A STUDY OF INEQUALITIES ON NEW GENERALIZED INFORMATION DIVERGENCE MEASURE AND THEIR APPLICATIONS

This thesis is submitted as a partial fulfillment of the degree of

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by

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to the

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DEDICATED

TO

MY LOVING, CARING MOTHER

CERTIFICATE

I feel immense pleasure in certifying that Mr. Praphull Chhabra has worked under my supervision for the award of Doctor of Philosophy in Mathematics on the topic "A Study of Inequalities on New Generalized Information Divergence Measure and Their Applications ".
It is further certified that no part of this thesis has been submitted to any University/Institute, in part or full, for the award of any degree or diploma and that the above candidate has carried out the research work under my guidance at the Department of Mathematics, Malaviya National Institute of Technology, Jaipur.

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DECLARATION

I herewith declare that the complete work reported in the Ph.D thesis entitled "A Study of Inequalities on New Generalized Information Divergence Measure and Their Applications"submitted at the Department of Mathematics, Malaviya National Institute of Technology, Jaipur (Rajasthan), India, is an authentic record of my work carried out under the supervision of Prof. K.C. Jain.

I have not submitted this work elsewhere for any other degree or diploma.

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ABSTRACT

Many authors did a lot of study regarding information divergence measures and applied these divergences in several fields in information theory. Jain and Saraswat introduced new generalized divergence (2012) and did a detail work. Now in this thesis, we extends that work with new information inequalities and their applications. The summary of the thesis is as follows:

Chapter 1 introduces the whole thesis.

Chapter 2 introduces several new information inequalities on new generalized divergence together with their applications and numerical verification.

Chapter 3 introduces new divergence measures of Csiszar's class, their bounds and their applications.

Chapter 4 introduces and characterize new series of divergences, intra relations and their applications.

Chapter 5 introduces several important and interesting relations among several new divergences and several well known divergences.

Chapter 6 introduces new generalized divergence for comparing finite number of discrete probability distributions.

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INTRODUCTION

1.1 Historical Background

One of the most prominent features of 20th century technology has been the development and exploitation of new communication media. Concurrent with the growth of devices for transmitting and processing information, a unifying theory known as Information Theory was initiated primarily by one man the U.S. electrical Engineer Claude E. Shannon, whose initial ideas appeared in an article "A Mathematical Theory of Communication"in the Bell System Tech. J. [83]. The term "Information Theory"does not possess a unique definition. Broadly speaking, information theory deals with the study of problems concern information processing, information storage, information retrieval and decision making.

The first person who studied all this was Harry Nyquist in 1924 [72], 1928 [73] and by Hartley in 1928 [36] who discovered the logarithmic nature of the measure of information. Harry Nyquist published the paper "Certain Factors Affecting Telegraph Speed"in which he gave the relation $W = K \log m$, where W is the speed of transmission of intelligence, m is the number of different voltage levels to choose from at each time step and K is a constant. He quantified "Intelligence"and the "Line Speed"by which it can be transmitted by communication system. In 1928 Ralph Hartley published a paper titled "Transmission of Information"[36] and used the word information as a measurable quantity and quantifying information as $H = \log S^n = n \log S$, where S was the number of possible symbols and n the number of symbols in a transmission. Around that time, only Wiener [114] also came up with results similar to those of Shannon.

This field is the intersection of Mathematics, Statistics, Computer science, Physics, Neurobiology, and Electronics engineering. Its impact has been crucial to the success of the Voyager missions to deep space, the invention of the compact disc, the feasibility of mobile phones, the development of the Internet, the study of linguistics and of human perception, the understanding of black holes, and numerous other fields. Important subfields of information theory are source coding, channel coding, algorithmic complexity theory, algorithmic information theory, and measures of information.

In 1974, Dutta [32] in his paper showed that information theory can also be applied in Number Theory, Quantum Mechanics, Qualitative Dynamics and Approximation Theory. The concepts introduced by Shannon, have also been applied with enormous degree of success in a number of fields such as Biology, Psychology, Economics, Statistics, Thermodynamics, Language Questionnaire theory, Probability theory, Communication theory, Cybernetics and many more. Since its inception it has broadened to find applications in many other areas,including

Statistical inference, Natural language processing, Cryptography, Networks other than communication networks as in neurobiology, the evolution and function of molecular codes, Model selection in ecology, Thermal physics, Quantum computing, Plagiarism detection and other forms of data analysis.

1.1.1 Shannon's entropy

Without essential loss of insight, we have restricted ourselves to discrete probability distributions, so let

$$\Gamma_n = \{P = (p_1, p_2, p_3, ..., p_n) : p_i > 0, \sum_{i=1}^{n} p_i = 1\}, n \geq 2 \tag{1.1.1}$$

be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3..., n$, then we have to suppose that $0f\left(0\right) = 0f\left(\frac{0}{0}\right) = 0$. Shannon [83] introduced the following measure of information for all $P \in \Gamma_n$

$$H\left(P\right) = -\sum_{i=1}^{n} p_i \log p_i. \tag{1.1.2}$$

The expression (1.1.2) is famous as Shannon's entropy or measure of uncertainty. This function $H\left(P\right)$ represents the expected value of uncertainty associated with the given probability scheme and it is uniquely determined by some rather natural postulates. The Shannon's entropy is the key concept in information theory. This entropy has found wide applications in different fields of science and technology (Bhattacharyya [10], Boekee [13], Denbibh etc. all [22], Gallager [33], Goldman [35], Horowitz [42], Majernik [68], Tverberg [110]). Applications of Shannon's entropy to music can be seen in (Siromoney and Rajagopalan [89]). Further this entropy has also been used extensively in the analysis of the structure of languages.

Many authors generalized the Shannon's entropy and obtained the interesting relations. Firstly, Renyi [79] introduced the generalization of Shannon's entropy. After that, Havrda and Charvat [37], Nath [69], Vajada [112], Arimoto [3], Kapur ([57], [58], [59]) etc., generalized it in different manners.

From 1961, more entropies had been introduced in the literature on information theory, generalizing Shannon's entropy. These are well known as parametric, trigonometric and weighted entropies. Renyi [79] for the first time gave the idea of parametric entropies. The idea of the trigonometric entropies were initiated by Aczel and Dacrozy [1] and the idea of weighted entropies were given by Belis and Guaisu [7]. Later Picard [76] extended it for generalized measures. The list of these generalized measures including their unified forms can be seen in Kapur [59] and Taneja [94].

1.1.2 Directed divergence and inaccuracy

Soloman Kullback and Richard-Leibler [63], two national security agency mathematicians, studied a measure of information, given by

$$K\left(P,Q\right)=\sum_{i=1}^{n}p_i\log\frac{p_i}{q_i} \tag{1.1.3}$$

for all $P,Q\in\Gamma_n$. This measure has many names given by different authors such as Relative information, Relative entropy, Directed Divergence, Cross entropy, Measure of discrimination etc. At the same time, Kullback and Leibler also studied a measure, called J-divergence, given by

$$J(P,Q) = K(P,Q) + K(Q,P) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}. \tag{1.1.4}$$

We can easily see that $K(P,Q)$ is non- symmetric whereas $J(P,Q)$ is symmetric with respect to probability distributions P and Q. The measure $J(P,Q)$ was already studied by Jeffrey [50].

Another important measure of information for a pair of probability distributions is the inaccuracy measure, introduced by Kerridge [61] and is given by

$$H^*(P,Q) = -\sum_{i=1}^{n} p_i \log q_i \tag{1.1.5}$$

for all $P, Q \in \Gamma_n$. When $p_i = q_i \; \forall \; i = 1, 2, ..., n$, the measure $H^*(P,Q)$ becomes the Shannon's entropy $H(P)$. Therefore Kerridge's inaccuracy is a generalization of Shannon's entropy. Also, we can see that $H(P)$, $K(P,Q)$, and $H^*(P,Q)$ satisfy a very interesting relationship given by

$$H^*(P,Q) = H(P) + K(P,Q). \tag{1.1.6}$$

Several authors presented alternative ways of generalizing Directed divergence. Some of those are as follows:

Directed divergence of order 'r'(Renyi [79])

$$K^r(P,Q) = (r-1)^{-1} \log \left(\sum_{i=1}^{n} p_i^r q_i^{1-r} \right), r \neq 1, r \geq 0. \tag{1.1.7}$$

Directed divergence of type 's'(Sharma and Autar [85])

$${}^1K_s(P,Q) = (s-1)^{-1} \left(\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1 \right), s \neq 1, s \geq 0. \tag{1.1.8}$$

The modified version of the measure (1.1.8) is given by

$$^{2}K_s(P,Q) = K_s(P,Q) = [s(s-1)]^{-1}\left(\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right), s \neq 0, 1. \qquad (1.1.9)$$

Particularly, we have

$$\lim_{r\to 1} K^r(P,Q) = \lim_{s\to 1}\ {}^{1}K_s(P,Q) = \lim_{s\to 1}\ {}^{2}K_s(P,Q) = K(P,Q)$$

and

$$\lim_{s\to 0}\ {}^{2}K_s(P,Q) = K(Q,P).$$

Some other important generalizations of Directed divergence can be seen in (Sharma [84], Theil [107]). The concept of weighted Directed divergence and weighted Inaccuracy were introduced by Taneja and Tuteja ([90], [91]). Further results in this direction can be seen in (Bhaker and Hooda [9]), (Hooda and Ram [40]), and (Hooda and Tuteja [41]). Similar generalizations of Kerridge's inaccuracy exist in the literatures (Kapur [59]), (Sharma and Mittal [86]), and (Taneja [94]).

1.2 A Review of Information and Divergence Measures

As a generalization of the uncertainty theory based on the notion of possibility, information theory consider the uncertainty of randomness perfectly. As pointed out by Renyi [79] in his fundamental paper on generalized information measures, in other short of problems other quantities may serve just as well, or even better, as measures of information. This should be supported either by their operational significance or by a set of natural postulates characterizing them or preferably by both. Thus the idea of generalized entropies arises in the literature. It started with Renyi [79] who characterized scalar parametric entropy, which includes Shannon entropy as a limiting case.

To design a communication system with a specific message handling capability, we need a measure of information content to be transmitted. Divergence measures are for quantifying the dissimilarity among probability distributions. Divergence measures are basically measures of distance between two probability distributions or compare two probability distributions. It means that any divergence measure must take its minimum value zero when probability distributions are equal. So, any divergence measure must increase as probability distributions move apart. As to the divergence and inaccuracy of information,Kullback and Leibler [63] studied a measure of information from statistical aspects of view involving two probability distributions associated with the same experiment, calling discrimination function, later different authors named as cross entropy, relative information etc. It is a non-symmetric measure of two probability distributions P and Q. At the same time they also developed the idea of the Harold invariant, famous as J-divergence. Kerridge [61] studied a different kind of measure calling inaccuracy measure involving again two probability distributions.

Sibson [88] studied another divergence measure involving two probability distributions, using mainly the concavity property of Shannon's entropy, calling information radius. Later, Burbea and Rao ([15], [16]) studied extensively the information radius and its parametric generalization, calling this measure as Jensen difference measure. Taneja ([96], [98]) studied a new measure of divergence and its two parametric generalizations involving two probability distributions based on arithmetic and geometric mean inequality.

Sant'anna and Taneja [81] and Sharma and Taneja [87] studied trigonometric entropies from different aspects. The idea of weighted entropies started by Belis and Guaisu [7], later Picard [76] extended it for generalized measures. After Renyi [79], other researchers such as Havrda and Charvat [37], Arimoto [3], Sharma and Mittal [86] etc. interested towards other kinds of expressions generalizing Shannon's entropy. Taneja [92] unified some of these. Taneja [93] introduced a new divergence measure called arithmetic geometric mean divergence measure.

Since our work deals with measures involving two probability distributions, our focus is more on these measures and generalizations. One of the important issues in many applications of Statistics and Probability Theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. Depending upon the nature of the problem, different divergence measures are suitable. So it is always desirable to develop a new divergence measure. A number of divergence measures for this purpose have been proposed and extensively studied. Divergence measures have been demonstrated very useful in a variety of disciplines such as Bayesian model validation (Tumer and Ghosh [109]), quantum information theory (Lamberti etc. all [67], Nielsen and Chuang [71]), model validation (Benveniste etc. all [8]), robust detection (Poor [78]), economics and political science (Theil ([107], [108])), biology (Pielou [77]), analysis of contingency tables (Gokhale and Kullback [34]), approximation of probability distributions (Chow and Lin [19], Kazakos and Cotsidas [60]), signal processing (Kadota and Shepp [54], Kailath [56]), pattern recognition (Bassat [5], Boekee [13], Chen [18], Jones and Byrne [53]), color image segmentation (Nielsen and Boltz [70]), 3D image segmentation and word alignment (Taskar etc. all [106]), cost- sensitive classification for medical diagnosis (Santos-Rodriguez etc. all [82]), magnetic resonance image analysis (Vemuri etc. all [113]) etc.

Also we can use divergence measures in fuzzy mathematics as fuzzy directed divergences and fuzzy entropies (Bajaj and Hooda [4], Hooda [39], Jha and Mishra [52]), which are very useful to find the amount of average ambiguity or difficulty in making a decision whether an element belongs to a set or not. Fuzzy information measures have recently found applications to fuzzy aircraft control, fuzzy traffic control, engineering, medicines, computer science, management and decision making etc. Divergence measures are also very useful to find the utility of an event (Bhullar etc. all [12], Taneja and Tuteja [91]), i.e., an event is how much useful compare to other event. Also Bhatia and Singh [11], Jain and Chhabra [45] etc. have developed metric with the help of divergence measures.

1.2.1 Csiszar's generalized divergence and properties

We start this subsection with a very important function, convex function. Convex functions play a very important role for information divergence measures in information theory. Several information inequalities had been introduced for convex functions and keep going in this thesis as well. Many authors had introduced generalized divergences like: Csiszar's divergence (Ali and Silvey [2], Csiszar [20]), Bregman's divergence (Bregman [14]), Burbea- Rao's divergence (Burbea and Rao [16]), Renyi's like divergence (Renyi [79]), M- divergence (Salicru [80]), New generalized divergence (Jain and Saraswat [48]) etc., where they had considered f as a real, continuous, and convex function on $(0, \infty)$. By putting suitable convex function in these generalized divergences, we can obtain several divergence measures. So it is very necessary to understand first the definition of convex function, as follows.

Definition 1.2.1. *Convex function: A function $f(t)$ is said to be convex over an interval (a, b) if for every $t_1, t_2 \in (a, b)$ and $0 \leq \lambda \leq 1$, we have*

$$f[\lambda t_1 + (1-\lambda) t_2] \leq \lambda f(t_1) + (1-\lambda) f(t_2),$$

and said to be strictly convex if equality does not hold only if $\lambda \neq 0$ or $\lambda \neq 1$. Geometrically, it means that if A, B, C are three distinct points on the graph of convex function f with B between A and C, then B is on or below chord AC.

Furthermore, let C be the set of convex functions $f : [0, \infty) \to (-\infty, \infty)$ continuous at 0, i.e., $f(0) = \lim_{t \to 0} f(t)$, also f is normalized, i.e., $f(1) = 0$. Further, let $f^* \in C$, defined by

$$f^*(t) = t f\left(\frac{1}{t}\right), t \in (0, \infty)$$

the $*$- conjugate convex function of f, let a function $f \in C$ satisfying $f^* \equiv f$ be called the $*$- self conjugate.

In order to avoid meaningless expressions in the sequel, let us agree in the following notational conventions.

$$0f^{*}\left(\frac{t}{0}\right)=tf\left(\frac{0}{t}\right)=tf\left(0\right),t\in\left(0,\infty\right).$$
$$0f\left(\frac{t}{0}\right)=tf^{*}\left(\frac{0}{t}\right)=tf\left(0\right),t\in\left(0,\infty\right).$$
$$0f\left(\frac{0}{0}\right)=0f^{*}\left(\frac{0}{0}\right)=0.$$
$$0f\left(\frac{a}{0}\right)=\lim_{\epsilon\to 0^{+}}\epsilon f\left(\frac{a}{\epsilon}\right)=a\lim_{t\to\infty}\frac{f\left(t\right)}{t},a>0.$$

Csiszar's divergence ([2], [20]) and Jain- Saraswat's divergence [48] are widely used due to its compact nature, Specially Csiszar's divergence, which is given by

$$C_{f}\left(P,Q\right)=\sum_{i=1}^{n}q_{i}f\left(\frac{p_{i}}{q_{i}}\right), \tag{1.2.1}$$

where $f:(0,\infty)\to R$ (set of real no.) is real, continuous, and convex function and $P,Q\in\Gamma_{n}$. $C_{f}\left(P,Q\right)$ is a natural distance measure from a true probability distribution P to an arbitrary probability distribution Q. Typically P represents observations or a precise calculated probability distribution, whereas Q represents a model, a description or an approximation of P. We note that $C_{f}\left(P,Q\right)=C_{f^{*}}\left(Q,P\right)$ and $C_{f}\left(P,Q\right)+C_{f^{*}}\left(P,Q\right)$ will be a symmetric generalized information divergence measure.

The properties (Uniqueness theorem, Symmetry theorem, Range of values theorem and Characterization theorem) of Csiszar's generalized divergence can be seen in literature by Osterreicher [74]. Osterreicher has discussed axiomatic properties and some important classes of generalized divergence measures. Now we are discussing the following fundamental properties of $C_{f}\left(P,Q\right)$, which are being used in this thesis.

Proposition 1.2.1. *(Non negativity) Let $f:(0,\infty)\to R$ be a real, convex function and $(P,Q)\in\Gamma_n\times\Gamma_n$, then we have*

$$C_f(P,Q)\geq f(1). \tag{1.2.2}$$

If f is normalized, i.e., $f(1)=0$ then $C_f(P,Q)\geq 0$ and $C_f(P,Q)=0$ if and only if $P=Q$, and f is strictly convex.

Proposition 1.2.2. *(Convexity) If the function f is convex and normalized, i.e., $f''(t)\geq 0\ \forall\ t>0$ and $f(1)=0$ respectively, then $C_f(P,Q)$ and $C_f(Q,P)$ are both non-negative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.*

Proposition 1.2.3. *(Linearity) If f_1 and f_2 are two convex functions such that $F=af_1+bf_2$ then $C_F(P,Q)=aC_{f_1}(P,Q)+bC_{f_2}(P,Q)$, where a and b are constants and $(P,Q)\in\Gamma_n\times\Gamma_n$.*

Proof: Let $F=af_1+bf_2$, then

$$\begin{aligned}C_F(P,Q)&=\sum_{i=1}^{n}q_iF\left(\frac{p_i}{q_i}\right)=\sum_{i=1}^{n}q_i\left(af_1+bf_2\right)\left(\frac{p_i}{q_i}\right)\\&=a\sum_{i=1}^{n}q_if_1\left(\frac{p_i}{q_i}\right)+b\sum_{i=1}^{n}q_if_2\left(\frac{p_i}{q_i}\right)=aC_{f_1}(P,Q)+bC_{f_2}(P,Q).\end{aligned}$$

Dragomir [26] introduced the following generalized divergence measure for comparing finite discrete probability distributions, given by

$$C_f^n(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n)=\sum_{i=1}^{m}\sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}q_{i2}...q_{in}f\left(\frac{\frac{p_{i1}}{q_{i1}}+\frac{p_{i2}}{q_{i2}}+...+\frac{p_{in}}{q_{in}}}{n}\right). \tag{1.2.3}$$

Ciszar's divergence measure is a particular case of this measure for comparing two discrete probability distributions. Following relation can be seen as well in the same literature

$$\begin{aligned}&C_f^1(P_1,Q_1)\geq C_f^2(P_1,P_2,Q_1,Q_2)\geq...\geq C_f^n(P_1,...,P_n,Q_1,...,Q_n)\\&\geq C_f^{n+1}(P_1,...,P_{n+1},Q_1,...,Q_{n+1})\geq f(1).\end{aligned} \tag{1.2.4}$$

Divergences for comparing more than two probability distributions are useful for discrimination and taxonomy.

Many authors introduced several divergence measures. These divergences are very useful in information theory for comparing discrete probability distributions. These are defined as follows.

