,times},\ldots\ldots,\underbrace{(1-\delta_{k},p_{k}),\ldots,(1-\delta_{k},p_{k})}_{r_k\,times};$$

$$n^*=\sum_{i=1}^{h}s_{i}+1 \qquad and \qquad q^*=\sum_{j=1}^{k}r_{j}+2. \tag{2.2.7}$$

*Also, let condition* (1.2.2) *is satisfied then the E-function can be written as follows:*

$${}_{\tau}E^{h}_{k}\left[z\left|\begin{matrix}(\rho,a);(\gamma_{1},q_{1},s_{1}),\ldots,(\gamma_{h},q_{h},s_{h})\\(\alpha,\beta);(\delta_{1},p_{1},r_{1}),\ldots,(\delta_{k},p_{k},r_{k})\end{matrix}\right.\right]=z^{\tau}\frac{\prod_{m=1}^{k}[\Gamma(\delta_{m})]^{r_m}}{\prod_{l=1}^{h}[\Gamma(\gamma_{l})]^{s_l}}$$

$$\times\overline{H}^{1,h+1}_{h+1,k+2}\left[(-1)^{\rho}(-z^{a})\left|\begin{matrix}(0,1;1),(1-\gamma_{i},q_{i};s_{i})_{1}^{h};\text{---}\\(0,1);(1-\beta,\alpha;1),\left(1-\delta_{j},p_{j};r_{j}\right)_{1}^{k}\end{matrix}\right.\right]. \tag{2.2.8}$$

**Proof.** Using (1.3.1) the $E$-function ${}_{\tau}E^{h}_{k}[z]$ can be written as follows

$${}_{\tau}E^{h}_{k}\left[z\left|\begin{matrix}(\rho,a);(\gamma_{1},q_{1},s_{1}),\ldots,(\gamma_{h},q_{h},s_{h})\\(\alpha,\beta);(\delta_{1},p_{1},r_{1}),\ldots,(\delta_{k},p_{k},r_{k})\end{matrix}\right.\right]$$

$$= \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}} \frac{z^{\tau}}{2\pi i} \int_{\mathcal{L}} g(\zeta) \left\{(-1)^{\rho}(-z^{a})\right\}^{-\zeta} d\zeta, \tag{2.2.9}$$

where

$$g(\zeta) = \frac{\Gamma(0+\zeta)\Gamma\{1-0-\zeta\} \prod_{i=1}^{h} [\Gamma\{1-(1-\gamma_i)-q_i\zeta\}]^{s_i}}{\Gamma\{1-(1-\beta)-\alpha\zeta\} \prod_{j=1}^{k} \left[\Gamma\left\{1-\left(1-\delta_j\right)-p_j\zeta\right\}\right]^{r_j}}. \tag{2.2.10}$$

Now by comparing (2.2.9) with definition of $H$-function (2.1.1), we get

$$L.H.S. = z^{\tau} \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}} H_{n^*,q^*}^{1,n^*} \left[ (-1)^{\rho}(-z^{a}) \,\middle|\, \begin{matrix} (0,1),(A,B) \\ (0,1),(1-\beta,\alpha),(C,D) \end{matrix} \right], \tag{2.2.11}$$

where

$$(A,B) = \overbrace{(1-\gamma_1,q_1),\ldots,(1-\gamma_1,q_1)}^{s_1\, times},\ldots\ldots,\overbrace{(1-\gamma_h,q_h),\ldots,(1-\gamma_h,q_h)}^{s_h\, times};$$

$$(C,D) = \underbrace{(1-\delta_1,p_1),\ldots,(1-\delta_1,p_1)}_{r_1\, times},\ldots\ldots,\underbrace{(1-\delta_k,p_k),\ldots,(1-\delta_k,p_k)}_{r_k\, times};$$

$$n^* = \sum_{i=1}^{h} s_i + 1 \qquad \text{and} \qquad q^* = \sum_{j=1}^{k} r_j + 2. \tag{2.2.12}$$

Again by comparing (2.2.9) with definition of $\overline{H}$-function (2.1.5), we get

$$L.H.S. = z^{\tau} \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}}$$

$$\times \overline{H}^{1,h+1}_{h+1,k+2}\left[(-1)^{\rho}(-z^{a})\left|\begin{matrix}(0,1;1),(1-\gamma_i,q_i;s_i)_1^h;\text{——}\\(0,1);(1-\beta,\alpha;1),\left(1-\delta_j,p_j;r_j\right)_1^k\end{matrix}\right.\right]. \tag{2.2.13}$$

**Theorem 3.** (Wright function) *Let condition* (1.2.2) *is satisfied with restriction* $s_i, r_j \in \mathbb{N}_0$, $i = 1, \ldots, h; j = 1, \ldots, k$ *then the E-function can be written as follows:*

$${}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right]=z^{\tau}\frac{\prod\limits_{m=1}^{k}\left[\Gamma(\delta_m)\right]^{r_m}}{\prod\limits_{l=1}^{h}\left[\Gamma(\gamma_l)\right]^{s_l}}$$

$$\times {}_{p^*}\Psi_{q^*}\left[\begin{matrix}(1,1),\overbrace{(\gamma_1,q_1),\ldots,(\gamma_1,q_1)}^{s_1\,times},\ldots\ldots,\overbrace{(\gamma_h,q_h),\ldots,(\gamma_h,q_h)}^{s_h\,times};\\ \\(\beta,\alpha),\underbrace{(\delta_1,p_1),\ldots,(\delta_1,p_1)}_{r_1\,times},\ldots\ldots,\underbrace{(\delta_k,p_k),\ldots,(\delta_k,p_k)}_{r_k\,times};\end{matrix}\;(-1)^{\rho}z^{a}\right] \tag{2.2.14}$$

*where* $\quad p^* = \sum\limits_{i=1}^{h} s_i + 1 \quad$ *and* $\quad q^* = \sum\limits_{j=1}^{k} r_j + 1.$

**Proof.** The $E$-function is defined by (1.2.1) as follows

$${}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{\rho n}\prod\limits_{i=1}^{h}\left[(\gamma_i)_{q_i n}\right]^{s_i}}{\Gamma(\alpha n+\beta)\prod\limits_{j=1}^{k}\left[(\delta_j)_{p_j n}\right]^{r_j}}z^{an+\tau} \tag{2.2.15}$$

$$= z^{\tau}\frac{\prod_{m=1}^{k}[\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h}[\Gamma(\gamma_l)]^{s_l}}\sum_{n=0}^{\infty}\frac{(-1)^{\rho n}\,\Gamma(1+n)\prod_{i=1}^{h}[\Gamma(\gamma_i+q_i n)]^{s_i}}{\Gamma(\alpha n+\beta)\prod_{j=1}^{k}\left[\Gamma\left(\delta_j+p_j n\right)\right]^{r_j}}\frac{z^{an}}{n!}\cdot \tag{2.2.16}$$

Now comparing (2.2.16) with definition of Fox-Wright function [34, p. 183], we get

$$L.H.S. = z^{\tau}\frac{\prod_{m=1}^{k}[\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h}[\Gamma(\gamma_l)]^{s_l}}$$

$$\times {}_{p^*}\Psi_{q^*}\left[\begin{matrix}(1,1),\overbrace{(\gamma_1,q_1),\ldots,(\gamma_1,q_1)}^{s_1\,times},\ldots\ldots,\overbrace{(\gamma_h,q_h),\ldots,(\gamma_h,q_h)}^{s_h\,times};\\ \\ (\beta,\alpha),\underbrace{(\delta_1,p_1),\ldots,(\delta_1,p_1)}_{r_1\,times},\ldots\ldots,\underbrace{(\delta_k,p_k),\ldots,(\delta_k,p_k)}_{r_k\,times};\end{matrix}\ (-1)^{\rho}z^{a}\right], \tag{2.2.17}$$

where $\quad p^* = \sum_{i=1}^{h} s_i + 1 \quad$ and $\quad q^* = \sum_{j=1}^{k} r_j + 1.$

## 2.3 MITTAG-LEFFLER FUNCTIONS AS SPECIAL CASES OF THE $E$-FUNCTION

1. Put $h=1, s_1=0;\ k=1, r_1=0;\ a=1; \rho=0; \beta=1; \tau=0$ in (1.2.1), then we get Mittag-Leffler function $E_{\alpha}(z)$ defined in (0.7.1), as

$${}_0E_1^1\left[z\left|\begin{matrix}(0,1);(\gamma_1,q_1,0)\\ \\ (\alpha,1);(\delta_1,p_1,0)\end{matrix}\right.\right] = E_{\alpha}(z). \tag{2.3.1}$$

2. Put $h = 1, s_1 = 0$; $k = 1, r_1 = 0$; $a = 1; \rho = 0; \tau = 0$ in (1.2.1), then we get generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined in (0.7.2), as

$$ {}_0E_1^1 \left[ z \middle| \begin{matrix} (0,1);(\gamma_1, q_1, 0) \\ (\alpha,\beta);(\delta_1, p_1, 0) \end{matrix} \right] = E_{\alpha,\beta}(z). \qquad (2.3.2) $$

3. Put $h = 1, s_1 = 0$; $k = 1, r_1 = 0$;$a = 1; \rho = 0$; $\tau = \frac{\beta-1}{\alpha}$ in (1.2.1), then we get Mittag-Leffler type function $E_{\alpha,\beta}(z)$ defined in (0.7.3), as

$$ {}_{\frac{\beta-1}{\alpha}}E_1^1 \left[ z \middle| \begin{matrix} (0,1);(\gamma_1, q_1, 0) \\ (\alpha,\beta);(\delta_1, p_1, 0) \end{matrix} \right] = E^*_{\alpha,\beta}(z). \qquad (2.3.3) $$

4. Put $h = 1, s_1 = 0$; $k = 1, r_1 = 1, \delta_1 = \beta_2, p_1 = \alpha_2$;$a = 1; \rho = 0$; $\tau = 0; \alpha = \alpha_1; \beta = \beta_1$ in (1.2.1), then we get Mittag-Leffler type function $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ defined in (0.7.4), as

$$ {}_0E_1^1 \left[ z \middle| \begin{matrix} (0,1);(\gamma_1, q_1, 0) \\ (\alpha_1,\beta_1);(\beta_2, \alpha_2, 1) \end{matrix} \right] = \Gamma(\beta_2)\, E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z). \qquad (2.3.4) $$

5. Put $h = 1, s_1 = 1, \gamma_1 = \gamma, q_1 = 1$; $k = 1, r_1 = 1, \delta_1 = 1, p_1 = 1$;$a = 1; \rho = 0$; $\tau = 0$; in (1.2.1), then we get $E^{\gamma}_{\alpha,\beta}(z)$ defined in (0.7.5), as

$$ {}_0E_1^1 \left[ z \middle| \begin{matrix} (0,1);(\gamma, 1, 1) \\ (\alpha,\beta);(1, 1, 1) \end{matrix} \right] = E^{\gamma}_{\alpha,\beta}(z). \qquad (2.3.5) $$

6. Put $h = 1, s_1 = 0$; $k = m-1, r_1 = \ldots = r_{m-1} = 1, \delta_1 = \mu_1, \ldots, \delta_{m-1} = \mu_{m-1}, p_1 = 1/\rho_1, \ldots, p_{m-1} = 1/\rho_{m-1}$;$a = 1; \rho = 0$; $\tau = 0; \alpha = 1/\rho_m; \beta = \mu_m$

in (1.2.1), then we get $E_{\left(1/\rho_i\right),\left(\mu_i\right)}(z)$ defined in (0.7.7), as

$$
{}_0E^1_{m-1}\left[z\left|\begin{array}{c}(0,1);(\gamma_1,q_1,0)\\ \\ (1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1)\end{array}\right.\right]
$$

$$
=\Gamma(\mu_1)\ldots\Gamma(\mu_{m-1})\,E_{\left(1/\rho_i\right),\left(\mu_i\right)}(z). \tag{2.3.6}
$$

7. Put $h=1, s_1=1, \gamma_1=\gamma, q_1=q;\ k=1, r_1=1, \delta_1=1, p_1=1; a=1; \rho=0;\ \tau=0$ in (1.2.1), then we get $E^{\gamma,q}_{\alpha,\beta}(z)$ defined in (0.7.9), as

$$
{}_0E^1_1\left[z\left|\begin{array}{c}(0,1);(\gamma,q,1)\\ \\ (\alpha,\beta);(1,1,1)\end{array}\right.\right]=E^{\gamma,q}_{\alpha,\beta}(z). \tag{2.3.7}
$$

8. Put $h=1, s_1=1, \gamma_1=\gamma, q_1=\delta;\ k=1, r_1=1, \delta_1=1, p_1=1; a=1; \rho=0;\ \tau=0$ in (1.2.1), then we get function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ defined in (0.7.10), as

$$
{}_0E^1_1\left[z\left|\begin{array}{c}(0,1);(\gamma,\delta,1)\\ \\ (\alpha,\beta);(1,1,1)\end{array}\right.\right]=\breve{E}^{\gamma,\delta}_{\alpha,\beta}(z). \tag{2.3.8}
$$

9. Put $h=1, s_1=1, \gamma_1=\gamma, q_1=K,\ k=m, r_1=\ldots=r_m=1, \delta_1=\beta_1,\ldots,\ \delta_m=\beta_m, p_1=\alpha_1,\ldots,p_m=\alpha_m; a=1; \rho=0;\ \tau=0; \alpha=1; \beta=1$ in (1.2.1), then we get $E_{\gamma,K}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\right]$ defined in (0.7.11), as

$$
{}_0E^1_m\left[z\left|\begin{array}{c}(0,1);(\gamma,K,1)\\ \\ (1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1)\end{array}\right.\right]
$$

$$
=\Gamma(\beta_1)\ldots\Gamma(\beta_m)\,E_{\gamma,K}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\right]. \tag{2.3.9}
$$

10. Put $h = 1, s_1 = 0; k = v-1, r_1 = \ldots = r_{v-1} = 1, \delta_1 = 1+\mu_1, \ldots, \delta_{v-1} = 1 + \mu_{v-1}, p_1 = \lambda_1, \ldots, p_{v-1} = \lambda_{v-1}; a = \sum_{i=1}^{\nu} \lambda_i = \Lambda; \rho = 1; \tau = \sum_{i=1}^{\nu} \mu_i = M;\ \alpha = \lambda_v; \beta = 1 + \mu_v$ and replace z by $\frac{z}{\Lambda}$ in (1.2.1), then we get $HE_{\mu_1,\ldots,\mu_v}^{\lambda_1,\ldots,\lambda_v}(z)$ defined in (0.7.13), as

$$ {}_{M}E_{v-1}^{1}\left[\frac{z}{\Lambda}\middle|\begin{array}{c}(1,\Lambda);(\gamma_1,q_1,0)\\ (\lambda_v,1+\mu_v);(1+\mu_1,\lambda_1,1),\ldots,(1+\mu_{v-1},\lambda_{v-1},1)\end{array}\right] $$

$$ = \Gamma(1+\mu_1)\ldots\Gamma(1+\mu_{v-1})\, HE_{\mu_1,\ldots,\mu_v}^{\lambda_1,\ldots,\lambda_v}(z). \qquad (2.3.10) $$

11. Put $h = 1, s_1 = 1, \gamma_1 = \gamma, q_1 = q;\ k = 1, r_1 = 1, \delta_1 = \delta, p_1 = p;$ $a = 1; \rho = 0;\ \tau = 0$ in (1.2.1), then we get function $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)$ defined in (0.7.14), as

$$ {}_{0}E_{1}^{1}\left[z\middle|\begin{array}{c}(0,1);(\gamma,q,1)\\ (\alpha,\beta);(\delta,p,1)\end{array}\right] = E_{\alpha,\beta,p}^{\gamma,\delta,q}(z). \qquad (2.3.11) $$

## 2.4 OTHER SPECIAL FUNCTIONS AS SPECIAL CASES OF THE $E$-FUNCTION

1. Put $h = 1, s_1 = 0;\ k = 1, r_1 = 1, \delta_1 = 1, p_1 = 1; a = 1; \rho = 1; \alpha = 1; \beta = \nu+1;\ \tau = \frac{v}{2}$ and replace $z$ by $\frac{z^2}{4}$ in (1.2.1), then we get Bessel function $J_\nu(z)$ (0.9.1), as

$$ {}_{\frac{v}{2}}E_{1}^{1}\left[\frac{z^2}{4}\middle|\begin{array}{c}(1,1);(\gamma_1,q_1,0)\\ (1,\nu+1);(1,1,1)\end{array}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu} = J_\nu(z). \qquad (2.4.1) $$

2. Put $h=1, s_1=0;\ k=1, r_1=1, \delta_1=1, p_1=1; a=1; \rho=1; \alpha=\mu; \beta=\nu+1;\ \tau=0$ in (1.2.1), then we get Bessel Maitland function $J_\nu^\mu(z)$ (0.9.2), as

$$ {}_0E_1^1\left[z\left|\begin{matrix}(1,1);(\gamma_1,q_1,0)\\ \\ (\mu,\nu+1);(1,1,1)\end{matrix}\right.\right]=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{(1)_n\,\Gamma(n\mu+\nu+1)}=J_\nu^\mu(z). \tag{2.4.2}$$

3. Put $h=1, s_1=0;\ k=1, r_1=1, \delta_1=\lambda+1, p_1=1; a=1; \rho=1; \alpha=\mu; \beta=\nu+\lambda+1;\ \tau=\frac{\nu+2\lambda}{2}$ and replace $z$ by $\frac{z^2}{4}$ in (1.2.1), then we get generalized Bessel Maitland function $J_{\nu,\lambda}^\mu(z)$ (0.9.3), as

$$ {}_{\frac{\nu+2\lambda}{2}}E_1^1\left[\frac{z^2}{4}\left|\begin{matrix}(1,1);(\gamma_1,q_1,0)\\ \\ (\mu,\nu+\lambda+1);(\lambda+1,1,1)\end{matrix}\right.\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(\lambda+1)\left(\frac{z}{2}\right)^{\nu+2\lambda+2n}}{\Gamma(n+\lambda+1)\,\Gamma(n\mu+\nu+\lambda+1)}=\Gamma(\lambda+1)\,J_{\nu,\lambda}^\mu(z). \tag{2.4.3}$$

4. Put $h=1, s_1=0;\ k=1, r_1=1, \delta_1=1, p_1=1; a=1; \rho=0; \alpha=1; \beta=m+1;\ \tau=0$ in (1.2.1), then we get Bessel Clifford function $C_m(z)$ (0.9.4), as

$$ {}_0E_1^1\left[z\left|\begin{matrix}(0,1);(\gamma_1,q_1,0)\\ \\ (1,m+1);(1,1,1)\end{matrix}\right.\right]=\sum_{n=0}^{\infty}\frac{z^n}{(1)_n\,\Gamma(n+m+1)}=C_m(z). \tag{2.4.4}$$

5. Put $h=1, s_1=1, \gamma_1=1, q_1=1;\ k=2, r_1=1, r_2=1, \delta_1=\frac{\mu-\nu+3}{2}, \delta_2=\frac{\mu+\nu+3}{2}, p_1=1, p_2=1; a=2; \rho=1;\ \tau=\mu+1; \alpha=1; \beta=1$ and replace

$z$ by $\frac{z}{2}$ in (1.2.1), then we get Lommel function $s_{\mu,\nu}(z)$ (0.9.5), as

$$
{}_{\mu+1}E_2^1\left[\frac{z}{2}\left|\begin{matrix}(1,2);(1,1,1)\\ (1,1);\left(\frac{\mu-\nu+3}{2},1,1\right),\left(\frac{\mu+\nu+3}{2},1,1\right)\end{matrix}\right.\right]
$$

$$
=\sum_{n=0}^{\infty}\frac{(1)_n(-1)^n\left(\frac{z}{2}\right)^{2n+\mu+1}}{\left(\frac{\mu-\nu+3}{2}\right)_n\left(\frac{\mu+\nu+3}{2}\right)_n\Gamma(n+1)}=\frac{(\mu-\nu+1)(\mu+\nu+1)}{2^{\mu+1}}s_{\mu,\nu}(z). \tag{2.4.5}
$$

6. Put $h=2, s_1=\rho, s_2=1, \gamma_1=\nu, \gamma_2=\beta, q_1=1, q_2=\alpha;\ k=1, r_1=\rho, \delta_1=\nu+1, p_1=1;\ a=0; \rho=0;\ \tau=0$ in (1.2.1), then we get Hurwitz zeta function $\zeta(\rho,\nu)$ (0.9.6), as

$$
{}_0E_1^2\left[z\left|\begin{matrix}(0,0);(\nu,1,\rho),(\beta,\alpha,1)\\ (\alpha,\beta);(\nu+1,1,\rho)\end{matrix}\right.\right]=\frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{[(\nu)_n]^\rho}{[(\nu+1)_n]^\rho}
$$

$$
=\frac{\nu^\rho}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{1}{(n+\nu)^\rho}=\frac{\nu^\rho}{\Gamma(\beta)}\zeta(\rho,\nu). \tag{2.4.6}
$$

7. Put $h=1, s_1=-\nu, \gamma_1=2, q_1=1;\ k=1, r_1=-\nu, \delta_1=1, p_1=1; a=0; \rho=0;\ \tau=0; \alpha=0; \beta=1$ in (1.2.1), then we get Riemann zeta function $\zeta(\nu)$ (0.9.7), as

$$
{}_0E_1^1\left[z\left|\begin{matrix}(0,0);(2,1,-\nu)\\ (0,1);(1,1,-\nu)\end{matrix}\right.\right]=\sum_{n=0}^{\infty}(n+1)^{-\nu}=\zeta(\nu). \tag{2.4.7}
$$

8. Put $h=1, s_1=0;\ k=1, r_1=1, \delta_1=\frac{3}{2}, p_1=1; a=2; \rho=1; \alpha=1; \beta=\nu+\frac{3}{2};\ \tau=\nu+1$ and replace $z$ by $\frac{z}{2}$ in (1.2.1), then we get Struve

function $H_\nu(z)$ (0.9.8), as

$$ {}_{\nu+1}E_1^1\left[\frac{z}{2}\left|\begin{matrix}(1,2);(\gamma_1,q_1,0)\\ \left(1,\nu+\frac{3}{2}\right);\left(\frac{3}{2},1,1\right)\end{matrix}\right.\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(\frac{3}{2}\right)_n\Gamma(n+\nu+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+\nu+1} $$

$$ =\frac{\sqrt{\pi}}{2}H_\nu(z). \tag{2.4.8} $$

9. Put $h=1,s_1=0;\ k=1,r_1=1,\delta_1=\frac{3}{2},p_1=1;a=2;\rho=0;\alpha=1;\beta=\nu+\frac{3}{2};\ \tau=\nu+1$ and replace $z$ by $\frac{z}{2}$ in (1.2.1), then we get modified Struve function $L_\nu(z)$ (0.9.9), as

$$ {}_{\nu+1}E_1^1\left[\frac{z}{2}\left|\begin{matrix}(0,2);(\gamma_1,q_1,0)\\ \left(1,\nu+\frac{3}{2}\right);\left(\frac{3}{2},1,1\right)\end{matrix}\right.\right]=\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)_n\Gamma(n+\nu+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+\nu+1} $$

$$ =\frac{\sqrt{\pi}}{2}L_\nu(z). \tag{2.4.9} $$

10. Put $h=2,s_1=1,s_2=1,\gamma_1=\nu,\gamma_2=\sigma,q_1=1,q_2=\frac{\omega}{\mu};\ k=1,r_1=1,\delta_1=\theta,p_1=\frac{\omega}{\mu};a=1;\rho=0;\ \tau=0;\alpha=1;\beta=1$ in (1.2.1), then we get Dotsenko function ${}_2R_1^{\frac{\omega}{\mu}}(\nu,\sigma;\theta,\omega;\mu;z)$ (0.9.10), as

$$ {}_0E_1^2\left[z\left|\begin{matrix}(0,1);(\nu,1,1),\left(\sigma,\frac{\omega}{\mu},1\right)\\ (1,1);\left(\theta,\frac{\omega}{\mu},1\right)\end{matrix}\right.\right]=\frac{\Gamma(\theta)}{\Gamma(\nu)\Gamma(\sigma)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n)\Gamma\left(\sigma+\frac{\omega}{\mu}n\right)z^n}{\Gamma\left(\theta+\frac{\omega}{\mu}n\right)n!} $$

$$ ={}_2R_1^{\frac{\omega}{\mu}}(\nu,\sigma;\theta,\omega;\mu;z). \tag{2.4.10} $$

11. Put $h=1,s_1=0;\ k=1,r_1=0;\ a=\nu+1;\rho=0;\tau=\nu;\alpha=\nu+1;\beta=\nu+1$ and replace $z$ by $t\zeta^{\frac{1}{\nu+1}}$ in (1.2.1), then we get Rabotnov's function

$R_\nu(\zeta, t)$ (0.9.11), as

$$ {}_\nu E_1^1 \left[ t\zeta^{\frac{1}{\nu+1}} \middle| \begin{matrix} (0, \nu+1); (\gamma_1, q_1, 0) \\ (\nu+1, \nu+1); (\delta_1, p_1, 0) \end{matrix} \right] = t^\nu \zeta^{\frac{\nu}{\nu+1}} \sum_{n=0}^{\infty} \frac{\zeta^n t^{(\nu+1)n}}{\Gamma\{(\nu+1)(n+1)\}} $$

$$ = \zeta^{\frac{\nu}{\nu+1}} R_\nu(\zeta, t). \tag{2.4.11} $$

12. Put $h = 1, s_1 = 0;\ k = 1, r_1 = 0; a = 1; \rho = 0; \alpha = 1; \beta = \nu + 1; \tau = \nu$ and replace $z$ by $bt$ in (1.2.1), then we get Mellin-Ross function $E_t(\nu, b)$ (0.9.12), as

$$ {}_\nu E_1^1 \left[ bt \middle| \begin{matrix} (0, 1); (\gamma_1, q_1, 0) \\ (1, \nu+1); (\delta_1, p_1, 0) \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(bt)^{n+\nu}}{\Gamma(\nu+n+1)} = b^\nu E_t(\nu, b). \tag{2.4.12} $$

# CHAPTER 3

# MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS INVOLVING MULTIVARIABLE POLYNOMIAL AND MITTAG-LEFFLER TYPE $E-$FUNCTION

**Publications:**

**1. Fractional integral operators involving Mittag-Leffler type $E$-function,** ***Journal of Rajasthan Academy of Physical Sciences*** **14, No. 3 & 4(2015), 309-322.**

**2. Composition formulae for the multidimensional fractional integral operators involving Mittag-Leffler type $E$-function,** ***Communicated.***

In this chapter, we define two fractional integral operators whose kernels involve generalized multivariable polynomial $S_V^{U_1,...,U_k}(x_1, ..., x_k)$ and the $E$-

function.

In the first section, we define a pair of multidimensional fractional integral operators $I_x$ and $J_x$ and give the conditions of existence. Then under these operators we obtain images of important functions. After this, we prove two theorems connecting the multidimensional generalized Stieltjes transform and here defined integral operators. Then, we establish Mellin transform, Mellin convolutions and inversion formulae of these operators. Finally, we study three composition formulae of the multidimensional fractional integral operators and obtain two dimensional analogue of second composition formula.

The pair of multidimensional fractional integral operators $I_x$ and $J_x$ defined in this chapter are generalized integral operators and these are extensions and unifications of many results of earlier defined fractional integral operators.

The kernels of multidimensional fractional integral operators involve generalized multivariable polynomial $S_V^{U_1,...,U_k}(x_1, ..., x_k)$ and Mittag-Leffler type $E$-function are general in nature and our work yields a number of corresponding earlier derived results by many authors with simpler polynomials and functions.

The results due to Erdélyi [33], Goyal and Jain [57], Goyal, Jain and Gaur [58], Raina [154], and many others can be obtained as special cases of three composition formulae.

## 3.1 DEFINITIONS

### 3.1.1 The General Multivariable Polynomials

Srivastava and Garg [195, p. 686, Eq. (1.4)] defined the multivariable polynomial $S_V^{U_1,...,U_k}(x_1,...,x_k)$ as follows:

$$S_V^{U_1,...,U_k}[x_1,...,x_k] = \sum_{R_1,...,R_k=0}^{\sum_{i=1}^{k} U_i R_i \leq V} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V, R_1,...,R_k) \frac{x_i^{R_i}}{R_i!}, \qquad (3.1.1)$$

where V= 0,1,...; $U_1,...,U_k$ are arbitrary positive integers and the coefficients $A(V, R_1,...,R_k)$ are arbitrary constants (real or complex). Several single and general multivariable polynomial can be obtained as special cases of general multivariable polynomial $S_V^{U_1,...,U_k}(x_1,...,x_k)$ by replacing coefficients $A(V, R_1,...,R_k)$ occuring in (3.1.1) with a suitable function. Further detail of this polynomial and its special cases can be seen in Appendix B.

### 3.1.2 The $\overline{H}$-Function

In 1987, Inayat Hussain [80] defined the $\overline{H}$-function by Mellin-Barnes type contour integral as follows:

$$\overline{H}_{P,Q}^{M,N}\left[z \mid \begin{array}{c} (\varepsilon_j, \omega_j; \Upsilon_j)_1^N, (\varepsilon_j, \omega_j)_{N+1}^P \\ (b_j, \vartheta_j)_1^M, (b_j, \vartheta_j; B_j)_{M+1}^Q \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \overline{\phi}(\xi) z^{\xi} d\xi, \qquad (3.1.2)$$

where

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j - \vartheta_j \xi) \prod_{j=1}^{N}\left[\Gamma(1-\varepsilon_j + \omega_j \xi)\right]^{\Upsilon_j}}{\prod_{j=M+1}^{Q}\left[\Gamma(1-b_j+\vartheta_j\xi)\right]^{B_j} \prod_{j=N+1}^{P}\Gamma(\varepsilon_j - \omega_j \xi)}, \qquad (3.1.3)$$

where $M, N, P$ and, $Q$ are non-negative integers satisfying $0 \leq N \leq P$, $0 \leq M \leq Q$ and empty products are taken as unity. Also, $\Upsilon_j (j = 1, \ldots, P)$ and $B_j (j = 1, \ldots, Q)$ are positive real numbers for standardization purpose, $\varepsilon_j (j = 1, \ldots, P)$ and $b_j (j = 1, \ldots, Q)$ are complex numbers such that the points $\xi = \frac{b_j + k}{\vartheta_j}$ $(j = 1, ..., M; k = 0, 1, ...)$ which are the poles of $\Gamma(b_j - \vartheta_j \xi)(j = 1, \ldots, M)$ and the points $\xi = \frac{\varepsilon_j - 1 - k}{\omega_j}$ $(j = 1, ..., N; k = 0, 1, ...)$ which are the singularities of $\left[\Gamma\left(1 - \varepsilon_j + \omega_j \xi\right)\right]^{\Upsilon_j} (j = 1, \ldots, N)$ do not coincide.

The contour $\mathfrak{L}$ is the line from $c - i\infty$ to $c + i\infty$ suitably intended to keep the poles of $\Gamma(b_j - \vartheta_j \xi)(j = 1, \ldots, M)$ to the right of the path and the singularities of $\left[\Gamma\left(1 - \varepsilon_j + \omega_j \xi\right)\right]^{\Upsilon_j}$ $(j = 1, ..., N)$ to the left of the path. If $\Upsilon_i = B_j = 1 (i = 1, ..., N; j = M + 1, ..., Q)$ the $\overline{H}$-function reduces to the familiar Fox $H$-function.

Gupta, Jain and Agrawal [67] have been given the sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$-function given by (3.1.2), as follows:

$$\left.\begin{array}{l} (i)\ |\arg(z)| < \frac{1}{2}\pi\Omega \qquad \text{and } \Omega > 0; \\ (ii)\ |\arg(z)| = \frac{1}{2}\pi\Omega \qquad \text{and } \Omega \geq 0; \\ \text{and} \\ (a) \mu \neq 0 \text{ and the contour } \mathfrak{L} \text{ is so chosen that } (c\mu + \lambda + 1) < 0; \\ (b)\ \mu = 0 \text{ and } (\lambda + 1) < 0, \end{array}\right\} \tag{3.1.4}$$

where

$$\left.\begin{aligned} \Omega &= \sum_{1}^{M}\vartheta_j + \sum_{1}^{N}\omega_j\Upsilon_j - \sum_{1}^{Q}\vartheta_j B_j - \sum_{N+1}^{P}\omega_j \\ \mu &= \sum_{1}^{N}\omega_j\Upsilon_j + \sum_{N+1}^{P}\omega_j - \sum_{1}^{M}\vartheta_j - \sum_{M+1}^{Q}\vartheta_j B_j \\ \lambda &= Re\left(\sum_{1}^{M}b_j + \sum_{M+1}^{Q}b_j B_j - \sum_{1}^{N}\varepsilon_j\Upsilon_j - \sum_{N+1}^{P}\varepsilon_j\right) \\ &\quad +\tfrac{1}{2}\left(-M - \sum_{M+1}^{Q}B_j + \sum_{1}^{N}\Upsilon_j + P - N\right). \end{aligned}\right\} \tag{3.1.5}$$

The series representation of the $\overline{H}$-function was given by Rathie [158]:

$$\overline{H}_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (\varepsilon_j,\,\omega_j;\,\Upsilon_j)_1^N;(\varepsilon_j,\,\omega_j)_{N+1}^P \\ (b_j,\,\vartheta_j)_1^M;(b_j,\,\vartheta_j;\,B_j)_{M+1}^Q \end{matrix}\right] = \sum_{\nu=1}^{M}\sum_{D=0}^{\infty}\overline{\theta}\left(S_{D,\nu}\right) z^{S_{D,\nu}}, \tag{3.1.6}$$

where $S_{D,\nu} = \frac{b_\nu + D}{\vartheta_\nu}$ and $\overline{\theta}\left(S_{D,\nu}\right)$

$$= \frac{\prod_{j=1,j\neq\nu}^{M}\Gamma\left(b_j - \vartheta_j S_{D,\nu}\right)\prod_{j=1}^{N}\left[\Gamma\left(1-\varepsilon_j+\omega_j S_{D,\nu}\right)\right]^{\Upsilon_j}(-1)^{D}}{\prod_{j=M+1}^{Q}\left[\Gamma\left(1-b_j+\vartheta_j S_{D,\nu}\right)\right]^{B_j}\prod_{j=N+1}^{P}\Gamma\left(\varepsilon_j-\omega_j S_{D,\nu}\right)D!\vartheta_\nu}, \tag{3.1.7}$$

for small and large values of $z$ the behavior of the $\overline{H}$-function is given by Saxena [172, p. 112, Eqs. (2.3) & (2.4)] as follows:

$\overline{H}_{P,Q}^{M,N}[z] = O\left[|z|^{\triangle}\right]$ for small $z$, where

$$\triangle = \min_{1\leq j\leq M} Re\left(\frac{b_j}{\vartheta_j}\right). \tag{3.1.8}$$

$\overline{H}_{P,Q}^{M,N}[z] = O\left[|z|^{\nabla}\right]$ for large $z$, where

$$\nabla = \max_{1\leq j\leq N} Re\left[\Upsilon_j\left(\frac{\varepsilon_j - 1}{\omega_j}\right)\right]. \tag{3.1.9}$$

details of series representation of the $\overline{H}$-function can be seen in Appendix-A.

### 3.1.3 $I$ and $J-$ Integral Operators

In this chapter we assume that $f(t_1,...,t_s) \in$ A represents the class of functions $f(t_1,...,t_s)$ for which $\int \cdots \int_{\Omega_s} |f(t_1,...,t_s)|\, dt_1...dt_s < \infty$ for every bounded $s$-dimensional region $\Omega_s$ excluding the origin and

$$f(t_1,...,t_s) = \begin{cases} O\prod_{j=1}^{s}\left(|t_j|^{\psi_j}\right) & \max\{|t_j|\} \to 0 \\ O\prod_{j=1}^{s}\left(|t_j|^{-\zeta_j} e^{-W_j|t_j|}\right) & \min\{|t_j|\} \to \infty \end{cases} \quad ; \quad j = 1,...,s. \tag{3.1.10}$$

Now, we define a pair of multidimensional fractional integral operators with kernels involving multivariable polynomial $S_V^{U_1,...,U_k}(x_1,...,x_k)$ and the $E$-function having general arguments as follows:

$$\begin{aligned} I_x[f(t_1,...,t_s)] &= I_{x;U,V;z}^{\pi,\sigma;e,f;\eta,\lambda}[f(t_1,...,t_s); x_1,...,x_s] \\ &= \left(\prod_{j=1}^{s} x_j^{-\pi_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left[\prod_{j=1}^{s} t_j^{\pi_j}(x_j-t_j)^{\sigma_j-1}\right] \\ &\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},...,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right] \\ &\times {}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j} \middle| \begin{matrix}(\rho,a);(\gamma_i,q_i,d_i)_{1,h}\\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{matrix}\right] f(t_1,...,t_s)\,dt_1...dt_s, \end{aligned} \tag{3.1.11}$$

where

$$\left.\begin{aligned} &(1)\min Re(e_j,f_j,\eta_j a,\lambda_j a) \geq 0 \text{ not all zero simultaneously;}\\ &(2)\min Re[1+\pi_j+\eta_j\tau+\psi_j] > 0, \min Re[\sigma_j+\lambda_j\tau] > 0,\\ &\qquad \text{where } j = 1,...,s. \end{aligned}\right\} \tag{3.1.12}$$

$$J_x\left[f\left(t_1,...,t_s\right)\right]=J^{\pi,\sigma;e,f;\eta,\lambda}_{x;U,V;z}\left[f\left(t_1,...,t_s\right);x_1,...,x_s\right]$$
$$=\left(\prod_{j=1}^{s}x_j^{\pi_j}\right)\int_{x_1}^{\infty}...\int_{x_s}^{\infty}\left[\prod_{j=1}^{s}t_j^{-\pi_j-\sigma_j}\left(t_j-x_j\right)^{\sigma_j-1}\right]$$
$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{e_1}\left(1-\frac{x_1}{t_1}\right)^{f_1},...,E_s\left(\frac{x_s}{t_s}\right)^{e_s}\left(1-\frac{x_s}{t_s}\right)^{f_s}\right]$$
$$\times {}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{\eta_j}\left(1-\frac{x_j}{t_j}\right)^{\lambda_j}\;\middle|\;\begin{matrix}(\rho,a);(\gamma_i,q_i,d_i)_{1,h}\\ \\(\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right]f\left(t_1,...,t_s\right)dt_1...dt_s\,, \tag{3.1.13}$$

where

$$\left.\begin{matrix}(1)\min Re(e_j,f_j,\eta_j a,\lambda_j a)\geq 0 \text{ not all zero simultaneously,}\\ (2)\min Re[\pi_j+\eta_j\tau+\zeta_j]>0,\ \min Re\left[\sigma_j+\lambda_j\tau\right]>0,\ Re\left[W_j\right]=0\\ \text{or } Re\left[W_j\right]>0,\min Re\left[\sigma_j+\lambda_j\tau\right]>0,\ \text{where } j=1,...,s.\end{matrix}\right\} \tag{3.1.14}$$

## 3.2 IMAGES OF INTEGRAL OPERATORS

Here we evaluate images of some functions $\prod_{j=1}^{s}t_j^{\nu_j}\left(h_j+t_j\right)^{-\varphi_j}$ under the operators defined by (3.1.11) and (3.1.13) as follows:

$$I_x\left[\prod_{j=1}^{s}t_j^{\nu_j}\left(h_j+t_j\right)^{-\varphi_j}\right]=z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}}\sum_{R_1,...,R_s=0}^{\sum_{i=1}^{s}U_iR_i\leq V}\left(-V\right)_{\sum_{i=1}^{s}U_iR_i}A\left(V,R_1,...,R_s\right)$$
$$\times\frac{E_i^{R_i}}{R_i!}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(n+1\right)}\prod_{j=1}^{s}\left(-\frac{x_j}{h_j}\right)^{n}\left(\frac{x_j^{\nu_j}}{h_j^{\varphi_j}}\right)\left(1+\frac{x_j}{h_j}\right)^{\sigma_j+f_jR_j+\lambda_j\tau-\varphi_j}$$
$$\times\overline{H}^{1,h+3s+1}_{h+3s+1,k+2s+2}\left[\left(-1\right)^{\rho}\left(-z^{a}\right)\left(1+\tfrac{x_j}{h_j}\right)^{\lambda_j a}\;\middle|\;\begin{matrix}A^*\\ \\B^*\end{matrix}\right], \tag{3.2.1}$$

where

$$A^* = \left(-\pi_j - \nu_j - e_j R_j - \eta_j \tau, \eta_j a; 1\right)_{1,s}, \left(1 - \sigma_j - f_j R_j - \lambda_j \tau - n, \lambda_j a; 1\right)_{1,s},$$

$$\left(\varphi_j - \sigma_j - \pi_j - \nu_j - \left(e_j + f_j\right) R_j - \left(\lambda_j + \eta_j\right) \tau - n, \left(\lambda_j + \eta_j\right) a; 1\right)_{1,s},$$

$$(0, 1; 1), \left(1 - \gamma_i, q_i; d_i\right)_{1,h}; \text{---} \tag{3.2.2}$$

and

$$B^* = (0, 1); \left(\varphi_j - \sigma_j - \pi_j - \nu_j - \left(e_j + f_j\right) R_j - \left(\lambda_j + \eta_j\right) \tau, \left(\lambda_j + \eta_j\right) a; 1\right)_{1,s},$$

$$\left(-\sigma_j - \pi_j - \nu_j - \left(e_j + f_j\right) R_j - \left(\lambda_j + \eta_j\right) \tau - n, \left(\lambda_j + \eta_j\right) a; 1\right)_{1,s},$$

$$\left(1 - \delta_j, p_j; r_j\right)_{1,k}, (1 - \beta, \alpha; 1) \tag{3.2.3}$$

provided that

$$\min Re\left(e_j, f_j, \eta_j a, \lambda_j a\right) \geq 0 \text{ not all zero simultaneously,}$$

$$\min Re\left[1 + \pi_j + \eta_j \tau + \nu_j\right] > 0, \ \min Re\left[\sigma_j + \lambda_j \tau\right] > 0, \ (j = 1, ..., s).$$

Also

$$J_x\left[\prod_{j=1}^{s} t_j^{\nu_j} \left(h_j + t_j\right)^{-\varphi_j}\right] = z^{\tau} \frac{\prod_{m=1}^{k} \left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h} \left[\Gamma\left(\gamma_l\right)\right]^{d_l}} \sum_{R_1, \ldots, R_s = 0}^{\sum_{i=1}^{s} U_i R_i \leq V} (-V)_{\sum_{i=1}^{s} U_i R_i} A\left(V, R_1, ..., R_s\right)$$

$$\times \frac{E_i^{R_i}}{R_i!} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)} \prod_{j=1}^{s} \left(x_j^{\nu_j - \varphi_j}\right) \left(-\frac{h_j}{x_j}\right)^n \left(1 + \frac{h_j}{x_j}\right)^{\sigma_j + f_j R_j + \lambda_j \tau - \varphi_j}$$

$$\times \overline{H}^{1, h+3s+1}_{h+3s+1, k+2s+2}\left[(-1)^{\rho}\left(-z^a\right)\left(1 + \frac{h_j}{x_j}\right)^{\lambda_j a} \ \middle| \ \begin{matrix} A^{**} \\ \\ B^{**} \end{matrix}\right], \tag{3.2.4}$$

where

$$A^{**}=(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_ja;1)_{1,s},(1-\pi_j+\nu_j-e_jR_j-\eta_j\tau-\varphi_j,\eta_ja;1)_{1,s},$$
$$(1-\sigma_j-\pi_j-(e_j+f_j)R_j+\nu_j-(\lambda_j+\eta_j)\tau-n,(\lambda_j+\eta_j)a;1)_{1,s},$$
$$(0,1;1),(1-\gamma_i,q_i;d_i)_{1,h};\text{——} \tag{3.2.5}$$

and

$$B^{**}=(0,1);(1-\sigma_j-\pi_j+\nu_j-(e_j+f_j)R_j-(\lambda_j+\eta_j)\tau,(\lambda_j+\eta_j)a;1)_{1,s},$$
$$(1-\sigma_j-\pi_j-(e_j+f_j)R_j+\nu_j-(\lambda_j+\eta_j)\tau-n,(\lambda_j+\eta_j)a;1)_{1,s},$$
$$\left(1-\delta_j,p_j;r_j\right)_{1,k},(1-\beta,\alpha;1) \tag{3.2.6}$$

provided that

$\min Re\,(e_j,f_j,\eta_ja,\lambda_ja)\geq 0$ not all zero simultaneously,

$\min Re\,[\pi_j+\eta_j\tau+\varphi_j-\nu_j]>0$, $\min Re\,[\sigma_j+\lambda_j\tau]>0$, $(j=1,...,s)$.

**Proof:** To prove (3.2.1), we write down the $I$-operator in the integral form as defined in equation (3.1.11). After this, we write down multivariable polynomial $S_V^{U_1,...,U_k}(x_1,...,x_k)$ as defined in the series form (3.1.1). Then, interchange the series and $t_j$-integrals and now using (1.3.1), we express the Mittag-Leffler type $E$-function as Mellin Barnes type contour integral. Now interchange the order of $\xi$ and $t_j$-integrals $(j=1,...,s)$ (which is permissible under the earlier stated conditions) then we arrived at the following form (say $\Delta$)

$$\Delta = z^{\tau}\frac{\prod\limits_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod\limits_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}}\sum_{R_1,...,R_s=0}^{\sum\limits_{i=1}^{s}U_iR_i\leq V}\left(-V\right)_{\sum\limits_{i=1}^{s}U_iR_i}A\left(V,R_1,...,R_s\right)\frac{E_i^{R_i}}{R_i!}$$
$$\times\frac{1}{2\pi i}\int_{\mathcal{L}}\overline{\theta}\left(\xi\right)\left(-1\right)^{\rho\xi}\left(-z^a\right)^{\xi}\prod_{j=1}^{s}x_j^{-\sigma_j-\pi_j-(e_j+f_j)R_j-(\lambda_j+\eta_j)\tau-(\lambda_j+\eta_j)a\xi}$$
$$\times\left\{\int_0^{x_1}...\int_0^{x_s}\prod_{j=1}^{s}t_j^{\pi_j+\nu_j+e_jR_j+\eta_j\tau+\eta_ja\xi}\left(x_j-t_j\right)^{\sigma_j+f_jR_j+\lambda_j\tau+\lambda_ja\xi-1}\right.$$
$$\left.\times\left(h_j+t_j\right)^{-\varphi_j}dt_1...dt_s\right\}d\xi. \tag{3.2.7}$$

Now, using known result [60, p. 287, Eq. 3.197(8)], we calculate the $t_j$-integral, then we get

$$I_x\left[\prod_{j=1}^{s}t_j^{\nu_j}\left(h_j+t_j\right)^{-\varphi_j}\right]$$
$$= z^{\tau}\frac{\prod\limits_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod\limits_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}}\sum_{R_1,...,R_s=0}^{\sum\limits_{i=1}^{s}U_iR_i\leq V}\left(-V\right)_{\sum\limits_{i=1}^{s}U_iR_i}A\left(V,R_1,...,R_s\right)\frac{E_i^{R_i}}{R_i!}$$
$$\times\prod_{j=1}^{s}\left(x_j^{\nu_j-\varphi_j}\right)\left(\frac{h_j}{x_j}\right)^{-\varphi_j}\frac{1}{2\pi i}\int_{\mathcal{L}}\overline{\theta}\left(\xi\right)\left(-1\right)^{\rho\xi}\left(-z^a\right)^{\xi}$$
$$\times B\left(\sigma_j+f_jR_j+\lambda_ja\xi+\lambda_j\tau,\pi_j+\nu_j+e_jR_j+\eta_ja\xi+\eta_j\tau+1\right)$$
$$\times{}_2F_1\left[\begin{matrix}\varphi_j,\pi_j+\nu_j+e_jR_j+\eta_j\tau+\eta_ja\xi+1\\ \sigma_j+\pi_j+\nu_j+\left(f_j+e_j\right)R_j+\left(\lambda_j+\eta_j\right)\tau+\left(\lambda_j+\eta_j\right)a\xi+1\end{matrix};\left(-\frac{x_j}{h_j}\right)\right]d\xi, \tag{3.2.8}$$

where

$\left|\arg\left(\frac{x_j}{h_j}\right)\right|<\pi$, $Re\left(\pi_j+\eta_j\tau+\nu_j+e_jR_j+\eta_ja\xi+1\right)>0$,

$Re\left(\sigma_j+\lambda_j\tau+f_jR_j+\lambda_ja\xi\right)>0$, for $j=1,...,s$.

Finally, with the help of transformation formula [156, p. 60, Eq. (5)] and then reinterpreting the result thus arrived in terms of the $\overline{H}$-function (3.1.2), and after a little simplification we easily achive the desired final result (3.2.1).

The proof of (3.2.1) can be done easily on the similar lines as given above.

## 3.3 THE MULTIDIMENSIONAL GENERALIZED STIELTJES TRANSFORM WITH $I$ AND $J-$ INTEGRAL OPERATORS

The multidimensional generalized Stieltjes transform of a function $\phi(t_1, ..., t_s)$ is defined as

$$S_{w_1,...,w_s}(\phi)(h_1, ..., h_s) = \int_0^\infty ... \int_0^\infty \phi(t_1, ..., t_s) \prod_{j=1}^{s} (t_j + h_j)^{-w_j} dt_1 ... dt_s \,, \tag{3.3.1}$$

provided that the integral exists.