Symmetric divergence measures

Symmetric divergence measures are those measures that are symmetric with respect to probability distributions $P, Q \in \Gamma_n$. These measures are as follows.

Triangular discrimination (Dacunha- Castelle etc. all [21])

$$\Delta(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}. \tag{1.2.5}$$

Hellinger discrimination (Hellinger [38])

$$h(P,Q) = \sum_{i=1}^{n} \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2}. \tag{1.2.6}$$

Variational distance or l_1 distance (Kolmogorov [62])

$$V(P,Q) = \sum_{i=1}^{n} |p_i - q_i|. \tag{1.2.7}$$

Jain and Srivastava divergence (Jain and Srivastava [49])

$$E^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{\sqrt{p_i q_i}}. \tag{1.2.8}$$

Symmetric Chi- square divergence (Dragomir etc. all [31])

$$\psi(P,Q) = \chi^2(P,Q) + \chi^2(Q,P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i}, \tag{1.2.9}$$

where $\chi^2(P,Q)$ is the Chi- square divergence (1.2.19).

J- divergence (Jeffrey [50], Kullback and Leibler [63])

$$J(P,Q) = K(P,Q) + K(Q,P) = J_R(P,Q) + J_R(Q,P) = \sum_{i=1}^{n}(p_i - q_i)\log\frac{p_i}{q_i}, \tag{1.2.10}$$

where $K(P,Q)$ and $J_R(P,Q)$ are the Relative entropy (1.2.18) and Relative J- divergence (1.2.22), respectively.

Arithmetic- Geometric Mean divergence (Taneja [93])

$$T(P,Q) = \frac{1}{2}\left[G(P,Q) + G(Q,P)\right] = \sum_{i=1}^{n}\left(\frac{p_i + q_i}{2}\right)\log\frac{p_i + q_i}{2\sqrt{p_i q_i}}, \tag{1.2.11}$$

where $G(P,Q)$ is the Relative AG divergence (1.2.20).

Jensen- Shannon divergence (Burbea and Rao [16], Sibson [88])

$$I(P,Q) = \frac{1}{2}\left[F(P,Q) + F(Q,P)\right] = \frac{1}{2}\left[\sum_{i=1}^{n} p_i \log\frac{2p_i}{p_i + q_i} + \sum_{i=1}^{n} q_i \log\frac{2q_i}{p_i + q_i}\right], \tag{1.2.12}$$

where $F(P,Q)$ is the Relative JS divergence (1.2.21).

Kumar and Chhina divergence (Kumar and Chhina [64])

$$S^*(P,Q) = \sum_{i=1}^{n}\frac{(p_i + q_i)(p_i - q_i)^2}{p_i q_i}\log\frac{p_i + q_i}{2\sqrt{p_i q_i}}. \tag{1.2.13}$$

Kumar and Hunter divergence (Kumar and Hunter [65])

$$L(P,Q) = \sum_{i=1}^{n}\frac{(p_i - q_i)^2}{p_i + q_i}\log\frac{p_i + q_i}{2\sqrt{p_i q_i}}. \tag{1.2.14}$$

Kumar and Johnson divergence (Kumar and Johnson [66])

$$\psi_M(P,Q) = \sum_{i=1}^{n}\frac{(p_i^2 - q_i^2)^2}{2(p_i q_i)^{\frac{3}{2}}}. \tag{1.2.15}$$

d- Divergence (Basseville [6])

$$d\left(P,Q\right)=1-\sum_{i=1}^{n}\sqrt{\frac{p_i+q_i}{2}}\left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right). \tag{1.2.16}$$

Jain and Mathur divergence (Jain and Mathur [46])

$$P^{*}\left(P,Q\right)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^4\left(p_i+q_i\right)\left(p_i^2+q_i^2\right)}{p_i^3q_i^3}. \tag{1.2.17}$$

Non- symmetric divergence measures

Non- symmetric divergence measures are those measures that are not symmetric with respect to probability distributions $P,Q\in\Gamma_n$. These measures are as follows.

Relative entropy or Kullback- Leibler distance (Kullback and Leibler [63])

$$K\left(P,Q\right)=\sum_{i=1}^{n}p_i\log\frac{p_i}{q_i}. \tag{1.2.18}$$

Chi- square divergence or Pearson divergence (Pearson [75])

$$\chi^2\left(P,Q\right)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^2}{q_i}. \tag{1.2.19}$$

Relative Arithmetic- Geometric divergence (Taneja [93])

$$G\left(P,Q\right)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2p_i}\right). \tag{1.2.20}$$

Relative Jensen- Shannon divergence (Sibson [88])

$$F\left(P,Q\right)=\sum_{i=1}^{n}p_i\log\frac{2p_i}{p_i+q_i}. \tag{1.2.21}$$

Relative J- Divergence (Dragomir etc. all [28])

$$J_R\left(P,Q\right)=\sum_{i=1}^{n}\left(p_i-q_i\right)\log\left(\frac{p_i+q_i}{2q_i}\right). \tag{1.2.22}$$

Parametric symmetric and non- symmetric divergence measures

Some authors defined parametric divergence measures, by which we obtain some well known existing divergences for some values of parameter. These measures are as follows.

Relative information of type $'s'$ (Taneja and Kumar [104])

$$K_s(P,Q) = [s(s-1)]^{-1}\left(\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right), s \neq 0,1$$

$$= \begin{cases} \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i} = \frac{1}{2}\chi^2(P,Q) = K_2(P,Q) & \text{if } s=2 \\ \sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} = K(P,Q) = \lim_{s\to 1} K_s(P,Q) & \text{if } s=1 \\ 4\sum_{i=1}^{n}\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2} = 4h(P,Q) = K_{\frac{1}{2}}(P,Q) & \text{if } s=\frac{1}{2} \\ \sum_{i=1}^{n} q_i \log\frac{q_i}{p_i} = K(Q,P) = \lim_{s\to 0} K_s(P,Q) & \text{if } s=0 \end{cases}. \tag{1.2.23}$$

Unified Relative Jensen- Shannon and Arithmetic- Geometric divergence of type $'s'$ (Taneja and Kumar [105])

$$\Phi_s(P,Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i\left(\frac{p_i+q_i}{2p_i}\right)^s - 1\right], s \neq 0,1$$

$$= \begin{cases} \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{4}\Delta(P,Q) = \Phi_{-1}(P,Q) & \text{if } s=-1 \\ \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i+q_i}\right) = F(P,Q) = \lim_{s\to 0}\Phi_s(P,Q) & \text{if } s=0 \\ \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2p_i}\right) = G(P,Q) = \lim_{s\to 1}\Phi_s(P,Q) & \text{if } s=1 \\ \frac{1}{8}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i} = \frac{1}{8}\chi^2(Q,P) = \Phi_2(P,Q) & \text{if } s=2 \end{cases}. \tag{1.2.24}$$

Relative J- divergence of type $'s'$ (Taneja and Kumar [105])

$$\tau_s(P,Q) = (s-1)^{-1}\sum_{i=1}^{n}\left(\frac{p_i-q_i}{2}\right)\left(\frac{p_i+q_i}{2q_i}\right)^{s-1}, s \neq 1$$

$$= \begin{cases} \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{2}\Delta(P,Q) = \tau_0(P,Q) & \text{if } s=0 \\ \frac{1}{2}\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) = \frac{1}{2}J_R(P,Q) = \lim_{s\to 1}\tau_s(P,Q) & \text{if } s=1 \\ \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i} = \frac{1}{4}\chi^2(P,Q) = \tau_2(P,Q) & \text{if } s=2 \end{cases}. \tag{1.2.25}$$

Generalized Jensen- Shannon and Arithmetic Geometric mean divergence (Taneja [95])

$$\Omega_s(P,Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n}\left(\frac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\frac{p_i+q_i}{2}\right)^s - 1\right], s \neq 0,1, s \in R$$

$$= \begin{cases} \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{4}\Delta(P,Q) = \Omega_{-1}(P,Q) & \text{if } s=-1 \\ \frac{1}{2}\left[\sum_{i=1}^{n} p_i \log\frac{2p_i}{p_i+q_i} + \sum_{i=1}^{n} q_i \log\frac{2q_i}{p_i+q_i}\right] = I(P,Q) = \lim_{s\to 0}\Omega_s(P,Q) & \text{if } s=0 \\ 4\left[1-\sum_{i=1}^{n}\sqrt{\frac{p_i+q_i}{2}}\left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)\right] = 4d(P,Q) = \Omega_{\frac{1}{2}}(P,Q) & \text{if } s=\frac{1}{2} \\ \sum_{i=1}^{n}\frac{p_i+q_i}{2}\log\frac{p_i+q_i}{2\sqrt{p_iq_i}} = T(P,Q) = \lim_{s\to 1}\Omega_s(P,Q) & \text{if } s=1 \\ \frac{1}{16}\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i+q_i)}{p_iq_i} = \frac{1}{16}\psi(P,Q) = \Omega_2(P,Q) & \text{if } s=2 \end{cases} \tag{1.2.26}$$

Renyi's $'a'$ order entropy (Renyi [79])

$$R_a(P,Q) = \sum_{i=1}^{n}\frac{p_i^a}{q_i^{a-1}}, a > 1. \tag{1.2.27}$$

Series of symmetric and non- symmetric divergence measures

These divergence measures are basically series of measures and we obtain infinite divergences by putting the particular value of parameter.

Jain and Srivastava divergences (Jain and Srivastava [49])

$$E_m^*(P,Q) = \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_iq_i)^{\frac{2m-1}{2}}}, m = 1,2,3,..., \tag{1.2.28}$$

where $E_1^*(P,Q) = E^*(P,Q)$ is a particular case at $m=1$, given by (1.2.8).

Jain and Srivastava divergences (Jain and Srivastava [49])

$$J_m^*(P,Q) = \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_iq_i)^{\frac{2m-1}{2}}}\exp\frac{(p_i-q_i)^2}{p_iq_i}, m = 1,2,3,.... \tag{1.2.29}$$

Puri and Vineze divergences (Kafka etc. all [55])

$$\Delta_m(P,Q) = \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_i+q_i)^{2m-1}}, m = 1,2,3..., \tag{1.2.30}$$

where $\Delta_1(P,Q) = \Delta(P,Q)$ is a particular case at $m=1$, given by (1.2.5).

Chi- m divergences (Vajda [111])

$$\chi^{2m}(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{2m}}{q_i^{2m-1}}, m = 1,2,3..., \tag{1.2.31}$$

where $\chi^2(P,Q)$ is a particular case at $m=1$, given by (1.2.19).

Now, let us define some means for $a,b>0$ that can be seen in literature (Taneja [101]). The following means are being used in many new relations and for making new divergences.

$$S(a,b) = \sqrt{\frac{a^2+b^2}{2}} = \text{Root mean square.} \tag{1.2.32}$$

$$H(a,b) = \frac{2ab}{a+b} = \text{Harmonic mean.} \tag{1.2.33}$$

$$A(a,b) = \frac{a+b}{2} = \text{Arithmetic mean.} \tag{1.2.34}$$

$$N_1(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = \text{Square root mean.} \tag{1.2.35}$$

$$N_2(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\sqrt{\frac{a+b}{2}} = N_2 \text{ mean.} \tag{1.2.36}$$

$$N_3(a,b) = \frac{a+\sqrt{ab}+b}{3} = \text{Heronian mean.} \tag{1.2.37}$$

$$L_*(a,b) = \frac{a-b}{\log a - \log b}, a \neq b = \text{Logarithmic mean.} \tag{1.2.38}$$

$$B(a,b) = \sqrt{ab} = \text{Geometric mean.} \tag{1.2.39}$$

$$C(a,b) = \frac{a^2+b^2}{a+b} = \text{Contra harmonic mean.} \tag{1.2.40}$$

$$R(a,b) = \frac{2}{3}\frac{a^2+ab+b^2}{a+b} = \text{Centroidal mean.} \tag{1.2.41}$$

Now for $P, Q \in \Gamma_n$, put $a = p_i$ and $b = q_i$ in above means and then sum over all $i = 1, 2, ..., n$, we obtain

$$S(P,Q) = \sum_{i=1}^{n} \sqrt{\frac{p_i^2 + q_i^2}{2}}, H(P,Q) = \sum_{i=1}^{n} \frac{2p_iq_i}{p_i + q_i}, A(P,Q) = \sum_{i=1}^{n} \frac{p_i + q_i}{2} = 1,$$
$$N_1(P,Q) = \sum_{i=1}^{n} \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2}\right)^2, N_2(P,Q) = \sum_{i=1}^{n} \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2}\right) \sqrt{\frac{p_i + q_i}{2}},$$
$$N_3(P,Q) = \sum_{i=1}^{n} \frac{p_i + \sqrt{p_iq_i} + q_i}{3}, L_*(P,Q) = \sum_{i=1}^{n} \frac{p_i - q_i}{\log p_i - \log q_i}, p_i \neq q_i,$$
$$B(P,Q) = \sum_{i=1}^{n} \sqrt{p_iq_i}, C(P,Q) = \sum_{i=1}^{n} \frac{p_i^2 + q_i^2}{p_i + q_i}, R(P,Q) = \frac{2}{3} \sum_{i=1}^{n} \frac{p_i^2 + p_iq_i + q_i^2}{p_i + q_i}$$

respectively. Here $B(P,Q)$ is the well known Bhattacharya distance (Bhattacharyya [10]). Also we note a good inequality relation among these, by (5.2.47), which is

$$H(P,Q) \leq B(P,Q) \leq N_3(P,Q) \leq A(P,Q)$$
$$\leq R(P,Q) \leq S(P,Q) \leq C(P,Q).$$

We can see some small equality relations as well among Triangular discrimination, Hellinger discrimination and above defined quantities. These are as follows:

$$\Delta(P,Q) = 3[C(P,Q) - R(P,Q)] = 2[A(P,Q) - H(P,Q)] = 2[C(P,Q) - A(P,Q)]$$
$$= 6[R(P,Q) - A(P,Q)] = \frac{3}{2}[R(P,Q) - H(P,Q)]$$

and

$$h(P,Q) = 3[A(P,Q) - N_3(P,Q)] = [A(P,Q) - B(P,Q)] = \frac{3}{2}[N_3(P,Q) - B(P,Q)].$$

Now we define some other divergences (Taneja [101]), as follows:

Square root- arithmetic mean divergence

$$M_{SA}(P,Q) = S(P,Q) - A(P,Q) = \sum_{i=1}^{n} \sqrt{\frac{p_i^2 + q_i^2}{2}} - 1. \tag{1.2.42}$$

Square root- geometric mean divergence

$$M_{SB}(P,Q) = S(P,Q) - B(P,Q) = \sum_{i=1}^{n} \left(\sqrt{\frac{p_i^2 + q_i^2}{2}} - \sqrt{p_i q_i} \right). \tag{1.2.43}$$

Square root- harmonic mean divergence

$$M_{SH}(P,Q) = S(P,Q) - H(P,Q) = \sum_{i=1}^{n} \left(\sqrt{\frac{p_i^2 + q_i^2}{2}} - \frac{2p_i q_i}{p_i + q_i} \right). \tag{1.2.44}$$

Some difference of particular divergences can be seen in literature (Taneja [100]), which are as follows.

$$\begin{aligned} D_{\psi T}(P,Q) &= \frac{1}{16}\psi(P,Q) - T(P,Q) \\ &= \frac{1}{2}\sum_{i=1}^{n} \left[\frac{(p_i - q_i)^2}{8p_i q_i} - \log \frac{p_i + q_i}{2\sqrt{p_i q_i}} \right](p_i + q_i). \end{aligned} \tag{1.2.45}$$

$$\begin{aligned} D_{\psi J}(P,Q) &= \frac{1}{2}\psi(P,Q) - J(P,Q) \\ &= \sum_{i=1}^{n} \left[\frac{(p_i^2 - q_i^2)}{2p_i q_i} - \log \frac{p_i}{q_i} \right](p_i - q_i). \end{aligned} \tag{1.2.46}$$

1.2.2 New generalized divergence

We did a detail study about Csiszar's divergence in previous subsection. Similarly, Jain and Saraswat [48] introduced and characterized a new generalized divergence, given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right), \tag{1.2.47}$$

where $f : (0,\infty) \to R$ (set of real no.) is real, continuous, and convex function and $P, Q \in \Gamma_n$. We can obtain Several well known divergences by suitably defining the convex function in (1.2.47). For example:

If we take $f(t) = (t-1)\log t$ in (1.2.47), we obtain

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i + q_i}{2q_i}\right) = \frac{1}{2}J_R(P,Q).$$

If we take $f(t) = \frac{(t-1)^2}{t}$ in (1.2.47), we obtain

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i - q_i)^2}{p_i + q_i} = \frac{1}{2}\Delta(P,Q).$$

If we take $f_m(t) = \frac{(t-1)^{2m}}{t^{2m-1}}\exp\frac{(t-1)^2}{t^2}$, m=1,2,3... in (1.2.47), we obtain

$$S_f(P,Q) = \sum_{i=1}^{n}\frac{(p_i - q_i)^{2m}}{(p_i + q_i)^{2m-1}}\exp\frac{(p_i - q_i)^2}{(p_i + q_i)^2} = N_m^*(P,Q), \text{(Jain and Saraswat [47]).} \tag{1.2.48}$$

and many more. Where $\Delta(P,Q)$ and $J_R(P,Q)$ are already defined by (1.2.5) and (1.2.22) respectively.

Like the fundamental properties of $C_f(P,Q)$, There are the following fundamental properties of $S_f(P,Q)$.

Proposition 1.2.4. *(Non negativity) Let $f : (0,\infty) \to R$ be a real, convex function and $(P,Q) \in \Gamma_n \times \Gamma_n$, then we have*

$$S_f(P,Q) \geq f(1). \tag{1.2.49}$$

If f is normalized, i.e., $f(1) = 0$ then $S_f(P,Q) \geq 0$ and $S_f(P,Q) = 0$ if and only if $P = Q$, and f is strictly convex..

Proposition 1.2.5. *(Convexity) If the function f is convex and normalized, i.e., $f''(t) \geq 0\ \forall\ t > 0$ and $f(1) = 0$ respectively, then $S_f(P,Q)$ and $S_f(Q,P)$ are both non-negative and convex in the pair of probability distribution $(P,Q) \in \Gamma_n \times \Gamma_n$.*

Proposition 1.2.6. *(Linearity) If f_1 and f_2 are two convex functions such that $F = af_1 + bf_2$ then $S_F(P,Q) = aS_{f_1}(P,Q) + bS_{f_2}(P,Q)$, where a and b are constants and $(P,Q) \in \Gamma_n \times \Gamma_n$.*

Proof: Let $F = af_1 + bf_2$, then

$$S_F(P,Q) = \sum_{i=1}^{n} q_i F\left(\frac{p_i + q_i}{2q_i}\right) = \sum_{i=1}^{n} q_i (af_1 + bf_2)\left(\frac{p_i + q_i}{2q_i}\right)$$
$$= a\sum_{i=1}^{n} q_i f_1\left(\frac{p_i + q_i}{2q_i}\right) + b\sum_{i=1}^{n} q_i f_2\left(\frac{p_i + q_i}{2q_i}\right) = aS_{f_1}(P,Q) + bS_{f_2}(P,Q).$$

Now, we define a relation between Jain and Saraswat's divergence measure $S_f(P,Q)$ and Csiszar's divergence measure $C_f(P,Q)$. This relation can also be seen in literature (Jain and Saraswat [48]).

Theorem 1.2.1. *Let $f : (0,\infty) \to R$ be the differentiable convex function, i.e., $f''(t) \geq 0 \;\forall\; t > 0$ and normalized, i.e., $f(1) = 0$. Then for $P, Q \in \Gamma_n$, we have the following relation*

$$S_f(P,Q) \leq \frac{1}{2} C_f(P,Q).$$

Proof: Apply Jensen inequality (1.3.3) for the domain $I \subset (0,\infty)$, by putting $\lambda_1 = \lambda_2 = \frac{1}{2}, \lambda_3 = ... = \lambda_n = 0$ in (1.3.3), we get

$$f\left(\frac{t_1 + t_2}{2}\right) \leq \frac{1}{2}\left[f(t_1) + f(t_2)\right].$$

Now put $t_1 = t$ and $t_2 = 1$ in above inequality, we obtain

$$f\left(\frac{t+1}{2}\right) \leq \frac{1}{2} f(t).$$

Now take $t = \frac{p_i}{q_i}$ in above inequality, multiply with q_i for each i and then summation over from $i = 1$ to $i = n$, we obtain the required relation.