The multidimensional generalized Stieltjes transform of the $I$ and $J-$integral operators can be obtained as follows:

**Theorem 1.** *Let* $\phi(t_1, ..., t_s) \in A$*,* $min\, Re\,(e_j, f_j, \eta_j a, \lambda_j a) \geq 0$ *not all zero simultaneously and* $min\, Re\,[\sigma_j + \lambda_j \tau] > 0 \; (j = 1, ..., s)$*, then*

(a) *For* $min\, Re\,[\pi_j + \eta_j \tau + w_j] > 0 \; (j = 1, ..., s)$*, we have*

$$\begin{aligned} & S_{w_1,...,w_s}(I_t \phi)(h_1, ..., h_s) \\ & = \int_0^\infty ... \int_0^\infty \phi(x_1, ..., x_s)\, \psi_1(x_1, ..., x_s; h_1, ..., h_s)\, dx_1 ... dx_s \,, \end{aligned} \tag{3.3.2}$$

(b) *For* $min\,Re\,[1+\pi_j+\eta_j\tau]>0,\ (j=1,...,s),$ *we have*

$$S_{w_1,...,w_s}\left(J_t\phi\right)\left(h_1,...,h_s\right)$$
$$=\int_0^\infty ... \int_0^\infty \phi\left(x_1,...,x_s\right)\psi_2\left(x_1,...,x_s;h_1,...,h_s\right)dx_1...dx_s\,, \tag{3.3.3}$$

*where*

$$\psi_1\left(x_1,...,x_s;h_1,...,h_s\right)=J_x\left[\prod_{j=1}^{s}\left(h_j+t_j\right)^{-w_j}\right]=z^{\tau}\frac{\prod\limits_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod\limits_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}}$$
$$\times\sum_{R_1,...,R_s=0}^{\sum\limits_{i=1}^{s}U_iR_i\le V}\left(-V\right)_{\sum\limits_{i=1}^{s}U_iR_i}A\left(V,R_1,...,R_s\right)\frac{E_i^{R_i}}{R_i!}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(n+1\right)}\prod_{j=1}^{s}\left(x_j^{-w_j}\right)\left(-\frac{h_j}{x_j}\right)^{n}$$
$$\times\left(1+\frac{h_j}{x_j}\right)^{\sigma_j+f_jR_j+\lambda_j\tau-w_j}\overline{H}^{1,h+3s+1}_{h+3s+1,k+2s+2}\left[\left(-1\right)^{\rho}\left(-z^{a}\right)\left(1+\tfrac{h_j}{x_j}\right)^{\lambda_j a}\ \middle|\ \begin{matrix}A^*\\ B^*\end{matrix}\right], \tag{3.3.4}$$

*here*

$$A^*=\left(1-\sigma_j-\pi_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},$$
$$\left(1-\pi_j-e_jR_j-\eta_j\tau-w_j,\eta_ja;1\right)_{1,s},\left(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_ja;1\right)_{1,s},$$
$$\left(1-\gamma_i,q_i;d_i\right)_{1,h},\left(0,1;1\right);—— \tag{3.3.5}$$

*and*

$$B^*=\left(0,1\right);\left(1-\sigma_j-\pi_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},$$
$$\left(1-\sigma_j-\pi_j+w_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},$$
$$\left(1-\delta_j,p_j;r_j\right)_{1,k},\left(1-\beta,\alpha;1\right) \tag{3.3.6}$$

*Also*

$$\psi_2(x_1,...,x_s;h_1,...,h_s)=I_x\left[\prod_{j=1}^{s}(h_j+t_j)^{-w_j}\right]=z^{\tau}\frac{\prod_{m=1}^{k}[\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h}[\Gamma(\gamma_l)]^{d_l}}$$

$$\times\sum_{R_1,...,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}(-V)_{\sum_{i=1}^{s}U_iR_i}A(V,R_1,...,R_s)\frac{E_i^{R_i}}{R_i!}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(h_j^{-w_j}\right)\left(-\frac{x_j}{h_j}\right)^n$$

$$\times\left(1+\frac{x_j}{h_j}\right)^{\sigma_j+f_jR_j+\lambda_j\tau-w_j}\overline{H}_{h+3s+1,k+2s+2}^{1,h+3s+1}\left[(-1)^{\rho}(-z^a)\left(1+\frac{x_j}{h_j}\right)^{\lambda_j a}\;\Big|\;\begin{matrix}A^{**}\\ B^{**}\end{matrix}\right],\tag{3.3.7}$$

*here*

$$\begin{aligned}A^{**}=&(w_j-\sigma_j-\pi_j-(e_j+f_j)R_j-(\lambda_j+\eta_j)\tau-n,(\lambda_j+\eta_j)a;1)_{1,s},\\&(1-\gamma_i,q_i;d_i)_{1,h},(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_ja;1)_{1,s},\\&(-\pi_j-e_jR_j-\eta_j\tau,\eta_ja;1)_{1,s},(0,1;1);\text{——}\end{aligned}\tag{3.3.8}$$

*and*

$$\begin{aligned}B^{**}=&(0,1);(w_j-\sigma_j-\pi_j-(e_j+f_j)R_j-(\lambda_j+\eta_j)\tau,(\lambda_j+\eta_j)a;1)_{1,s},\\&(-\sigma_j-\pi_j-(e_j+f_j)R_j-(\lambda_j+\eta_j)\tau-n,(\lambda_j+\eta_j)a;1)_{1,s},\\&\left(1-\delta_j,p_j;r_j\right)_{1,k},(1-\beta,\alpha;1)\end{aligned}\tag{3.3.9}$$

*The integrals on the right hand side of equations (3.3.4) and (3.3.7) are assumed to be exist.*

**Proof:** By the definitions of $I_x-$operator and of multidimensional Stieltjes transform given by (3.1.11) and (3.3.1) respectively, the LHS of (3.3.3) can be obtained as follows:

$$= \int_0^\infty \cdots \int_0^\infty \left[ \left( \prod_{j=1}^s t_j^{-\pi_j-\sigma_j} \right) \int_0^{t_1} \cdots \int_0^{t_s} \prod_{j=1}^s x_j^{\pi_j} (t_j - x_j)^{\sigma_j-1} \right.$$
$$\times S_V^{U_1,\dots,U_s} \left[ E_1 \left( \frac{x_1}{t_1} \right)^{e_1} \left( 1 - \frac{x_1}{t_1} \right)^{f_1}, \dots, E_s \left( \frac{x_s}{t_s} \right)^{e_s} \left( 1 - \frac{x_s}{t_s} \right)^{f_s} \right]$$
$$\times {}_\tau E_k^h \left[ z \prod_{j=1}^s \left( \frac{x_j}{t_j} \right)^{\eta_j} \left( 1 - \frac{x_j}{t_j} \right)^{\lambda_j} \Big| \begin{matrix} (\rho, a) ; (\gamma_i, q_i, d_i)_{1,h} \\ (\alpha, \beta) ; (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right]$$
$$\left. \times \phi (x_1, \dots, x_s) \, dx_1 \dots dx_s \right] \prod_{j=1}^s \left\{ (t_j + h_j)^{-w_j} \right\} dt_1 \dots dt_s \,. \tag{3.3.10}$$

By interchanging the order of $t_j$ and $x_j$ integrals (under the conditions stated with the theorem), we get

$$= \int_0^\infty \cdots \int_0^\infty \left[ \prod_{j=1}^s \left( x_j^{\pi_j} \right) \phi (x_1, \dots, x_s) \int_{x_1}^\infty \cdots \int_{x_s}^\infty \prod_{j=1}^s \left( t_j^{-\pi_j-\sigma_j} \right) (t_j - x_j)^{\sigma_j-1} \right.$$
$$\times S_V^{U_1,\dots,U_s} \left[ E_1 \left( \frac{x_1}{t_1} \right)^{e_1} \left( 1 - \frac{x_1}{t_1} \right)^{f_1}, \dots, E_s \left( \frac{x_s}{t_s} \right)^{e_s} \left( 1 - \frac{x_s}{t_s} \right)^{f_s} \right]$$
$$\times {}_\tau E_k^h \left[ z \prod_{j=1}^s \left( \frac{x_j}{t_j} \right)^{\eta_j} \left( 1 - \frac{x_j}{t_j} \right)^{\lambda_j} \Big| \begin{matrix} (\rho, a) ; (\gamma_i, q_i, d_i)_{1,h} \\ (\alpha, \beta) ; (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right]$$
$$\left. \times \prod_{j=1}^s \left\{ (t_j + h_j)^{-w_j} \right\} dt_1 \dots dt_s \right] dx_1 \dots dx_s \,. \tag{3.3.11}$$

By writing the $t_j$-integrals in terms of the operator defined by (3.1.13), the above result (3.3.11) can be transform as follows

$$= \int_0^\infty \cdots \int_0^\infty \phi (x_1, \dots, x_s) \, J_x \left[ \prod_{j=1}^s (t_j + h_j)^{-w_j} \right] dx_1 \dots dx_s \,. \tag{3.3.12}$$

To find the value of $J_x \left[ \prod_{j=1}^s (t_j + h_j)^{-w_j} \right]$, we use the result (3.2.4) with $\nu_j = 0$, then we achive the right hand side of (3.3.2).

The proof of part (b) of Theorem 1 can be easily developed on the similar lines as given above, with the help of definition of $J_x-$operator defined by (3.1.13) and the result (3.2.1) with $\nu_j = 0$.

The $I$ and $J-$integral operators of multidimensional generalized Stieltjes transform can be obtained as follows:

**Theorem 2.** *If $\phi(t_1, ..., t_s) \in A$, $min\,Re\,(e_j, f_j, \eta_j a, \lambda_j a) \geq 0$ $(j = 1, ..., s)$ not all zero simultaneously, then*

1. *For $min\,Re\,[1 + \pi_j + \eta_j \tau] > 0$, $(j = 1, ..., s)$*

$$I_y \left[S_{w_1,...,w_s} \phi(t_1, ..., t_s)(x_1, ..., x_s)\right]$$
$$= \int_0^\infty ... \int_0^\infty \phi(t_1, ..., t_s)\, \psi_2(t_1, ..., t_s; x_1, ..., x_s)\, dt_1 ... dt_s\,, \qquad (3.3.13)$$

2. *For $min\,Re\,[\pi_j + \eta_j \tau + w_j] > 0$, $(j = 1, ..., s)$*

$$J_y \left[S_{w_1,...,w_s} \phi(t_1, ..., t_s)(x_1, ..., x_s)\right]$$
$$= \int_0^\infty ... \int_0^\infty \phi(t_1, ..., t_s)\, \psi_1(t_1, ..., t_s; x_1, ..., x_s)\, dt_1 ... dt_s\,, \qquad (3.3.14)$$

*where $\psi_1(t_1, ..., t_s; x_1, ..., x_s)$ and $\psi_2(t_1, ..., t_s; x_1, ..., x_s)$ are as given in (3.3.4) and (3.3.7) respectively, provided that the integrals in the R.H.S. of equations (3.3.13) and (3.3.14) exist.*

**Proof:** Results (3.3.13) and (3.3.14) of Theorem 2 can be obtained on the similar lines to the proof of Theorem 1.

Moreover, the one dimensional analogues of the Theorem 1 and 2 can be easily derived.

## 3.4 MELLIN TRANSFORMS, INVERSION FORMULAS AND CONVOLUTION

Srivastava and Panda [199, part I, p. 125, Eq. (3.5)] defined the multidimensional Mellin transform of the function $f(t_1,...,t_s) \in A$ as follows:

$$M[f(t_1,...,t_s);\theta_1,...,\theta_s] = \int_0^\infty ... \int_0^\infty \prod_{j=1}^{s} t_j^{\theta_j-1} f(t_1,...,t_s)\, dt_1...dt_s\,, \tag{3.4.1}$$

provided that the integral exists.

The multidimensional Mellin transforms, corresponding inversion formulas and convolutions of the $I$ and $J-$fractional integral operators defined by (3.1.11) and (3.1.13) respectively, can be obtained as follows:

**Result 1**

*If the conditions of the existence of the operator* $I^{\pi,\sigma;e,f;\eta,\lambda}_{x;U,V;z}[f(t_1,...,t_s)]$ *are satisfied and* $M[I_x\{f(t_1,...,t_s);\theta_1,...,\theta_s\}]$ *exists, then*

$$M[I_x\{f(t_1,...,t_s);\theta_1,...,\theta_s\}] = M[f(t_1,...,t_s);\theta_1,...,\theta_s]\,\chi(\theta_1,...,\theta_s)\,, \tag{3.4.2}$$

*where*

$$\chi(\theta_1,...,\theta_s) = z^{\tau}\frac{\prod_{m=1}^{k}[\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h}[\Gamma(\gamma_l)]^{d_l}} \sum_{R_1,...,R_s=0}^{\sum_{i=1}^{s}U_iR_i \le V} (-V)_{\sum_{i=1}^{s}U_iR_i}\, A(V,R_1,...,R_s)\frac{E_i^{R_i}}{R_i!}$$

$$\times\, \overline{H}^{1,h+2s+1}_{h+2s+1,k+s+2}\left[(-1)^{\rho}(-z^{a}) \,\middle|\, \begin{matrix} C^* \\ D^* \end{matrix}\right], \tag{3.4.3}$$

*here*

$$C^* = \left(1-\sigma_j - f_j R_j - \lambda_j \tau, \lambda_j a; 1\right)_{1,s}, \left(-\pi_j + \theta_j - e_j R_j - \eta_j \tau, \eta_j a; 1\right)_{1,s},$$

$$\left(1-\gamma_i, q_i; d_i\right)_{1,h}, (0,1;1); \text{——} \tag{3.4.4}$$

*and* $$D^* = (0,1); \left(-\sigma_j - \pi_j + \theta_j - (e_j + f_j) R_j - (\lambda_j + \eta_j)\tau, (\lambda_j + \eta_j) a; 1\right)_{1,s},$$

$$\left(1-\delta_j, p_j; r_j\right)_{1,k}, (1-\beta, \alpha; 1) \tag{3.4.5}$$

**Result 2**

*If the conditions of the existence of the operator* $J_{x;U,V;z}^{\pi,\sigma;e,f;\eta,\lambda}\left[f(t_1,...,t_s)\right]$ *are satisfied and* $M\left[J_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]$ *exists, then*

$$M\left[J_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right] = M\left[f(t_1,...,t_s);\theta_1,...,\theta_s\right]\chi(1-\theta_1,...,1-\theta_s), \tag{3.4.6}$$

*where* $\chi(1-\theta_1,...,1-\theta_s)$ *can be determined by (3.4.3).*

**Proof:** The multidimensional Mellin transform of the $I$-operator can be obtained using equations (3.4.1) and (3.1.11), as follows:

$$\begin{aligned}
&M\left[I_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right] \\
&= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^s \left(x_j^{\theta_j-1}\right)\left[\prod_{j=1}^s \left(x_j^{-\pi_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\prod_{j=1}^s t_j^{\pi_j}(x_j-t_j)^{\sigma_j-1}\right. \\
&\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},...,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right] \\
&\times {}_\tau E_k^h\left[z\prod_{j=1}^s\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j} \left| \begin{matrix}(\rho,a);\left(\gamma_i,q_i,d_i\right)_{1,h} \\ (\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right] \\
&\times \left. f(t_1,...,t_s)\,dt_1...dt_s\right]dx_1...dx_s.
\end{aligned} \tag{3.4.7}$$

Now, by interchanging the orders of $t_j$ and $x_j$ integrals (which is permissible under the conditions stated above), we get the RHS of (3.4.7) as follows

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{s} \left(t_j^{\pi_j}\right) f\left(t_1, ..., t_s\right) \left[\int_{t_1}^\infty \cdots \int_{t_s}^\infty \prod_{j=1}^{s} \left(x_j^{\theta_j - \pi_j - \sigma_j - 1}\right) \left(x_j - t_j\right)^{\sigma_j - 1}\right.$$
$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1}, ..., E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right]$$
$$\left.\times \, {}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j} \Big| \begin{matrix} (\rho, a);(\gamma_i, q_i, d_i)_{1,h} \\ (\alpha, \beta);(\delta_j, p_j, r_j)_{1,k} \end{matrix}\right] dx_1...dx_s\right] dt_1...dt_s\,. \tag{3.4.8}$$

By applying the definition (3.1.13), the above expression reduces to

$$\int_0^\infty \cdots \int_0^\infty f\left(t_1, ..., t_s\right) J_t\left(\prod_{j=1}^{s} x_j^{\theta_j - 1}\right) dt_1...dt_s\,. \tag{3.4.9}$$

Again by using the result

$$I_x\left[\prod_{j=1}^{s} t_j^{\nu_j}\left(x_j - t_j\right)^{\delta_j}\right] = \left(\prod_{j=1}^{s} x_j^{2\nu_j + \delta_j + 1}\right) J_x\left[\prod_{j=1}^{s} t_j^{-(1+\nu_j+\delta_j)}\left(t_j - x_j\right)^{\delta_j}\right]. \tag{3.4.10}$$

The integral (3.4.9) reduces to

$$\int_0^\infty \cdots \int_0^\infty f\left(t_1, ..., t_s\right) \prod_{j=1}^{s}\left(t_j^{2\theta_j - 1}\right) I_t\left(\prod_{j=1}^{s} x_j^{-\theta_j}\right) dt_1...dt_s\,, \tag{3.4.11}$$

with the help of (3.2.1), we evaluate $I_t\left[\prod_{j=1}^{s} x_j^{-\theta_j}\right]$ and then arrived at (3.4.2)

.

The proof of result 2 can be developed on similar lines.

### 3.4.1 Inversion Formulas

The inversion formulas for $I$ and $J-$operators (3.1.11) and (3.1.13) respectively, can be obtained with the help of the inversion theorems for the multidimensional Mellin transform (3.4.1), given by Srivastava and Panda [199, part I, p. 125, Lemma 2] as follows:

**Result 3**

$$f\left(t_1,...,t_s\right)=\frac{1}{\left(2\pi i\right)^s}$$
$$\times\int_{c_1-i\infty}^{c_1+i\infty}...\int_{c_s-i\infty}^{c_s+i\infty}\frac{\prod\limits_{j=1}^{s}t_j^{-\theta_j}}{\chi\left(\theta_1,...,\theta_s\right)}M\left[I_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]d\theta_1...d\theta_s\,, \tag{3.4.12}$$

**Result 4**

$$f\left(t_1,...,t_s\right)=\frac{1}{\left(2\pi i\right)^s}$$
$$\times\int_{c_1-i\infty}^{c_1+i\infty}...\int_{c_s-i\infty}^{c_s+i\infty}\frac{\prod\limits_{j=1}^{s}t_j^{-\theta_j}}{\chi\left(1-\theta_1,...,1-\theta_s\right)}M\left[J_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]d\theta_1...d\theta_s\,, \tag{3.4.13}$$

*where* $\chi\left(\theta_1,...,\theta_s\right)$ *and* $\chi\left(1-\theta_1,...,1-\theta_s\right)$ *can be easily derived by (3.4.3).*

The conditions of validity for the inversion formulas (3.4.12) and (3.4.13) can be easily obtained from conditions of existence of multidimensional fractional integral operators and their multidimensional Mellin transforms defined earlier.

### 3.4.2 Mellin Convolutions

The multidimensional Mellin convolutions of two functions $f(t_1, ..., t_s)$ and $g(t_1, ..., t_s)$ is defined as follows:

$$(f \star g)(t_1, ..., t_s) = (g \star f)(t_1, ..., t_s)$$
$$= \int_0^\infty ... \int_0^\infty \left(\prod_{j=1}^{s} x_j^{-1}\right) f\left(\frac{t_1}{x_1}, ..., \frac{t_s}{x_s}\right) g(x_1, ..., x_s)\, dx_1 ... dx_s\,, \quad (3.4.14)$$

provided that the multiple integrals in right hand side involved in (3.4.14) exist.

Let $f(t_1, ..., t_s) \in A$, then the $I$ and $J-$fractional integral operators defined by (3.1.11) and (3.1.13) respectively, can be written easily as multidimensional Mellin convolutions in the following forms:

**Result 5**

$$I_{x;U,V;z}^{\pi,\sigma;e,f;\eta,\lambda} f(t_1, ..., t_s) = (I_{\pi,\sigma,e,f;\eta,\lambda;x;U,V;z} \star f)(x_1, ..., x_s)\,, \quad (3.4.15)$$

*where*

$$I_{\pi,\sigma,e,f;\eta,\lambda;x;U,V;z} = \left(\prod_{j=1}^{s} x_j^{-\pi_j-\sigma_j} (x_j - 1)^{\sigma_j - 1}\, \Theta(x_j - 1)\right)$$
$$\times S_V^{U_1,...,U_s}\left[E_1 (x_1)^{-e_1-f_1} (x_1 - 1)^{f_1}, ..., E_s (x_s)^{-e_s-f_s} (x_s - 1)^{f_s}\right]$$
$$\times {}_{\tau}E_k^h \left[ z \prod_{j=1}^{s} (x_j)^{-\eta_j-\lambda_j} (x_j - 1)^{\lambda_j} \;\middle|\; \begin{matrix} (\rho, a); (\gamma_i, q_i, d_i)_{1,h} \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right], \quad (3.4.16)$$

*here $\Theta(x)$ is the Heaviside unit function .*

**Result 6**

$$J_{x;U,V;z}^{\pi,\sigma;e,f;\eta,\lambda} f\left(t_1,...,t_s\right) = \left(J_{\pi,\sigma,e,f;\eta,\lambda;x;U,V;z} \star f\right)\left(x_1,...,x_s\right), \qquad (3.4.17)$$

*where*

$$J_{\pi,\sigma;e,f;\eta,\lambda;x;U,V;z} = \left(\prod_{j=1}^{s} x_j^{\pi_j}\left(1-x_j\right)^{\sigma_j-1}\Theta\left(1-x_j\right)\right)$$
$$\times\, S_V^{U_1,...,U_s}\left[E_1\left(x_1\right)^{e_1}\left(1-x_1\right)^{f_1},...,E_s\left(x_s\right)^{e_s}\left(1-x_s\right)^{f_s}\right]$$
$$\times\, {}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(x_j\right)^{\eta_j}\left(1-x_j\right)^{\lambda_j} \,\middle|\, \begin{matrix}(\rho,a);\left(\gamma_i,q_i,d_i\right)_{1,h}\\ (\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right], \qquad (3.4.18)$$

*here* $\Theta(x)$ *is the Heaviside unit function* .

**Proof:** Result 5, can be proved by writing the $I$-operator defined by (3.1.11) in the following form using the Heaviside's unit function:

$$I_{x;U,V;z}^{\pi,\sigma;e,f;\eta,\lambda} f\left(t_1,...,t_s\right)$$
$$= \int_0^{\infty}...\int_0^{\infty}\left(\prod_{j=1}^{s} t_j^{-1}\right)\left\{\prod_{j=1}^{s}\left[\left(\frac{x_j}{t_j}\right)^{-\pi_j-\sigma_j}\left(\frac{x_j}{t_j}-1\right)^{\sigma_j-1}\Theta\left(\frac{x_j}{t_j}-1\right)\right]\right\}$$
$$\times\, S_V^{U_1,...,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{-e_1-f_1}\left(\frac{x_1}{t_1}-1\right)^{f_1},...,E_s\left(\frac{x_s}{t_s}\right)^{-e_s-f_s}\left(\frac{x_s}{t_s}-1\right)^{f_s}\right]$$
$$\times\, {}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{-\eta_j-\lambda_j}\left(\frac{x_j}{t_j}-1\right)^{\lambda_j} \,\middle|\, \begin{matrix}(\rho,a);\left(\gamma_i,q_i,d_i\right)_{1,h}\\ (\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right]$$
$$\times\, f\left(t_1,...,t_s\right)dt_1...dt_s\,. \qquad (3.4.19)$$

The result 5, can be easily deduced with the help of the equation (3.4.16) and the definition of the Mellin convolutions given by (3.4.14) in the above equation. The proof of the result 6 can be developed on the similar lines.

## 3.5 COMPOSITION FORMULAE FOR THE MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS INVOLVING MITTAG-LEFFLER TYPE $E$-FUNCTION

**Result 7**

$$I_{x;U,V;z}^{\pi,\sigma;e,f;\Lambda,\theta}\left\{J_{y;U',V';z'}^{\pi',\sigma';e',f';\Lambda',\theta'}\left[f\left(t_1,...,t_s\right)\right]\right\}=\left(\prod_{j=1}^{s}x_j^{-\pi_j-\theta_j\tau-1}\right)$$
$$\times\int_0^{x_1}...\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\pi_j+\theta_j\tau}\right)G\left(\frac{t_1}{x_1},...,\frac{t_s}{x_s}\right)f\left(t_1,...,t_s\right)dt_1...dt_s$$
$$+\left(\prod_{j=1}^{s}x_j^{\pi_j'+\theta_j'\tau'}\right)\int_{x_1}^{\infty}...\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{-\pi_j'-\theta_j'\tau'-1}\right)G'\left(\frac{x_1}{t_1},...,\frac{x_s}{t_s}\right)$$
$$\times f\left(t_1,...,t_s\right)dt_1...dt_s\,, \tag{3.5.1}$$

*where*

$$G\left(t_1,...,t_s\right)=\frac{\left(z^{\tau}\right)\left(z'\right)^{\tau'}\prod\limits_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}\prod\limits_{m'=1}^{k'}\left[\Gamma\left(\delta'_{m'}\right)\right]^{r'_{m'}}}{\prod\limits_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}\prod\limits_{l'=1}^{h'}\left[\Gamma\left(\gamma'_{l'}\right)\right]^{d'_{l'}}}$$
$$\times\sum_{R_1,...,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}\left(-V\right)_{\sum_{i=1}^{s}U_iR_i}A\left(V,R_1,...,R_s\right)\frac{E_i^{R_i}}{R_i!}$$
$$\times\sum_{R_1',...,R_s'=0}^{\sum_{i=1}^{s}U_i'R_i'\le V'}\left(-V'\right)_{\sum_{i=1}^{s}U_i'R_i'}A\left(V',R_1',...,R_s'\right)\frac{E_i'^{R_i'}}{R_i'!}$$
$$\times\sum_{D'=0}^{\infty}\overline{\theta}\left(S_{D',1}\right)\left[(-1)^{\rho'}\left\{-\left(z'\right)^{a'}\right\}\right]^{S_{D',1}}\left(1-t_j\right)^{\sigma_j+\theta_j\tau+\sigma_j'+\theta_j'\tau'+f_jR_j+f_j'R_j'+\theta_j'a'S_{D',1}-1}$$
$$\times t_j^{e_jR_j+n}\,\overline{H}_{h+2s+1,k+2s+2}^{1,h+2s+1}\left[(-1)^{\rho+1}z^a\prod_{j=1}^{s}t_j^{\Lambda_ja}\left(1-t_j\right)^{\theta_ja}\;\Big|\;\begin{matrix}A^*\\ B^*\end{matrix}\right], \tag{3.5.2}$$

*here*

$$A^{*}=\left(-\pi_j-\Lambda_j\tau-\pi_j^{'}-\Lambda_j^{'}\tau^{'}-e_jR_j-e_j^{'}R_j^{'}-\Lambda_j^{'}a^{'}S_{D',1},\Lambda_ja;1\right)_{1,s},$$
$$\left(1-\gamma_i,q_i;d_i\right)_{1,h},\left(1-\pi_j-\Lambda_j\tau-\pi_j^{'}-\Lambda_j^{'}\tau^{'}-\sigma_j-\theta_j\tau-\sigma_j^{'}-\theta_j^{'}\tau^{'}-n\right.$$
$$\left.-\left(e_j+f_j\right)R_j-\left(e_j^{'}+f_j^{'}\right)R_j^{'}-\left(\theta_j^{'}+\Lambda_j^{'}\right)a^{'}S_{D',1},\left(\theta_j+\Lambda_j\right)a;1\right)_{1,s},$$
$$(0,1;1)\,;—— \tag{3.5.3}$$

*and*

$$B^{*}=(0,1)\,;\left(1-\delta_j,p_j;r_j\right)_{1,k},(1-\beta,\alpha;1)\,,\left(-\pi_j^{'}-\Lambda_j^{'}\tau^{'}-\pi_j-\Lambda_j\tau\right.$$
$$\left.-\sigma_j^{'}-\theta_j^{'}\tau^{'}-n-e_jR_j-\left(e_j^{'}+f_j^{'}\right)R_j^{'}-\left(\theta_j^{'}+\Lambda_j^{'}\right)a^{'}S_{D',1},\Lambda_ja;1\right)_{1,s},$$
$$\left(1-\pi_j-\Lambda_j\tau-\pi_j^{'}-\Lambda_j^{'}\tau^{'}-\sigma_j-\theta_j\tau-\sigma_j^{'}-\theta_j^{'}\tau^{'}-\left(e_j+f_j\right)R_j\right.$$
$$\left.-\left(e_j^{'}+f_j^{'}\right)R_j^{'}-\left(\theta_j^{'}+\Lambda_j^{'}\right)a^{'}S_{D',1},\left(\theta_j+\Lambda_j\right)a;1\right)_{1,s} \tag{3.5.4}$$

*it is assumed that the composite operator defined by the L.H.S. of (3.5.1) exists, $f(t_1,...,t_s)\in A$ and $G^{'}(t_1,...,t_s)$ can be written from $G(t_1,...,t_s)$ from (3.5.2) by interchanging the parameters with dashes with those without dashes also $S_{D',1}$ and $\overline{\theta}\left(S_{D',1}\right)$ can be obtained from (3.1.7) by replacing parameters with suitable parameters with dashes and the following conditions are satisfied*:

$$\left.\begin{array}{l}(1)\,min\ \mathrm{Re}[\pi_j+\Lambda_j\tau]>-1,\ min\,Re\left[\pi_j^{'}+\Lambda_j^{'}\tau^{'}+\psi_j\right]>-1;\\ (2)\,min\,Re[\sigma_j+\theta_j\tau]>0,\ min\,Re\left[\sigma_j^{'}+\theta_j^{'}\tau^{'}\right]>0;\\ (3)\,Re[W_j]>0\ \text{or}\ min\,Re\left[1+\pi_j+\sigma_j+\zeta_j+\Lambda_j\tau+\theta_j\tau\right]>0,\\ \qquad Re[W_j]=0,\ where\,(j=1,...,s)\,.\end{array}\right\} \tag{3.5.5}$$

**Result 8**

$$I_{x;U,V;z}^{\pi,\sigma;e,f;\Lambda,\theta}\left\{I_{y;U',V';z'}^{\pi',\sigma';e',f';\Lambda',\theta'}\left[f\left(t_1,...,t_s\right)\right]\right\}=\left(\prod_{j=1}^{s}x_j^{-\pi_j'-1}\right)$$
$$\times\int_0^{x_1}...\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\pi_j'}\right)G\left(\frac{t_1}{x_1},...,\frac{t_s}{x_s}\right)f\left(t_1,...,t_s\right)dt_1...dt_s\,, \qquad (3.5.6)$$

*where*

$$G\left(t_1,...,t_s\right)=\frac{\left(z^{\tau}\right)\left(z'\right)^{\tau'}\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}\prod_{m'=1}^{k'}\left[\Gamma\left(\delta'_{m'}\right)\right]^{r'_{m'}}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}\prod_{l'=1}^{h'}\left[\Gamma\left(\gamma'_{l'}\right)\right]^{d'_{l'}}}$$
$$\times\sum_{R_1,...,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}\left(-V\right)_{\sum_{i=1}^{s}U_iR_i}A\left(V,R_1,...,R_s\right)\frac{E_i^{R_i}}{R_i!}$$
$$\times\sum_{R'_1,...,R'_s=0}^{\sum_{i=1}^{s}U'_iR'_i\le V'}\left(-V'\right)_{\sum_{i=1}^{s}U'_iR'_i}A\left(V',R'_1,...,R'_s\right)\frac{E_i'^{R'_i}}{R'_i!}$$
$$\times\sum_{D'=0}^{\infty}\overline{\theta}\left(S_{D',1}\right)\left[(-1)^{\rho'}\left\{-\left(z'\right)^{a'}\right\}\right]^{S_{D',1}}t_j^{e'_jR'_j+\Lambda'_j\tau'+\Lambda'_ja'S_{D',1}}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\sigma'_j+\theta'_j\tau'+f'_jR'_j+\theta'_ja'S_{D',1}\right)}{\Gamma\left(n+1\right)}\left(1-t_j\right)^{\sigma_j+\theta_j\tau+\sigma'_j+\theta'_j\tau'+f_jR_j+f'_jR'_j+\theta'_ja'S_{D',1}+n-1}$$
$$\times\overline{H}_{h+2s+1,k+2s+2}^{s+1,h+s+1}\left[(-1)^{\rho}\left(-z^{a}\right)\prod_{j=1}^{s}\left(1-t_j\right)^{\theta_ja}\;\Big|\;\begin{matrix}C^{*}\\ \\ D^{*}\end{matrix}\right], \qquad (3.5.7)$$

*here*

$$C^{*}=\left(0,1;1\right),\left(1-\gamma_i,q_i;d_i\right)_{1,h},\left(1-\sigma_j-\theta_j\tau-f_jR_j-n,\theta_ja;1\right)_{1,s};$$
$$\left(-\pi_j-\Lambda_j\tau-e_jR_j+\sigma'_j+\left(e'_j+f'_j\right)R'_j+\pi'_j+\left(\Lambda'_j+\theta'_j\right)\tau'\right.$$
$$\left.+\left(\Lambda'_j+\theta'_j\right)a'S_{D',1},\Lambda_ja\right)_{1,s} \qquad (3.5.8)$$

*and*

$$D^* = (0,1), \left(-\pi_j - \Lambda_j \tau - e_j R_j + \sigma_j^{'} + \left(e_j^{'} + f_j^{'}\right) R_j^{'} + \pi_j^{'} + n + \left(\Lambda_j^{'} + \theta_j^{'}\right) \tau^{'} \right.$$
$$\left. + \left(\Lambda_j^{'} + \theta_j^{'}\right) a^{'} S_{D^{'},1}, \Lambda_j a\right)_{1,s}; (1-\beta, \alpha; 1), \left(1-\delta_j, p_j; 1\right)_{1,k},$$
$$\left(1 - \sigma_j - \sigma_j^{'} - \theta_j \tau - \theta_j^{'} \tau^{'} - n - f_j R_j - f_j^{'} R_j^{'} - \theta_j^{'} a^{'} S_{D^{'},1}, \theta_j a; 1\right)_{1,s} \tag{3.5.9}$$

*where* $S_{D^{'},1}$ *and* $\overline{\theta}\left(S_{D^{'},1}\right)$ *can be obtained from (3.1.7) by replacing parameters with suitable parameters with dashes and the following conditions are satisfied*:

$$\left.\begin{aligned} &(1)\, min\, Re\left[1+\pi_j+\Lambda_j\tau\right] > 0, min\, Re\left[1+\pi_j^{'}+\Lambda_j^{'}\tau^{'}+\psi_j\right] > 0,\\ &(2)\, min\, Re\left[\sigma_j+\theta_j\tau\right] > 0, min\, Re\left[\sigma_j^{'}+\theta_j^{'}\tau^{'}\right] > 0,\ where\ j=1,...,s. \end{aligned}\right\} \tag{3.5.10}$$

**Result 9**

$$J_{x;U^{'},V^{'};z^{'}}^{\pi^{'},\sigma^{'};e^{'},f^{'};\Lambda^{'},\theta^{'}} \left\{J_{y;U,V;z}^{\pi,\sigma;e,f;\Lambda,\theta}\left[f\left(t_1,...,t_s\right)\right]\right\} = \left(\prod_{j=1}^{s} x_j^{\pi_j^{'}}\right)$$
$$\times \int_{x_1}^{\infty} ... \int_{x_s}^{\infty} \left(\prod_{j=1}^{s} t_j^{-\pi_j^{'}-1}\right) G\left(\frac{x_1}{t_1},...,\frac{x_s}{t_s}\right) f\left(t_1,...,t_s\right) dt_1...dt_s, \tag{3.5.11}$$

*where* $f\left(t_1,...,t_s\right) \in A$*, the operator defined by the L.H.S. of (3.5.11) exists,* $G\left(t_1,...,t_s\right)$ *is given by (3.5.7) and following conditions are satisfied*:

$$\left.\begin{aligned} &(1) Re\left[W_j\right] > 0\ or\ Re\left[W_j\right] = 0,\ min\, Re\left[1+\pi_j+\Lambda_j\tau+\sigma_j+\theta_j\tau+\zeta_j\right] > 0,\\ &(2) min\ Re\left[\sigma_j+\theta_j\tau\right] > 0,\ min\ Re\left[\sigma_j^{'}+\theta_j^{'}\tau^{'}\right] > 0,\quad where\ j=1,...,s. \end{aligned}\right\} \tag{3.5.12}$$

**Proof:** To prove result 7, first of all we write $I$ and $J-$multidimensional fractional integral operators involved in the L.H.S. of equation (3.5.1), in the integral form by using the definition of $I$ and $J-$fractional integral operators (3.1.11) and (3.1.13) respectively, then we get the following integral:

$$I^{\pi,\sigma;e,f;\Lambda,\theta}_{x;U,V;z}\left\{J^{\pi',\sigma';e',f';\Lambda',\theta'}_{y;U',V';z'}\left[f\left(t_1,...,t_s\right)\right]\right\}$$
$$=\left(\prod_{j=1}^{s}x_j^{-\pi_j-\sigma_j}\right)\int_0^{x_1}...\int_0^{x_s}\left[\prod_{j=1}^{s}y_j^{\pi_j}\left(x_j-y_j\right)^{\sigma_j-1}\right]$$
$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},...,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]$$
$$\times\,{}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\Lambda_j}\left(1-\frac{y_j}{x_j}\right)^{\theta_j}\right]\left(\prod_{j=1}^{s}y_j^{\pi_j'}\right)\int_{y_1}^{\infty}...\int_{y_s}^{\infty}\prod_{j=1}^{s}t_j^{-\pi_j'-\sigma_j'}\left(t_j-y_j\right)^{\sigma_j'-1}$$
$$\times S_{V'}^{U_1',...,U_s'}\left[E_1'\left(\frac{y_1}{t_1}\right)^{e_1'}\left(1-\frac{y_1}{t_1}\right)^{f_1'},...,E_s'\left(\frac{y_s}{t_s}\right)^{e_s'}\left(1-\frac{y_s}{t_s}\right)^{f_s'}\right]$$
$$\times\,{}_{\tau'}E_{k'}^{h'}\left[z'\prod_{j=1}^{s}\left(\frac{y_j}{t_j}\right)^{\Lambda_j'}\left(1-\frac{y_j}{t_j}\right)^{\theta_j'}\right]f\left(t_1,...,t_s\right)dt_1...dt_sdy_1...dy_s\,. \tag{3.5.13}$$

After this, by interchanging the order of $t_j$ and $y_j$ integrals (which is permissible under the conditions stated) we get

$$I^{\pi,\sigma;e,f;\Lambda,\theta}_{x;U,V;z}\left\{J^{\pi',\sigma';e',f';\Lambda',\theta'}_{y;U',V';z'}\left[f\left(t_1,...,t_s\right)\right]\right\}$$
$$=\int_0^{x_1}...\int_0^{x_s}\left\{\int_0^{t_1}...\int_0^{t_s}\Omega dy_1...dy_s\right\}f\left(t_1,...,t_s\right)dt_1...dt_s$$
$$+\int_{x_1}^{\infty}...\int_{x_s}^{\infty}\left\{\int_0^{x_1}...\int_0^{x_s}\Omega dy_1...dy_s\right\}f\left(t_1,...,t_s\right)dt_1...dt_s$$
$$=\int_0^{x_1}...\int_0^{x_s}I_1f\left(t_1,...,t_s\right)dt_1...dt_s+\int_{x_1}^{\infty}...\int_{x_s}^{\infty}I_2f\left(t_1,...,t_s\right)dt_1...dt_s\,, \tag{3.5.14}$$

where

$$\Omega = \left( \prod_{j=1}^{s} x_j^{-\pi_j - \sigma_j} t_j^{-\pi_j^{'} - \sigma_j^{'}} y_j^{\pi_j + \pi_j^{'}} (x_j - y_j)^{\sigma_j - 1} (t_j - y_j)^{\sigma_j^{'} - 1} \right)$$
$$\times S_V^{U_1,...,U_s} \left[ E_1 \left( \frac{y_1}{x_1} \right)^{e_1} \left( 1 - \frac{y_1}{x_1} \right)^{f_1}, ..., E_s \left( \frac{y_s}{x_s} \right)^{e_s} \left( 1 - \frac{y_s}{x_s} \right)^{f_s} \right]$$
$$\times S_{V^{'}}^{U_1^{'},...,U_s^{'}} \left[ E_1^{'} \left( \frac{y_1}{t_1} \right)^{e_1^{'}} \left( 1 - \frac{y_1}{t_1} \right)^{f_1^{'}}, ..., E_s^{'} \left( \frac{y_s}{t_s} \right)^{e_s^{'}} \left( 1 - \frac{y_s}{t_s} \right)^{f_s^{'}} \right]$$
$$\times {}_{\tau}E_k^h \left[ z \prod_{j=1}^{s} \left( \frac{y_j}{x_j} \right)^{\Lambda_j} \left( 1 - \frac{y_j}{x_j} \right)^{\theta_j} \right] {}_{\tau^{'}}E_{k^{'}}^{h^{'}} \left[ z^{'} \prod_{j=1}^{s} \left( \frac{y_j}{t_j} \right)^{\Lambda_j^{'}} \left( 1 - \frac{y_j}{t_j} \right)^{\theta_j^{'}} \right] dy_1...dy_s \tag{3.5.15}$$

and

$$I_1 = \int_0^{t_1} ... \int_0^{t_s} \Omega dy_1...dy_s, I_2 = \int_0^{x_1} ... \int_0^{x_s} \Omega dy_1...dy_s \,. \tag{3.5.16}$$

Now to find $I_1$, involved in the integral on the R.H.S. of (3.5.14), we write both the multivariable polynomials $S_V^{U_1,...,U_s}$, $S_{V^{'}}^{U_1^{'},...,U_s^{'}}$ and the ${}_{\tau^{'}}E_{k^{'}}^{h^{'}}$-function involved in terms of their series expansion using equations (3.1.1) and (1.2.1) respectively, the $E$-function is expressed in terms of the Mellin-Barne's type contour integral form defined by (1.3.1). Then interchanging the order of summations and Mellin-Barne's type contour integral with $y_j$-integral and further, evaluating the $y_j$-integral, we have

$$I_1 = \int_0^{t_1} ... \int_0^{t_s} \Omega dy_1...dy_s = \frac{(z^{\tau}) \left( z^{'} \right)^{\tau^{'}} \prod_{m=1}^{k} \left[ \Gamma \left( \delta_m \right) \right]^{r_m} \prod_{m^{'}=1}^{k^{'}} \left[ \Gamma \left( \delta_{m^{'}}^{'} \right) \right]^{r_{m^{'}}^{'}}}{\prod_{l=1}^{h} \left[ \Gamma \left( \gamma_l \right) \right]^{d_l} \prod_{l^{'}=1}^{h^{'}} \left[ \Gamma \left( \gamma_{l^{'}}^{'} \right) \right]^{d_{l^{'}}^{'}}}$$

$$\times \sum_{R_1,\ldots,R_s=0}^{\sum_{i=1}^{s} U_i R_i \le V} (-V)_{\sum_{i=1}^{s} U_i R_i} A\left(V, R_1, \ldots, R_s\right) \frac{E_i^{R_i}}{R_i!}$$

$$\times \sum_{R_1^{'},\ldots,R_s^{'}=0}^{\sum_{i=1}^{s} U_i^{'} R_i^{'} \le V^{'}} \left(-V^{'}\right)_{\sum_{i=1}^{s} U_i^{'} R_i^{'}} A\left(V^{'}, R_1^{'}, \ldots, R_s^{'}\right) \frac{E_i^{'R_i^{'}}}{R_i^{'}!}$$

$$\times \frac{1}{2\pi i} \int_{\mathcal{L}} \overline{\phi}\left(\xi\right) \left\{(-1)^{\rho} \left(-z^{a}\right)\right\}^{\xi} \sum_{D^{'}=0}^{\infty} \overline{\theta}\left(S_{D^{'},1}\right) \left[(-1)^{\rho^{'}} \left\{-\left(z^{'}\right)^{a^{'}}\right\}\right]^{S_{D^{'},1}}$$

$$\times \left(\prod_{j=1}^{s} t_j^{-e_j^{'} R_j^{'} - f_j^{'} R_j^{'} - \Lambda_j^{'} a^{'} S_{D^{'},1} - \theta_j^{'} a^{'} S_{D^{'},1}} x_j^{-e_j R_j - f_j R_j - \Lambda_j a\xi - \theta_j a\xi}\right)$$

$$\times \int_0^{t_1} \ldots \int_0^{t_s} \prod_{j=1}^{s} y_j^{\pi_j + \Lambda_j \tau + \pi_j^{'} + \Lambda_j^{'} \tau^{'} + e_j R_j + e_j^{'} R_j^{'} + \Lambda_j a\xi + \Lambda_j^{'} a^{'} S_{D^{'},1}}$$

$$\times \left(x_j - y_j\right)^{\sigma_j + \theta_j \tau + f_j R_j + \theta_j a\xi - 1} \left(t_j - y_j\right)^{\sigma_j^{'} + \theta_j^{'} \tau^{'} + f_j^{'} R_j^{'} + \theta_j^{'} a^{'} S_{D^{'},1} - 1} dy_1 \ldots dy_s d\xi \,. \tag{3.5.17}$$

After this, we put $y_j = t_j u_j$ in (3.5.17) and integrate it with the help of the result [195, p. 47, Th. 1.6] we obtain the following equation:

$$I_1 = \frac{\left(z^{\tau}\right) \left(z^{'}\right)^{\tau^{'}} \prod_{m=1}^{k} \left[\Gamma\left(\delta_m\right)\right]^{r_m} \prod_{m^{'}=1}^{k^{'}} \left[\Gamma\left(\delta_{m^{'}}^{'}\right)\right]^{r_{m^{'}}^{'}}}{\prod_{l=1}^{h} \left[\Gamma\left(\gamma_l\right)\right]^{d_l} \prod_{l^{'}=1}^{h^{'}} \left[\Gamma\left(\gamma_{l^{'}}^{'}\right)\right]^{d_{l^{'}}^{'}}}$$

$$\times \sum_{R_1,\ldots,R_s=0}^{\sum_{i=1}^{s} U_i R_i \le V} (-V)_{\sum_{i=1}^{s} U_i R_i} A\left(V, R_1, \ldots, R_s\right) \frac{E_i^{R_i}}{R_i!}$$

$$\times \sum_{R_1^{'},\ldots,R_s^{'}=0}^{\sum_{i=1}^{s} U_i^{'} R_i^{'} \le V^{'}} \left(-V^{'}\right)_{\sum_{i=1}^{s} U_i^{'} R_i^{'}} A\left(V^{'}, R_1^{'}, \ldots, R_s^{'}\right) \frac{E_i^{'R_i^{'}}}{R_i^{'}!}$$

$$\times \sum_{D^{'}=0}^{\infty} \overline{\theta}\left(S_{D^{'},1}\right) \left[(-1)^{\rho^{'}} \left\{-\left(z^{'}\right)^{a^{'}}\right\}\right]^{S_{D^{'},1}} \frac{1}{2\pi i} \int_{\mathcal{L}} \overline{\phi}\left(\xi\right)$$

$$\times \left\{(-1)^{\rho} \left(-z^{a}\right)\right\}^{\xi} t_j^{\pi_j + \Lambda_j \tau + e_j R_j + \Lambda_j a\xi} x_j^{-\pi_j - \Lambda_j \tau - e_j R_j - \Lambda_j a\xi - 1}$$

$$\times \frac{\Gamma\left(\pi_j + \Lambda_j \tau + \pi_j' + \Lambda_j' \tau' + e_j R_j + e_j' R_j' + \Lambda_j' a' S_{D',1} + \Lambda_j a\xi + 1\right)}{\Gamma\left(\pi_j + \Lambda_j \tau + \pi_j' + \Lambda_j' \tau' + \sigma_j' + \theta_j' \tau' + e_j R_j\right.}$$

$$\frac{\Gamma\left(\sigma_j' + \theta_j' \tau' + f_j' R_j' + \theta_j' a' S_{D',1}\right)}{\left. + \left(e_j' + f_j'\right) R_j' + \left(\Lambda_j' + \theta_j'\right) a' S_{D',1} + 1\right)}$$

$$\times {}_2F_1\left[\begin{array}{c} 1 - \sigma_j - \theta_j \tau + f_j R_j + \theta_j a\xi, 1 + \pi_j + \Lambda_j \tau + \pi_j' \\ 1 + \pi_j + \Lambda_j \tau + \pi_j' + \Lambda_j' \tau' + \sigma_j' + \theta_j' \tau' + e_j R_j \end{array}\right.$$

$$\left.\begin{array}{c} + \Lambda_j' \tau' + e_j R_j + e_j' R_j' + \Lambda_j' a' S_{D',1} + \Lambda_j a\xi \\ + \left(e_j' + f_j'\right) R_j' + \left(\Lambda_j' + \theta_j'\right) a' S_{D',1} + \Lambda_j a\xi \end{array} ; \frac{t}{x}\right] d\xi \,. \tag{3.5.18}$$

Finally, we transform RHS of (3.5.18), using the following result [156, p. 60, Eq. (5)]

$$ {}_2F_1\left(a, b; c; z\right) = \left(1 - z\right)^{c-a-b} {}_2F_1\left(c - a, c - b; c; z\right), \qquad |z| < 1 \tag{3.5.19}$$

and expand the ${}_2F_1$ thus deduced in the series form and re-arranging the result in terms of $\overline{H}$-function we obtain the solution of $I_1$.

To find $I_2 = \int_0^{x_1} \ldots \int_0^{x_s} \Omega dy_1 \ldots dy_s$, we follow the same procedure as it is mentioned above with the only difference that we substitute $y_j = x_j u_j$ in the corresponding expression to (3.5.17). By writing the values of $I_1$ and $I_2$ in (3.5.14 ), we get the required result (3.5.1).

To prove (3.5.6), we express the $I$ operator present in the LHS of (3.5.6) in the integral form using the equation (3.1.11), we have

$$I_{x;U,V;z}^{\pi,\sigma;e,f;\Lambda,\theta}\left\{I_{y;U',V';z'}^{\pi',\sigma';e',f';\Lambda',\theta'}\left[f\left(t_1,...,t_s\right)\right]\right\}$$
$$=\left(\prod_{j=1}^{s}x_j^{-\pi_j-\sigma_j}\right)\int_0^{x_1}...\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\pi_j'}\right)f\left(t_1,...,t_s\right)\Delta dt_1...dt_s\,, \quad (3.5.20)$$

where

$$\Delta=\int_{t_1}^{x_1}...\int_{t_s}^{x_s}\left[\prod_{j=1}^{s}y_j^{\pi_j-\pi_j'-\sigma_j'}\left(x_j-y_j\right)^{\sigma_j-1}\left(y_j-t_j\right)^{\sigma_j'-1}\right]$$
$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},...,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]$$
$$\times S_{V'}^{U_1',...,U_s'}\left[E_1'\left(\frac{t_1}{y_1}\right)^{e_1'}\left(1-\frac{t_1}{y_1}\right)^{f_1'},...,E_s'\left(\frac{t_s}{y_s}\right)^{e_s'}\left(1-\frac{t_s}{y_s}\right)^{f_s'}\right]$$
$$\times\,{}_{\tau}E_k^h\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\Lambda_j}\left(1-\frac{y_j}{x_j}\right)^{\theta_j}\right]{}_{\tau'}E_{k'}^{h'}\left[z'\prod_{j=1}^{s}\left(\frac{t_j}{y_j}\right)^{\Lambda_j'}\left(1-\frac{t_j}{y_j}\right)^{\theta_j'}\right]$$
$$\times\,dy_1...dy_s\,. \quad (3.5.21)$$

To find $\Delta$, first of all we express both the multivariable polynomials $S_V^{U_1,...,U_s}$, $S_{V'}^{U_1',...,U_s'}$ and ${}_{\tau'}E_{k'}^{h'}$-function involved in terms of their respective series with the help of equations (3.1.11) and (3.1.13) respectively, and express the $E$-function in terms of the Mellin-Barnes type contour integral by using (3.1.1). Then interchanging the order of summations and Mellin-Barnes contour integral with $y_j$-integral, we get

$$\Delta=\frac{\left(z^{\tau}\right)\left(z'\right)^{\tau'}\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}\prod_{m'=1}^{k'}\left[\Gamma\left(\delta_{m'}'\right)\right]^{r_{m'}'}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}\prod_{l'=1}^{h'}\left[\Gamma\left(\gamma_{l'}'\right)\right]^{d_{l'}'}}$$

$$\times \sum_{R_1,...,R_s=0}^{\sum_{i=1}^{s} U_i R_i \le V} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V, R_1, ..., R_s) \frac{E_i^{R_i}}{R_i!}$$
$$\times \sum_{R_1',...,R_s'=0}^{\sum_{i=1}^{s} U_i' R_i' \le V'} \left(-V'\right)_{\sum_{i=1}^{s} U_i' R_i'} A\left(V', R_1', ..., R_s'\right) \frac{E_i'^{R_i'}}{R_i'!}$$
$$\times \frac{1}{2\pi i}\int_{\mathcal{L}} \overline{\phi}(\xi)\{(-1)^{\rho}(-z^{a})\}^{\xi} \sum_{D'=0}^{\infty} \overline{\theta}(S_{D',1}) \left[(-1)^{\rho'}\left\{-\left(z'\right)^{a'}\right\}\right]^{S_{D',1}}$$
$$\times \prod_{j=1}^{s} x_j^{-\pi_j-\sigma_j-\Lambda_j\tau-\theta_j\tau-e_jR_j-f_jR_j-\theta_j a\xi-\Lambda_j a\xi} \int_0^{x_1}\cdots\int_0^{x_s} \prod_{j=1}^{s} t_j^{\pi_j'+\Lambda_j'\tau'+e_j'R_j'+\Lambda_j'a' S_{D',1}}$$
$$\times \left[\int_{t_1}^{x_1}\cdots\int_{t_s}^{x_s} \prod_{j=1}^{s} y_j^{\pi_j-\pi_j'-\sigma_j'+\Lambda_j\tau-\Lambda_j'\tau'-\theta_j'\tau'+e_jR_j-e_j'R_j'-f_j'R_j'+\Lambda_j a\xi-\Lambda_j'a' S_{D',1}-\theta_j'a' S_{D',1}}\right.$$
$$\left.\times (x_j-y_j)^{\sigma_j+\theta_j\tau+f_jR_j+\theta_j a\xi-1}(y_j-t_j)^{\sigma_j'+\theta_j'\tau'+f_j'R_j'+\theta_j'a' S_{D',1}-1}\, dy_1...dy_s\right] d\xi\,. \tag{3.5.22}$$

Now we put $\frac{x_j-y_j}{x_j-t_j} = u_j$ in (3.5.22) and calculate the $u_j$ integral thus obtained by using the following result [60, p. 287, Eq. 3.197(8)]

$$\int_0^1 x^{\nu-1}(x+a)^{\lambda}(1-x)^{\mu-1}\,dx = a^{\lambda}B(\mu,\nu)\,{}_2F_1\left(-\lambda,\nu;\mu+\nu;-\frac{1}{a}\right). \tag{3.5.23}$$

Now rearranging the result thus obtained in terms of the $\overline{H}-$function and substituting the value in (3.5.20), then we get the result (3.5.6), after little arrangements.