Now for a differentiable function $f : (0,\infty) \to R$, consider the associated functions $g : (0,\infty) \to R$ and $h : (0,\infty) \to R$, are given by

$$g(t) = (t-1) f'(t) \tag{1.2.50}$$

and

$$h(t) = (t-1) f'\left(\frac{t+1}{2}\right). \tag{1.2.51}$$

For $g(t)$ and $h(t)$, we get the followings respectively.

$$S_g(P,Q) = E_{S_{f'}}(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i - q_i}{2}\right) f'\left(\frac{p_i + q_i}{2q_i}\right), \tag{1.2.52}$$

$$S_h(P,Q) = E^*_{S_{f'}}(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i - q_i}{2}\right) f'\left(\frac{p_i + 3q_i}{4q_i}\right), \tag{1.2.53}$$

$$C_g(P,Q) = E_{C_{f'}}(P,Q) = \sum_{i=1}^{n} (p_i - q_i) f'\left(\frac{p_i}{q_i}\right), \tag{1.2.54}$$

and

$$C_h(P,Q) = E^*_{C_{f'}}(P,Q) = \sum_{i=1}^{n} (p_i - q_i) f'\left(\frac{p_i + q_i}{2q_i}\right), \tag{1.2.55}$$

where $C_f(P,Q)$ and $S_f(P,Q)$ are given by (1.2.1) and (1.2.47) respectively.

1.3 Information Inequalities and Definitions

In recent years, several mathematicians, like: S.S. Dragomir, I.J. Taneja and many more, introduced and characterized information inequalities for comparing probability distributions. Also they derived many relations among several divergences, mean divergences and evaluated bounds by using that information inequalities. Specially Taneja ([97], [101], [102], [103]) did a lot of work in inequalities involving several means, mean divergences and difference of means.

There are many information inequalities in information and statistics theory. By defining two of them, we are just giving an idea how these inequalities relate divergence measures for probability distributions. We start with the following information inequality involving Csiszar's f- divergence (1.2.1) and Kullback-Leibler divergence (1.2.18). This inequality is given in literature (Cerone etc. all [17]) but obtained by using the inequality given by Dragomir [27].

Proposition 1.3.1. *Let $f : [\alpha, \beta] \subset (0, \infty) \to (-\infty, \infty)$ be an absolutely continuous and convex function with $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$ and $\alpha \le \frac{p_i}{q_i} \le \beta \; \forall \; i = 1, 2, ..., n$. Then we have the following inequality involving $C_f(P, Q)$ and $K(P, Q)$ between probability distributions $P, Q \in \Gamma_n$:*

$$\left| C_f(P,Q) - \frac{1}{\beta - \alpha} \int_\alpha^\beta \frac{f(t)}{t} dt \right| \le \frac{2}{\beta - \alpha} \left[K(P,Q) - \log B + A - 1 \right] \| f' l - f \|_\infty, \tag{1.3.1}$$

where $B \equiv B(\alpha, \beta) = \sqrt{\alpha\beta}$ and $A \equiv A(\alpha, \beta) = \frac{\alpha+\beta}{2}$ are Geometric and Arithmetic mean of α and β respectively, l is the identity function, i.e., $l(x) = x \; \forall \; x \in [\alpha, \beta]$ and

$$\| f' l - f \|_\infty = \sup_{t \in [\alpha, \beta]} |(f' l - f)(t)| < \infty.$$

We note that, several results can be obtained in terms of Kullback Leibler divergence by using inequality (1.3.1) for appropriate convex function f, for example:

For function $\frac{1}{2}\left(1 - \sqrt{t}\right)^2$, we obtain $C_f(P, Q) = h(P, Q)$ (1.2.6), similarly for the functions $\left(\frac{t+1}{2}\right) \log \frac{t+1}{2t}$, $(t-1)^2$, $t \log \frac{2t}{t+1}$, $(t-1)\log t$, $\frac{t+1}{2} \log \frac{t+1}{2\sqrt{t}}$, $\frac{(t-1)^2(t+1)}{t}$ etc., we obtain corresponding divergence measures $G(P, Q)$ (1.2.20), $\chi^2(P, Q)$ (1.2.19), $F(P, Q)$ (1.2.21), $J(P, Q)$ (1.2.10), $T(P, Q)$ (1.2.11), $\psi(P, Q)$ (1.2.9) etc. In this way, we obtain many relations between Csiszar's family member and Kullback Leibler divergence measure.

Dragomir [24] given the following information inequalities as well, which relate Csiszar's family members to the well known Chi- square divergence measure (1.2.19).

Proposition 1.3.2. *Let $f : [\alpha, \beta] \subset (0, \infty) \to R$ be a real, convex differentiable function with $0 < \alpha \leq 1 \leq \beta < \infty$, $\alpha \neq \beta$.*
If $P, Q \in \Gamma_n$ and satisfying the assumption $\alpha \leq \frac{p_i}{q_i} \leq \beta$, $\forall\, i = 1, 2, 3, ..., n$, then we have the following inequalities

$$0 \leq B_f(\alpha, \beta) - C_f(P, Q) \leq \frac{f'(\beta) - f'(\alpha)}{\beta - \alpha}\left[(\beta - 1)(1 - \alpha) - \chi^2(P, Q)\right] \leq A_f(\alpha, \beta), \tag{1.3.2}$$

where

$$B_f(\alpha, \beta) = \frac{(\beta - 1) f(\alpha) + (1 - \alpha) f(\beta)}{\beta - \alpha},$$

$$A_f(\alpha, \beta) = \frac{1}{4}(\beta - \alpha)\left[f'(\beta) - f'(\alpha)\right].$$

So, by defining suitable convex function like defined for inequality (1.3.1), we obtain relations between a particular divergence of Csiszar's family and Chi-square divergence.

Apart from all above, now we are giving some definitions. We start with well known Jensen's inequality. Jensen [51] introduced and characterized the following fundamental inequality which are very useful in statistics and probability theory.

Definition 1.3.1. *(Jensen inequality): Let $f : I \subset R \to R$ be differentiable convex on I^0 (I^0 is the interior of the interval I), $t_i \in I^0, \lambda_i > 0\ \forall\, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} \lambda_i = 1$, then we have the following inequality.*

$$f\left(\sum_{i=1}^{n} \lambda_i t_i\right) \leq \sum_{i=1}^{n} \lambda_i f(t_i). \tag{1.3.3}$$

If function is concave, then Jensen's inequality will be reversed.

Corollary 1.3.1. *We obtain the Propositions 1.2.1 and 1.2.4 after replacing λ_i with q_i as $\sum_{i=1}^{n} q_i = 1$ and t_i with $\frac{p_i}{q_i}$ and $\frac{p_i + q_i}{2q_i}$ respectively for each $i = 1, ..., n$ in Jensen's inequality, by assuming that the function is normalized, i.e., $f(1) = 0$.*

Definition 1.3.2. *(Absolutely continuous) Let $I \subset R$ be an interval. A function $f : I \to R$ is said to be absolutely continuous on I if for every $\epsilon > 0 \; \exists \; \delta > 0$ such that*

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \leq \epsilon$$

for every finite number of non overlapping intervals $(a_i, b_i), i = 1, 2..., n$ with $[a_i, b_i] \subset I$ and

$$\sum_{i=1}^{n} |(b_i - a_i)| \leq \delta.$$

Further f is said to be locally absolutely continuous if it is absolutely continuous in $[a, b]$ for every interval $[a, b] \subset I$.

Definition 1.3.3. *(Total variation and Bounded variation) The total variation of a real valued or more generally complex valued function f, defined on an interval $[a, b] \subset R$ is the quantity*

$$A_a^b(f) = \sup_{P \in B} \sum_{i=0}^{np-1} |f(x_{i+1}) - f(x_i)|,$$

where the supremum is taken over the set $B = \{P = (x_0, ..., x_{n_p}) : P \text{ is a partition of } [a, b]\}$ of all partitions of the interval considered. Further, if f is differentiable and its derivative is Riemann- integrable, its total variation is the vertical component of the arc length of its graph, i.e.,

$$A_a^b(f) = \int_a^b |f'(x)| \, dx.$$

Now, a real valued function f on the real line is said to be of bounded variation on a chosen interval $[a, b] \subset R$ if its total variation is finite, i.e.,

$$A_a^b(f) = \int_a^b |f'(x)| \, dx < \infty.$$

Now, we give a brief idea of the chapters of this work. Chapter 2 introduces several new information inequalities on new generalized divergence in different aspects together with their applications in obtaining new relations and bounds together with numerical verification. Chapter 3 introduces new divergence measures of Csiszar's class, their bounds by using existing information inequalities and their applications. Chapter 4 introduces and characterize new series of divergences, intra relations and their applications. Chapter 5 introduces several important and interesting relations among several new divergences and several well known divergences by helping out some algebraic and exponential inequalities. Chapter 6 introduces new generalized divergence for comparing finite number of discrete probability distributions.

Lastly further scope of the work, references and candidate's academic and research profile.

2

NEW INFORMATION INEQUALITIES AND APPLICATIONS

2.1 Introduction

Information inequalities play a very important role in information and probability theory. Such inequalities are for instance needed in order to calculate the relative efficiency of two or more divergences. Most of the achievable limits are thus stated in the form of inequalities involving fundamental measures of information such as: entropy and information divergence measures.

Ali- Silvey [2] and Csiszar's [20] introduced the generalized divergence measure, given by

$$C_f\left(P,Q\right)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right). \tag{2.1.1}$$

Similarly, Jain and Saraswat [48] introduced new generalized divergence, given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right), \tag{2.1.2}$$

where $f : (0, \infty) \to R$ is real, continuous, and convex function and $P = (p_1, p_2, ..., p_n), Q = (q_1, q_2, ..., q_n) \in \Gamma_n$ (Discrete probability distributions). Many divergence measures can be obtained from this generalized measure by suitably defining the function.

In this chapter, we obtain various new information inequalities on $S_f(P,Q)$. The complete chapter is organized as follows: After this introduction section 2.1, we introduce the new information inequalities in section 2.2 and get the bounds of the new divergence measure. In section 2.3 , 2.4, and 2.5 new inequalities in terms of the Chi- square divergence, Variational distance and the Unified Relative Jensen- Shannon and Arithmetic- Geometric divergence measure of type $'s'$ are introduced respectively, with applications. Further, new information inequalities on absolute functions are obtained in section 2.6 and lastly section 2.7 introduces new information inequalities in a different manner on $S_f(P,Q)$ by using Ostrowski's inequalities. Section 2.8 concludes the whole chapter.

2.2 On New Generalized Divergence Measure and Applications

In this section, we obtain upper and lower bounds of a new non- symmetric divergence measure in terms of the well known divergence measures $\Delta(P,Q)$, $\chi^2(P,Q)$, $J_R(P,Q)$, $G(P,Q)$, and $F(P,Q)$ by using new information inequalities on $S_f(P,Q)$.

2.2.1 New information inequalities

The following theorem or inequalities are introduced which relate $S_f(P,Q)$ for two different convex functions. The results are on similar lines to the results presented by Taneja [95].

Theorem 2.2.1. *Let $f_1, f_2 : I \subset (0,\infty) \to R$ be two convex and normalized functions, i.e., $f_1''(t), f_2''(t) \geq 0\ \forall\, t > 0$ and $f_1(1) = f_2(1) = 0$ respectively and suppose the following assumptions.*
(i) f_1 and f_2 are twice differentiable on (α,β), $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$.
(ii) There exists the real constants m, M such that $m < M$ and

$$m \leq \frac{f_1''(t)}{f_2''(t)} \leq M,\ f_2''(t) \neq 0\ \forall\, t \in (\alpha,\beta). \tag{2.2.1}$$

If $P, Q \in \Gamma_n$, then we have the following inequalities

$$mS_{f_2}(P,Q) \leq S_{f_1}(P,Q) \leq MS_{f_2}(P,Q). \tag{2.2.2}$$

Proof:Let us consider two functions

$$F_m(t) = f_1(t) - mf_2(t) \tag{2.2.3}$$

and

$$F_M(t) = Mf_2(t) - f_1(t), \tag{2.2.4}$$

where m and M are the minimum and maximum values of the function $\frac{f_1''(t)}{f_2''(t)}$ $\forall\, t \in (\alpha, \beta)$.

Since

$$f_1(1) = f_2(1) = 0 \Rightarrow F_m(1) = F_M(1) = 0 \tag{2.2.5}$$

and the functions $f_1(t)$ and $f_2(t)$ are twice differentiable. Then in view of (2.2.1), we have

$$F_m''(t) = f_1''(t) - mf_2''(t) = f_2''(t)\left[\frac{f_1''(t)}{f_2''(t)} - m\right] \geq 0 \tag{2.2.6}$$

and

$$F_M''(t) = Mf_2''(t) - f_1''(t) = f_2''(t)\left[M - \frac{f_1''(t)}{f_2''(t)}\right] \geq 0. \tag{2.2.7}$$

In view (2.2.5), (2.2.6) and (2.2.7), we can say that the functions $F_m(t)$ and $F_M(t)$ are convex and normalized on (α, β).

Now, with the help of linearity property and non- negativity, we have

$$S_{F_m}(P,Q) = S_{f_1 - mf_2}(P,Q) = S_{f_1}(P,Q) - mS_{f_2}(P,Q) \geq 0 \tag{2.2.8}$$

and

$$S_{F_M}(P,Q) = S_{Mf_2 - f_1}(P,Q) = MS_{f_2}(P,Q) - S_{f_1}(P,Q) \geq 0. \tag{2.2.9}$$

From (2.2.8) and (2.2.9), we get the result (2.2.2).

2.2.2 New divergence measure and properties

Divergence measures are basically measures of distance between two probability distributions or compare two probability distributions. Depending on the nature of the problem, different divergence measures are suitable. So it is always desirable to develop a new divergence measure.

Let $f : (0, \infty) \to R$ be a mapping, defined as

$$f(t) = f_1(t) = \frac{(t-1)^2}{\sqrt{t}}, \forall\, t \in (0, \infty). \tag{2.2.10}$$

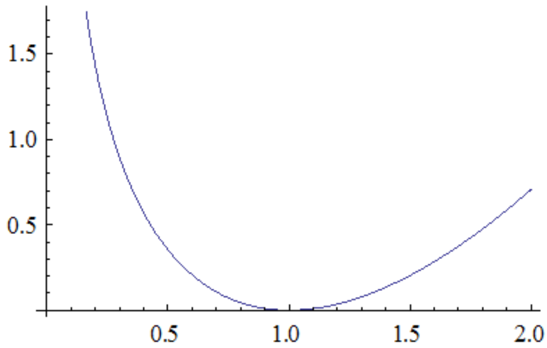

Figure 2.1: Convex function $f_1(t)$

For this function, we obtain

$$S_f(P, Q) = S_{f_1}(P, Q) = L^*(P, Q) = \frac{1}{2}\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{\sqrt{2q_i(p_i + q_i)}} \tag{2.2.11}$$

and

$$f_1'(t) = \frac{(t-1)(3t+1)}{2t^{\frac{3}{2}}}, f_1''(t) = \frac{3t^2 + 2t + 3}{4t^{\frac{5}{2}}}. \tag{2.2.12}$$

Since $f_1''(t) > 0$ and $f_1(1) = 0$, therefore $f_1(t)$ is strictly convex and normalized respectively.

Moreover by the properties of $S_f(P, Q)$, we see that $L^*(P, Q) > 0$ and convex in the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$ and $L^*(P, Q) = 0$ (Non-degeneracy) if $P = Q$ or attains its minimum value when $p_i = q_i$. We can also see that $L^*(P, Q)$ is non- symmetric divergence w.r.t. P and Q as $L^*(P, Q) \neq L^*(Q, P)$.

2.2.3 Bounds of new divergence measure

Now, we evaluate bounds of $L^*(P,Q)$ in terms of other symmetric and non-symmetric divergence measures by using new information inequalities (2.2.2).

Proposition 2.2.1. *For $P,Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*

$$\frac{\sqrt{\alpha}\left(3\alpha^2+2\alpha+3\right)}{16}\Delta(P,Q) \leq L^*(P,Q) \leq \frac{\sqrt{\beta}\left(3\beta^2+2\beta+3\right)}{16}\Delta(P,Q), \tag{2.2.13}$$

where $\Delta(P,Q)$ is defined by (1.2.5).

Proof: Let us consider

$$f_2(t) = \frac{(t-1)^2}{t}, t \in (0,\infty)$$

and

$$f_2'(t) = \frac{t^2-1}{t^2}, f_2''(t) = \frac{2}{t^3}. \tag{2.2.14}$$

Since $f_2''(t) > 0 \; \forall \, t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we get

$$S_{f_2}(P,Q) = \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{2}\Delta(P,Q). \tag{2.2.15}$$

Now, let $g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{\sqrt{t}\left(3t^2+2t+3\right)}{8}$ and $g'(t) = \frac{3\left(5t^2+2t+1\right)}{16\sqrt{t}} > 0 \; \forall \, t > 0$.

It is clear that $g(t)$ is always strictly increasing in $(0,\infty)$, so

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{\sqrt{\alpha}\left(3\alpha^2+2\alpha+3\right)}{8}. \tag{2.2.16}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = g(\beta) = \frac{\sqrt{\beta}\left(3\beta^2+2\beta+3\right)}{8}. \tag{2.2.17}$$

The result (2.2.13) is obtained by using (2.2.11), (2.2.15), (2.2.16),and (2.2.17) in (2.2.2).

Proposition 2.2.2. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*

$$\frac{3\beta^2 + 2\beta + 3}{32\beta^{\frac{5}{2}}}\chi^2(P,Q) \leq L^*(P,Q) \leq \frac{3\alpha^2 + 2\alpha + 3}{32\alpha^{\frac{5}{2}}}\chi^2(P,Q), \tag{2.2.18}$$

where $\chi^2(P,Q)$ is defined by (1.2.19).

Proof: Let us consider

$$f_2(t) = (t-1)^2, t \in (0,\infty)$$

and

$$f_2'(t) = 2(t-1), f_2''(t) = 2. \tag{2.2.19}$$

Since $f_2''(t) > 0 \; \forall \; t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we have

$$S_{f_2}(P,Q) = \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i - q_i)^2}{q_i} = \frac{1}{4}\chi^2(P,Q). \tag{2.2.20}$$

Now, let $g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{3t^2+2t+3}{8t^{\frac{5}{2}}}$ and $g'(t) = -\frac{3\left(t^2+2t+5\right)}{16t^{\frac{7}{2}}} < 0 \; \forall \; t > 0$.
It is clear that $g(t)$ is always strictly decreasing in $(0,\infty)$, so

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(\beta) = \frac{3\beta^2 + 2\beta + 3}{8\beta^{\frac{5}{2}}}. \tag{2.2.21}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{3\alpha^2 + 2\alpha + 3}{8\alpha^{\frac{5}{2}}}. \tag{2.2.22}$$

The result (2.2.18) is obtained by using (2.2.11), (2.2.20), (2.2.21),and (2.2.22) in (2.2.2).

Proposition 2.2.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have* *(i) If $0 < \alpha < 1$, then*

$$\frac{1}{2} J_R(P,Q) \leq L^*(P,Q) \leq \frac{1}{8} max\left[\frac{3\alpha^2+2\alpha+3}{\alpha^{\frac{1}{2}}(1+\alpha)}, \frac{3\beta^2+2\beta+3}{\beta^{\frac{1}{2}}(1+\beta)}\right] J_R(P,Q). \tag{2.2.23}$$

(ii) If $\alpha = 1$, then

$$\frac{1}{2} J_R(P,Q) \leq L^*(P,Q) \leq \frac{3\beta^2+2\beta+3}{8\beta^{\frac{1}{2}}(1+\beta)} J_R(P,Q), \tag{2.2.24}$$

where $J_R(P,Q)$ is defined by (1.2.22).