On the similar lines, the proof of result 9 can be developed, so we omit the details.

## 3.6 SPECIAL CASE OF COMPOSITION FORMULAE

Here, we show a two dimensional analogue of second composition formula.

By putting $s = 2$ and assuming the generalized class of polynomials as unity, we get

$$
\begin{aligned}
& I_{x,y;z}^{\pi,m,\sigma,n;\eta,\vartheta,\lambda,\mu}\left\{I_{s,t;z'}^{\pi',m',\sigma',n';\eta',\vartheta',\lambda',\mu'}\left[f\left(u,v\right)\right]\right\} \\
& = I_{x,y;z}^{\pi,m,\sigma,n;\eta,\vartheta,\lambda,\mu}\left\{s^{-\pi'-\sigma'}t^{-m'-n'}\int_0^s\int_0^t\left(s-u\right)^{\sigma'-1}\left(t-v\right)^{n'-1}\right. \\
& \times {}_{\tau'}E_{k'}^{h'}\left[z'\left(\frac{u}{s}\right)^{\eta'}\left(\frac{v}{t}\right)^{\vartheta'}\left(1-\frac{u}{v}\right)^{\lambda'}\left(1-\frac{v}{t}\right)^{\mu'}\right]u^{\pi'}v^{m'}f\left(u,v\right)dudv\left.\right\} \\
& = \frac{\left(z^{\tau}\right)\left(z'\right)^{\tau'}\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}\prod_{m'=1}^{k'}\left[\Gamma\left(\delta'_{m'}\right)\right]^{r'_{m'}}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{d_l}\prod_{l'=1}^{h'}\left[\Gamma\left(\gamma'_{l'}\right)\right]^{d'_{l'}}}\sum_{D'=0}^{\infty}\overline{\theta}\left(S_{D',1}\right)\left[\left(-1\right)^{\rho'}\left\{-\left(z'\right)^{a'}\right\}\right]^{S_{D',1}} \\
& \times \sum_{l=0}^{\infty}\frac{x^{-\sigma-\lambda\tau-\sigma'-\lambda'\tau'-\pi'-\eta'\tau'-l-\left(\eta'+\lambda'\right)a'S_{D',1}}}{\Gamma\left(l+1\right)}y^{-n-\mu\tau-n'-\mu'\tau'-m'-\vartheta'\tau'-l-\left(\vartheta'+\mu'\right)a'S_{D',1}} \\
& \times \Gamma\left(\sigma'_j+\lambda'\tau'+\lambda'a'S_{D',1}\right)\Gamma\left(n'+\mu'\tau'+\mu'a'S_{D',1}\right)\int_0^x\int_0^y u^{\pi'+\eta'\tau'+\eta'a'S_{D',1}} \\
& \times \left(x-u\right)^{\sigma+\sigma'+\lambda\tau+\lambda'\tau'+\lambda'a'S_{D',1}+l-1}\left(y-v\right)^{n+n'+\mu\tau+\mu'\tau'+\mu'a'S_{D',1}+l-1}v^{m'+\vartheta'\tau'+\vartheta'a'S_{D',1}} \\
& \times \overline{H}_{h+5,k+6}^{3,h+3}\left[\left(-1\right)^{\rho}\left(-z^{a}\right)\left(1-\frac{u}{x}\right)^{\lambda a}\left(1-\frac{v}{y}\right)^{\mu a}\ \middle|\ \begin{matrix}A^{**}\\ B^{**}\end{matrix}\right]f\left(u,v\right)dudv\,,
\end{aligned}
\tag{3.6.1}
$$

where

$$
\begin{aligned}
A^{**} = & \left(1-\sigma-\lambda\tau-l,\lambda a;1\right),\left(1-n-\mu\tau-l,\mu'a';1\right),\left(1-\gamma_i,q_i;d_i\right)_{1,h}, \\
& \left(0,1;1\right);\left(-m+m'+n'-\vartheta\tau+\left(\vartheta'+\mu'\right)\tau'+\left(\vartheta'+\mu'\right)a'S_{D',1},\vartheta a\right), \\
& \left(\pi'-\pi+\sigma'-\eta\tau+\left(\eta'+\lambda'\right)\tau'+\left(\eta'+\lambda'\right)a'S_{D',1},\eta a\right)
\end{aligned}
\tag{3.6.2}
$$

and

$$B^{**} = \left(m^{'} + n^{'} - m - \vartheta\tau + l + \left(\vartheta^{'} + \mu^{'}\right)\tau^{'} + \left(\vartheta^{'} + \mu^{'}\right)a^{'}S_{D',1}, \vartheta a\right),$$
$$\left(\pi^{'} - \pi + \sigma^{'} - \eta\tau + l + \left(\eta^{'} + \lambda^{'}\right)\tau^{'} + \left(\eta^{'} + \lambda^{'}\right)a^{'}S_{D',1}, \eta a\right), (0,1);$$
$$\left(1 - \sigma - \sigma^{'} - \lambda\tau - \lambda^{'}\tau^{'} - l - \lambda^{'}a^{'}S_{D',1}, \lambda a; 1\right), \left(1 - \delta_{j}, p_{j}; 1\right)_{1,k},$$
$$(1 - \beta, \alpha; 1), \left(1 - n - n^{'} - \mu\tau - \mu^{'}\tau^{'} - l - \mu^{'}a^{'}S_{D',1}, \mu a; 1\right) \tag{3.6.3}$$

the appropriate conditions can be found from conditions (3.5.10).

## 3.7 CONCLUSIONS AND FUTURE WORK

From results 7 and 9, similar two dimensional formulae can be deduced. By taking the $E$-function to unity, these formulae can be reduced to the results derived by Raina [154, p. 511-513, Eqs. (2.8), (2.9) & (2.15)].

If we reduce both the generalized class of polynomials and the $E$-function to unity, in these composition formula then we obtain the multidimensional analogue introduced by Erdélyi [33, p. 166, Eq. (6.2); p. 167, Eq. (6.3)]. Also we can obtain the corresponding result derived by Goyal and Jain [57, p. 253, Eq. (2.4); p. 254, Eq. (2.7); p. 255, Eq. (2.12)] by reducing the generalized class of polynomials to unity and the $E$-function to the generalized hypergeometric function.

# CHAPTER 4

# FRACTIONAL INTEGRAL TRANSFORMATIONS OF THE $E$-FUNCTION

**Publications:**

**1. Fractional integral transformations of Mittag-Leffler type $E$-function,** ***South East Asian Journal of Mathematics and Mathematical Sciences*** **11, No. 1(2015), 31-38.**

**2. The Mellin-Barnes type contour integral representation of a new Mittag-Leffler type $E$-function,** ***American Journal of Mathematical Science and Applications*** **2, No. 2(2014), 137-141.**

An integral transform is useful if it allows to turn a complicated problem into a simpler one. To be definite suppose that we want to solve a differential equation, with unknown function $f$. One first applies the transform to the differential equation to turn it into an equation one can solve easily often an algebraic equation for the transform $F$ of $f$. One then solves this equation for $F$ and finally applies the inverse transform to find $f$.

In this chapter, we study various fractional integral transformations of the $E$-function [11]. First we establish Riemann-Liouville fractional integral transformation of the $E$-function then obtain various special cases. Further establish Erdélyi-Kober and generalized fractional integral transformation of the $E$-function then obtain various special cases. Finally discuss second form of Mellin-Barnes type contour integral representation of the $E$-function then obtain various special cases.

## 4.1 DEFINITIONS

### 4.1.1 Riemann-Liouville Fractional Integral Transform

The Riemann-Liouville fractional integral transform $\left(I_{c+}^{\theta}\Psi\right)(x)$ [164] is defined as follows:

$$\left(I_{c+}^{\theta}\Psi\right)(x) = \frac{1}{\Gamma(\theta)}\int_{c}^{x}(x-t)^{\theta-1}\,\Psi(t)\,dt, \tag{4.1.1}$$

where $\theta \in \mathbb{C}$ and $\Re(\theta) > 0$.

### 4.1.2 Erdélyi-Kober Fractional Integral Transform

The Erdélyi-Kober fractional integral transform $\left(\Xi_{0+}^{\eta,\theta}f\right)(x)$ [164] is defined as follows:

$$\left(\Xi_{0+}^{\eta,\theta}f\right)(x) = \frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_{0}^{x}(x-t)^{\eta-1}\,t^{\theta}f(t)\,dt, \tag{4.1.2}$$

where $\eta, \theta \in \mathbb{C}; \Re(\eta) > 0$ and $\Re(\theta) > 0$.

## 4.2 THE IMAGE OF $E$-FUNCTION UNDER THE RIEMANN-LIOUVILLE (R-L) OPERATOR $I^{\theta}_{c+}$

**Theorem 1.** *If convergence conditions (1.2.2) are satisfied also $\theta \in \mathbb{C}$ and $\Re(\theta) > 0$ then the R-L transform $I^{\theta}_{c+}$ of the E-function is*

$$\left(I^{\theta}_{c+}\left[{}_{\tau}E^{h}_{k}(t-c)\right]\right)(x) = \frac{1}{(\tau+1)_{\theta}}$$
$$\times\, {}_{\theta+\tau}E^{h+1}_{k+1}\left[(x-c)\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+1,a,1)\\ (\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k},(\tau+\theta+1,a,1)\end{matrix}\right]. \quad (4.2.1)$$

**Proof.** We obtain the R-L transform $I^{\theta}_{c+}$ of the $E$-function as follows

$$\left(I^{\theta}_{c+}\left[{}_{\tau}E^{h}_{k}(t-c)\right]\right)(x) = \frac{1}{\Gamma(\theta)}\int_{c}^{x}(x-t)^{\theta-1}\sum_{n=0}^{\infty}\Phi(n)\,(t-c)^{an+\tau}\,dt, \quad (4.2.2)$$

where

$$\Phi(n) = \frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\cdots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\cdots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)}. \quad (4.2.3)$$

Then

$$\left(I^{\theta}_{c+}\left[{}_{\tau}E^{h}_{k}(t-c)\right]\right)(x) = \frac{1}{(\tau+1)_{\theta}}\sum_{n=0}^{\infty}\Phi(n)\frac{(\tau+1)_{an}}{(\tau+\theta+1)_{an}}(x-c)^{an+\theta+\tau}$$

$$= \frac{1}{(\tau+1)_{\theta}}\,{}_{\theta+\tau}E^{h+1}_{k+1}\left[(x-c)\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+1,a,1)\\ (\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k},(\tau+\theta+1,a,1)\end{matrix}\right]. \quad (4.2.4)$$

### 4.2.1 Special Cases of Theorem 1

1. R-L transform $I^{\theta}_{c+}$ of the M-L type function (0.7.7)

$$\left[I^{\theta}_{c+}\left\{E_{(1/\rho_i),(\mu_i)}(t)\right\}\right](x) = \frac{1}{(\theta)!\prod_{j=1}^{m-1}\Gamma\left(\mu_j\right)}$$

$$\times_{\theta}E^{1}_{m}\left[(x-c)\left|\begin{array}{c}(0,1)\,;(1,1,1)\\ (1/\rho_m,\mu_m)\,;(\mu_1,1/\rho_1,1)\,,\ldots,(\mu_{m-1},1/\rho_{m-1},1)\,,(\theta+1,1,1)\end{array}\right.\right]. \tag{4.2.5}$$

2. R-L transform $I^{\theta}_{0+}$ of the M-L type function (0.7.11)

$$\left[I^{\theta}_{0+}\left\{E_{\gamma,\kappa}\left[(\alpha_1,\beta_1)\,,\ldots,(\alpha_m,\beta_m)\,;t\right]\right\}\right](x) = \frac{1}{(\theta)!\prod_{j=1}^{m}\Gamma\left(\beta_j\right)}$$

$$\times\,_{\theta}E^{2}_{m+1}\left[x\left|\begin{array}{c}(0,1)\,;(\gamma,\kappa,1)\,,(1,1,1)\\ (1,1)\,;(\beta_1,\alpha_1,1)\,,\ldots,(\beta_m,\alpha_m,1)\,,(\theta+1,1,1)\end{array}\right.\right]. \tag{4.2.6}$$

3. R-L transform $I^{\theta}_{0+}$ of the M-L type function (0.7.13)

$$\left[I^{\theta}_{0+}\left\{HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu}(t)\right\}\right](x) = \frac{x^{\theta}}{(M+1)_{\theta}\prod_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)}$$

$$\times\,_{M}E^{1}_{\nu}\left[\frac{x}{\Lambda}\left|\begin{array}{c}(1,\Lambda)\,;(M+1,\Lambda,1)\\ (\lambda_\nu,1+\mu_\nu)\,;(1+\mu_i,\lambda_i,1)_{1,\nu-1}\,,(M+\theta+1,\Lambda,1)\end{array}\right.\right]. \tag{4.2.7}$$

## 4.3 THE IMAGE OF $E$-FUNCTION UNDER THE ERDÉLYI-KOBER (E-K) OPERATOR $\Xi_{0+}^{\eta,\theta}$

**Theorem 2.** *If convergence conditions (1.2.2) are satisfied also* $\eta,\theta\in\mathbb{C},\Re(\eta)>0\,and\,\Re(\theta)>0$, *then the E-K transform* $\Xi_{0+}^{\eta,\theta}$ *of the E-function is*

$$\left(\Xi_{0+}^{\eta,\theta}\left[{}_{\tau}E_k^h(t)\right]\right)(x)=\frac{1}{(\tau+\theta+1)_{\eta}}\times{}_{\tau}E_{k+1}^{h+1}\left[x\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\theta+1,a,1)\\(\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k},(\tau+\eta+\theta+1,a,1)\end{matrix}\right.\right]. \tag{4.3.1}$$

**Proof.** We obtain the E-K transform $\Xi_{0+}^{\eta,\theta}$ of the $E$-function as follows

$$\left(\Xi_{0+}^{\eta,\theta}\left[{}_{\tau}E_k^h(t)\right]\right)(x)=\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{\theta}\sum_{n=0}^{\infty}\Phi(n)\,t^{an+\tau}dt, \tag{4.3.2}$$

where

$$\Phi(n)=\frac{\left[(\gamma_1)_{q_1n}\right]^{s_1}\left[(\gamma_2)_{q_2n}\right]^{s_2}\dots\left[(\gamma_h)_{q_hn}\right]^{s_h}(-1)^{\rho n}}{\left[(\delta_1)_{p_1n}\right]^{r_1}\left[(\delta_2)_{p_2n}\right]^{r_2}\dots\left[(\delta_k)_{p_kn}\right]^{r_k}\Gamma(\alpha n+\beta)}. \tag{4.3.3}$$

Then

$$\left(\Xi_{0+}^{\eta,\theta}\left[{}_{\tau}E_k^h(t)\right]\right)(x)=\frac{1}{(\tau+\theta+1)_{\eta}}\sum_{n=0}^{\infty}\Phi(n)\frac{(\tau+\theta+1)_{an}}{(\tau+\theta+\eta+1)_{an}}x^{an+\tau}$$

$$=\frac{1}{(\tau+\theta+1)_{\eta}}{}_{\tau}E_{k+1}^{h+1}\left[x\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\theta+1,a,1)\\(\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k},(\tau+\eta+\theta+1,a,1)\end{matrix}\right.\right]. \tag{4.3.4}$$

### 4.3.1 Special Cases of Theorem 2

1. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (0.7.7)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{(1/\rho_i),(\mu_i)}(t)\right\}\right](x)=\frac{1}{(\theta+1)_\eta\prod_{j=1}^{m-1}\Gamma\left(\mu_j\right)}$$

$$\times{}_0E_m^1\left[x\left|\begin{matrix}(0,1);(\theta+1,1,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+1,1,1)\end{matrix}\right.\right].\tag{4.3.5}$$

2. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (0.7.11)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);t\right]\right\}\right](x)=\frac{1}{(\theta+1)_\eta\prod_{j=1}^{m}\Gamma\left(\beta_j\right)}$$

$$\times{}_0E_{m+1}^2\left[x\left|\begin{matrix}(0,1);(\gamma,\kappa,1),(\theta+1,1,1)\\(1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\eta+\theta+1,1,1)\end{matrix}\right.\right].\tag{4.3.6}$$

3. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (0.7.13)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{HE_{\mu_1,\ldots,\mu_\nu}^{\lambda_1,\ldots,\lambda_\nu}(t)\right\}\right](x)=\frac{1}{(M+\theta+1)_\eta\prod_{j=1}^{m}\Gamma\left(1+\mu_j\right)}$$

$$\times{}_ME_\nu^1\left[\frac{x}{\Lambda}\left|\begin{matrix}(1,\Lambda);(M+\theta+1,\Lambda,1)\\(\lambda_\nu,1+\mu_\nu);(1+\mu_i,\lambda_i,1)_{1,\nu-1},(M+\eta+\theta+1,\Lambda,1)\end{matrix}\right.\right].\tag{4.3.7}$$

## 4.4 THE IMAGE OF $E$-FUNCTION UNDER THE GENERALIZED INTEGRAL OPERATOR

**Theorem 3.** *If convergence conditions (1.2.2) are satisfied also* $\eta, \theta, \sigma \in \mathbb{C}, \Re(\eta) > 0, \Re(\theta) > 0$, $\Re(\sigma) > 0$, *and* $t, x, v \in \mathbb{R}$, *then*

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\,{}_{\tau}E_k^h\left\{v(s-t)^{\sigma}\right\}ds = (x-t)^{\eta+\theta-1}B(\theta+\sigma\tau,\eta)$$

$$\times\,{}_{\tau}E_{k+1}^{h+1}\left[v(x-t)^{\sigma}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau,\sigma a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau,\sigma a,1)\end{matrix}\right.\right]. \quad (4.4.1)$$

**Corollary 1.** *If convergence conditions (1.2.2) are satisfied also* $\eta, \theta, \sigma \in \mathbb{C}, \Re(\eta) > 0, \Re(\theta) > 0$, $\Re(\sigma) > 0$, *and* $x, v \in \mathbb{R}$, *then*

$$\int_0^x (x-s)^{\eta-1}s^{\theta-1}\,{}_{\tau}E_k^h\left\{vs^{\sigma}\right\}ds = x^{\eta+\theta-1}B(\theta+\sigma\tau,\eta)$$

$$\times\,{}_{\tau}E_{k+1}^{h+1}\left[vx^{\sigma}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau,\sigma a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau,\sigma a,1)\end{matrix}\right.\right]. \quad (4.4.2)$$

**Corollary 2.** *If convergence conditions (1.2.2) are satisfied also* $\theta, \sigma \in \mathbb{C}, \Re(\theta) > 0, \Re(\sigma) > 0$, *and* $x, v \in \mathbb{R}$, *then*

$$\int_0^x s^{\theta-1}\,{}_{\tau}E_k^h\left(vs^{\sigma}\right)ds = \left(\frac{x^{\theta}}{\sigma\tau+\theta}\right)$$

$$\times\,{}_{\tau}E_{k+1}^{h+1}\left[vx^{\sigma}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau,\sigma a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\theta+\sigma\tau+1,\sigma a,1)\end{matrix}\right.\right]. \quad (4.4.3)$$

**Proof.** We prove the theorem as follows

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\,{}_{\tau}E_k^h\{v(s-t)^\sigma\}\,ds$$

$$=\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\sum_{n=0}^{\infty}\Phi(n)\;v^{an+\tau}\{(s-t)^\sigma\}^{an+\tau}\,ds, \tag{4.4.4}$$

where

$$\Phi(n)=\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\ldots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\ldots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)}. \tag{4.4.5}$$

Then

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\,{}_{\tau}E_k^h\{v(s-t)^\sigma\}\,ds$$

$$=\frac{\Gamma(\eta)(x-t)^{\eta+\theta-1}}{(\tau+1)_\theta}\sum_{n=0}^{\infty}\Phi(n)\frac{(\theta+\sigma\tau)_{\sigma a n}}{(\theta+\sigma\tau+\eta)_{\sigma a n}}\;\{v(x-t)^\sigma\}^{an+\tau}$$

$$=(x-t)^{\eta+\theta-1}B(\theta+\sigma\tau,\eta)$$

$$\times\,{}_{\tau}E_{k+1}^{h+1}\left[v(x-t)^\sigma \;\middle|\; \begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau,\sigma a,1)\\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau,\sigma a,1)\end{matrix}\right]. \tag{4.4.6}$$

### 4.4.1 Special Cases of Theorem 3

1. General integral transform of the M-L type function (0.7.7)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}E_{(1/\rho_i),(\mu_i)}\left[v(s-t)^\sigma\right]ds=\frac{(x-t)^{\eta+\theta-1}B(\theta,\eta)}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}$$

$$\times\,{}_0E_m^1\left[v(x-t)^\sigma \;\middle|\; \begin{matrix}(0,1);(\theta,\sigma,1)\\ (1/\rho_m,\mu_m);(\mu_i,1/\rho_i,1)_{1,m-1},(\eta+\theta,\sigma,1)\end{matrix}\right]. \tag{4.4.7}$$

2. General integral transform of the M-L type function (0.7.11)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1} E_{\gamma,\kappa}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);(s-t)\right] ds$$

$$= \frac{(x-t)^{\eta+\theta-1} B(\theta,\eta)}{\prod_{j=1}^{m}\Gamma(\beta_j)} {}_0E^2_{m+1}\left[(x-t)\left|\begin{matrix}(0,1);(\gamma,\kappa,1),(\theta,1,1)\\(1,1);(\beta_i,\alpha_i,1)_{1,m},(\eta+\theta,1,1)\end{matrix}\right.\right]. \tag{4.4.8}$$

3. General integral transform of the M-L type function (0.7.13)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1} HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu}\left[v(s-t)^\sigma\right] ds = \frac{(x-t)^{\eta+\theta-1} B(\theta+\sigma M,\eta)}{\prod_{j=1}^{\nu-1}\Gamma(1+\mu_j)}$$

$$\times {}_ME^1_\nu\left[\frac{v(x-t)^\sigma}{\Lambda}\left|\begin{matrix}(1,\Lambda);(\theta+\sigma M,\sigma\Lambda,1)\\(\lambda_\nu,1+\mu_\nu);(1+\mu_i,\lambda_i,1)_{1,\nu-1},(\eta+\theta+\sigma M,\sigma\Lambda,1)\end{matrix}\right.\right]. \tag{4.4.9}$$

## 4.5 MELLIN-BARNES TYPE CONTOUR INTEGRAL OF THE $E$-FUNCTION

**Theorem 4.** *Let convergence conditions (1.2.2) are satisfied then the $E$-function ${}_\tau E^h_k[z]$ can be represented as the Mellin-Barnes type contour integral as follows:*

$${}_\tau E^h_k\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right.\right] = \frac{\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}}$$

$$\times \frac{(\rho+1)\, z^{\tau}}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left[(\rho+1)\,\zeta\right]\Gamma\left[1-(\rho+1)\,\zeta\right]\prod_{i=1}^{h}\left[\Gamma\left(\gamma_i - q_i\zeta\right)\right]^{s_i}}{\Gamma\left(\beta-\alpha\zeta\right)\prod_{j=1}^{k}\left[\Gamma\left(\delta_j - p_j\zeta\right)\right]^{r_j}} \left(-z^{a}\right)^{-\zeta} d\zeta, \tag{4.5.1}$$

*where $\mathcal{L}$ is a suitable contour of integration that runs from $c - i\infty$ to $c + i\infty, c \in \mathbb{R}$ and intended to separate the poles of the integrand at $\zeta = -\frac{n}{\rho+1}$ for all $n \in \mathbb{N}_0\,(to\,the\,left)$ from those at $\zeta = \frac{n+1}{\rho+1}$ and at $\zeta = \frac{\gamma_i+n}{q_i}, i = 1, \ldots, h;$ for all $n \in \mathbb{N}_0\,(to\,the\,right)$.*

**Proof.** The proof can be done similarly to that of Theorem 1 of chapter 1. □

### 4.5.1 Special Cases of Theorem 4

1. Put $h = 1, s_1 = 0;\ k = 1, r_1 = 0;\ a = 1; \rho = 0; \beta = 1; \tau = 0$ in (4.5.1), then we get M-L function $E_\alpha(z)$ defined in 1903 by Gösta Mittag-Leffler [133], as

$$ {}_0E_1^1 \left[ z \left| \begin{matrix} (0,1)\,;(\gamma_1, q_1, 0) \\ (\alpha,1)\,;(\delta_1, p_1, 0) \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(\zeta)\,\Gamma(1-\zeta)}{\Gamma(1-\alpha\zeta)} (-z)^{-\zeta} d\zeta = E_\alpha(z). \tag{4.5.2}$$

2. Put $h = 1, s_1 = 0;\ k = 1, r_1 = 0;\ a = 1; \rho = 0; \tau = 0$ in (4.5.1), then we get generalized M-L function $E_{\alpha,\beta}(z)$ defined in 1905 by Wiman [215], as

$$ {}_0E_1^1 \left[ z \left| \begin{matrix} (0,1)\,;(\gamma_1, q_1, 0) \\ (\alpha,\beta)\,;(\delta_1, p_1, 0) \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(\zeta)\Gamma(1-\zeta)}{\Gamma(\beta-\alpha\zeta)}(-z)^{-\zeta}\,d\zeta = E_{\alpha,\beta}(z). \tag{4.5.3}$$

3. Put $h=1, s_1=0$; $k=m-1, r_1=\ldots=r_{m-1}=1, \delta_1=\mu_1,\ldots,\delta_{m-1}=\mu_{m-1}, p_1={}^1/\rho_1,\ldots,p_{m-1}={}^1/\rho_{m-1}; a=1; \rho=0;\ \tau=0; \alpha={}^1/\rho_m; \beta=\mu_m$ in (4.5.1), then we get $E_{({}^1/\rho_i),(\mu_i)}(z)$ defined in 2000 by Kiryakova [95], as

$$ {}_0E^1_{m-1}\left[z\left|\begin{matrix}(0,1);(\gamma_1,q_1,0)\\ ({}^1/\rho_m,\mu_m);(\mu_1,{}^1/\rho_1,1),\ldots,(\mu_{m-1},{}^1/\rho_{m-1},1)\end{matrix}\right.\right]$$

$$= \prod_{v=1}^{m-1}\left[\Gamma(\mu_\nu)\right]\sum_{n=0}^{\infty}\frac{1}{\Gamma(\mu_1+{}^n/\rho_1)\ldots\Gamma(\mu_m+{}^n/\rho_m)}z^n$$

$$= \frac{\prod_{v=1}^{m-1}\left[\Gamma(\mu_\nu)\right]}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(\zeta)\Gamma(1-\zeta)}{\prod_{j=1}^{m}\left[\Gamma\left(\mu_j-\frac{1}{\rho_j}\zeta\right)\right]}(-z)^{-\zeta}\,d\zeta = \prod_{v=1}^{m-1}\left[\Gamma(\mu_\nu)\right]E_{({}^1/\rho_i),(\mu_i)}(z). \tag{4.5.4}$$

4. Put $h=1, s_1=1, \gamma_1=\gamma, q_1=\delta$; $k=1, r_1=1, \delta_1=1, p_1=1; a=1; \rho=0;\ \tau=0$ in (4.5.1), then we get M-L type function $\breve{E}^{\gamma,\delta}_{\alpha,\beta}(z)$ defined in 2009 by Srivastava and Tomovski [204], as

$$ {}_0E^1_1\left[z\left|\begin{matrix}(0,1);(\gamma,\delta,1)\\ (\alpha,\beta);(1,1,1)\end{matrix}\right.\right] = \sum_{n=0}^{\infty}\frac{(\gamma)_{n\delta}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$$

$$= \frac{1}{\Gamma(\gamma)\,2\pi i}\int_{\mathcal{L}}\frac{\Gamma(\zeta)\Gamma(\gamma-\delta\zeta)}{\Gamma(\beta-\alpha\zeta)}(-z)^{-\zeta}\,d\zeta = \breve{E}^{\gamma,\delta}_{\alpha,\beta}(z). \tag{4.5.5}$$

# CHAPTER 5

# FRACTIONAL DIFFERENTIAL CALCULUS OF THE $E$-FUNCTION

**Publications:**

**1. Fractional differential calculus of Mittag-Leffler type the $E$-function (Communicated).**

In this chapter, we study fractional differential calculus of the $E$-function [11]. First we discuss essentials of fractional calculus [132] then give definition of fractional derivative in the Riemann-Liouville and the Caputo sense. Next we mention a generalized Saigo fractional derivative operator and operate upon $S_V^{U_1,\ldots,U_s}\,{}_\tau E_k^h\,[zt]$. Finally establish some important theorems on fractional differentiation of the $E$-function.

## 5.1 DEFINITIONS

Here we provide the essentials of fractional calculus:

The following equation demonstrate the formula usually attributed to

Cauchy for evaluating the $n^{th}$ integration of the function $f(t)$

$$\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n \quad times} f(\tau)\underbrace{d\tau d\tau ... d\tau}_{n \quad times} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau. \qquad (5.1.1)$$

Let us first define the Riemann-Liouville fractional integral operator ${}_tJ^{\mu}$ of order $\mu > 0$

$$ {}_tJ^{\mu} f(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1} f(\tau)d\tau, \quad t > 0. \qquad (5.1.2)$$

By convention ${}_tJ^0 = I$ (Identity operator). We can prove

$$ {}_tJ^{\mu}{}_tJ^{\nu} = {}_tJ^{\nu}{}_tJ^{\mu} = {}_tJ^{\mu+\nu}, \; \mu,\nu > 0, \qquad (Semigroup\ Property) \quad (5.1.3)$$

$$ {}_tJ^{\mu}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}t^{\gamma+\mu}, \mu \geq 0, \quad \gamma > -1, \; t > 0. \qquad (5.1.4)$$

The fractional derivative of order $\mu > 0$ in the Riemann-Liouville sense, is defined as the operator ${}_tD^{\mu}$

$$ {}_tD^{\mu}{}_tJ^{\mu} = I, \quad \mu > 0. \qquad (5.1.5)$$

If $m$ denotes the positive integer such that $m-1 < \mu \leq m$, we can obtain

$$ {}_tD^{\mu} f(t) = {}_t D^{m}{}_tJ^{m-\mu} f(t), \qquad t > 0 \qquad (5.1.6)$$

hence

$$ {}_tD^{\mu} f(t) = \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\mu)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\mu+1-m}}\right], & m-1 < \mu < m, \\ \frac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \qquad (5.1.7)$$

For completion ${}_tD^0 = I$. The semigroup property is no longer valid but

$$ {}_tD^\mu t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} t^{\gamma-\mu}, \qquad \mu \geq 0, \quad \gamma > -1, \ t > 0. \tag{5.1.8}$$

However the property ${}_tD^\mu = {}_tJ^{-\mu}$ is not generally valid. An alternative definition of fractional derivative, which is due to Caputo, is

$$ {}_tD_*^\mu f(t) = {}_tJ^{m-\mu}{}_tD^m f(t). \tag{5.1.9}$$

We note in general that

$$ {}_tD^m{}_tJ^{m-\mu} f(t) \neq {}_tJ^{m-\mu}{}_tD^m f(t). \tag{5.1.10}$$

**1. Generalized Saigo Fractional Derivative Operator**

Let $0 \leq \alpha < 1$, $\beta$, $\eta$, $x \in \Re$, $m \in \mathbb{N}$ then the generalized modified fractional derivative operator due to Saigo [160] is defined as

$$ D_{0,x,m}^{\alpha,\beta,\eta} f(x) = \frac{d}{dx}\left( \frac{x^{m(\beta-\eta)}}{\Gamma(1-\alpha)} \int_0^x (x^m - t^m)^{-\alpha} {}_2F_1 \left[ \begin{matrix} \beta-\alpha; 1-\eta; & \\ & 1-\frac{t^m}{x^m} \\ 1-\alpha; & \end{matrix} \right] f(t)\, dt^m \right). \tag{5.1.11}$$

The multiplicity of $(x^m - t^m)^{-\alpha}$ in equation (5.1.11) is removed by requiring $\log(x^m - t^m)^{-\alpha}$ to be real when $(x^m - t^m) > 0$, and is assumed to be well defined in the unit disk. When $m = 1$ then the above operator reduces to Saigo derivative operator $D_{0,x}^{\alpha,\beta,\eta}$ and $D_{0,x}^{\alpha,\alpha,\eta} f(x) = D_x^\alpha f(x)$.

On putting $\alpha = \beta$ and $m = 1$, in (5.1.11), it reduces to the Riemann-Liouville fractional derivative operator given by Miller and Ross [132].

- Results Required

We will use following relations in establishing our results

$$ {}_{\alpha}D_{x}^{\mu}\left(x^{\mu-1}\right)=\frac{d^{\alpha}x^{\mu-1}}{dx^{\alpha}}=\frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}x^{\mu-\alpha-1},\qquad \alpha\neq\mu \tag{5.1.12} $$

$$ D_{k,\alpha,x}^{m}\left(x^{\mu}\right)=\prod_{p=0}^{m-1}\frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk+1-\alpha)}x^{\mu+km},\qquad \alpha\neq\mu+1 \tag{5.1.13} $$

where $\alpha$ and $k$ are not necessarily integers.

## 5.2 MAIN THEOREMS

**Theorem 1.** *If convergence conditions (1.2.2) are satisfied, then*

$$ D_{l,\lambda-\mu,t}^{m}\left\{t^{\lambda-1}S_{V}^{U_1,\ldots,U_s}\left[w_1t^{\varphi_1},\ldots,w_st^{\varphi_s}\right]{}_{\tau}E_k^h[zt]\,f(xt)\right\} $$

$$ =\sum_{R_1,\ldots,R_s=0}^{\sum_{i=1}^{s}U_iR_i\leq V}(-V)_{\sum_{i=1}^{s}U_iR_i}A(V,R_1,\ldots,R_s)\frac{w_i^{R_i}}{R_i!}\sum_{c=0}^{\infty}\Phi(c)\frac{z^{ac+\tau}}{\Gamma(\alpha c+\beta)} $$

$$ \times t^{\lambda+\Delta+ml-1}\prod_{\vartheta=0}^{m-1}\frac{\Gamma(\lambda+\Delta+\vartheta l)}{\Gamma(\mu+\Delta+\vartheta l)}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}D_x^n\{f(x)\} $$

$$ \times\ {}_{m+1}F_m\left[\begin{matrix}-n,\lambda+\Delta,\ldots,\lambda+\Delta+(m-1)l;\\ \mu+\Delta,\ldots,\mu+\Delta+(m-1)l;\end{matrix}\ t\right], \tag{5.2.1} $$

*where*

$$ \Phi(c)=\frac{\left[(\gamma_1)_{q_1c}\right]^{s_1}\left[(\gamma_2)_{q_2c}\right]^{s_2}\ldots\left[(\gamma_h)_{q_hc}\right]^{s_h}(-1)^{\rho c}}{\left[(\delta_1)_{p_1c}\right]^{r_1}\left[(\delta_2)_{p_2c}\right]^{r_2}\ldots\left[(\delta_k)_{p_kc}\right]^{r_k}\Gamma(\alpha c+\beta)}, $$

$$ \Re(\mu+\Delta+\vartheta l)>0,\Re(\lambda+\Delta+\vartheta l)>0,|t|<1, $$

*here* $\Delta=\varphi_1R_1+\ldots+\varphi_sR_s+ac+\tau$, $\vartheta=0,\ldots,m-1;c=0,1,\ldots.$

**Theorem 2.** *If convergence conditions (1.2.2) are satisfied, then*

$$D^m_{l,\lambda-\mu,t}\left\{t^\lambda S_V^{U_1,\ldots,U_s}\left[w_1t^{\varphi_1},\ldots,w_st^{\varphi_s}\right]{}_\tau E_k^h\left[zt\right]f\left(xt\right)\right\}$$

$$=\sum_{R_1,\ldots,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}\left(-V\right)_{\sum_{i=1}^{s}U_iR_i}A\left(V,R_1,\ldots,R_s\right)\frac{w_i^{R_i}}{R_i!}\sum_{c=0}^{\infty}\Phi\left(c\right)\frac{z^{ac+\tau}}{\Gamma\left(\alpha c+\beta\right)}$$

$$\times t^{\lambda+\Delta+ml-1}\sum_{n=0}^{\infty}\frac{\left(-t\right)^{-n}}{n!}D_x^n\left\{x^nf\left(x\right)\right\}\prod_{\vartheta=0}^{m-1}\frac{\Gamma\left(\lambda+\Delta+\vartheta l\right)\left(1-\mu-\Delta-\vartheta l\right)_n}{\Gamma\left(\mu+\Delta+\vartheta l\right)\left(1-\lambda-\Delta-\vartheta l\right)_n}$$

$$\times\ {}_{m+1}F_m\left[\begin{array}{r}-n,\lambda+\Delta-n,\ldots,\lambda+\Delta+\left(m-1\right)l-n;\\ t\\ \mu+\Delta-n,\ldots,\mu+\Delta+\left(m-1\right)l-n;\end{array}\right],\qquad(5.2.2)$$

*where*

$$\Phi\left(c\right)=\frac{\left[\left(\gamma_1\right)_{q_1c}\right]^{s_1}\left[\left(\gamma_2\right)_{q_2c}\right]^{s_2}\ldots\left[\left(\gamma_h\right)_{q_hc}\right]^{s_h}\left(-1\right)^{\rho c}}{\left[\left(\delta_1\right)_{p_1c}\right]^{r_1}\left[\left(\delta_2\right)_{p_2c}\right]^{r_2}\ldots\left[\left(\delta_k\right)_{p_kc}\right]^{r_k}\Gamma\left(\alpha c+\beta\right)},$$

$$\Re\left(\mu+\Delta+\vartheta l-n\right)>0,\Re\left(\lambda+\Delta+\vartheta l-n\right)>0,\left|t\right|<1,$$

*here* $\Delta=\varphi_1R_1+\ldots+\varphi_kR_s+ac+\tau$, $\vartheta=0,\ldots,m-1;c=0,1,\ldots$.

**Proof.** Let us consider the well-known Taylor's expansion

$$f\left(xt\right)=\sum_{n=0}^{\infty}\frac{\left(t-1\right)^n}{n!}x^nD_x^n\left\{f\left(x\right)\right\}.\qquad(5.2.3)$$

Multiplying both sides of (5.2.3) by $t^{\lambda-1}S_V^{U_1,\ldots,U_s}\left[w_1t^{\varphi_1},\ldots,w_st^{\varphi_s}\right]{}_\tau E_k^h\left[zt\right]$ and applying the operator $D^m_{l,\lambda-\mu,t}$ both sides, we get

$$D^m_{l,\lambda-\mu,t}\left\{t^{\lambda-1}S_V^{U_1,\ldots,U_s}\left[w_1t^{\varphi_1},\ldots,w_st^{\varphi_s}\right]{}_\tau E_k^h\left[zt\right]f\left(xt\right)\right\}$$

$$= D^m_{l,\lambda-\mu,t}\left\{t^{\lambda-1}S_V^{U_1,\dots,U_s}\left[w_1t^{\varphi_1},\dots,w_st^{\varphi_s}\right]{}_\tau E^h_k\left[zt\right]\sum_{n=0}^{\infty}\frac{(t-1)^n}{n!}x^nD_x^n\left\{f\left(x\right)\right\}\right\}. \tag{5.2.4}$$

Then we express $S_V^{U_1,\dots,U_s}$ and ${}_\tau E^h_k\left[zt\right]$ in its series form with the help of (0.5.1) and (1.2.1) respectively, also expand $(t-1)^n$ using binomial expansion and changing the order of operator and summation, we obtain

$$D^m_{l,\lambda-\mu,t}\left\{t^{\lambda-1}S_V^{U_1,\dots,U_s}\left[w_1t^{\varphi_1},\dots,w_st^{\varphi_s}\right]{}_\tau E^h_k\left[zt\right]f\left(xt\right)\right\}$$

$$= \sum_{R_1,\dots,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}\left(-V\right)_{\sum_{i=1}^{s}U_iR_i}A\left(V,R_1,\dots,R_s\right)\frac{w_i^{R_i}}{R_i!}\sum_{n=0}^{\infty}\sum_{h=0}^{n}\frac{(-1)^n\left(-n\right)_h x^n}{n!\ h!}$$

$$\times\sum_{c=0}^{\infty}\Phi\left(c\right)\frac{z^{ac+\tau}}{\Gamma\left(\alpha c+\beta\right)}D_x^n\left\{f\left(x\right)\right\}D^m_{l,\lambda-\mu,t}\left\{t^{\lambda+h+\varphi_1R_1+\dots+\varphi_sR_s+ac+\tau-1}\right\}. \tag{5.2.5}$$

Now using (5.1.13) in the RHS of (5.2.5), we get the following form

$$= \sum_{R_1,\dots,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}\left(-V\right)_{\sum_{i=1}^{s}U_iR_i}A\left(V,R_1,\dots,R_s\right)\frac{w_i^{R_i}}{R_i!}\sum_{n=0}^{\infty}\sum_{h=0}^{n}\frac{(-1)^n\left(-n\right)_h x^n}{n!\ h!}$$

$$\times\sum_{c=0}^{\infty}\Phi\left(c\right)\frac{z^{ac+\tau}}{\Gamma\left(\alpha c+\beta\right)}t^{\lambda+h+\varphi_1R_1+\dots+\varphi_sR_s+ac+\tau+ml-1}$$

$$\times\prod_{\vartheta=0}^{m-1}\frac{\Gamma\left(\lambda+h+\varphi_1R_1+\dots+\varphi_sR_s+ac+\tau+\vartheta l\right)}{\Gamma\left(\mu+h+\varphi_1R_1+\dots+\varphi_sR_s+ac+\tau+\vartheta l\right)}D_x^n\left\{f\left(x\right)\right\}. \tag{5.2.6}$$

Further, recombining above result in terms of generalized hypergeometric function ${}_PF_Q$ we get the RHS of (5.2.1).

Theorem 2 can be proved similarly by using the following expansion [27]

$$tf\left(xt\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(1-\frac{1}{t}\right)^nD_x^n\left\{x^nf\left(x\right)\right\}. \tag{5.2.7}$$

## 5.3 THEOREMS ON FRACTIONAL DIFFERENTIATION

**Theorem 3.** *If convergence conditions (1.2.2) are satisfied then for* $a, \tau, u \in \mathbb{R}$, $\mu \in \mathbb{C}$; *such that* $\tau + u + 1 - \mu \in \mathbb{C} \setminus Z_0^-$ *and* $\tau + u + an + 1 \neq 0, -1, ...$; $n \in \mathbb{N}_0$, *we have*

$$ {}_zD^{\mu}\left( z^u \, {}_{\tau}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h} \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right. \right] \right) = \frac{\Gamma(\tau + u + 1)}{\Gamma(\tau + u + 1 - \mu)}$$

$$\times \, {}_{\tau+u-\mu}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h}, (\tau + u + 1, a, 1) \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k}, (\tau + u + 1 - \mu, a, 1) \end{matrix} \right. \right]. \qquad (5.3.1)$$

**Corollary 1.** *If convergence conditions (1.2.2) are satisfied then for* $a, \tau \in \mathbb{R}, \mu \in \mathbb{C}$; *such that* $\tau + 1 - \mu \in \mathbb{C} \setminus Z_0^-$ *and* $an + \tau + 1 \neq 0, -1, ...$; $n \in \mathbb{N}_0$, *we have*

$$ {}_zD^{\mu}\left( {}_{\tau}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h} \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right. \right] \right) = \frac{\Gamma(\tau + 1)}{\Gamma(\tau + 1 - \mu)}$$

$$\times \, {}_{\tau-\mu}E_{k+1}^{h+1} \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h}, (\tau + 1, a, 1) \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k}, (\tau + 1 - \mu, a, 1) \end{matrix} \right. \right]. \qquad (5.3.2)$$

**Proof.** Let the convergence conditions (1.2.2) are satisfied and $a, \tau, u \in \mathbb{R}$, $\mu \in \mathbb{C}$; such that $\tau + u + 1 - \mu \in \mathbb{C} \setminus Z_0^-$ and $an + \tau + u + 1 \neq 0, -1, ...$; $n \in \mathbb{N}_0$, we have

$$ {}_zD^{\mu}\left( z^u \, {}_{\tau}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h} \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right. \right] \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\rho n} \prod_{i=1}^{h} \left[ (\gamma_i)_{q_i n} \right]^{s_i}}{\Gamma(\alpha n + \beta) \prod_{j=1}^{k} \left[ (\delta_j)_{p_j n} \right]^{r_j}} {}_z D^{\mu} \left( z^{an+\tau+u} \right) \tag{5.3.3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\rho n} \prod_{i=1}^{h} \left[ (\gamma_i)_{q_i n} \right]^{s_i}}{\Gamma(\alpha n + \beta) \prod_{j=1}^{k} \left[ (\delta_j)_{p_j n} \right]^{r_j}} \frac{\Gamma(an+\tau+u+1)}{\Gamma(an+\tau+u+1-\mu)} z^{an+\tau+u-\mu} \tag{5.3.4}$$

$$= \frac{\Gamma(\tau+u+1)}{\Gamma(\tau+u+1-\mu)} \sum_{n=0}^{\infty} \frac{(-1)^{\rho n} \prod_{i=1}^{h} \left[ (\gamma_i)_{q_i n} \right]^{s_i}}{\Gamma(\alpha n + \beta) \prod_{j=1}^{k} \left[ (\delta_j)_{p_j n} \right]^{r_j}} \frac{(\tau+u+1)_{an}}{(\tau+u+1-\mu)_{an}} z^{an+\tau+u-\mu} \tag{5.3.5}$$

$$= \frac{\Gamma(\tau+u+1)}{\Gamma(\tau+u+1-\mu)} {}_{\tau+u-\mu}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h}, (\tau+u+1, a, 1) \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k}, (\tau+u+1-\mu, a, 1) \end{matrix} \right. \right]. \tag{5.3.6}$$

**Theorem 4.** *If convergence conditions (1.2.2) are satisfied then for* $m, a, \tau, u \in \mathbb{N}$, *such that* $a(c-1)+\tau+u < m \leq ac+\tau+u$ *where* $c \in \mathbb{N}$, *we have*

$${}_z D^m \left( e^{-z} z^u {}_{\tau}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h} \\ (\alpha, \beta); (\delta_j, p_j, r_j)_{1,k} \end{matrix} \right. \right] \right)$$

$$= (-1)^m e^{-z} {}_{\tau+u}E_k^h(z) + \frac{\prod_{i=1}^{h} \left[ (\gamma_i)_{q_i c} \right]^{s_i}}{\prod_{j=1}^{k} \left[ (\delta_j)_{p_j c} \right]^{r_j}} \frac{e^{-z} (-1)^{\rho c} (ac+\tau+u)!}{(ac+\tau+u-m)!}$$

$$\times {}_{ac+\tau+u-m}E_{k+1}^{h+1}\left[z\left|\begin{matrix}(\rho,a);(\gamma_{i}+cq_{i},q_{i},s_{i})_{1,h},(1+ac+\tau+u,a,1)\\(\alpha,\alpha c+\beta);\left(\delta_{j}+cp_{j},p_{j},r_{j}\right)_{1,k},(1+ac+\tau+u-m,a,1)\end{matrix}\right.\right]. \tag{5.3.7}$$

**Corollary 1.** *If convergence conditions (1.2.2) are satisfied then for* $m,a,\tau\in\mathbb{N}$, *such that* $a(c-1)+\tau<m\leq ac+\tau$ *where* $c\in\mathbb{N}$, *we have*

$${}_{z}D^{m}\left({}_{\tau}E_{k}^{h}\left[z\left|\begin{matrix}(\rho,a);(\gamma_{i},q_{i},s_{i})_{1,h}\\(\alpha,\beta);\left(\delta_{j},p_{j},r_{j}\right)_{1,k}\end{matrix}\right.\right]\right)=\frac{\prod_{i=1}^{h}\left[(\gamma_{i})_{q_{i}c}\right]^{s_{i}}}{\prod_{j=1}^{k}\left[(\delta_{j})_{p_{j}c}\right]^{r_{j}}}\frac{(-1)^{\rho c}(ac+\tau)!}{(ac+\tau-m)!}$$

$$\times {}_{ac+\tau-m}E_{k+1}^{h+1}\left[z\left|\begin{matrix}(\rho,a);(\gamma_{i}+cq_{i},q_{i},s_{i})_{1,h},(1+ac+\tau,a,1)\\(\alpha,\alpha c+\beta);\left(\delta_{j}+cp_{j},p_{j},r_{j}\right)_{1,k},(1+ac+\tau-m,a,1)\end{matrix}\right.\right]. \tag{5.3.8}$$