Proof: Let us consider

$$f_2(t) = (t-1)\log t, t \in (0,\infty)$$

and

$$f_2'(t) = \frac{(t-1)}{t} + \log t, f_2''(t) = \frac{1+t}{t^2}. \tag{2.2.25}$$

Since $f_2''(t) > 0 \; \forall \; t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we obtain

$$S_{f_2}(P,Q) = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) = \frac{1}{2} J_R(P,Q). \tag{2.2.26}$$

Now, let $g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{3t^2+2t+3}{4t^{\frac{1}{2}}(1+t)}$ and $g'(t) = \frac{(3t+1)(t-1)(t+3)}{8t^{\frac{3}{2}}(1+t)^2}$, $g''(t) = \frac{\left(-3t^4-12t^3+42t^2+28t+9\right)}{16t^{\frac{5}{2}}(1+t)^3}$.
If $g'(t) = 0 \Rightarrow t = 1, -3, -\frac{1}{3}$.
It is clear that $g'(t) < 0$ in $(0,1)$ and ≥ 0 in $[1,\infty)$, i.e., $g(t)$ is strictly decreasing in $(0,1)$ and increasing in $[1,\infty)$. So $g(t)$ has a minimum value at $t = 1$ because $g''(1) = \frac{1}{2} > 0$. So

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = 1. \tag{2.2.27}$$

(i) If $0 < \alpha < 1$, then

$$M = \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] = max\left[\frac{3\alpha^2+2\alpha+3}{4\alpha^{\frac{1}{2}}(1+\alpha)}, \frac{3\beta^2+2\beta+3}{4\beta^{\frac{1}{2}}(1+\beta)}\right]. \tag{2.2.28}$$

(ii) If $\alpha = 1$, then

$$M = \sup_{t\in(1,\beta)} g(t) = g(\beta) = \frac{3\beta^2+2\beta+3}{4\beta^{\frac{1}{2}}(1+\beta)}. \tag{2.2.29}$$

The inequalities (2.2.23) and (2.2.24) are obtained by using (2.2.11), (2.2.26), (2.2.27), (2.2.28) and (2.2.29) in (2.2.2).

Proposition 2.2.4. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*
(i) If $1 < \beta \leq 2.09$, then

$$1.678G(Q,P) \leq L^*(P,Q) \leq \frac{3\alpha^2+2\alpha+3}{4\alpha^{\frac{3}{2}}}G(Q,P). \tag{2.2.30}$$

(ii) If $\beta > 2.09$, then

$$\begin{aligned} &min\left[\frac{3\alpha^2+2\alpha+3}{4\alpha^{\frac{3}{2}}}, \frac{3\beta^2+2\beta+3}{4\beta^{\frac{3}{2}}}\right] G(Q,P) \leq L^*(P,Q) \\ &\leq max\left[\frac{3\alpha^2+2\alpha+3}{4\alpha^{\frac{3}{2}}}, \frac{3\beta^2+2\beta+3}{4\beta^{\frac{3}{2}}}\right] G(Q,P), \end{aligned} \tag{2.2.31}$$

where $G(P,Q)$ is defined by (1.2.20).

Proof: Let us consider

$$f_2(t) = t\log t, t \in (0,\infty)$$

and

$$f_2'(t) = 1 + \log t, f_2''(t) = \frac{1}{t}. \tag{2.2.32}$$

Since $f_2''(t) > 0\ \forall\ t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we have

$$S_{f_2}(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \log \frac{p_i + q_i}{2q_i} = G(Q,P). \tag{2.2.33}$$

Now, let $g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{3t^2+2t+3}{4t^{\frac{3}{2}}}$ and $g'(t) = \frac{\left(3t^2-2t-9\right)}{8t^{\frac{5}{2}}}$, $g''(t) = \frac{3\left(-t^2+2t+15\right)}{16t^{\frac{7}{2}}}$.

If $g'(t) = 0 \Rightarrow t = 2.09, -1.430$.

It is clear that $g'(t) < 0$ in $(0, 2.09)$ and > 0 in $(2.09, \infty)$, i.e., $g(t)$ is strictly decreasing in $(0, 2.09)$ and strictly increasing in $(2.09, \infty)$. So $g(t)$ has a minimum value at $t = 2.09$ because $g''(2.09) = 0.210 > 0$. So

(i) If $1 < \beta \leq 2.09$, then

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(2.09) = 1.678. \tag{2.2.34}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{3\alpha^2 + 2\alpha + 3}{4\alpha^{\frac{3}{2}}}. \tag{2.2.35}$$

(ii) If $\beta > 2.09$, then

$$m = \inf_{t\in(\alpha,\beta)} g(t) = min\left[g(\alpha), g(\beta)\right] = min\left[\frac{3\alpha^2 + 2\alpha + 3}{4\alpha^{\frac{3}{2}}}, \frac{3\beta^2 + 2\beta + 3}{4\beta^{\frac{3}{2}}}\right]. \tag{2.2.36}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] = max\left[\frac{3\alpha^2 + 2\alpha + 3}{4\alpha^{\frac{3}{2}}}, \frac{3\beta^2 + 2\beta + 3}{4\beta^{\frac{3}{2}}}\right]. \tag{2.2.37}$$

The inequalities (2.2.30) and (2.2.31) are obtained by using (2.2.11), (2.2.33), (2.2.34), (2.2.35), (2.2.36) and (2.2.37) in (2.2.2).

By the similar approach, we obtain the bounds of $L^*(P,Q)$ in terms of divergence measure $F(P,Q)$. The result is as follows:

For $f_2(t) = -\log t$, we obtain

(i) If $0 < \alpha \leq 0.47$, then

$$1.678F(Q,P) \leq L^*(P,Q) \leq max\left[\frac{3\alpha^2+2\alpha+3}{4\alpha^{\frac{1}{2}}}, \frac{3\beta^2+2\beta+3}{4\beta^{\frac{1}{2}}}\right]F(Q,P). \tag{2.2.38}$$

(ii) If $0.47 < \alpha \leq 1$, then

$$\frac{3\alpha^2+2\alpha+3}{4\alpha^{\frac{1}{2}}}F(Q,P) \leq L^*(P,Q) \leq \frac{3\beta^2+2\beta+3}{4\beta^{\frac{1}{2}}}F(Q,P), \tag{2.2.39}$$

where $F(P,Q)$ is defined by (1.2.21).

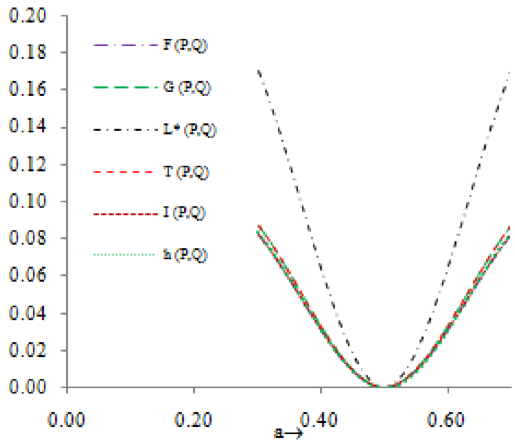

Figure 2.2: Comparison of the well known divergences with $L^*(P,Q)$

Figure 2.2 shows the behavior of $L^*(P,Q)$, $F(P,Q)$, $G(P,Q)$, $T(P,Q)$, $I(P,Q)$, and $h(P,Q)$. We have considered $p_i = (a, 1-a)$, $q_i = (1-a, a)$, where $a \in (0,1)$. It is clear from Figure that the $L^*(P,Q)$ has a steeper slope than all others.

2.3 New Information Inequalities in Terms of Chi- square Divergence and Applications

In this section, new information inequality is introduced and characterized on $S_f(P,Q)$ in terms of the well known Chi- square divergence and this inequality is taken for evaluating the relations among some standard divergences with the Chi-square divergence. Numerical verifications of the obtained relations are done as well by considering two discrete probability distributions: Binomial and Poisson. Now the following lemma is important for proving the upcoming new information inequality. This lemma has been obtained from literature (Dragomir etc. all [30]).

Lemma 2.3.1. *Let $\psi : [a,b] \subset R \to R$ be an absolutely continuous and differentiable function, and there exists the constants $m, M \in R$, such that*

$$m \leq \psi'(t) \leq M \; \forall \, t \in [a,b].$$

Then, we have

$$\left| \frac{\psi(a)+\psi(b)}{2} - \frac{1}{b-a}\int_a^b \psi(t)\,dt \right| \leq \frac{1}{8}(b-a)(M-m). \tag{2.3.1}$$

Proof: We start with the following identity that is obvious by using integration by parts.

$$\frac{\psi(a)+\psi(b)}{2} - \frac{1}{b-a}\int_a^b \psi(t)\,dt = \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)\psi'(t)\,dt.$$

We observe that

$$\frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)\psi'(t)\,dt = \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)\left(\psi'(t) - \frac{m+M}{2}\right)dt$$

and since

$$\left|\psi'(t) - \frac{m+M}{2}\right| \leq \frac{M-m}{2} \; \forall \, t \in [a,b].$$

So we deduce that

$$\frac{1}{b-a}\left|\int_a^b\left(t-\frac{a+b}{2}\right)\left(\psi'(t)-\frac{m+M}{2}\right)dt\right| \leq \frac{1}{b-a}\frac{M-m}{2}\int_a^b\left|t-\frac{a+b}{2}\right|dt$$
$$= \frac{M-m}{8}(b-a).$$

2.3.1 New information inequalities

Now, we introduce new information inequality in terms of the well known Chi-square divergence measure. The results are on similar lines to the results presented by (Dragomir etc. all [30]).

Theorem 2.3.1. *Let $f:(0,\infty)\to R$ be a mapping which is normalized, i.e., $f(1)=0$ and f' is locally absolutely continuous on $[\alpha,\beta]\subset(0,\infty)$ then there exists the constants $m, M\in R$ with $m<M$, such that*

$$m \leq f''(t) \leq M \;\forall\, t\in(\alpha,\beta).$$

If $P, Q\in\Gamma_n$ and $0<\alpha\leq 1\leq\beta<\infty$ with $\alpha\neq\beta$, then we have the following inequality

$$\left|S_f(P,Q)-\frac{1}{2}E_{S_{f'}}(P,Q)\right| \leq \frac{1}{32}(M-m)\chi^2(P,Q), \tag{2.3.2}$$

where $E_{S_{f'}}(P,Q)$ is defined by (1.2.52).

Proof:Put $\psi(t)=f'(t), b=t\in(\alpha,\beta)$ and $a=1$ in (2.3.1), we get

$$\left|\frac{f'(1)+f'(t)}{2}-\frac{1}{t-1}\int_1^t f'(t)\,dt\right| \leq \frac{1}{8}(t-1)(M-m).$$

Or

$$\left|f(t)-\frac{1}{2}(t-1)\left[f'(t)+f'(1)\right]\right| \leq \frac{1}{8}(t-1)^2(M-m). \tag{2.3.3}$$

Now put $t=\frac{p_i+q_i}{2q_i}, i=1,2,3...,n$ in (2.3.3), we obtain

$$\left|f\left(\frac{p_i+q_i}{2q_i}\right)-\frac{(p_i-q_i)}{4q_i}\left[f'\left(\frac{p_i+q_i}{2q_i}\right)+f'(1)\right]\right| \leq \frac{1}{32}(M-m)\frac{(p_i-q_i)^2}{q_i^2}.$$

Now multiply the above expressions by q_i and sum over all $i = 1, 2, 3..., n$ by taking into account $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, we obtain

$$\left| \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right) - \frac{1}{2} \sum_{i=1}^{n} \frac{(p_i - q_i)}{2} f'\left(\frac{p_i + q_i}{2q_i}\right) \right| \leq \frac{1}{32}(M - m) \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}.$$

Or

$$\left| S_f(P,Q) - \frac{1}{2} E_{S_{f'}}(P,Q) \right| \leq \frac{1}{32}(M - m)\chi^2(P,Q).$$

Hence prove the inequality (2.3.2).

2.3.2 Application of new information inequalities

Now, we obtain relations among standard divergence measures: Relative JS divergence, Relative AG divergence, Relative J- divergence, and Triangular discrimination with Chi- square divergence by using new inequalities (2.3.2) (taking only convex functions here).

Proposition 2.3.1. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, we have*

$$|G(P,Q) - F(P,Q)| \leq \frac{1}{16}\left(\frac{\beta - \alpha}{\alpha\beta}\right)\chi^2(Q,P), \tag{2.3.4}$$

where $\chi^2(P,Q)$, $G(P,Q)$, and $F(P,Q)$ are already defined by (1.2.19), (1.2.20), and (1.2.21) respectively.

Proof: Let us consider

$$f(t) = t\log t, t > 0, f(1) = 0, f'(t) = 1 + \log t \text{ and } f''(t) = \frac{1}{t}.$$

Since $f''(t) > 0 \;\forall\; t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we get the followings.

$$S_f(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i+q_i}{2}\right) \log \frac{p_i+q_i}{2q_i} = G(Q,P). \tag{2.3.5}$$

$$\begin{aligned}
E_{S_{f'}}(P,Q) =& \sum_{i=1}^{n} \left(\frac{p_i-q_i}{2}\right) \left[1+\log \frac{p_i+q_i}{2q_i}\right] = \sum_{i=1}^{n} \left(\frac{p_i-q_i}{2}\right) \log \frac{p_i+q_i}{2q_i} \\
=& \sum_{i=1}^{n} \left(\frac{q_i-p_i}{2}\right) \log \frac{2q_i}{p_i+q_i} = \sum_{i=1}^{n} \left(q_i - \frac{p_i+q_i}{2}\right) \log \frac{2q_i}{p_i+q_i} \\
=& \sum_{i=1}^{n} \left[q_i \log \frac{2q_i}{p_i+q_i} - \left(\frac{p_i+q_i}{2}\right) \log \frac{2q_i}{p_i+q_i}\right] \\
=& \sum_{i=1}^{n} \left[q_i \log \frac{2q_i}{p_i+q_i} + \left(\frac{p_i+q_i}{2}\right) \log \frac{p_i+q_i}{2q_i}\right] \\
=& F(Q,P) + G(Q,P).
\end{aligned} \tag{2.3.6}$$

Now, let $g(t) = f''(t) = \frac{1}{t}$ and $g'(t) = -\frac{1}{t^2} < 0$.

It is clear that $g(t)$ is always strictly decreasing in $(0,\infty)$, so

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(\beta) = \frac{1}{\beta}. \tag{2.3.7}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{1}{\alpha}. \tag{2.3.8}$$

The result (2.3.4) is obtained by using (2.3.5), (2.3.6), (2.3.7) and (2.3.8) in (2.3.2), after interchanging P and Q.

Proposition 2.3.2. *For $P,Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$, we have*

$$|J_R(P,Q) - \Delta(P,Q)| \le \frac{1}{8} \frac{(\beta-\alpha)(\alpha+\beta+\alpha\beta)}{\alpha^2\beta^2} \chi^2(P,Q), \tag{2.3.9}$$

where $\Delta(P,Q)$, $\chi^2(P,Q)$, and $J_R(P,Q)$ are defined by (1.2.5), (1.2.19), and (1.2.22) respectively.

Proof: Let us consider

$$f(t) = (t-1)\log t, t > 0, f(1) = 0, f'(t) = \frac{t-1}{t} + \log t \text{ and } f''(t) = \frac{1+t}{t^2}.$$

Since $f''(t) > 0 \; \forall \; t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we get the followings.

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) = \frac{1}{2}J_R(P,Q). \tag{2.3.10}$$

$$\begin{aligned} E_{S_{f'}}(P,Q) =& \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) + \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} \\ =& \frac{1}{2}\left[J_R(P,Q) + \Delta(P,Q)\right]. \end{aligned} \tag{2.3.11}$$

Now, let $g(t) = f''(t) = \frac{1+t}{t^2}$ and $g'(t) = -\frac{2+t}{t^3} < 0$.

It is clear that $g(t)$ is always strictly decreasing in $(0, \infty)$, so

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(\beta) = \frac{1+\beta}{\beta^2}. \tag{2.3.12}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{1+\alpha}{\alpha^2}. \tag{2.3.13}$$

The result (2.3.9) is obtained by using (2.3.10), (2.3.11), (2.3.12) and (2.3.13) in (2.3.2).

2.3.3 Numerical verification of obtained results

By using new information inequalities (2.3.2), relations among well known divergences have been obtained mathematically with Chi- square divergence in last subsection. Now, in this subsection, we give an example for calculating the divergences $G(P,Q)$, $F(P,Q)$, $\Delta(P,Q)$, $J_R(P,Q)$, $\chi^2(P,Q)$, $\chi^2(Q,P)$ and verify the inequalities (2.3.4) and (2.3.9), numerically.

Example 2.3.1. *Let P be the Binomial probability distribution with parameters $(n = 10, p = 0.7)$ and Q its approximated Poisson probability distribution with parameter $(\lambda = np = 7)$ for the random variable X , then we have*

x_i	0	1	2	3	4	5	6	7	8	9	10
$p_i \approx$	.0000059	.000137	.00144	.009	.036	.102	.200	.266	.233	.121	.0282
$q_i \approx$	.000911	.00638	.022	.052	.091	.177	.199	.149	.130	.101	.0709
$\frac{p_i+q_i}{2q_i} \approx$	.503	.510	.532	.586	.697	.788	1.002	1.392	1.396	1.099	.698

Table 2.1: Evaluation of Binomial and Poisson probability distributions

By using Table 2.1, we obtain the followings.

$$\alpha\,(= .503) \leq \frac{p_i + q_i}{2q_i} \leq \beta\,(= 1.396)\,. \tag{2.3.14}$$

$$G\,(P, Q) = \sum_{i=1}^{11} \left(\frac{p_i + q_i}{2}\right) \log\left(\frac{p_i + q_i}{2p_i}\right) \approx .0746. \tag{2.3.15}$$

$$F\,(P, Q) = \sum_{i=1}^{11} p_i \log\left(\frac{2p_i}{p_i + q_i}\right) \approx .0842. \tag{2.3.16}$$

$$\Delta\,(P, Q) = \sum_{i=1}^{11} \frac{(p_i - q_i)^2}{p_i + q_i} \approx .1812. \tag{2.3.17}$$

$$J_R\,(P, Q) = \sum_{i=1}^{11} (p_i - q_i) \log\left(\frac{p_i + q_i}{2q_i}\right) \approx .1686. \tag{2.3.18}$$

$$\chi^2\,(P, Q) = \sum_{i=1}^{11} \frac{(p_i - q_i)^2}{q_i} \approx .3298. \tag{2.3.19}$$

$$\chi^2\,(Q, P) = \sum_{i=1}^{11} \frac{(p_i - q_i)^2}{p_i} \approx 1.2260. \tag{2.3.20}$$

Put all approximated numerical values from (2.3.14) to (2.3.20) in results (2.3.4) and (2.3.9), we obtain the followings numerical results

$$.0096 \leq .0973 \text{ and } .0126 \leq .1942$$

respectively. Hence verified the new inequalities numerically for $p = 0.7$.

Remark 2.3.1. *In a similar manner, we can verify the inequalities for different values of p and q and for other discrete probability distributions as well, like; Negative binomial, Geometric, Uniform etc.*

2.4 New Information Inequalities in Terms of Variational Distance and Applications

In previous section, new information inequality has been obtained in terms of $\chi^2(P, Q)$. Now, in this section, new information inequality is introduced and characterized on $S_f(P, Q)$ in terms of the well known Variational or l_1 distance $V(P, Q)$ and this inequality is taken for evaluating the relations among standard divergences with the Variational distance.

Let us begin with an important proposition for proving the new information inequality. This proposition with proof can be seen in literature (Dragomir etc. all [29]).

Proposition 2.4.1. *Let $\psi : [a, b] \subset R \to R$ be a differentiable function and is of bounded variation on $[a, b]$, i.e., $A_a^b(\psi) = \int_a^b |\psi'(t)|\, dt < \infty$. Then for all $u \in [a, b]$, we have*

$$\left| \int_a^b \psi(t)\, dt - \psi(u)(b-a) \right| \le \left(\frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right) A_a^b(\psi). \tag{2.4.1}$$

Now for all $u_1, u_2 \in [a, b]$, if we put $u = u_i$ and summing over i, we get the following

$$\left| \int_a^b \psi(t)\, dt - \left(\frac{b-a}{2} \right) \sum_{i=1}^{2} \psi(u_i) \right| \le \left(\frac{b-a}{2} + \frac{1}{2} \sum_{i=1}^{2} \left| u_i - \frac{a+b}{2} \right| \right) A_a^b(\psi). \tag{2.4.2}$$

2.4.1 New information inequalities

Now we introduce new information inequality on $S_f(P,Q)$ in terms of well known Variational distance. The results are on similar lines to the results presented by (Dragomir etc. all [29]).

Theorem 2.4.1. *Let $f : [\alpha,\beta] \subset (0,\infty) \to R$ be a twice differentiable function which is normalized, i.e., $f(1) = 0$ and f' is of bounded variation on $[\alpha,\beta]$, i.e., $A_\alpha^\beta(f') = \int_\alpha^\beta |f''(t)|\,dt < \infty$.*
If $P,Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty$ with $\alpha \neq \beta$, then we have the following inequality

$$\left| S_f(P,Q) - \frac{1}{2}E_{S_{f'}}(P,Q) \right| \le \frac{1}{2}V(P,Q)\,A_\alpha^\beta(f'), \tag{2.4.3}$$

where $E_{S_{f'}}(P,Q)$ is defined by (1.2.52).

Proof:

Case I (for $1 \le u$): Put $\psi = f', u_1 = a = 1$, and $u_2 = b = u \in [\alpha,\beta]$ in (2.4.2), we obtain

$$\begin{aligned}&\left| \int_1^u f'(t)\,dt - \left(\frac{u-1}{2}\right)(f'(1) + f'(u)) \right| \\ &\le \left[\frac{u-1}{2} + \frac{1}{2}\left(\left|1 - \frac{u+1}{2}\right| + \left|u - \frac{u+1}{2}\right| \right) \right] A_1^u(f').\end{aligned}$$

Or

$$\left| f(u) - f(1) - \left(\frac{u-1}{2}\right)(f'(1) + f'(u)) \right| \le \left(\frac{u-1}{2} + \left|\frac{u-1}{2}\right| \right) A_1^u(f').$$

Or

$$\left| f(u) - \left(\frac{u-1}{2}\right)(f'(1) + f'(u)) \right| \le (u-1)\,A_1^u(f') \le (u-1)\,A_\alpha^\beta(f'). \tag{2.4.4}$$

Case II (for $u < 1$): Put $\psi = f', u_1 = a = u \in [\alpha, \beta]$, and $u_2 = b = 1$ in (2.4.2), we get similarly

$$\left| -f(u) - \left(\frac{1-u}{2}\right)(f'(1) + f'(u)) \right| \leq (1-u) A_u^1(f') \leq (1-u) A_\alpha^\beta(f').$$

Or

$$\left| f(u) - \left(\frac{u-1}{2}\right)(f'(1) + f'(u)) \right| \leq (1-u) A_u^1(f') \leq (1-u) A_\alpha^\beta(f'). \quad (2.4.5)$$

By using together (2.4.4) and (2.4.5), we obtain respectively

$$\left| f(u) - \left(\frac{u-1}{2}\right)(f'(1) + f'(u)) \right| \leq |u-1| A_\alpha^\beta(f'). \quad (2.4.6)$$

Now put $u = \frac{p_i+q_i}{2q_i}, i = 1, 2, 3..., n$ in (2.4.6), we obtain

$$\left| f\left(\frac{p_i+q_i}{2q_i}\right) - \left(\frac{p_i-q_i}{4q_i}\right)\left[f'\left(\frac{p_i+q_i}{2q_i}\right) + f'(1)\right] \right| \leq \left|\frac{p_i-q_i}{2q_i}\right| A_\alpha^\beta(f').$$

Now multiply the above expression by q_i and sum over all $i = 1, 2, 3..., n$ by taking into account $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$, we get the desire result (2.4.3).

2.4.2 Relations with the Variational distance

In this subsection, we obtain relations among the standard divergence measures: $F(P,Q)$, $G(P,Q)$, $J_R(P,Q)$, and $\Delta(P,Q)$ by using new inequality (2.4.3), with $V(P,Q)$ (taking only convex functions here), where $V(P,Q)$ is defined by (1.2.7).

Proposition 2.4.2. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, we have*

$$|G(P,Q) - F(P,Q)| \leq \log\left(\frac{\beta}{\alpha}\right) V(P,Q). \quad (2.4.7)$$

Proof: Let us consider

$$f(t) = t\log t, t > 0, f(1) = 0, f'(t) = 1 + \log t \text{ and } f''(t) = \frac{1}{t}.$$

Since $f''(t) > 0 \;\forall\; t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we obtain the followings respectively.

$$S_f(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \log \frac{p_i + q_i}{2q_i} = G(Q,P). \tag{2.4.8}$$

$$\begin{aligned} E_{S_{f'}}(P,Q) &= \sum_{i=1}^{n} \left(\frac{p_i - q_i}{2}\right) \left[1 + \log \frac{p_i + q_i}{2q_i}\right] = \sum_{i=1}^{n} \left(\frac{p_i - q_i}{2}\right) \log \frac{p_i + q_i}{2q_i} \\ &= \sum_{i=1}^{n} \left(\frac{q_i - p_i}{2}\right) \log \frac{2q_i}{p_i + q_i} = \sum_{i=1}^{n} \left(q_i - \frac{p_i + q_i}{2}\right) \log \frac{2q_i}{p_i + q_i} \\ &= \sum_{i=1}^{n} \left[q_i \log \frac{2q_i}{p_i + q_i} - \left(\frac{p_i + q_i}{2}\right) \log \frac{2q_i}{p_i + q_i}\right] \\ &= \sum_{i=1}^{n} \left[q_i \log \frac{2q_i}{p_i + q_i} + \left(\frac{p_i + q_i}{2}\right) \log \frac{p_i + q_i}{2q_i}\right] = F(Q,P) + G(Q,P). \end{aligned} \tag{2.4.9}$$

$$A_\alpha^\beta(f') = \int_\alpha^\beta |f''(t)|\, dt = \int_\alpha^\beta \left|\frac{1}{t}\right| dt = \int_\alpha^\beta \frac{1}{t} dt = \log\left(\frac{\beta}{\alpha}\right). \tag{2.4.10}$$

The result (2.4.7) is obtained by using (2.4.8), (2.4.9), and (2.4.10) in (2.4.3), after interchanging P and Q.

Proposition 2.4.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$, we have*

$$|J_R(P,Q) - \Delta(P,Q)| \le 2\left(\frac{\beta - \alpha}{\alpha\beta} + \log\frac{\beta}{\alpha}\right) V(P,Q). \tag{2.4.11}$$

Proof: Let us consider

$$f(t) = (t-1)\log t, t > 0, f(1) = 0, f'(t) = \frac{t-1}{t} + \log t \text{ and } f''(t) = \frac{1+t}{t^2}.$$

Since $f''(t) > 0 \; \forall \; t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we get the followings respectively.

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) = \frac{1}{2}J_R(P,Q). \tag{2.4.12}$$

$$\begin{aligned} E_{S_{f'}}(P,Q) =& \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) + \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} \\ =& \frac{1}{2}\left[J_R(P,Q) + \Delta(P,Q)\right]. \end{aligned} \tag{2.4.13}$$

$$A_\alpha^\beta(f') = \int_\alpha^\beta |f''(t)|\, dt = \int_\alpha^\beta \left|\frac{1+t}{t^2}\right| dt = \int_\alpha^\beta \frac{1+t}{t^2} dt = \frac{\beta-\alpha}{\alpha\beta} + \log\frac{\beta}{\alpha}. \tag{2.4.14}$$

The result (2.4.11) is obtained by using (2.4.12), (2.4.13) and (2.4.14) in (2.4.3).

2.4.3 Numerical verification of obtained results

Now, in this subsection, we give an example for verifying the new results (2.4.7) and (2.4.11), numerically.

Example 2.4.1. *Here, we are considering the example same as example 2.3.1 (subsection- 2.3.3). The observations for Binomial and Poisson distributions are also same as Table 2.1, so we are skipping the repetition. Now, by Table 2.1, we get the value of $V(P,Q)$, as follows.*

$$V(P,Q) = \sum_{i=1}^{11}|p_i - q_i| \approx .4844. \tag{2.4.15}$$

Put the approximated numerical values from (2.3.14) to (2.3.18) and (2.4.15) into (2.4.7) and (2.4.11), we get

$$9.6 \times 10^{-3} \leq .4944 \text{ and } .0126 \leq 2.22098.$$

respectively and hence verified the inequalities (2.4.7) and (2.4.11) for $p = 0.7$.

Remark 2.4.1. *We can verify the obtained new information inequalities (2.4.7) and (2.4.11) numerically in a similar manner like subsection 2.3.3, by considering two different discrete probability distributions for different values of probability of success and probability of failure.*

2.5 New Information Inequalities in Terms of One Parametric Divergence Measure and Applications

In this section, we are going to introduce new information inequalities on $S_f(P,Q)$ in terms of the well known one parametric divergence $\Phi_s(Q,P)$, designated as adjoint of the Unified relative Jensen- Shannon (JS) and Arithmetic- Geometric (AG) divergence measure of type s, where $s \in R - \{0,1\}$ is parameter here. $\Phi_s(P,Q)$ is already defined by (1.2.24) in introduction chapter. Also introduce the new information divergence measure, characterize it and obtain the mathematical relations with other divergences: $F(P,Q)$, $G(P,Q)$, $J_R(P,Q)$, and $\chi^2(P,Q)$. Further we obtain bounds of the new divergence as an application of new information inequalities, together with numerical verification.

2.5.1 New information inequalities

In this part of the section, we introduce two new information inequalities (Theorems 2.5.1 and 2.5.2) on $S_f(P,Q)$; one of them is in terms of one parametric divergence $\Phi_s(Q,P)$. The results are on similar lines to the results presented by Dragomir [24].

Theorem 2.5.1. *Let $f : (0, \infty) \to R$ be a real, convex function on $(\alpha, \beta) \subset (0, \infty)$ with $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$. If $P, Q \in \Gamma_n$, then we have the following inequality*

$$S_f (P, Q) \leq B_f (\alpha, \beta), \tag{2.5.1}$$

where

$$B_f (\alpha, \beta) = \frac{(\beta - 1) f (\alpha) + (1 - \alpha) f (\beta)}{\beta - \alpha}. \tag{2.5.2}$$

Proof: Since f is convex on $(0, \infty)$, therefore we can write the following for $(\alpha, \beta) \in (0, \infty) \times (0, \infty), \lambda \in [0, 1]$ by the definition of convex function

$$f [\lambda\alpha + (1 - \lambda) \beta] \leq \lambda f (\alpha) + (1 - \lambda) f (\beta). \tag{2.5.3}$$

Now assume $\lambda = \frac{\beta - x}{\beta - \alpha}$ for $x \in (\alpha, \beta)$ in (2.5.3), we get

$$f (x) \leq \frac{(\beta - x) f (\alpha) + (x - \alpha) f (\beta)}{\beta - \alpha}. \tag{2.5.4}$$

Now put $x = \frac{p_i + q_i}{2q_i}$ in (2.5.4), multiply by q_i and then sum over all $i = 1, 2, 3, ..., n$, we obtain the require inequality (2.5.1).

Theorem 2.5.2. *Let $f : (0, \infty) \to R$ be a real, convex and twice differentiable function on $(\alpha, \beta) \subset (0, \infty)$ with $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$. If there exists the real constants m, M with $m < M$ and*

$$m \leq t^{2-s} f'' (t) \leq M \; \forall \; t \in (\alpha, \beta), s \in R - \{0, 1\}.$$

If $P, Q \in \Gamma_n$, then we have

$$m [B_{\phi_s} (\alpha, \beta) - \Phi_s (Q, P)] \leq B_f (\alpha, \beta) - S_f (P, Q) \leq M [B_{\phi_s} (\alpha, \beta) - \Phi_s (Q, P)], \tag{2.5.5}$$

where

$$\Phi_s(P,Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i \left(\frac{p_i+q_i}{2p_i}\right)^s - 1\right], s \neq 0,1$$

$$= \begin{cases} \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{4}\Delta(P,Q) = \Phi_{-1}(P,Q) & \textit{if } s=-1 \\ \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i+q_i}\right) = F(P,Q) = \lim_{s\to 0}\Phi_s(P,Q) & \textit{if } s=0 \\ \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2p_i}\right) = G(P,Q) = \lim_{s\to 1}\Phi_s(P,Q) & \textit{if } s=1 \\ \frac{1}{8}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i} = \frac{1}{8}\chi^2(Q,P) = \Phi_2(P,Q) & \textit{if } s=2 \end{cases}.$$

given by (1.2.24) and

$$B_f(\alpha,\beta) = \frac{(\beta-1)f(\alpha)+(1-\alpha)f(\beta)}{\beta-\alpha}. \tag{2.5.6}$$

$$B_{\phi_s}(\alpha,\beta) = \frac{(\beta-1)\phi_s(\alpha)+(1-\alpha)\phi_s(\beta)}{\beta-\alpha}. \tag{2.5.7}$$

Proof: Let us define a function $F_m : (0,\infty) \to R$ as

$$F_m(t) = f(t) - m[s(s-1)]^{-1}(t^s-1) = f(t) - m\phi_s(t), \tag{2.5.8}$$

where

$$\phi_s(t) = [s(s-1)]^{-1}(t^s-1), s \in R - \{0,1\}. \tag{2.5.9}$$

Since $f(t)$ and $\phi_s(t)$ are both twice differentiable, therefore $F_m(t)$ is twice differentiable as well, So

$$F_m''(t) = f''(t) - mt^{s-2} = t^{s-2}\left[t^{2-s}f''(t) - m\right] \geq 0.$$

Since $F_m''(t) \geq 0 \; \forall \; t \in (\alpha,\beta) \subset (0,\infty)$, therefore $F_m(t)$ is convex as well.

Now we write inequality (2.5.1) for the function $F_m(t)$, we obtain

$$S_{F_m}(P,Q) \leq B_{F_m}(\alpha,\beta), \text{ i.e.,}$$

$$\sum_{i=1}^{n} q_i f\left(\frac{p_i+q_i}{2q_i}\right) - m\left[s\left(s-1\right)\right]^{-1} \sum_{i=1}^{n} q_i \left(\frac{p_i+q_i}{2q_i}\right)^{s} - 1$$

$$\leq \frac{\left(\beta-1\right) f\left(\alpha\right)+\left(1-\alpha\right) f\left(\beta\right)}{\beta-\alpha} - m\left[s\left(s-1\right)\right]^{-1} \frac{\left(\beta-1\right)\left(\alpha^{s}-1\right)+\left(1-\alpha\right)\left(\beta^{s}-1\right)}{\beta-\alpha}, \text{ i.e.,}$$

$$S_f\left(P,Q\right) - m\Phi_s\left(Q,P\right) \leq B_f\left(\alpha,\beta\right) - mB_{\phi_s}\left(\alpha,\beta\right), \text{ i.e.,}$$

$$m\left[B_{\phi_s}\left(\alpha,\beta\right) - \Phi_s\left(Q,P\right)\right] \leq B_f\left(\alpha,\beta\right) - S_f\left(P,Q\right).$$

Hence prove the first inequality of (2.5.5).

The second inequality of (2.5.5) obtains by a similar approach for the function

$$F_m\left(t\right) = M\left[s\left(s-1\right)\right]^{-1}\left(t^{s}-1\right) - f\left(t\right).$$

We omit the details.

2.5.2 New information divergence measure, properties and relations

In this subsection, we introduce new information divergence measure of class $S_f\left(P,Q\right)$. Properties and relations of this new divergence with other divergences are also given.

Now, let $f:(0,\infty)\to R$ be a mapping defined as

$$f\left(t\right) = \left(\frac{t+1}{2}\right)\log\left(\frac{t+1}{2t}\right), \tag{2.5.10}$$

$$f'\left(t\right) = \frac{1}{2}\left[\log\left(\frac{t+1}{2t}\right) - \frac{1}{t}\right], f\left(1\right)=0$$

and

$$f''\left(t\right) = \frac{1}{2t^{2}\left(t+1\right)}. \tag{2.5.11}$$

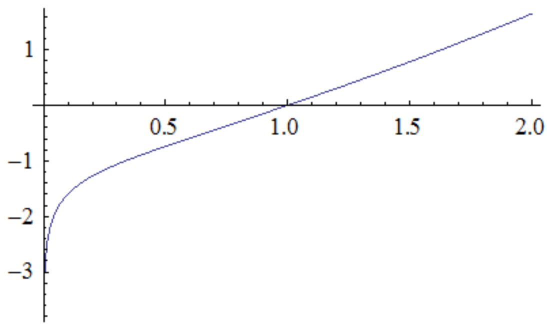

Figure 2.3: Convex function $f(t)$

Since $f''(t) > 0 \; \forall \; t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively. Now for $f(t)$, we have the following new divergence measure

$$S_f(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i + 3q_i}{4} \right) \log \left[\frac{p_i + 3q_i}{2(p_i + q_i)} \right] = M^*(P,Q). \tag{2.5.12}$$

Moreover by properties of $S_f(P,Q)$, we see that $M^*(P,Q) > 0$ and convex in the pair of probability distribution $(P,Q) \in \Gamma_n \times \Gamma_n$ and $M^*(P,Q) = 0$ (Non- degeneracy) if $P = Q$ or attains its minimum value when $p_i = q_i$. We can also see that $M^*(P,Q)$ is non- symmetric divergence w.r.t. P and Q as $M^*(P,Q) \neq M^*(Q,P)$.

Now we are giving the following theorem, statement of which is being used for obtaining upcoming new relation. This theorem with proof can be seen in literature (Jain and Saraswat [48]).

Theorem 2.5.3. *Let $f : (0,\infty) \to R$ be a convex and normalized function, i.e., $f''(t) \geq 0 \; \forall \; t > 0$ and $f(1) = 0$ respectively, then for $(P,Q) \in \Gamma_n \times \Gamma_n$, we have*

$$S_f(P,Q) \leq C_f(P,Q) \leq E_{C_{f'}}(P,Q), \tag{2.5.13}$$

where $S_f(P,Q)$, $C_f(P,Q)$, and $E_{C_{f'}}(P,Q)$ have their usual meanings.

Now, we derive a simple relation (Proposition 2.5.1) for $M^*(P,Q)$ in terms of $F(P,Q), G(P,Q), J_R(P,Q)$, and $\chi^2(P,Q)$.

Proposition 2.5.1. *Let $(P,Q) \in \Gamma_n \times \Gamma_n$, then we have the following new inter relation*

$$\frac{1}{4}\left[F(Q,P) - G(Q,P)\right] \leq M^*(P,Q) \leq \frac{1}{2}\left[J_R(Q,P) + \chi^2(Q,P)\right], \tag{2.5.14}$$

where $F(P,Q), G(P,Q), J_R(P,Q)$, and $\chi^2(P,Q)$ have their usual meanings, respectively.