**Proof.** Let the convergence conditions (1.2.2) are satisfied and $m,a,\tau,u\in\mathbb{N}$, such that $a(c-1)+\tau+u<m\leq ac+\tau+u$ where $c\in\mathbb{N}$, we have

$${}_{z}D^{m}\left(e^{-z}z^{u}{}_{\tau}E_{k}^{h}\left[z\left|\begin{matrix}(\rho,a);(\gamma_{i},q_{i},s_{i})_{1,h}\\(\alpha,\beta);\left(\delta_{j},p_{j},r_{j}\right)_{1,k}\end{matrix}\right.\right]\right)=(-1)^{m}e^{-z}{}_{\tau+u}E_{k}^{h}(z)$$

$$+e^{-z}\sum_{n=0}^{\infty}\frac{(-1)^{\rho(n+c)}\prod_{i=1}^{h}\left[(\gamma_{i})_{q_{i}(n+c)}\right]^{s_{i}}}{\Gamma\{\alpha(n+c)+\beta\}\prod_{j=1}^{k}\left[(\delta_{j})_{p_{j}(n+c)}\right]^{r_{j}}}{}_{z}D^{m}\left(z^{a(n+c)+\tau+u}\right) \tag{5.3.9}$$

$$=(-1)^{m}e^{-z}{}_{\tau+u}E_{k}^{h}(z)+e^{-z}\left(\sum_{n=0}^{\infty}\frac{(-1)^{\rho(n+c)}\prod_{i=1}^{h}\left[(\gamma_{i})_{q_{i}(n+c)}\right]^{s_{i}}}{\Gamma\{\alpha(n+c)+\beta\}\prod_{j=1}^{k}\left[(\delta_{j})_{p_{j}(n+c)}\right]^{r_{j}}}\right.$$

$$\times \frac{(an+ac+\tau+u)!}{(an+ac+\tau+u-m)!} z^{an+ac+\tau+u-m}\Bigg) \tag{5.3.10}$$

$$= (-1)^m e^{-z} {}_{\tau+u}E_k^h(z) + \frac{\prod_{i=1}^{h}\left[(\gamma_i)_{q_i c}\right]^{s_i}}{\prod_{j=1}^{k}\left[(\delta_j)_{p_j c}\right]^{r_j}} \frac{e^{-z}(-1)^{\rho c}(ac+\tau+u)!}{(ac+\tau+u-m)!}$$

$$\times {}_{ac+\tau+u-m}E_{k+1}^{h+1}\left[z\left|\begin{matrix}(\rho,a);(\gamma_i+cq_i,q_i,s_i)_{1,h},(1+ac+\tau+u,a,1)\\(\alpha,\alpha c+\beta);\left(\delta_j+cp_j,p_j,r_j\right)_{1,k},(1+ac+\tau+u-m,a,1)\end{matrix}\right.\right]. \tag{5.3.11}$$

**Theorem 5.** *If convergence conditions (1.2.2) are satisfied then*

$$\beta{}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta+1);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]+\frac{\alpha z}{a}\frac{d}{dz}\left({}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta+1);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]\right)$$

$$-\frac{\alpha\tau}{a}{}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta+1);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]={}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]. \tag{5.3.12}$$

**Proof.** Let the convergence conditions (1.2.2) are satisfied then

$$\beta{}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta+1);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]+\frac{\alpha z}{a}\frac{d}{dz}\left({}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta+1);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]\right)$$

$$-\frac{\alpha\tau}{a}{}_\tau E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta+1);\left(\delta_j,p_j,r_j\right)_{1,k}\end{matrix}\right.\right]=\beta\sum_{n=0}^{\infty}\Phi(n)\frac{z^{an+\tau}}{(\alpha n+\beta)\Gamma(\alpha n+\beta)}$$

$$+\frac{\alpha}{a}\sum_{n=0}^{\infty}\Phi(n)\frac{(an+\tau)z^{an+\tau}}{(\alpha n+\beta)\Gamma(\alpha n+\beta)}-\frac{\alpha\tau}{a}\sum_{n=0}^{\infty}\Phi(n)\frac{z^{an+\tau}}{(\alpha n+\beta)\Gamma(\alpha n+\beta)}, \tag{5.3.13}$$

where

$$\Phi(n) = \frac{(-1)^{\rho n} \prod_{i=1}^{h} \left[ (\gamma_i)_{q_i n} \right]^{s_i}}{\prod_{j=1}^{k} \left[ (\delta_j)_{p_j n} \right]^{r_j}}.$$

Then (5.3.13) can be written as

$$L.H.S. = \sum_{n=0}^{\infty} \frac{(-1)^{\rho n} \prod_{i=1}^{h} \left[ (\gamma_i)_{q_i n} \right]^{s_i}}{\prod_{j=1}^{k} \left[ (\delta_j)_{p_j n} \right]^{r_j}} \frac{z^{an+\tau}}{\Gamma(\alpha n + \beta)} = {}_{\tau}E_k^h \left[ z \left| \begin{matrix} (\rho, a); (\gamma_i, q_i, s_i)_{1,h} \\ (\alpha, \beta); \left(\delta_j, p_j, r_j\right)_{1,k} \end{matrix} \right. \right]. \tag{5.3.14}$$

## CONCLUDING REMARKS

The present chapter provides a scope of defining M-L function of many parameters as a MATLAB function. At present MATLAB provides MLF-FIT2.M [183], in which the M-L function in two parameters are used.

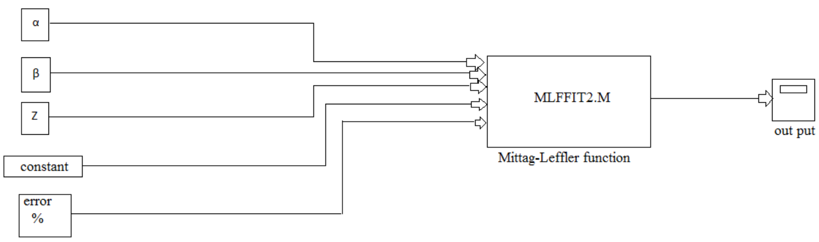


Figure : MATLAB Simulation

# Appendix A

## THE $\overline{H}$-FUNCTION

In the present Appendix, we shall define a function which is more general than well known Fox $H$-function. We also mention some special cases of this function which are not particular cases of Fox $H$-function but have practical applications. We shall denote this function by the symbol $\overline{H}$. The $\overline{H}$-function was introduced by Inayat Hussain [80] and later studied by Buschman and Srivastava [14] and many others.

The $\overline{H}$-function is defined and represented by Mellin-Barnes type contour integral as follows:

$$\overline{H}_{P,Q}^{M,N}\left[z \mid \begin{matrix} \left(a_j, \alpha_j; A_j\right)_1^N, \left(a_j, \alpha_j\right)_{N+1}^P \\ \left(b_j, \beta_j\right)_1^M, \left(b_j, \beta_j; B_j\right)_{M+1}^Q \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}} \overline{\phi}\left(\xi\right) z^{\xi} d\xi, \qquad \text{(A-1)}$$

where

$$\overline{\phi}\left(\xi\right) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j - \beta_j\xi\right)\prod_{j=1}^{N}\left[\Gamma\left(1 - a_j + \alpha_j\xi\right)\right]^{A_j}}{\prod_{j=M+1}^{Q}\left[\Gamma\left(1 - b_j + \beta_j\xi\right)\right]^{B_j}\prod_{j=N+1}^{P}\Gamma\left(a_j - \alpha_j\xi\right)}, \qquad \text{(A-2)}$$

$M, N, P$ and, $Q$ are non-negative integers satisfying $0 \leq N \leq P, 0 \leq M \leq Q$ and empty products are taken as unity. Also, $A_j(j = 1, \ldots, P)$ and $B_j(j = 1, \ldots, Q)$ are positive real numbers for standardization purpose, $a_j(j = 1, \ldots, P)$ and $b_j(j = 1, \ldots, Q)$ are complex numbers such that the points $\xi = \frac{b_j+k}{\beta_j}$ $(j = 1, ..., M; k = 0, 1, ...)$ which are the poles of $\Gamma(b_j - B_j s)$ and the points $\xi = \frac{a_j-1-k}{\alpha_j}$ $(j = 1, ..., N; k = 0, 1, ...)$ which are the singularities of $\left[\Gamma\left(1 - a_j + \alpha_j \xi\right)\right]^{A_j}$ do not coincide.

The contour $\mathcal{L}$ is the line from $\mathbb{C} - i\infty$ to $\mathbb{C} + i\infty$ suitably intended to keep the poles of $\Gamma(b_j - B_j s)$ $(j = 1, ..., M)$ to the right of the path and the singularities of $\left[\Gamma\left(1 - a_j + \alpha_j \xi\right)\right]^{A_j}$ $(j = 1, ..., N)$ to the left of the path. If $A_i = B_j = 1(i = 1, ..., N; j = M + 1, ..., Q)$ the $\overline{H}$-function reduces to the familiar Fox $H$-function.

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$-function given by (A-1) have been recently given by Gupta, Jain and Agrawal [67]

$$\left.\begin{aligned} &(i)\ \ |\arg(z)| < \tfrac{1}{2}\pi\Omega \text{ and } \Omega > 0; \\ &(ii)\ \ |\arg(z)| = \tfrac{1}{2}\pi\Omega \text{ and } \Omega \geq 0; \end{aligned}\right\} \tag{A-3}$$

and

$(a)\mu \neq 0$ and the contour $\mathcal{L}$ is so chosen that $(c\mu + \lambda + 1) < 0$;

$(b)$ $\mu = 0$ and $(\lambda + 1) < 0$,

where

$$\left.\begin{aligned} \Omega &= \sum_{1}^{M}\beta_j + \sum_{1}^{N}\alpha_j A_j - \sum_{M+1}^{Q}\beta_j B_j - \sum_{n+1}^{P}\alpha_j \\ \mu &= \sum_{1}^{N}\alpha_j A_j + \sum_{n+1}^{P}\alpha_j - \sum_{1}^{M}\beta_j - \sum_{M+1}^{Q}\beta_j B_j \\ \lambda &= Re\left(\sum_{1}^{M}b_j + \sum_{M+1}^{Q}b_j B_j - \sum_{1}^{N}a_j A_j - \sum_{N+1}^{P}a_j\right) \\ &+\tfrac{1}{2}\left(-M - \sum_{M+1}^{Q}B_j + \sum_{1}^{N}A_j + P - N\right). \end{aligned}\right\} \tag{A-4}$$

The series representation of the $\overline{H}$-function was given by Rathie [158] and Saxena [165] has been used in the present work:

$$\overline{H}_{P,Q}^{M,N}\left[z \mid \begin{matrix} (a_j,\, \alpha_j;\, A_j)_1^N, (a_j,\, \alpha_j)_{N+1}^P \\ (b_j,\, \beta_j)_1^M, (b_j,\, \beta_j;\, B_j)_{M+1}^Q \end{matrix}\right] = \sum_{\nu=1}^{M}\sum_{\pi=0}^{\infty}\overline{\theta}\left(S_{\pi,\nu}\right) z^{S_{\pi,\nu}}, \tag{A-5}$$

where

$$\overline{\theta}\left(S_{\pi,\nu}\right) = \frac{\prod\limits_{j=1,j\neq\nu}^{M}\Gamma\left(b_j - \beta_j S_{\pi,\nu}\right)\prod\limits_{j=1}^{N}\left[\Gamma\left(1 - a_j + \alpha_j S_{\pi,\nu}\right)\right]^{A_j}(-1)^{\pi}}{\prod\limits_{j=M+1}^{Q}\left[\Gamma\left(1 - b_j + \beta_j S_{\pi,\nu}\right)\right]^{B_j}\prod\limits_{j=N+1}^{P}\Gamma\left(a_j - \alpha_j S_{\pi,\nu}\right)\pi!\beta_\nu},$$

$$S_{\pi,\nu} = \frac{b_\nu + \pi}{\beta_\nu}. \tag{A-6}$$

The following behaviour of the $\overline{H}_{P,Q}^{M,N}[z]$ function for small and large values of $z$ as recorded by Saxena et al. [172, p. 112, Eqs. (2.3) & (2.4)]

$\overline{H}_{P,Q}^{M,N}[z] = O\left[|z|^{\alpha}\right]$ for small $z$, where

$$\alpha = \min_{1\leq j\leq M} Re\left(\frac{b_j}{\beta_j}\right), \tag{A-7}$$

$\overline{H}_{P,Q}^{M,N}[z] = O\left[|z|^{\beta}\right]$ for large $z$, where

$$\beta = \max_{1\le j\le N} Re\left[A_j\left(\frac{a_j-1}{\alpha_j}\right)\right], \tag{A-8}$$

provided that either of the following conditions are satisfied:

$$\left.\begin{array}{l}(i)\ \mu<0 \text{ and } 0<|z|<\infty;\\ (ii)\ \mu=0 \text{ and } 0<|z|<\delta^{-1},\end{array}\right\} \tag{A-9}$$

where

$$\left.\begin{array}{l}(i)\ \mu=\sum_{1}^{N}\alpha_j A_j+\sum_{N+1}^{P}\alpha_j-\sum_{1}^{M}\beta_j-\sum_{M+1}^{Q}\beta_j B_j,\\ (ii)\ \delta=\prod_{1}^{N}(\alpha_j)^{\alpha_j A_j}\prod_{N+1}^{P}(\alpha_j)^{\alpha_j}\prod_{1}^{M}(\beta_j)^{-\beta_j}\prod_{M+1}^{Q}(\beta_j)^{-\beta_j B_j}.\end{array}\right\} \tag{A-10}$$

**Special Cases**

**1. The Fox $H$-Function**

If $A_i = B_j = 1$ $(i = 1, ..., N; j = M+1, ..., Q)$, the $\overline{H}$-function reduces to the familiar Fox $H$-function [196]:

$$H_{P,Q}^{M,N}\left[z \mid \begin{array}{c}(a_j,\alpha_j)_1^P\\ (b_j,\beta_j)_1^Q\end{array}\right] = \overline{H}_{P,Q}^{M,N}\left[z \mid \begin{array}{c}(a_j,\alpha_j;1)_1^N,(a_j,\alpha_j)_{N+1}^P\\ (b_j,\beta_j)_1^M,(b_j,\beta_j;1)_{M+1}^Q\end{array}\right]. \tag{A-11}$$

The following special functions which are quite general in nature and of our interest, are particular cases of $\overline{H}$-function but not of Fox $H$-function:

**2. The Generalized Wrignt Hypergeometric Function** [68, p. 271, Eq. (7)]:

$$ {}_P\overline{\Psi}_Q\left[\begin{array}{l}\left(a_j,\,\alpha_j;A_j\right)_1^P\,;\\ \left(b_j,\,\beta_j;B_j\right)_1^Q\,;\end{array} z\right] = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^{P}\left\{\Gamma\left(a_j+\alpha_j r\right)\right\}^{A_j}}{\prod_{j=1}^{Q}\left\{\Gamma\left(b_j+\beta_j r\right)\right\}^{B_j}}\frac{z^r}{r!} $$

$$ = \overline{H}^{1,P}_{P,Q+1}\left[-z\,\middle|\,\begin{array}{c}\left(1-a_j,\,\alpha_j;A_j\right)_1^P\\ (0,\,1)\,,\left(1-b_j,\,\beta_j;B_j\right)_1^Q\end{array}\right]. \qquad \text{(A-12)} $$

The function ${}_P\overline{\Psi}_Q$ reduces to ${}_P\Psi_Q$, the familiar Wright's generalized hypergeometric function [196, p. 19, Eq. (2.6.11)], for $A_j = 1$(j=1,...,P), $B_j = 1$(j=1,...,Q).

**3. A Generalization of the Generalized Hypergeometric Function** [68, p. 271, Eq. (9)]:

$$ {}_P\overline{F}_Q\left[\begin{array}{l}\left(a_j,\,1;A_j\right)_1^P\,;\\ \left(b_j,\,1;B_j\right)_1^Q\,;\end{array} z\right] $$

$$ = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^{P}\left\{\left(a_j\right)_r\right\}^{A_j}}{\prod_{j=1}^{Q}\left\{\left(b_j\right)_r\right\}^{B_j}}\frac{z^r}{r!} = \frac{\prod_{j=1}^{Q}\left\{\Gamma(b_j)\right\}^{B_j}}{\prod_{j=1}^{P}\left\{\Gamma(a_j)\right\}^{A_j}}\overline{H}^{1,P}_{P,Q+1}\left[-z\,\middle|\,\begin{array}{c}\left(1-a_j,\,1;A_j\right)_1^P\\ (0,\,1)\,,\left(1-b_j,\,1;B_j\right)_1^Q\end{array}\right] $$

$$ = \frac{\prod_{j=1}^{Q}\left\{\Gamma\left(b_j\right)\right\}^{B_j}}{\prod_{j=1}^{P}\left\{\Gamma\left(a_j\right)\right\}^{A_j}}\,{}_P\overline{\Psi}_Q\left[\begin{array}{l}\left(a_j,\,1;A_j\right)_1^P\,;\\ \left(b_j,\,1;B_j\right)_1^Q\,;\end{array} z\right]. \qquad \text{(A-13)} $$

The function ${}_P\overline{F}_Q$ reduces to well known ${}_PF_Q$ for $A_j = 1$(j=1,...,P), $B_j = 1$(j=1,...,Q) in it.

**4. Generalized Wright Bessel Function** [68, p. 271, Eq. (8)]:

$$\overline{J}_{\lambda}^{\nu,\mu}(z) = \sum_{r=0}^{\infty} \frac{(-z)^r}{r!\,\{\Gamma(1+\lambda+\nu r)\}^{\mu}}$$

$$= \overline{H}_{0,2}^{1,0}\left[ z \mid \begin{matrix} \text{—} \\ (0,\,1),(-\lambda,\,\nu;\mu) \end{matrix} \right]. \tag{A-14}$$

The function $\overline{J}_{\lambda}^{\nu,\mu}(z)$ reduces to the Wright's generalized Bessel function [196, p. 19, Eq. (2.6.10)] for $\mu = 1$.

**5. The Generalized Riemann Zeta Function** [34, p. 27, §1.11, Eq. (1); 47, p. 314-315, Eqs. (1.6) & (1.7)]:

$$\phi(z,p,\eta) = \sum_{r=0}^{\infty} \frac{z^r}{(\eta+r)^p}$$

$$= \overline{H}_{2,2}^{1,2}\left[ -z \mid \begin{matrix} (0,1;1),(1-\eta,\,1;p) \\ (0,\,1),(-\eta,\,1;p) \end{matrix} \right] = \eta^{-p}\,{}_2\overline{F}_1\left[ \begin{matrix} (1,1),(\eta,p); \\ (1+\eta,p); \end{matrix} \; z \right]. \tag{A-15}$$

On taking $z = 1$, in (A-15) the above function reduces to well known Hurwitz zeta function $\zeta(p,n)$ [34, p. 24, §1.10, Eq. (1)]:

$$\zeta(p,n) = \phi(1,p,\eta) = \sum_{r=0}^{\infty} \frac{1}{(\eta+r)^p}, \tag{A-16}$$

and further on taking $\eta = 1$ in (A-16) it reduces to the Riemann zeta function $\zeta(p)$ [34, p. 32, §1.12, Eq. (1)]:

$$\zeta(p) = \zeta(p,1) = \phi(1,p,1) = \sum_{r=0}^{\infty} \frac{1}{(1+r)^p} = \sum_{r=1}^{\infty} \frac{1}{r^p}. \tag{A-17}$$

**6. The Polylogarithm of Order p** [34, p. 30, §1.11, Eq. (14); 47, p. 315, Eq. (1.9)]:

$$F(z,p)=\sum_{r=1}^{\infty}\frac{z^r}{r^p}=z\phi(z,p,1)=-\overline{H}_{1,2}^{1,1}\left[-z\mid\begin{matrix}(1,1;p+1)\\(1,\ 1),(0,\ 1;p)\end{matrix}\right]$$

$$=z\overline{H}_{1,2}^{1,1}\left[-z\mid\begin{matrix}(0,1;p+1)\\(0,\ 1),(-1,\ 1;p)\end{matrix}\right]=z_1\overline{F}_1\left[\begin{matrix}(1,p+1);\\(2,p);\end{matrix}\ z\right].\quad\text{(A-18)}$$

The above function reduces into Euler's dilogarithm [34, p. 31, §1.11.1, Eq. (22)], for p=2:

$$L_2(z)=F(z,2)=\sum_{r=1}^{\infty}\frac{z^r}{r^2}\,.\qquad\text{(A-19)}$$

**7. The $g_1$-Function over the d-Dimensional Space** [80, p. 4125, Eq. (20); 71, p. 98, Eq. (1.3)]:

$$g_1=(-1)^m g(\gamma,\eta,\tau,m,z)=\frac{\Gamma(m+1)\Gamma\left(\frac{1+\tau}{2}\right)}{\pi^{d/2}2^{m+d}\Gamma\left(\frac{d-1}{2}\right)\Gamma(\gamma)\Gamma\left(\gamma-\frac{\tau}{2}\right)}$$

$$\times\overline{H}_{3,3}^{1,3}\left[-z\mid\begin{matrix}(1-\gamma,1;1),\left(1-\gamma+\frac{\tau}{2},1;1\right),(1-\eta,1;1+m)\\(0,\ 1),\left(-\frac{\tau}{2},1;1\right),(-\eta,\ 1;1+m)\end{matrix}\right].\qquad\text{(A-20)}$$

Further if we take $\gamma=1+\tau/2$ in Eq.(A-20), we have:

$$g_1\left(1+\frac{\tau}{2},\eta,\tau,m,z\right)=\frac{\Gamma(m+1)\Gamma\left(\frac{1+\tau}{2}\right)}{\pi^{d/2}2^{m+d}\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(1+\frac{\tau}{2}\right)}\phi(z,m+1,\eta)\,.\qquad\text{(A-21)}$$

## 8. The Function Associated with Gaussian Model Free Energy

[80, p. 4126, 4127, Eqs. (23) & (28); 71, p. 98, Eq. (1.4)]:

$$\beta F\left(d;\varepsilon\right)=\frac{-1}{4\pi^{\mathrm{d}/2}\left(1+\varepsilon\right)^{2}}\overline{H}_{2,2}^{1,2}\left[-\frac{1}{\left(1+\varepsilon\right)^{2}}\;\middle|\;\begin{matrix}\left(0,1;2\right),\left(-1/2,1;d\right)\\ \left(0,\,1\right),\left(-1,1;1+d\right)\end{matrix}\right]$$

$$=\frac{-1}{2^{2+d}\left(1+\varepsilon\right)^{2}}{}_{2}\overline{F}_{1}\left[\begin{matrix}\left(1,1;2\right),\left(3/2,1;d\right)\\ \left(2,1;1+d\right)\end{matrix}\;;\frac{1}{\left(1+\varepsilon\right)^{2}}\right]. \qquad \text{(A-22)}$$

# Appendix B

## A GENERAL CLASS OF POLYNOMIALS

Srivastava [188] introduced the general class of polynomials (see also [189] and [194]) defined as follows:

$$S_V^U[x] = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} A_{V,R}}{R!} x^R, \qquad V = 0, 1, ...; \tag{B-1}$$

where U is an arbitrary positive integer, the coefficients $A_{V,R}$ are arbitrary constants, real or complex.

If $x = 0$, $A_{0,0} = 1$, then $S_V^U[x]$ reduces to unity.

## SPECIAL CASES OF THE POLYNOMIALS $S_V^U[x]$

On suitably specializing the coefficients $A_{V,R}$ occurring in (B-1), the general class of polynomials $S_V^U[x]$ can be reduced to the classical orthogonal polynomials and the generalized hypergeometric polynomials as cited in the papers referred to above.