Proof: Since we know that AM $\geq$ GM, i.e., for $a, b > 0$

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{2.5.15}$$

Now put $a = 1$ and $b = \frac{1}{t}$ in (2.5.15) for $t > 0$, we obtain

$$\frac{1+\frac{1}{t}}{2} \geq \sqrt{\frac{1}{t}} \Rightarrow \log\left(\frac{t+1}{2t}\right) \geq \frac{1}{2}\log\frac{1}{t}. \tag{2.5.16}$$

Now multiply (2.5.16) by $\frac{t+1}{2}$ for $t > 0$, we get

$$\frac{t+1}{2}\log\left(\frac{t+1}{2t}\right) \geq \frac{t+1}{4}\log\frac{1}{t}. \tag{2.5.17}$$

Now put $t = \frac{p_i+q_i}{2q_i}$ in (2.5.17), multiply by q_i and then sum over all $i = 1, 2, 3..., n$, we have

$$\sum_{i=1}^{n}\left(\frac{p_i+3q_i}{4}\right)\log\left[\frac{p_i+3q_i}{2(p_i+q_i)}\right] \geq \sum_{i=1}^{n}\left(\frac{p_i+3q_i}{8}\right)\log\left[\frac{2q_i}{(p_i+q_i)}\right], \text{ i.e.,}$$

$$M^*(P,Q) \geq \frac{1}{4}\sum_{i=1}^{n}\left(\frac{p_i+q_i+2q_i}{2}\right)\log\left[\frac{2q_i}{(p_i+q_i)}\right], \text{ i.e.,}$$

$$M^*(P,Q) \geq \frac{1}{4}\left[\sum_{i=1}^{n} q_i \log\frac{2q_i}{p_i+q_i} - \sum_{i=1}^{n}\frac{p_i+q_i}{2} log\frac{p_i+q_i}{2q_i}\right], \text{ i.e.,}$$

$$M^*(P,Q) \geq \frac{1}{4}\left[F(Q,P) - G(Q,P)\right]. \tag{2.5.18}$$

Since we know the following by (2.5.13).

$$S_f(P,Q) \leq E_{C_{f'}}(P,Q) \Rightarrow \sum_{i=1}^{n} q_i f\left(\frac{p_i+q_i}{2q_i}\right) \leq \sum_{i=1}^{n}(p_i - q_i) f'\left(\frac{p_i}{q_i}\right). \tag{2.5.19}$$

Now put $f(t)$ and $f'(t)$ in (2.5.19), we obtain

$$\sum_{i=1}^{n}\left(\frac{p_i+3q_i}{4}\right)\log\left[\frac{p_i+3q_i}{2(p_i+q_i)}\right] \leq \frac{1}{2}\sum_{i=1}^{n}(p_i-q_i)\left[\log\left(\frac{p_i+q_i}{2p_i}\right) - \frac{q_i}{p_i}\right], \text{ i.e.,}$$

$$M^*(P,Q) \leq \frac{1}{2}\left[\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2p_i}\right) + \sum_{i=1}^{n}\frac{q_i(q_i-p_i)}{p_i}\right], \text{ i.e.,}$$

$$M^*(P,Q) \leq \frac{1}{2}\left[J_R(Q,P) + \sum_{i=1}^{n}\frac{q_i(q_i-p_i)}{p_i}\right], \text{ i.e.,}$$

$$M^*(P,Q) \leq \frac{1}{2}\left[J_R(Q,P) + \left(\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i} + \frac{p_i(q_i-p_i)}{p_i}\right)\right], \text{ i.e.,}$$

$$M^*(P,Q) \leq \frac{1}{2}\left[J_R(Q,P) + \chi^2(Q,P)\right]. \tag{2.5.20}$$

Relations (2.5.18) and (2.5.20) together give the required relation (2.5.14).

2.5.3 Bounds of the new information divergence measure

In this subsection, we obtain bounds of the new information divergence measure $M^*(P,Q)$ in terms of one parametric generalized divergence measure $\Phi_s(Q,P)$ for different values of $'s'$, by using new information inequalities (2.5.5). Actually, this part is an application of obtained new inequalities (2.5.5).

Proposition 2.5.2. *For $P,Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, we have*
a. For $s \to 0,1$ and $s > 0$, we have

$$\begin{aligned} &\frac{1}{2\beta^s(\beta+1)}\left[B_{\phi_s}(\alpha,\beta) - \Phi_s(Q,P)\right] \leq B_f(\alpha,\beta) - M^*(P,Q) \\ &\leq \frac{1}{2\alpha^s(\alpha+1)}\left[B_{\phi_s}(\alpha,\beta) - \Phi_s(Q,P)\right]. \end{aligned} \tag{2.5.21}$$

b. For $s \leq -1$, we have

$$\begin{aligned} &\frac{1}{2\alpha^s(\alpha+1)}\left[B_{\phi_s}(\alpha,\beta)-\Phi_s(Q,P)\right] \leq B_f(\alpha,\beta)-M^*(P,Q) \\ &\leq \frac{1}{2\beta^s(\beta+1)}\left[B_{\phi_s}(\alpha,\beta)-\Phi_s(Q,P)\right], \end{aligned} \tag{2.5.22}$$

where $B_f(\alpha,\beta)$ and $B_{\phi_s}(\alpha,\beta)$ are evaluated below by equations (2.5.24) and (2.5.25) respectively.

Proof: For $f(t)$ (2.5.10) and $\phi_s(t)$ (2.5.9), we get the followings respectively.

$$S_f(P,Q) = \sum_{i=1}^{n}\left(\frac{p_i+3q_i}{4}\right)\log\left[\frac{p_i+3q_i}{2(p_i+q_i)}\right] = M^*(P,Q). \tag{2.5.23}$$

$$\begin{aligned} B_f(\alpha,\beta) &= \frac{(\beta-1)f(\alpha)+(1-\alpha)f(\beta)}{\beta-\alpha} \\ &= \frac{(\beta-1)(1+\alpha)\log\left(\frac{1+\alpha}{2\alpha}\right)+(1-\alpha)(\beta+1)\log\left(\frac{1+\beta}{2\beta}\right)}{2(\beta-\alpha)}. \end{aligned} \tag{2.5.24}$$

$$\begin{aligned} B_{\phi_s}(\alpha,\beta) =& \frac{(\beta-1)\phi_s(\alpha)+(1-\alpha)\phi_s(\beta)}{\beta-\alpha} \\ =& [s(s-1)]^{-1}\left[\frac{(\beta-1)(\alpha^s-1)+(1-\alpha)(\beta^s-1)}{\beta-\alpha}\right] \\ =& [s(s-1)]^{-1}\left[\frac{\beta^s-\alpha^s}{\beta-\alpha}-\frac{\alpha\beta(\beta^{s-1}-\alpha^{s-1})}{\beta-\alpha}-1\right], \end{aligned} \tag{2.5.25}$$

$s \in R - \{0,1\}$.

Now let us consider the function $g(t) = t^{2-s}f''(t) = \frac{1}{2t^s(t+1)}$, where $f''(t)$ is given by (2.5.11) and

$$g'(t) = -\left[\frac{(s+1)t+s}{2t^{s+1}(t+1)^2}\right] = \begin{cases} <0 & \text{if } s \geq 0 \\ >0 & \text{if } s \leq -1 \end{cases}. \tag{2.5.26}$$

So $g(t)$ is monotonically decreasing for $s \geq 0$ and monotonically increasing for $s \leq -1$. Therefore, we have

$$m = \inf_{t\in(\alpha,\beta)} = \begin{cases} g(\beta) = \frac{1}{2\beta^s(\beta+1)} & s \geq 0 \\ g(\alpha) = \frac{1}{2\alpha^s(\alpha+1)} & s \leq -1 \end{cases}. \tag{2.5.27}$$

$$M = \sup_{t\in(\alpha,\beta)} = \begin{cases} g(\alpha) = \frac{1}{2\alpha^s(\alpha+1)} & s \geq 0 \\ g(\beta) = \frac{1}{2\beta^s(\beta+1)} & s \leq -1 \end{cases}. \tag{2.5.28}$$

Thus, the inequalities (2.5.21) and (2.5.22) are obtained by using (2.5.23), (2.5.24), (2.5.25), (2.5.27), and (2.5.28) in (2.5.5).

Now we evaluate some special results of proposition 2.5.2 at $s = -1, s = 0, s = 1$, and at $s = 2$ for getting bounds of the new divergence measure $M^*(P,Q)$ in terms of other well known divergences $\Delta(P,Q)$ (1.2.5), $F(P,Q)$ (1.2.21), $G(P,Q)$ (1.2.20), and $\chi^2(P,Q)$ (1.2.19).

Result 2.5.1. *For $P,Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, we have*

$$\begin{aligned} &\frac{\alpha}{2(\alpha+1)}\left[B_{\phi_{-1}}(\alpha,\beta) - \frac{1}{4}\Delta(P,Q)\right] \leq B_f(\alpha,\beta) - M^*(P,Q) \\ &\leq \frac{\beta}{2(\beta+1)}\left[B_{\phi_{-1}}(\alpha,\beta) - \frac{1}{4}\Delta(P,Q)\right], \end{aligned} \tag{2.5.29}$$

where $B_{\phi_{-1}}(\alpha,\beta)$ is evaluated below in the proof.

Proof: We evaluate $\Phi_s(Q,P)$ and $B_{\phi_s}(\alpha,\beta)$ at $s = -1$, i.e.,

$$\begin{aligned} \Phi_{-1}(Q,P) =& \frac{1}{2}\left[\sum_{i=1}^{n} q_i\left(\frac{p_i+q_i}{2q_i}\right)^{-1} - 1\right] = \frac{1}{2}\left[\sum_{i=1}^{n}\frac{2q_i^2}{p_i+q_i} - \sum_{i=1}^{n} q_i\right] \\ =& \frac{1}{2}\sum_{i=1}^{n}\frac{q_i^2 - p_iq_i - p_iq_i + p_iq_i}{p_i+q_i} = \frac{1}{2}\left[\sum_{i=1}^{n}\frac{q_i(p_i+q_i)}{p_i+q_i} - \sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i}\right] \\ =& \frac{1}{2}\left[1 - \sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i}\right] = \frac{1}{2}\left[\sum_{i=1}^{n}\frac{p_i+q_i}{2} - \sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i}\right] \\ =& \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i+q_i)^2 - 4p_iq_i}{p_i+q_i} = \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{4}\Delta(P,Q). \end{aligned} \tag{2.5.30}$$

$$B_{\phi_{-1}}(\alpha,\beta) = \frac{(1-\beta)(\alpha-1)}{2\alpha\beta}. \tag{2.5.31}$$

After putting (2.5.30) and (2.5.31) together with (2.5.24) in (2.5.22) at $s = -1$, we get the result (2.5.29) in terms of Triangular discrimination.

Result 2.5.2. *For $P, Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$, we have*

$$\begin{aligned} &\frac{1}{2(\beta+1)}\left[B_{\phi_0}(\alpha,\beta) - F(Q,P)\right] \le B_f(\alpha,\beta) - M^*(P,Q) \\ &\le \frac{1}{2(\alpha+1)}\left[B_{\phi_0}(\alpha,\beta) - F(Q,P)\right], \end{aligned} \tag{2.5.32}$$

where $B_{\phi_0}(\alpha,\beta)$ is evaluated below in the proof.

Proof: We evaluate $\Phi_s(Q,P)$ and $B_{\phi_s}(\alpha,\beta)$ at $s \to 0$, i.e.,

$$\begin{aligned} \Phi_0(Q,P) &= \lim_{s\to 0} \Phi_s(Q,P) = \lim_{s\to 0}\left[s(s-1)\right]^{-1}\left[\sum_{i=1}^n q_i\left(\frac{p_i+q_i}{2q_i}\right)^s - 1\right] \\ &= \sum_{i=1}^n q_i \log\left(\frac{2q_i}{p_i+q_i}\right) = F(Q,P). \end{aligned} \tag{2.5.33}$$

$$B_{\phi_0}(\alpha,\beta) = \lim_{s\to 0}\left[\frac{(\beta-1)(\alpha^s-1) + (1-\alpha)(\beta^s-1)}{s(s-1)(\beta-\alpha)}\right] = \frac{0}{0}.$$

After applying D Hospital Rule, we obtain

$$\begin{aligned} B_{\phi_0}(\alpha,\beta) &= \lim_{s\to 0}\left[\frac{(\beta-1)(\alpha^s\log\alpha) + (1-\alpha)(\beta^s\log\beta)}{(2s-1)(\beta-\alpha)}\right] \\ &= \frac{(\alpha-1)\log\beta - (\beta-1)\log\alpha}{\beta-\alpha}. \end{aligned} \tag{2.5.34}$$

After putting (2.5.33) and (2.5.34) together with (2.5.24) in (2.5.21) at $s = 0$, we get the result (2.5.32) in terms of Relative Jensen- Shannon divergence.

Result 2.5.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$, we have*

$$\begin{aligned} &\frac{1}{2\beta(\beta+1)}\left[B_{\phi_1}(\alpha,\beta) - G(Q,P)\right] \le B_f(\alpha,\beta) - M^*(P,Q) \\ &\le \frac{1}{2\alpha(\alpha+1)}\left[B_{\phi_1}(\alpha,\beta) - G(Q,P)\right], \end{aligned} \tag{2.5.35}$$

where $B_{\phi_1}(\alpha,\beta)$ is evaluated below in the proof.

Proof: We evaluate $\Phi_s(Q,P)$ and $B_{\phi_s}(\alpha,\beta)$ at $s \to 1$, i.e.,

$$\begin{aligned}\Phi_1(Q,P) =& \lim_{s\to 1}\Phi_s(Q,P) = \lim_{s\to 1}[s(s-1)]^{-1}\left[\sum_{i=1}^{n} q_i\left(\frac{p_i+q_i}{2q_i}\right)^s - 1\right] \\ =& \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right) = G(Q,P).\end{aligned} \tag{2.5.36}$$

$$B_{\phi_1}(\alpha,\beta) = \lim_{s\to 1}\left[\frac{(\beta-1)(\alpha^s-1)+(1-\alpha)(\beta^s-1)}{s(s-1)(\beta-\alpha)}\right] = \frac{0}{0}.$$

After applying D Hospital Rule, we obtain

$$\begin{aligned}B_{\phi_1}(\alpha,\beta) =& \lim_{s\to 1}\left[\frac{(\beta-1)(\alpha^s\log\alpha)+(1-\alpha)(\beta^s\log\beta)}{(2s-1)(\beta-\alpha)}\right] \\ =& \frac{(1-\alpha)\beta\log\beta-(1-\beta)\alpha\log\alpha}{\beta-\alpha}.\end{aligned} \tag{2.5.37}$$

After putting (2.5.36) and (2.5.37) together with (2.5.24) in (2.5.21) at $s = 1$, we get the result (2.5.35) in terms of Relative Arithmetic- Geometric divergence.

Result 2.5.4. *For $P, Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$, we have*

$$\begin{aligned}&\frac{1}{2\beta^2(\beta+1)}\left[B_{\phi_2}(\alpha,\beta) - \frac{1}{8}\chi^2(P,Q)\right] \le B_f(\alpha,\beta) - M^*(P,Q) \\ &\le \frac{1}{2\alpha^2(\alpha+1)}\left[B_{\phi_2}(\alpha,\beta) - \frac{1}{8}\chi^2(P,Q)\right],\end{aligned} \tag{2.5.38}$$

where $B_{\phi_2}(\alpha,\beta)$ is evaluated below in the proof.

Proof: We evaluate $\Phi_s(Q,P)$ and $B_{\phi_s}(\alpha,\beta)$ at $s = 2$, i.e.,

$$\begin{aligned}\Phi_2(Q,P) =& \frac{1}{2}\left[\sum_{i=1}^{n} q_i\left(\frac{p_i+q_i}{2q_i}\right)^2 - 1\right] = \frac{1}{2}\left[\sum_{i=1}^{n}\frac{(p_i+q_i)^2}{4q_i} - \sum_{i=1}^{n} p_i\right] \\ =& \frac{1}{8}\left[\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}\right] = \frac{1}{8}\chi^2(P,Q).\end{aligned} \tag{2.5.39}$$

$$B_{\phi_2}(\alpha,\beta) = \frac{(1-\beta)(\alpha-1)}{2}. \tag{2.5.40}$$

After putting (2.5.39) and (2.5.40) together with (2.5.24) in (2.5.21) at $s = 2$, we get the result (2.5.38) in terms of Chi- Square divergence.

2.5.4 Numerical verification of obtained results

Now, in this subsection, we give an example for calculating the divergences $\Delta(P,Q)$ and $M^*(P,Q)$ and verify the inequalities (Or bounds of $M^*(P,Q)$ in terms of $\Delta(P,Q)$) (2.5.29) numerically.

Example 2.5.1. *Here, we are considering the example same as example 2.3.1 (subsection- 2.3.3). The observations for Binomial and Poisson distributions are also same as Table 2.1, so we are skipping the repetition. Now, by Table 2.1, we get the value of $M^*(P,Q)$, as follows.*

$$M^*(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i + 3q_i}{4}\right) \log\left[\frac{p_i + 3q_i}{2(p_i + q_i)}\right] \approx .0115412. \tag{2.5.41}$$

Put the approximated numerical values (2.3.14), (2.3.17), and (2.5.41) into (2.5.29), we obtain

$$.01586 \leq .031810 - .0115412 \, (= M^*(P,Q)) \leq .02762$$

and hence verified the inequalities (2.5.29) for $p = 0.7$.

Similarly, we can verify the other inequalities (2.5.32) (2.5.35), and (2.5.38) for different values of p and q.

2.6 New Information Inequalities on Absolute Form of The New Generalized Divergence and Applications

In this section, we introduce new information inequalities on $S_f(P,Q)$ by considering convex normalized functions in absolute form. Further, we apply these inequalities for getting relations among well known divergences, together with numerical verification.

2.6.1 New information inequalities and bounds

In this subsection, we introduce new information inequalities by theorem 2.6.1 on $S_f(P,Q)$ for functions in absolute form. The results are on similar line to the results presented by Dragomir [25]. Further, we obtain bounds of different divergences (in absolute form) in terms of the Variational distance by using this new inequalities as an application.

Theorem 2.6.1. *Let $f_1, f_2 : (\alpha, \beta) \subset (0, \infty) \to R$ be two real, convex and normalized differentiable functions, i.e., $f_1''(t), f_2''(t) \geq 0 \ \forall\ t > 0$ and $f_1(1) = f_2(1) = 0$ respectively with $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$. If there exists the real constants m, M such that $m < M$ and*

$$m \leq \frac{|f_1(t_1) - f_1(t_2)|}{|f_2(t_1) - f_2(t_2)|} \leq M, \tag{2.6.1}$$

i.e.,

$$m \leq \frac{|f_1'(t)|}{|f_2'(t)|} = \left|\frac{f_1'(t)}{f_2'(t)}\right| \leq M, \tag{2.6.2}$$

for all $t_1, t_2 \in (\alpha, \beta) \subset (0, \infty)$.
If $P, Q \in \Gamma_n$, then we have the following inequalities

$$mS_{|f_2|}(P,Q) \leq S_{|f_1|}(P,Q) \leq MS_{|f_2|}(P,Q), \tag{2.6.3}$$

where $S_f(P,Q)$ has its usual meaning.

Proof: Firstly, we can see that (2.6.2) is obtained from (2.6.1) by using Cauchy's theorem of calculus.

Now put $t_1 = \frac{p_i+q_i}{2q_i}$ and $t_2 = 1$ in (2.6.1), multiply with q_i and then sum over all

$i = 1, 2, 3..., n$, we get the desire result (2.6.3).

Now, let $f_2 : (0, \infty) \to R$ be a function defined as

$$f_2(t) = |t-1|, f_2(1) = 0, f_2'(t) = \begin{cases} -1 & \text{if } 0 < t < 1 \\ 1 & \text{if } 1 < t < \infty \end{cases},$$

$$f_2''(t) = 0 \; \forall \, t \in (0, \infty) \text{ but not at } t = 1 \text{ and}$$

$$|f_2'(t)| = 1. \tag{2.6.4}$$

Since $f_2''(t) \geq 0 \; \forall \, t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is convex and normalized function respectively. Now for $f_2(t)$, we have

$$S_{|f_2|}(P,Q) = \sum_{i=1}^{n} q_i \left| f_2\left(\frac{p_i + q_i}{2q_i}\right)\right| = \frac{1}{2}\sum_{i=1}^{n} |p_i - q_i| = \frac{1}{2}V(P,Q), \tag{2.6.5}$$

where $V(P,Q)$ (1.2.7) is well known Variational distance.

Now, the following propositions are presenting the bounds of different divergences in absolute form, in terms of Variational distance by using new obtained inequalities (2.6.3).

Proposition 2.6.1. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, we have*

$$\frac{1}{2\beta}V(P,Q) \leq |F|(P,Q) \leq \frac{1}{2\alpha}V(P,Q). \tag{2.6.6}$$

Proof: Let us consider

$$f_1(t) = -\log t, t > 0, f_1(1) = 0, f_1'(t) = -\frac{1}{t} \text{ and } f_1''(t) = \frac{1}{t^2}.$$

Since $f_1''(t) > 0 \; \forall \, t > 0$ and $f_1(1) = 0$, so $f_1(t)$ is strictly convex and normalized function respectively.

Now for $f_1(t)$, we have

$$S_{|f_1|}(P,Q) = \sum_{i=1}^{n} q_i \left|\log\left(\frac{2q_i}{p_i + q_i}\right)\right| = |F|(Q,P). \tag{2.6.7}$$

Now, let $g(t)=\left|\frac{f_1'(t)}{f_2'(t)}\right|=\left|-\frac{1}{t}\right|=\frac{1}{t}$, where $|f_2'(t)|=1$ and $g'(t)=-\frac{1}{t^2}<0$.

It is clear that $g(t)$ is always strictly decreasing in $(0,\infty)$, so

$$m=\inf_{t\in(\alpha,\beta)} g(t)=g(\beta)=|f_1'(\beta)|=\frac{1}{\beta}. \tag{2.6.8}$$

$$M=\sup_{t\in(\alpha,\beta)} g(t)=g(\alpha)=|f_1'(\alpha)|=\frac{1}{\alpha}. \tag{2.6.9}$$

The result (2.6.6) is obtained by using (2.6.5), (2.6.7), (2.6.8), and (2.6.9) in (2.6.3), after interchanging P and Q.

Proposition 2.6.2. *For $P,Q\in\Gamma_n$ and $0<\alpha\le 1\le\beta<\infty,\alpha\ne\beta$, we have*
(a). If $0<\alpha<1$, then

$$0\le\Delta(P,Q)\le\frac{1}{2}\left[\frac{\beta^2-\alpha^2}{\alpha^2\beta^2}+\left|\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}-2\right|\right]V(P,Q). \tag{2.6.10}$$

(b). If $\alpha=1$, then

$$0\le\Delta(P,Q)\le\frac{\beta^2-1}{\beta^2}V(P,Q). \tag{2.6.11}$$

Proof: Let us consider

$$f_1(t)=\frac{(t-1)^2}{t},t>0,f_1(1)=0,f_1'(t)=\frac{t^2-1}{t^2}\text{ and }f_1''(t)=\frac{2}{t^3}.$$

Since $f_1''(t)>0\;\forall\;t>0$ and $f_1(1)=0$, so $f_1(t)$ is strictly convex and normalized function respectively.

Now for $f_1(t)$, we have

$$S_{|f_1|}(P,Q)=\frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i}=\frac{1}{2}\Delta(P,Q). \tag{2.6.12}$$

Now, let

$$g(t)=\left|\frac{f_1'(t)}{f_2'(t)}\right|=\left|\frac{t^2-1}{t^2}\right|=\begin{cases}-\left(\frac{t^2-1}{t^2}\right) & \text{if } 0<t<1\\ \frac{t^2-1}{t^2} & \text{if } 1\le t<\infty\end{cases},$$

where $|f_2'(t)| = 1$ and $g'(t) = \begin{cases} -\frac{2}{t^3} < 0 & \text{if } 0 < t < 1 \\ \frac{2}{t^3} > 0 & \text{if } 1 \leq t < \infty \end{cases}$.

It is clear that $g'(t) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, i.e., $g(t)$ is strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$. So $g(t)$ has a minimum value at $t = 1$, therefore

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = |f_1'(1)| = 0. \tag{2.6.13}$$

$$\begin{aligned} M &= \sup_{t\in(\alpha,\beta)} g(t) \\ &= \begin{cases} max\left(|f_1'(\alpha)|, |f_1'(\beta)|\right) = \frac{|f_1'(\alpha)|+|f_1'(\beta)|+\left||f_1'(\alpha)|-|f_1'(\beta)|\right|}{2} & \text{if } 0 < \alpha < 1 \\ |f_1'(\beta)| & \text{if } \alpha = 1 \end{cases} \\ &= \begin{cases} \frac{1}{2}\left[\frac{\beta^2-\alpha^2}{\alpha^2\beta^2} + \left|\frac{\beta^2+\alpha^2}{\alpha^2\beta^2} - 2\right|\right] & \text{if } 0 < \alpha < 1 \\ \frac{\beta^2-1}{\beta^2} & \text{if } \alpha = 1 \end{cases}. \end{aligned} \tag{2.6.14}$$

The results (2.6.10) and (2.6.11) are obtained by using (2.6.5), (2.6.12), (2.6.13), and (2.6.14) in (2.6.3).

Proposition 2.6.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, we have*
(a). If $0 < \alpha \leq \frac{1}{e}$, then

$$0 \leq |G|(P,Q) \leq \frac{1}{2}\left[\log\sqrt{\frac{\beta}{\alpha}} + \left|\log e\sqrt{\alpha\beta}\right|\right] V(P,Q). \tag{2.6.15}$$

(b). If $\frac{1}{e} < \alpha \leq 1$, then

$$\frac{\log e\alpha}{2} V(P,Q) \leq |G|(P,Q) \leq \frac{\log e\beta}{2} V(P,Q). \tag{2.6.16}$$

Proof: Let us consider

$$f_1(t) = t\log t, t > 0, f_1(1) = 0, f_1'(t) = 1 + \log t \text{ and } f_1''(t) = \frac{1}{t}.$$

Since $f_1''(t) > 0 \;\forall\, t > 0$ and $f_1(1) = 0$, so $f_1(t)$ is strictly convex and normalized function respectively.

Now for $f_1(t)$, we have

$$S_{|f_1|}(P,Q) = \sum_{i=1}^{n} \left(\frac{p_i+q_i}{2}\right) \left|\log\left(\frac{p_i+q_i}{2q_i}\right)\right| = |G|(Q,P). \tag{2.6.17}$$

Now, let

$$g(t) = \left|\frac{f_1'(t)}{f_2'(t)}\right| = |1+\log t| = \begin{cases} -(1+\log t) & \text{if } 0<t\le\frac{1}{e} \\ 1+\log t & \text{if } \frac{1}{e}<t<\infty \end{cases},$$

where $|f_2'(t)| = 1$ and $g'(t) = \begin{cases} -\frac{1}{t}<0 & \text{if } 0<t\le\frac{1}{e} \\ \frac{1}{t}>0 & \text{if } \frac{1}{e}<t<\infty \end{cases}$.

It is clear that $g'(t)<0$ in $\left(0,\frac{1}{e}\right)$ and >0 in $\left(\frac{1}{e},\infty\right)$, i.e., $g(t)$ is strictly decreasing in $\left(0,\frac{1}{e}\right)$ and strictly increasing in $\left(\frac{1}{e},\infty\right)$. So $g(t)$ has a minimum value at $t=\frac{1}{e}$, therefore

$$m = \inf_{t\in(\alpha,\beta)} g(t) = \begin{cases} \left|f_1'\left(\frac{1}{e}\right)\right| = 0 & \text{if } 0<\alpha\le\frac{1}{e} \\ |f_1'(\alpha)| = 1+\log\alpha & \text{if } \frac{1}{e}<\alpha\le 1 \end{cases}. \tag{2.6.18}$$

$$\begin{aligned} M &= \sup_{t\in(\alpha,\beta)} g(t) \\ &= \begin{cases} max\left(|f_1'(\alpha)|, |f_1'(\beta)|\right) = \left[\log\sqrt{\frac{\beta}{\alpha}} + \left|\log e\sqrt{\alpha\beta}\right|\right] & \text{if } 0<\alpha\le\frac{1}{e} \\ |f_1'(\beta)| = 1+\log\beta & \text{if } \frac{1}{e}<\alpha\le 1 \end{cases}. \end{aligned} \tag{2.6.19}$$

The results (2.6.15) and (2.6.16) are obtained by using (2.6.5), (2.6.17), (2.6.18), and (2.6.19) in (2.6.3), after interchanging P and Q.

In a similar manner, we obtain the followings as well.

(a) For $f_1(t) = (t-1)\log t$, we obtain

(i) If $0<\alpha<1$, then

$$0 \le |J_R|(P,Q) \le \left[\log\sqrt{\frac{\beta}{\alpha}} + \frac{\beta-\alpha}{2\alpha\beta} + \left|\frac{\beta+\alpha}{2\alpha\beta} - \log e\sqrt{\alpha\beta}\right|\right] V(P,Q). \tag{2.6.20}$$

(ii) If $\alpha = 1$, then

$$0 \leq |J_R|(P,Q) \leq \left(\log e\beta - \frac{1}{\beta}\right) V(P,Q). \tag{2.6.21}$$

(b) For $f_1(t) = (t-1)^2$, we obtain

(i) If $0 < \alpha < 1$, then

$$0 \leq \chi^2(P,Q) \leq 2\left[\beta - \alpha + |2 - (\alpha+\beta)|\right] V(P,Q). \tag{2.6.22}$$

(ii) If $\alpha = 1$, then

$$0 \leq \chi^2(P,Q) \leq 4(\beta - 1) V(P,Q). \tag{2.6.23}$$

2.6.2 Numerical verification of obtained results

Now, in this subsection, we give an example for calculating the divergences $|F|(P,Q)$, $\Delta(P,Q)$, $|J_R|(P,Q)$, and $V(P,Q)$ and then verify the inequalities (2.6.6), (2.6.10), and (2.6.20), numerically.

Example 2.6.1. *Here, we are considering the example same as example 2.3.1 (subsection- 2.3.3). The observations for Binomial and Poisson distributions are also same as Table 2.1, so we are skipping the repetition. Now, by Table 2.1, we get the followings.*

$$\alpha(= .503) \leq \frac{p_i + q_i}{2q_i} \leq \beta(= 1.396). \tag{2.6.24}$$

$$|F|(P,Q) = \sum_{i=1}^{11} p_i \left|\log\left(\frac{2p_i}{p_i + q_i}\right)\right| \approx .21792. \tag{2.6.25}$$

$$\Delta(P,Q) = \sum_{i=1}^{11} \frac{(p_i - q_i)^2}{p_i + q_i} \approx .1812. \tag{2.6.26}$$

$$|J_R|(P,Q) = \sum_{i=1}^{11} \left|(p_i - q_i)\log\left(\frac{p_i + q_i}{2q_i}\right)\right| \approx .1686. \tag{2.6.27}$$

$$V(P,Q)=\sum_{i=1}^{11}|p_i-q_i|\approx .4844. \tag{2.6.28}$$

Put the approximated numerical values from (2.6.24) to (2.6.28) into (2.6.6), (2.6.10), and (2.6.20), we get

$$.1734\leq .2179\left(|F|\left(P,Q\right)\right)\leq .4815, 0\leq .1812\left(\Delta\left(P,Q\right)\right)\leq 1.4301,$$

and

$$0\leq .1686\left(|J_R|\left(P,Q\right)\right)\leq .8129$$

respectively and hence verified the inequalities (2.6.6), (2.6.10), and (2.6.20) for $p=0.7$.

Similarly, we can verify the other inequalities for different values of p and q and for other discrete probability distributions as well.

2.7 Ostrowski's Integral Inequalities on New Generalized Divergence and Applications

In this section, two different new information inequalities on $S_f(P,Q)$ are introduced. These inequalities are derived by using Ostrowski's inequalities. Further obtain the bounds of the well known divergences $F(P,Q)$, $\Delta(P,Q)$, $G(P,Q)$, $J_R(P,Q)$, and $h(P,Q)$ in terms of $\chi^2(P,Q)$ and $V(P,Q)$ separately in a different aspect, by using new information inequalities.

Firstly, following theorems 2.7.1 (Dragomir etc. all [28]) and 2.7.2 (Dragomir

[23]) are very important to introduce new information inequalities. We are considering the statements only of these theorems, detail prove can be seen in given literatures.

Theorem 2.7.1. *Let $f : (a, b) \subset R \to R$ be an absolutely continuous function with $a < b$ and $f' : (a, b) \to R$ is essentially bounded or $f' \in L_\infty(a, b)$, i.e.,*

$$\|f'\|_\infty = ess \sup_{t\in(a,b)} |f'(t)| < \infty,$$

then we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\,\|f'\|_\infty, \tag{2.7.1}$$

$\forall x, t \in (a, b)$.

Theorem 2.7.2. *Let $f : (a, b) \subset R \to R$ be differentiable function and is of bounded variation on (a, b), i.e.,*

$$A_a^b(f) = \int_a^b |f'(t)| < \infty,$$

then we have

$$\left| \int_a^b f(t)\,dt - f(x)(b-a) \right| \le \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] A_a^b(f), \tag{2.7.2}$$

$\forall x, t \in (a, b)$.

Inequalities (2.7.1) and (2.7.2) are well known Ostrowski's integral inequalities.

2.7.1 New information inequalities

Now, we obtain two new information inequalities in terms of the well known Chi- square divergence and Variational distance by helping above two Ostrowskis inequalities (2.7.1) and (2.7.2). The results are on similar lines to the results presented by Dragomir etc. all [28] and Dragomir [23], respectively.

Theorem 2.7.3. *Let $f : (\alpha, \beta) \subset (0, \infty) \to R$ be an absolutely continuous function with $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$ and $f' : (\alpha, \beta) \to R$ is essentially bounded or $f' \in L_\infty(\alpha, \beta)$, i.e.,*

$$\|f'\|_\infty = \sup_{t \in (\alpha,\beta)} |f'(t)| < \infty, \tag{2.7.3}$$

$\forall\, t \in (\alpha, \beta)$. If $P, Q \in \Gamma_n$, then we have the following inequality

$$\left|S_f(P,Q) - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)\,dt\right| \leq \frac{(\beta-\alpha)\|f'\|_\infty}{4}\left[2 + \frac{1}{(\beta-\alpha)^2}\chi^2(P,Q)\right], \tag{2.7.4}$$

where $S_f(P,Q)$ and $\chi^2(P,Q)$ have their usual meaning respectively.

Proof: Put $a = \alpha, b = \beta$, and $x = \frac{p_i+q_i}{2q_i}$ in inequality (2.7.1), multiply by q_i and then sum over all $i = 1, 2, 3..., n$, we get

$$\left|\sum_{i=1}^n q_i f\left(\frac{p_i+q_i}{2q_i}\right) - \frac{1}{(\beta-\alpha)}\int_\alpha^\beta f(t)\,dt \sum_{i=1}^n q_i\right|$$
$$\leq \left[\frac{1}{4}\sum_{i=1}^n q_i + \frac{1}{(\beta-\alpha)^2}\sum_{i=1}^n q_i\left(\frac{p_i+q_i}{2q_i} - \frac{\alpha+\beta}{2}\right)^2\right](\beta-\alpha)\|f'\|_\infty, \text{ i.e.,}$$
$$\left|S_f(P,Q) - \frac{1}{(\beta-\alpha)}\int_\alpha^\beta f(t)\,dt\right|$$
$$\leq \left[\frac{1}{4} + \frac{1}{4(\beta-\alpha)^2}\sum_{i=1}^n q_i\left(\frac{p_i+q_i}{q_i} - (\alpha+\beta)\right)^2\right](\beta-\alpha)\|f'\|_\infty$$
$$= \frac{(\beta-\alpha)\|f'\|_\infty}{4}\left[1 + \frac{1}{(\beta-\alpha)^2}\left(\sum_{i=1}^n \frac{(p_i+q_i)^2}{q_i} + (\alpha+\beta)^2\sum_{i=1}^n q_i - 2(\alpha+\beta)\sum_{i=1}^n (p_i+q_i)\right)\right]$$
$$= \frac{(\beta-\alpha)\|f'\|_\infty}{4}\left[1 + \frac{1}{(\beta-\alpha)^2}\left((\alpha+\beta)^2 - 4(\alpha+\beta) + 4 + \sum_{i=1}^n \frac{(p_i-q_i)^2}{q_i}\right)\right]$$
$$= \frac{(\beta-\alpha)\|f'\|_\infty}{4}\left[1 + \frac{1}{(\beta-\alpha)^2}\left((\alpha+\beta-2)^2 + \chi^2(P,Q)\right)\right]$$
$$= \frac{(\beta-\alpha)\|f'\|_\infty}{4}\left[2 + \frac{1}{(\beta-\alpha)^2}\chi^2(P,Q)\right], \left[\because (\alpha+\beta-2)^2 \leq (\beta-\alpha)^2\right].$$

Hence prove the inequality (2.7.4).

Theorem 2.7.4. *Let $f : (\alpha, \beta) \subset (0, \infty) \to R$ be a differentiable function and is of bounded variation on (α, β) with $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, i.e.,*

$$A_{\alpha}^{\beta}(f) = \int_{\alpha}^{\beta} |f'(t)| < \infty, \tag{2.7.5}$$

$\forall\, t \in (\alpha, \beta)$. If $P, Q \in \Gamma_n$, then we have the following inequality

$$\left| S_f(P,Q) - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t)\, dt \right| \leq \frac{A_{\alpha}^{\beta}(f)}{2} \left[2 + \frac{1}{(\beta - \alpha)} V(P,Q) \right], \tag{2.7.6}$$

where $V(P,Q)$ is the well known Variational distance.

Proof: Put $a = \alpha, b = \beta$, and $x = \frac{p_i + q_i}{2q_i}$ in inequality (2.7.2), multiply by q_i and then sum over all $i = 1, 2, 3..., n$, we get

$$\begin{aligned}
&\left| \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right) - \frac{1}{(\beta - \alpha)} \int_{\alpha}^{\beta} f(t)\, dt \sum_{i=1}^{n} q_i \right| \\
&\leq \left[\frac{1}{2} + \frac{1}{(\beta - \alpha)} \sum_{i=1}^{n} q_i \left| \frac{p_i + q_i}{2q_i} - \frac{\alpha + \beta}{2} \right| \right] A_{\alpha}^{\beta}(f), \text{ i.e.,} \\
&\left| S_f(P,Q) - \frac{1}{(\beta - \alpha)} \int_{\alpha}^{\beta} f(t)\, dt \right| \\
&\leq \left[\frac{1}{2} + \frac{1}{(\beta - \alpha)} \sum_{i=1}^{n} q_i \left| \frac{p_i + q_i}{2q_i} - 1 - \left(\frac{\alpha + \beta}{2} - 1\right) \right| \right] A_{\alpha}^{\beta}(f) \\
&= \left[\frac{1}{2} + \frac{1}{(\beta - \alpha)} \left(\sum_{i=1}^{n} q_i \left| \frac{p_i + q_i}{2q_i} - 1 \right| + \sum_{i=1}^{n} q_i \left| \frac{\alpha + \beta}{2} - 1 \right| \right) \right] A_{\alpha}^{\beta}(f) \\
&= \left[\frac{1}{2} + \frac{1}{(\beta - \alpha)} \left(\frac{1}{2} V(P,Q) + \left| \frac{\alpha + \beta}{2} - 1 \right| \right) \right] A_{\alpha}^{\beta}(f) \\
&= \frac{1}{2} \left[2 + \frac{1}{(\beta - \alpha)} V(P,Q) \right] A_{\alpha}^{\beta}(f), \left[\because \left| \frac{\alpha + \beta}{2} - 1 \right| \leq \frac{(\beta - \alpha)}{2} \right].
\end{aligned}$$

Hence prove the inequality (2.7.6).

2.7.2 Bounds of the well known divergences

Now, we derive bounds of the well known divergences in terms of the Chi- Square divergence and the Variational distance separately as an application of obtained new information inequalities (2.7.4) and (2.7.6) respectively. We are considering only convex functions, the inequalities hold good for concave functions as well.