We give below some of the important special cases of the Srivastava's polynomials $S_V^U[x]$:

## 1. Hermite Polynomial

If we take $U = 2$ and $A_{V,R} = (-1)^R$ in (B-1), we have

$$S_V^2[x] \to x^{V/2} H_V\left(\frac{1}{2\sqrt{x}}\right), \tag{B-2}$$

where $H_V(x)$ is the Hermite polynomial [208, p. 106, Eq. (5.5.4)], which is given by:

$$H_V[x] = \sum_{R=0}^{[V/2]} \frac{(-1)^R\, V!\, (2x)^{V-2R}}{R!\,(V-2R)!}$$

$$= (2x)^V\, {}_2F_0\left[\begin{matrix} \frac{-V}{2}, \frac{-V+1}{2} \\ - \end{matrix} ; -\frac{1}{x^2}\right].$$

## 2. The Jacobi Polynomial

On taking $U = 1$ and $A_{V,R} = \begin{pmatrix} V+\alpha \\ V \end{pmatrix} \frac{(\alpha+\beta+V+1)}{(\alpha+1)_R}$ in (B-1), we have

$$S_V^1[x] \to P_V^{(\alpha,\beta)}(1-2x), \tag{B-3}$$

where $P_V^{(\alpha,\beta)}$ is the Jacobi polynomial [208, p. 68, Eq. (4.3.2)], which is given by:

$$P_V^{(\alpha,\beta)}(x) = \sum_{R=0}^{V} \begin{pmatrix} V+\alpha \\ V-R \end{pmatrix} \begin{pmatrix} V+\beta \\ R \end{pmatrix} \left(\frac{x-1}{2}\right)^R \left(\frac{x+1}{2}\right)^{V-R}$$

$$= \frac{(1+\alpha)_V}{V!} \sum_{R=0}^{V} \frac{(-V)_R\,(1+\alpha+\beta+V)_R}{(1+\alpha)_R \quad R!} \left(\frac{1-x}{2}\right)^R.$$

Also the polynomials $S_V^U[x]$ defined by (B-1) can further be reduced to several special cases of the Jacobi polynomials $P_V^{(\alpha,\beta)}(x)$, for example, the Gegenbauer polynomial $C_V^v(x)$, the Legendre polynomials $P_V(x)$, the Tchebychef polynomials $T_V(x)$ and $U_V(x)$ of the first and second kinds

$$C_V^{\alpha+\frac{1}{2}}(x) = \begin{pmatrix} V+\alpha \\ V \end{pmatrix}^{-1} \begin{pmatrix} V+2\alpha \\ V \end{pmatrix} P_V^{(\alpha,\alpha)}(x) \tag{B-4}$$

$$P_V(x) = P_V^{(0,0)}(x) \tag{B-5}$$

$$T_V(x) = \begin{pmatrix} V-1/2 \\ V \end{pmatrix}^{-1} P_V^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \tag{B-6}$$

$$U_V(x) = \frac{1}{2} \begin{pmatrix} V+1/2 \\ V+1 \end{pmatrix}^{-1} P_V^{\left(\frac{1}{2},\frac{1}{2}\right)}(x). \tag{B-7}$$

**3. The Laguerre Polynomial**

On taking $U=1$ and $A_{V,R} = \begin{pmatrix} V+\alpha \\ V \end{pmatrix} \frac{1}{(\alpha+1)_R}$ in (B-1), we have

$$S_V^1[x] \to L_V^{(\alpha)}(x), \tag{B-8}$$

where $L_V^{(\alpha)}(x)$ is the Laguerre polynomial [208, p. 101, Eq. (5.1.6)], defined by:

$$L_V^{(\alpha)}(x) = \frac{(1+\alpha)_V}{V!} {}_1F_1[-V; 1+\alpha; x].$$

## 4. The Bessel Polynomial

Taking $U = 1$ and $A_{V,R} = (\alpha + V - 1)_R$ in (B-1), we have

$$S_V^1[x] \to y_V(-\beta x, \alpha, \beta), \tag{B-9}$$

where $y_V(x, \alpha, \beta)$ is the Bessel polynomial [101, p. 108, Eq. (34)], defined as follows:

$$y_V(x, \alpha, \beta) = \sum_{R=0}^{V} \frac{(-V)_R(\alpha + V - 1)_R}{R!} \left(-\frac{x}{\beta}\right)^R$$

$$\times {}_2F_0\left[-v; \alpha + V - 1; -; \frac{-x}{\beta}\right].$$

## 5. The Gould and Hopper Polynomial (Generalized Hermite Polynomial)

Taking $A_{V,R} = 1$ in (B-1), we have

$$S_V^U[x] \to \left(-\frac{x}{h}\right)^{V/U} g_V^U\left[\left(-\frac{h}{x}\right)^{1/U}, h\right], \tag{B-10}$$

where $g_V^U[x, h]$ is the Gould and Hopper polynomial [56, p. 58, Eq. (6.2)], given by:

$$g_V^U[x, h] = \sum_{R=0}^{[V/U]} \frac{V!}{R!(V - UR)!} h^R x^{V-UR}$$

$$= x^V {}_UF_0\left[\Delta(U; -V); -; h\left(\frac{-U}{x}\right)^U\right].$$

## 6. The Brafman Polynomial

Taking $A_{V,R} = \frac{(\alpha_1)_R \cdots (\alpha_p)_R}{(\beta_1)_R \cdots (\beta_q)_R}$ in (B-1), we have

$$S_V^U[x] \to \mathcal{B}_V^U\left[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; xU^U\right], \tag{B-11}$$

where $\mathcal{B}_V^U[\alpha_1, ..., \alpha_p; \beta_1, ...\beta_q : x]$ is the Brafman polynomial [13, p. 186], given by:

$$\mathcal{B}_V^U[\alpha_1, ..., \alpha_p; \beta_1, ...\beta_q : x] = {}_{U+p}F_q\left[\Delta(U; -V), \alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; x\right],$$

here $\Delta(U; V)$ abbreviates the array of $U$ parameters $\frac{V}{U}, \frac{V+1}{U}, \cdots, \frac{V+U-1}{U}$, $U \geq 1$ the set $\Delta(0, V)$being empty.

## 7. The Konhauser Biorthogonal Polynomial

If we take $U = 1$ and $A_{V,R} = \frac{1}{V!}\frac{\Gamma(1+\alpha+kV)}{\Gamma(1+\alpha+kR)}$ in (B-1), we have

$$S_V^1[x] \to Z_V^\alpha\left(x^{1/k}; k\right), \tag{B-12}$$

where $Z_V^\alpha(x; k)$ is the biorthogonal polynomial [99, p. 304, Eq. (5)], given by:

$$Z_V^\alpha(x; k) = \frac{\Gamma(1+\alpha+kV)}{V!}\sum_{R=0}^{V}(-1)^R\begin{pmatrix} V \\ R \end{pmatrix}\frac{x^{kR}}{\Gamma(1+\alpha+kR)}$$

$$= \frac{(1+\alpha)_{kV}}{V!}{}_1F_k\left[\begin{array}{c} -V; \\ \Delta(k; \alpha+1); \end{array} \left(\frac{x}{\beta}\right)^k\right].$$

## 8. Bedient Polynomials

(a) Taking $U=2$ and $A_{V,R}=\frac{(\beta)_V}{V!}\frac{(\lambda-\beta)_R}{(\lambda)_R(1-\beta-V)_R}$ in (B-1), we have

$$S_V^2\left[x\right] \;\to\; x^{V/2} R_V\left(\beta,\lambda;\frac{1}{2\sqrt{x}}\right), \tag{B-13}$$

where $R_V\left(\beta,\lambda;x\right)$ is the Bedient polynomial [198, p. 186, Eq. (48)], given by:

$$R_V\left(\beta,\lambda;x\right)=\frac{(\beta)_V}{V!}\left(2x\right)^V {}_3F_2\left[\begin{array}{l}\Delta\left(2;-V\right),\lambda-\beta;\\ \lambda,1-\beta-V;\end{array}\;\frac{1}{x^2}\right].$$

(b) Taking $U=2$ and $A_{V,R}=\frac{(\alpha)_V(\beta)_V}{V!(\alpha+\beta)_V}\frac{(1-\alpha-\beta-V)_R}{(\lambda)_R(1-\alpha-V)_R(1-\beta-V)_R}$ in (B-1), we have

$$S_V^2\left[x\right] \;\to\; x^{V/2} G_V\left(\alpha,\beta;\frac{1}{2\sqrt{x}}\right), \tag{B-14}$$

where $G_V\left(\alpha,\beta;x\right)$ is the Bedient polynomial [9, p. 15, Eq. (2.5) and p. 44, Eq. (3.4)], given by:

$$G_V\left(\alpha,\beta;x\right)=\frac{(\alpha)_V\left(\beta\right)_V}{V!\left(\alpha+\beta\right)_V}\left(2x\right)^V {}_3F_2\left[\begin{array}{l}\Delta\left(2;-V\right),1-\alpha-\beta-V;\\ 1-\alpha-V,1-\beta-V;\end{array}\;\frac{1}{x^2}\right].$$

## 9. Shively Polynomial

Taking $U = 1$, $A_{V,R} = \frac{(\lambda+V)_V}{V!} \frac{(\alpha_1)_R \ldots (\alpha_p)_R}{(\lambda+V)_R (\beta_1)_R \ldots (\beta_q)_R}$ in (B-1), we have

$$S_V^U [x] \rightarrow S_V^{(\lambda)} [x] , \tag{B-15}$$

where $S_V^{(\lambda)} [x]$ is the Shively polynomial [198, p. 187, Eq. (49); 179, p. 54], given by:

$$S_V^{(\lambda)} [x] = \frac{(\lambda + V)_V}{V!} {}_{p+1}F_{q+1} \left[ \begin{array}{c} -V, \alpha_1, ..., \alpha_p; \\ \lambda + V, \beta_1, ..., \beta_q; \end{array} x \right] .$$

## 10. Bateman Polynomials

(a) Taking $U = 1$ and $A_{V,R} = \frac{(1+V)_R}{R!\, R!}$ in (B-1), we have

$$S_V^1 [x] \rightarrow Z_V [x] , \tag{B-16}$$

where $Z_V [x]$ is the Bateman polynomial [198, p. 183, Eq. (42)], given by:

$$Z_V [x] = {}_2F_2 \left[ \begin{array}{c} -V, V + 1; \\ 1, 1; \end{array} x \right] .$$

(b) Taking $U=1$ and $A_{V,R}=\frac{\Gamma\left(\frac{\lambda}{2}+\sigma+V+1\right)}{V!\,\Gamma(\lambda+R+1)\,\Gamma\left(\frac{\lambda}{2}+\sigma+R+1\right)}$ in (B-1), we have

$$S_V^U[x]\rightarrow x^{-\lambda/2}J_V^{(\lambda,\sigma)}\left(\sqrt{x}\right), \tag{B-17}$$

where $J_V^{(\lambda,\sigma)}(x)$ is the Bateman polynomial [8, p. 574 & 575], given by:

$$J_V^{(\lambda,\sigma)}(x) = \begin{pmatrix} \frac{\lambda}{2}+\sigma+V \\ V \end{pmatrix} \frac{x^\lambda}{\Gamma(\lambda+1)} \ {}_1F_2\left[\begin{matrix} -V; \\ \lambda+1, \frac{\lambda}{2}+\sigma+1; \end{matrix} \ x^2\right].$$

## 11. Cesaro Polynomial

Taking $U=1$ and $A_{V,R}=\frac{(s+1)_V\ R!}{V!\,(-s-V)_R}$ in (B-1), we have

$$S_V^1[x]\rightarrow g_V^{(s)}(x), \tag{B-18}$$

where $g_V^{(s)}(x)$ is the cesaro polynomial, [198, p. 449, Eq. (20)], given by:

$$g_V^{(s)}(x) = \begin{pmatrix} s+V \\ V \end{pmatrix} \ {}_2F_1\left[\begin{matrix} -V, 1; \\ -s-V; \end{matrix} \ x\right].$$

## 12. Generalized Hypergeometric Polynomial by Fasenmyer

Taking $U=1$ and $A_{V,R}=\frac{(V+1)_R}{R!\,(1/2)_R}\frac{(\alpha_1)_R...(\alpha_p)_R}{(\beta_1)_R...(\beta_q)_R}$ in (B-1), we have

$$S_V^U[x]\rightarrow f_V(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x), \tag{B-19}$$

where $f_V(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x)$ is the generalized hypergeometric polynomial [198, p. 182, Eq. (41); 41, p. 806, Eq. (1)], given by:

$$f_V\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;x\right) = {}_{p+2}F_{q+2}\left[\begin{array}{c} -V, V+1, \alpha_1, ..., \alpha_p; \\ {}^1\!/_2, 1, \beta_1, ..., \beta_q; \end{array} x\right].$$

## 13. Krawtchouk Polynomial

Taking $U=1$ and $A_{V,R}=\frac{(-y)_R}{(-N)_R}$ in (B-1), we have

$$S_V^1\left[x\right] \to K_V\left(y, x^{-1}, N\right), \tag{B-20}$$

where $K_V\left(y,x,N\right)$ is the Krawtchouk polynomial [198, p. 75, Eq. (2)], given by:

$$K_V\left(y,x,N\right) = {}_2F_1\left[\begin{array}{c} -V, -y; \\ -N; \end{array} x^{-1}\right],$$

0<x<1,y=0,1,...,N.

## 14. Meixner Polynomial

Taking $U=1$ and $A_{V,R}=\frac{(-y)_R}{(-\beta)_R}$ in (B-1), we have

$$S_V^1\left[x\right] \to M_V\left(y;\beta,\left(1-x\right)^{-1}\right), \tag{B-21}$$

where $M_V\left(y,\beta,x\right)$ is the Meixner polynomial [198, p. 75, Eq. (3)], given by:

$$M_V\left(y,\beta,x\right) = {}_2F_1\left[\begin{array}{c} -V, -y; \\ \beta; \end{array} 1-x^{-1}\right],$$

$0<x<1$, $y=0,1,...,N,\beta>0$.

## 15. Gould's Polynomial

Taking $A_{V,R} = \frac{\Gamma(p+1)C^{p-V+(U-1)R}y^R U^{V-UR}}{V!\Gamma(p-V+(U-1)R+1)}$ in (B-1), we have

$$S_V^U[x] \rightarrow \left(-x^{1/U}\right)^V P_V\left(U, x^{-1/U}, y, p, C\right), \qquad \text{(B-22)}$$

where $P_V(U, x, y, p, C)$ is the Gould's polynomial [198, p. 77, Eq.(13); 55, p. 699], given by:

$$P_V(U, x, y, p, C) = \sum_{R=0}^{[V/U]} \begin{pmatrix} p \\ R \end{pmatrix} \begin{pmatrix} p-R \\ V-UR \end{pmatrix} C^{p-V+(U-1)R} y^R (-Ux)^{V-UR}.$$

## 16. Gottlieb Polynomial

Taking $U = 1$ and $A_{V,R} = \frac{(-y)_R}{R!}$ in (B-1), we have

$$S_V^1[x] \rightarrow (1-x)^V I_V(y; \log(1-x)), \qquad \text{(B-23)}$$

where $I_V(y,t)$ is the Gottlieb polynomial [198, p. 185, Eq. (47); 54, p. 454, Eq. (2.3)], given by:

$$I_V(y,t) = e^{-Vt}{}_2F_1\left[\begin{matrix} -V, -y; \\ 1; \end{matrix} \; 1-e^{-t}\right].$$

The polynomials $S_V^U[x]$ can be reduced to other hypergeometric polynomials such as extended Jacobi polynomials [201, part I, p. 24; 201, part II, p. 106, Eq. (1.3)] and their generalizations [200, p. 471, Eqs. (4.2) & (4.3)] and [201, part II, p. 107, Eq. (1.11); 201, part II, p. 108, Eq. (1.17)] etc. For details, one can refer to papers by Srivastava and Singh [203, p. 158-162] and Srivastava and Garg [195, p. 686].

## MULTIVARIABLE ANALOGUE OF $\mathbf{S_V^U}[\mathbf{x}]$

The generalized class of polynomials, $S_V^{U_1,...,U_k}(x_1,...,x_k)$ introduced by Srivastava and Garg [195, p. 686, Eq. (1.4)] is defined in the following manner:

$$S_V^{U_1,...,U_k}[x_1,...,x_k] = \sum_{R_1,...,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V,R_1,...,R_k) \frac{x_i^{R_i}}{R_i!}, \tag{B-24}$$

where $U_1,...,U_k$ are arbitrary positive integers, $V = 0,1,...$; and the coefficients $A(V,R_1,...,R_k)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A(V,R_1,...,R_k)$, occurring in (B-24), the class of multivariable polynomials can be reduced to several multivarialbe polynomials defined by different authors.

### (a) Multivariable Hypergeometric Polynomials $F_D^{(k)}$

In (B-24), if we take

$$A(V,R_1,...,R_k) = \frac{(\beta_1)_{R_1\phi_1} \cdots (\beta_k)_{R_k\phi_k}}{(\gamma)_{R_1\psi_1+...+R_k\psi_k}},$$

then

$$S_V^{U_1,...,U_k}[x_1,...,x_k] \to F_D^{(k)}[(-V:U_i):(\beta_i,\phi_i);(\gamma:\psi_i);x_1...x_k], \tag{B-25}$$

where $F_D^{(k)}$is the first class of multivariable hypergeometric polynomials defined by Carlitz and Srivastava [198, p. 462-463, Eq. 9.4(4)] and is given by:

$$F_D^{(k)}\left[(-V:U_i):(\beta_i,\phi_i);(\gamma:\psi_i);x_1...x_k\right]$$

$$=\sum_{R_1,...,R_k=0}^{\sum_{i=1}^{k}U_iR_i\leq V}(-V)_{\sum_{i=1}^{k}U_iR_i}\frac{(\beta_1)_{R_1\phi_1}...(\beta_k)_{R_k\phi_k}}{(\gamma)_{R_1\psi_1+...+R_k\psi_k}}\frac{x_1^{R_1}}{R_1!}...\frac{x_k^{R_k}}{R_k!}\cdot \tag{B-26}$$

## (b) Generalized Lauricella Polynomial

In (B-24), if we take

$$A(V,R_1,...,R_k)=\frac{\prod_{l=1}^{M}(\beta_l)_{\phi_l^{(1)}R_1+...+\phi_l^{(k)}R_k}\prod_{l=1}^{M_1}\left(\beta_l^{(1)}\right)_{R_1\delta_l^{(1)}}...\prod_{l=1}^{M_k}\left(\beta_l^{(k)}\right)_{R_k\delta_l^{(k)}}}{\prod_{l=1}^{N}(\gamma_l)_{\psi_l^{(1)}R_1+...+\psi_l^{(k)}R_k}\prod_{l=1}^{N_1}\left(\gamma_l^{(1)}\right)_{R_1\lambda_l^{(1)}}...\prod_{l=1}^{N_k}\left(\gamma_l^{(k)}\right)_{R_k\lambda_l^{(k)}}},$$

then

$$S_V^{U_1,...,U_k}[x_1,...,x_k]\to F_{N:N_1,...,N_k}^{M+1:M_1,...,M_k}\left[\begin{array}{c|l} x_1 & (-V:U_1,...,U_k), \\ \vdots & \left(\beta_l:\phi_l^{(1)},...,\phi_l^{(k)}\right)_{1,M}:\left(\beta_l^{(1)},\delta_l^{(1)}\right)_{1,M_1},...,\left(\beta_l^{(k)},\delta_l^{(k)}\right)_{1,M_k} \\ x_k & \left(\gamma_l:\psi_l^{(1)},...,\psi_l^{(k)}\right)_{1,N}:\left(\gamma_l^{(1)},\lambda_l^{(1)}\right)_{1,N_1},...,\left(\gamma_l^{(k)},\lambda_l^{(k)}\right)_{1,N_k}\end{array}\right], \tag{B-27}$$

where $F_{N:N_1,...,N_k}^{M+1:M_1,...,M_k}$ is the polynomial form of generalized Lauricella function of Srivastava and Daoust [194, p. 454], given by:

$$F_{N:N_1,...,N_k}^{M+1:M_1,...,M_k}\left[\begin{array}{c|l} x_1 & (-V:U_1,...,U_k), \\ \vdots & \\ x_k & \end{array}\right.$$

$$\left.\begin{array}{l} \left(\beta_l:\phi_l^{(1)},...,\phi_l^{(k)}\right)_{1,M}:\left(\beta_l^{(1)},\delta_l^{(1)}\right)_{1,M_1},...,\left(\beta_l^{(k)},\delta_l^{(k)}\right)_{1,M_k} \\ \\ \left(\gamma_l:\psi_l^{(1)},...,\psi_l^{(k)}\right)_{1,N}:\left(\gamma_l^{(1)},\lambda_l^{(1)}\right)_{1,N_1},...,\left(\gamma_l^{(k)},\lambda_l^{(k)}\right)_{1,N_k} \end{array}\right] \tag{B-28}$$

$$= \sum_{R_1,...,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} \frac{\prod_{l=1}^{M} (\beta_l)_{\phi_l^{(1)} R_1+...+\phi_l^{(k)} R_k}}{\prod_{l=1}^{N} (\gamma_l)_{\psi_l^{(1)} R_1+...+\psi_l^{(k)} R_k}}$$

$$\times \frac{\prod_{l=1}^{M_1}\left(\beta_l^{(1)}\right)_{R_1\delta_l^{(1)}} \cdots \prod_{l=1}^{M_k}\left(\beta_l^{(k)}\right)_{R_k\delta_l^{(k)}}}{\prod_{l=1}^{N_1}\left(\gamma_l^{(1)}\right)_{R_1\lambda_l^{(1)}} \cdots \prod_{l=1}^{N_k}\left(\gamma_l^{(k)}\right)_{R_k\lambda_l^{(k)}}} \prod_{i=1}^{k} \frac{x_i^{R_i}}{R_i!}. \tag{B-29}$$

### (c) Multivariable Jacobi Polynomial

In (B-24), if we take $U_1 = ... = U_k = 1$ and

$$A(V, R_1, ..., R_k) = \frac{\prod_{i=1}^{k} (1+\alpha_i)_V \prod_{i=1}^{k} (1+\alpha_i+\beta_i+V)_{R_i}}{(V!)^k \prod_{i=1}^{k} (1+\alpha_i)_{R_i}},$$

then

$$S_V^{1,...,1}[x_1, ..., x_k] \to P_V^{\alpha_1,\beta_1;...;\alpha_k,\beta_k}(1-2x_1, ..., 1-2x_k), \tag{B-30}$$

where $P_V^{\alpha_1,\beta_1;...;\alpha_k,\beta_k}$ is the Jacobi polynomial of k variables defined by Srivastava [205, p. 65, Eq. (14)] and is given by:

$$P_V^{\alpha_1,\beta_1;\ldots;\alpha_k,\beta_k}(x_1,\ldots,x_k) = \frac{\prod_{i=1}^{k}(1+\alpha_i)_V}{(V!)^k} \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i}$$

$$\times \frac{\prod_{i=1}^{k}(1+\alpha_i+\beta_i+V)_{R_i}}{\prod_{i=1}^{k}\left[(1+\alpha_i)_{R_i} R_i!\right]} \prod_{i=1}^{k}\left(\frac{1-x_i}{2}\right)^{R_i}. \tag{B-31}$$

## (d) Multivariable Bessel Polynomial

In (B-24), if we take $U_1 = \ldots = U_k = 1$ and

$$A(V, R_1, \ldots, R_k) = (1+\alpha_1+V)_{R_1} \prod_{i=2}^{k}(1+\alpha_i+n_i)_{R_i},$$

then

$$S_V^{1,\ldots,1}[x_1,\ldots,x_k] \to y_{V,n_2,\ldots,n_k}^{\alpha_1,\ldots,\alpha_k}(-2x_1,\ldots,-2x_k), \tag{B-32}$$

where $y_{V,n_2,\ldots,n_k}^{\alpha_1,\ldots,\alpha_k}$ is the Bessel polynomial of k variables [207, p. 164, Eq. (2.3)] and is given by:

$$y_{V,n_2,\ldots,n_k}^{\alpha_1,\ldots,\alpha_k}(x_1,\ldots,x_k) = \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} R_i \le V} (-V)_{\sum_{i=1}^{k} R_i} \frac{\prod_{i=1}^{k}(1+\alpha_1+V)_{R_1}}{R_1!\ldots R_k!}$$

$$\times \prod_{i=2}^{k}(1+\alpha_i+n_i)_{R_i} \prod_{i=1}^{k}\left(-\frac{x_i}{2}\right)^{R_i}. \tag{B-33}$$

## (e) Multivariable Hermite Polynomial

In (B-24) if we take $U_1 = ... = U_k = 2$ and

$$A(V, R_1, ..., R_k) = (-1)^{R_1 + ... + R_k},$$

then

$$S_V^{2,...,2}[x_1, ..., x_k] \to (x_1)^{V/2} H_V(X_1, ..., X_k), \tag{B-34}$$

where

$$X_1 = \frac{1}{2\sqrt{x_1}}, \qquad X_j = \frac{x_j}{x_1} \, (j = 2, ..., k)$$

and $H_V(X_1, ..., X_k)$ is the multivariable Hermite polynomial [206, p. 97, Eq. (24)], defined by:

$$H_V(x_1, ..., x_k) = x_1^V \sum_{R_1, ..., R_k = 0}^{\sum_{i=1}^{k} 2R_i \le V} (-V)_{\sum_{i=1}^{k} 2R_i} \frac{(2)^{V - 2(R_1 + ... + R_k)}}{R_1! ... R_k!}$$

$$\times \left(-\frac{1}{x_1^2}\right)^{R_1} \prod_{i=2}^{k} \left(-\frac{x_i}{x_1^2}\right)^{R_i}. \tag{B-35}$$

Many other special cases of $S_V^{U_1,...,U_k}[x_1, ..., x_k]$ can be obtained by specializing its parameters, but we do not record them here explicitly.

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## PUBLICATIONS

1. A family of Mittag-Leffler type functions and its properties, *Palestine Journal of Mathematics* **4**, No. 2(2015) 367-373.
2. A family of Mittag-Leffler type functions and its relation with basic special functions, *International Journal of Pure and Applied Mathematics* **101**, No. 3(2015), 369-379.
3. Mittag-Leffler type $E$-function and related functions, *International Journal of Mathematical Sciences and Enginnering Applications* **8**, No. 6(2014), 69-79.
4. Fractional integral transformations of Mittag-Leffler type $E$-function, *South East Asian Journal of Mathematics and Mathematical Sciences* **11**, No. 1(2015), 31-38.
5. The Mellin-Barnes type contour integral representation of a new Mittag-Leffler type $E$-function, *American Journal of Mathematical Science and Applications* **2**, No. 2(2014), 137-141.
6. Fractional integral operators involving Mittag-Leffler type $E$-function, *Journal of Rajasthan Academy of Physical Sciences* **14**, No. 3 & 4(2015), 309-322.
7. Composition formulae for the multidimensional fractional integral operators involving Mittag-Leffler type $E$-function, (Communicated).
8. Fractional differential calculus of Mittag-Leffler type $E$-function, (Communicated).

# BIO-DATA

**Sheikh Mohammed Faisal**

139, New colony behind

new middle school Kaithunipole

Kota Rajasthan India.

email:**sheikhmohammedf9@gmail.com**

**Research Area:** Special Functions, Fractional Calculus, Integral Transformations.

**Research Topic:** A Study of Differential and Integral Calculus of Arbitrary Order and New Generalized Mittag-Leffler Function with Applications.

**Academic Degrees:** B.Sc., M.Sc.(Mathematics), M.phil.(Mathematics).

**Core Knowledge and Skill Area:** Pure Mathematics, Applied Mathematics, Statistics.

**Computer Proficiency:** LyX, Latex, MS Office, Matlab.

**Strenghts:** Honest, Spiritual, Dedicated.

**Research and Teaching Experience:** 5 years experience of teachinng of U.G. and P.G. classes and 4 years research experience in domain of special function specially on Mittag-Leffler function.

**Others:** Research paper presented in "International Conference of Society of Special Function and Application-2013". Attended short term courses and workshops in many reputed institutions.