Proposition 2.7.1. *For $P, Q \in \Gamma_n$, we have*

$$\left|F(Q,P) + \frac{\beta\log\beta - \alpha\log\alpha}{\beta-\alpha} - 3\right| \leq F_1 \sup_{t\in(\alpha,\beta)} g_1(t) \tag{2.7.7}$$

and

$$\left|F(Q,P) + \frac{\beta\log\beta - \alpha\log\alpha}{\beta-\alpha} - 3\right| \leq F_2 A_\alpha^\beta(f), \tag{2.7.8}$$

where $F(P,Q)$, $\chi^2(P,Q)$, and $V(P,Q)$ have their usual meanings respectively. Also

$$F_1 \equiv \frac{(\beta-\alpha)}{4}\left[2 + \frac{1}{(\beta-\alpha)^2}\chi^2(P,Q)\right],$$

$$F_2 \equiv \frac{1}{2}\left[2 + \frac{1}{(\beta-\alpha)}V(P,Q)\right]$$

and $\sup_{t\in(\alpha,\beta)} g_1(t)$ and $A_\alpha^\beta(f)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = -\log t, t \in (0,\infty), f(1) = 0, f'(t) = -\frac{1}{t} \text{ and } f''(t) = \frac{1}{t^2}.$$

Since $f''(t) > 0 \ \forall\ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we obtain

$$S_f(P,Q) = \sum_{i=1}^n q_i \log\left(\frac{2q_i}{p_i+q_i}\right) = F(Q,P). \tag{2.7.9}$$

$$A_\alpha^\beta(f) = \int_\alpha^\beta |f'(t)|\,dt = \int_\alpha^\beta \left|-\frac{1}{t}\right| dt = \int_\alpha^\beta \frac{1}{t}dt = \log\beta - \log\alpha. \tag{2.7.10}$$

Now, let $g_1(t) = |f'(t)| = \left|-\frac{1}{t}\right| = \frac{1}{t}$, and $g_1'(t) = -\frac{1}{t^2} < 0$.

It is clear that $g_1(t)$ is always decreasing in $(0,\infty)$, so

$$\|f'\|_\infty = \sup_{t\in(\alpha,\beta)} |f'(t)| = \sup_{t\in(\alpha,\beta)} g_1(t) = g_1(\alpha) = \frac{1}{\alpha}. \tag{2.7.11}$$

The results (2.7.7) and (2.7.8) are obtained by using (2.7.9), (2.7.10), and (2.7.11) in inequalities (2.7.4) and (2.7.6) respectively.

Proposition 2.7.2. *For $P, Q \in \Gamma_n$, we have*

$$\left|\Delta(P,Q) + 4 - \frac{2(\log\beta - \log\alpha)}{\beta - \alpha} - (\alpha+\beta)\right| \le 2F_1 \sup_{t\in(\alpha,\beta)} g_2(t) \tag{2.7.12}$$

and

$$\left|\Delta(P,Q) + 4 - \frac{2(\log\beta - \log\alpha)}{\beta - \alpha} - (\alpha+\beta)\right| \le 2F_2 A_\alpha^\beta(f), \tag{2.7.13}$$

where $\Delta(P,Q)$ has its usual meaning. Also $\sup_{t\in(\alpha,\beta)} g_2(t)$ and $A_\alpha^\beta(f)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = \frac{(t-1)^2}{t}, t \in (0,\infty), f(1) = 0, f'(t) = \frac{t^2-1}{t^2} \text{ and } f''(t) = \frac{2}{t^3}.$$

Since $f''(t) > 0 \ \forall\ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we obtain

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^n \frac{(p_i - q_i)^2}{p_i + q_i} = \frac{1}{2}\Delta(P,Q). \tag{2.7.14}$$

$$A_\alpha^\beta(f) = \int_\alpha^\beta |f'(t)|\,dt = \int_\alpha^1 \frac{1-t^2}{t^2} + \int_1^\beta \frac{t^2-1}{t^2} = \frac{\alpha+\beta}{\alpha\beta} + \alpha + \beta - 4. \tag{2.7.15}$$

Now, let

$$g_2(t) = |f'(t)| = \left|\frac{t^2-1}{t^2}\right| = \begin{cases} -\frac{t^2-1}{t^2} & \text{if } 0 < t < 1 \\ \frac{t^2-1}{t^2} & \text{if } 1 \le t < \infty \end{cases},$$

and

$$g_2'(t) = \begin{cases} -\frac{2}{t^3} & \text{if } 0 < t < 1 \\ \frac{2}{t^3} & \text{if } 1 \le t < \infty \end{cases}.$$

It is clear that $g_2'(t) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, i.e., $g_2(t)$ is decreasing in $(0,1)$ and increasing in $(1,\infty)$, so

$$\begin{aligned}\|f'\|_\infty &= \sup_{t\in(\alpha,\beta)} |f'(t)| = \sup_{t\in(\alpha,\beta)} g_2(t)\\ &= \begin{cases} max\left[|f'(\alpha)|, |f'(\beta)|\right] = \frac{|f'(\alpha)|+|f'(\beta)|+||f'(\alpha)|-|f'(\beta)||}{2} & \text{if } 0<\alpha<1\\ |f'(\beta)| & \text{if } \alpha = 1\end{cases}\\ &= \begin{cases} \frac{1}{2}\left[\frac{\beta^2-\alpha^2}{\alpha^2\beta^2} + \left|\frac{\beta^2+\alpha^2}{\alpha^2\beta^2} - 2\right|\right] & \text{if } 0<\alpha<1\\ \frac{(\beta+1)(\beta-1)}{\beta^2} & \text{if } \alpha = 1\end{cases}.\end{aligned} \tag{2.7.16}$$

The results (2.7.12) and (2.7.13) are obtained by using (2.7.14), (2.7.15), and (2.7.16) in inequalities (2.7.4) and (2.7.6) respectively.

Proposition 2.7.3. *For $P, Q \in \Gamma_n$, we have*

$$\left|G(Q,P) + \frac{\alpha+\beta}{4} - \frac{\beta^2\log\beta - \alpha^2\log\alpha}{2(\beta-\alpha)}\right| \le F_1 \sup_{t\in(\alpha,\beta)} g_3(t) \tag{2.7.17}$$

and

$$\left|G(Q,P) + \frac{\alpha+\beta}{4} - \frac{\beta^2\log\beta - \alpha^2\log\alpha}{2(\beta-\alpha)}\right| \le F_2 A_\alpha^\beta(f), \tag{2.7.18}$$

where $G(P,Q)$ has its usual meaning. Also $\sup_{t\in(\alpha,\beta)} g_3(t)$ and $A_\alpha^\beta(f)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = t\log t, t\in(0,\infty), f(1) = 0, f'(t) = 1+\log t \text{ and } f''(t) = \frac{1}{t}.$$

Since $f''(t) > 0 \;\forall\; t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we obtain

$$S_f(P,Q) = \sum_{i=1}^n \frac{p_i+q_i}{2}\log\left(\frac{p_i+q_i}{2q_i}\right) = G(Q,P). \tag{2.7.19}$$

$$A_{\alpha}^{\beta}(f) = \int_{\alpha}^{\beta} |f'(t)|\, dt = \int_{\alpha}^{\frac{1}{e}} -(1+\log t)\, dt + \int_{\frac{1}{e}}^{\beta} (1+\log t)\, dt$$
$$= \alpha \log \alpha + \beta \log \beta + \frac{2}{e}. \tag{2.7.20}$$

Now, let

$$g_3(t) = |f'(t)| = |1+\log t| = \begin{cases} -1-\log t & \text{if } 0 < t \le \frac{1}{e} \\ 1+\log t & \text{if } \frac{1}{e} < t < \infty \end{cases},$$

and

$$g_3'(t) = \begin{cases} -\frac{1}{t} & \text{if } 0 < t \le \frac{1}{e} \\ \frac{1}{t} & \text{if } \frac{1}{e} < t < \infty \end{cases}.$$

It is clear that $g_3'(t) < 0$ in $\left(0, \frac{1}{e}\right)$ and > 0 in $\left(\frac{1}{e}, \infty\right)$, i.e., $g_3(t)$ is decreasing in $\left(0, \frac{1}{e}\right)$ and increasing in $\left(\frac{1}{e}, \infty\right)$, so

$$\|f'\|_{\infty} = \sup_{t\in(\alpha,\beta)} |f'(t)| = \sup_{t\in(\alpha,\beta)} g_3(t)$$
$$= \begin{cases} max\left[|f'(\alpha)|, |f'(\beta)|\right] = \left[\log\sqrt{\frac{\beta}{\alpha}} + \left|1+\log\sqrt{\alpha\beta}\right|\right] & \text{if } 0 < \alpha \le \frac{1}{e} \\ |f'(\beta)| = 1+\log\alpha & \text{if } \frac{1}{e} < \alpha \le 1 \end{cases}. \tag{2.7.21}$$

The results (2.7.17) and (2.7.18) are obtained by using (2.7.19), (2.7.20), and (2.7.21) in inequalities (2.7.4) and (2.7.6) respectively.

Proposition 2.7.4. *For $P, Q \in \Gamma_n$, we have*

$$\left|J_R(P,Q) - \left[\frac{\beta(\beta-2)\log\beta - \alpha(\alpha-2)\log\alpha}{\beta-\alpha} - \frac{\alpha+\beta}{2} + 2\right]\right| \le 2F_1 \sup_{t\in(\alpha,\beta)} g_4(t) \tag{2.7.22}$$

and

$$\left|J_R(P,Q) - \left[\frac{\beta(\beta-2)\log\beta - \alpha(\alpha-2)\log\alpha}{\beta-\alpha} - \frac{\alpha+\beta}{2} + 2\right]\right| \le 2F_2 A_{\alpha}^{\beta}(f), \tag{2.7.23}$$

where $J_R(P,Q)$ has its usual meaning. Also $\sup_{t\in(\alpha,\beta)} g_4(t)$ and $A_{\alpha}^{\beta}(f)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = (t-1)\log t, t \in (0,\infty), f(1) = 0, f'(t) = \frac{t-1}{t} + \log t \text{ and } f''(t) = \frac{t+1}{t^2}.$$

Since $f''(t) > 0 \ \forall \ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we obtain

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i + q_i}{2q_i}\right) = \frac{1}{2}J_R(P,Q). \tag{2.7.24}$$

$$\begin{aligned} A_\alpha^\beta(f) = \int_\alpha^\beta |f'(t)|\,dt = \int_\alpha^1 \left(-1 - \log t + \frac{1}{t}\right)dt + \int_1^\beta \left(1 + \log t - \frac{1}{t}\right)dt \\ = \alpha\log\alpha + \beta\log\beta - \log\alpha - \log\beta. \end{aligned} \tag{2.7.25}$$

Now, let

$$g_4(t) = |f'(t)| = \left|\frac{t-1}{t} + \log t\right| = \begin{cases} -1 - \log t + \frac{1}{t} & \text{if } 0 < t < 1 \\ 1 + \log t - \frac{1}{t} & \text{if } 1 \le t < \infty \end{cases},$$

and

$$g_4'(t) = \begin{cases} -\frac{t+1}{t^2} & \text{if } 0 < t < 1 \\ \frac{t+1}{t^2} & \text{if } 1 \le t < \infty \end{cases}.$$

It is clear that $g_4'(t) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, i.e., $g_4(t)$ is decreasing in $(0,1)$ and increasing in $(1,\infty)$, so

$$\begin{aligned} \|f'\|_\infty &= \sup_{t\in(\alpha,\beta)} |f'(t)| = \sup_{t\in(\alpha,\beta)} g_4(t) \\ &= \begin{cases} max\left[|f'(\alpha)|, |f'(\beta)|\right] = \left[\log\sqrt{\frac{\beta}{\alpha}} + \frac{\beta-\alpha}{2\alpha\beta} + \left|\frac{\beta+\alpha}{2\alpha\beta} - \log e\sqrt{\alpha\beta}\right|\right] & \text{if } 0 < \alpha < 1 \\ |f'(\beta)| = \left(\log e\beta - \frac{1}{\beta}\right) & \text{if } \alpha = 1 \end{cases}. \end{aligned} \tag{2.7.26}$$

The results (2.7.22) and (2.7.23) are obtained by using (2.7.24), (2.7.25), and (2.7.26) in inequalities (2.7.4) and (2.7.6) respectively.

Proposition 2.7.5. *For $P, Q \in \Gamma_n$, we have*

$$\left| h\left(\frac{P+Q}{2}, Q\right) + \frac{2\left(\beta^{\frac{3}{2}} - \alpha^{\frac{3}{2}}\right)}{3(\beta-\alpha)} - 1 \right| \leq F_1 \sup_{t\in(\alpha,\beta)} g_5(t) \tag{2.7.27}$$

and

$$\left| h\left(\frac{P+Q}{2}, Q\right) + \frac{2\left(\beta^{\frac{3}{2}} - \alpha^{\frac{3}{2}}\right)}{3(\beta-\alpha)} - 1 \right| \leq F_2 A_\alpha^\beta(f), \tag{2.7.28}$$

where $h(P,Q)$ has its usual meaning. Also $\sup_{t\in(\alpha,\beta)} g_5(t)$ and $A_\alpha^\beta(f)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = 1 - \sqrt{t}, t \in (0,\infty), f(1) = 0, f'(t) = -\frac{1}{2\sqrt{t}} \text{ and } f''(t) = \frac{1}{4t^{\frac{3}{2}}}.$$

Since $f''(t) > 0\ \forall\ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively.

Now for $f(t)$ and $f'(t)$, we obtain

$$\begin{aligned} S_f(P,Q) &= \sum_{i=1}^n q_i\left(1 - \sqrt{\frac{p_i+q_i}{2q_i}}\right) = 1 - \sum_{i=1}^n \sqrt{\frac{q_i(p_i+q_i)}{2}} \\ &= \frac{1}{2}\left[2 - 2\sum_{i=1}^n \sqrt{\frac{q_i(p_i+q_i)}{2}}\right] = \frac{1}{2}\left[\sum_{i=1}^n \frac{p_i+q_i}{2} + \sum_{i=1}^n q_i - \sqrt{2q_i(p_i+q_i)}\right] \\ &= \frac{1}{2}\sum_{i=1}^n \left(\sqrt{\frac{p_i+q_i}{2}} - \sqrt{q_i}\right)^2 = h\left(\frac{P+Q}{2}, Q\right). \end{aligned} \tag{2.7.29}$$

$$A_\alpha^\beta(f) = \int_\alpha^\beta |f'(t)|\, dt = \frac{1}{2}\int_\alpha^\beta \left|-\frac{1}{\sqrt{t}}\right| dt = \frac{1}{2}\int_\alpha^\beta \frac{1}{\sqrt{t}} dt = \sqrt{\beta} - \sqrt{\alpha}. \tag{2.7.30}$$

Now, let $g_5(t) = |f'(t)| = \frac{1}{2}\left|-\frac{1}{\sqrt{t}}\right| = \frac{1}{2\sqrt{t}}$, and $g_5'(t) = -\frac{1}{4t^{\frac{3}{2}}} < 0$.

It is clear that $g_5(t)$ is always decreasing in $(0,\infty)$, so

$$\|f'\|_\infty = \sup_{t\in(\alpha,\beta)} |f'(t)| = \sup_{t\in(\alpha,\beta)} g_5(t) = g_5(\alpha) = \frac{1}{2\sqrt{\alpha}}. \tag{2.7.31}$$

The results (2.7.27) and (2.7.28) are obtained by using (2.7.29), (2.7.30), and (2.7.31) in inequalities (2.7.4) and (2.7.6) respectively.

2.8 Conclusion

The study of information expressions and inequalities are of paramount importance in solving key results in information theory. In this chapter, we have derived some important information inequalities on new generalized divergence measure $S_f(P,Q)$ in terms of Variational distance, in terms of Chi- square divergence and many more. With the help of new information inequalities, we have obtained bounds of new divergences in terms of standard divergences and obtained many new inter relations among existing divergences as well.

3

NEW INFORMATION DIVERGENCE MEASURES OF CSISZAR'S CLASS AND APPLICATIONS

3.1 Introduction

In this chapter, we introduce two new information divergence measures of Csiszar's class and do a detail study regarding these measures.

This chapter is organized as follows: After introduction, section 3.2 introduce a new divergence measure which is exponential in nature and obtain bounds of this new divergence mathematically in terms of the other well known divergences, like: Kullback Leibler divergence (1.2.18), Triangular discrimination (1.2.5), Hellinger discrimination (1.2.6), Symmetric Chi- Square divergence (1.2.9) and many more, by using standard information inequalities on $C_f(P,Q)$, together with verification of obtained bounds numerically. In section 3.3, we introduce and characterize again a new divergence measure of Csiszar's class and obtain its bounds in terms of Relative information of type s, i.e., $K_s(P,Q)$ (1.2.23) by using well known information inequalities different from section 3.2, on $C_f(P,Q)$. Section 3.4 is the conclusion of the whole chapter.

3.2 New Exponential Divergence Measure, Properties and Bounds

We already discussed that Ali- Silvey [2] and Csiszar [20] introduced a generalized information divergence measure, given by

$$C_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right),$$

where $f:(0,\infty)\to R$ is real, continuous, and convex function and $P=(p_1,p_2,...,p_n), Q=(q_1,q_2,...,q_n)\in\Gamma_n$. Many divergence measures can be obtained from this generalized divergence measure by suitably defining the convex function.

We are taking here the following theorems 3.2.1 and 3.2.2 (statement only) for evaluating the bounds of the upcoming new exponential divergence measure in terms of several divergences of Csiszar's class and Variational distance separately. Actually Taneja [95] and Dragomir [25] gave the following theorems with their proofs respectively, which relate Csiszar's generalized divergence for two different convex functions. The results are on the similar lines to the results presented by Taneja [95] and Dragomir [25] separately.

Theorem 3.2.1. *Let $f_1,f_2 : I\subset(0,\infty)\to R$ be two convex differentiable and normalized functions, i.e., $f_1''(t), f_2''(t)\geq 0\ \forall\ t>0$ and $f_1(1)=f_2(1)=0$*

respectively and suppose the following assumptions.
(i) f_1 *and* f_2 *are twice differentiable on* (α, β), $0 < \alpha \leq 1 \leq \beta < \infty$ *with* $\alpha \neq \beta$.
(ii) There exists the real constants m, M *such that* $m < M$ *and*

$$m \leq \frac{f_1''(t)}{f_2''(t)} \leq M, f_2''(t) \neq 0 \; \forall \, t \in (\alpha, \beta). \tag{3.2.1}$$

If $P, Q \in \Gamma_n$, *then we have the following inequalities*

$$mC_{f_2}(P,Q) \leq C_{f_1}(P,Q) \leq MC_{f_2}(P,Q). \tag{3.2.2}$$

Theorem 3.2.2. *Let* $f_1, f_2 : (\alpha, \beta) \subset (0, \infty) \to R$ *be two real, convex and normalized differentiable functions, i.e.,* $f_1''(t), f_2''(t) \geq 0 \; \forall \; t > 0$ *and* $f_1(1) = f_2(1) = 0$ *respectively with* $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$. *If there exists the real constants* m, M *such that* $m < M$ *and*

$$m \leq \frac{|f_1(t_1) - f_1(t_2)|}{|f_2(t_1) - f_2(t_2)|} \leq M,$$

i.e.,

$$m \leq \frac{|f_1'(t)|}{|f_2'(t)|} = \left|\frac{f_1'(t)}{f_2'(t)}\right| \leq M, \tag{3.2.3}$$

for all $t_1, t_2 \in (\alpha, \beta) \subset (0, \infty)$.
If $P, Q \in \Gamma_n$ *is such that* $\alpha \leq \frac{p_i}{q_i} \leq \beta < \; \forall \; i = 1, 2, 3..., n$, *then we have the following inequalities*

$$mC_{|f_2|}(P,Q) \leq C_{|f_1|}(P,Q) \leq MC_{|f_2|}(P,Q). \tag{3.2.4}$$

3.2.1 New exponential divergence measure and properties

Let $f : (0, \infty) \to R$ be a real differentiable mapping, which is defined as

$$f(t) = f_1(t) = e^t(t-1), \forall \; t \in (0, \infty), \tag{3.2.5}$$

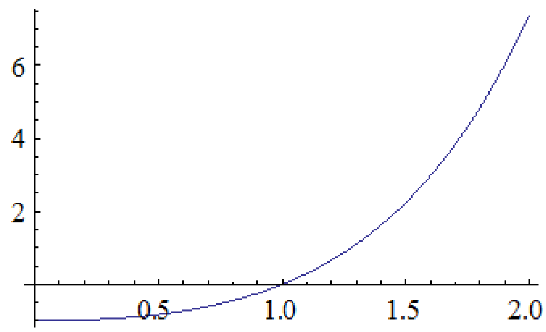

Figure 3.1: Convex function $f_1(t)$

$$f_1'(t) = te^t$$

and

$$f_1''(t) = e^t(t+1). \tag{3.2.6}$$

We can check that the function $f_1(t)$ is exponential in nature and strictly convex normalized because $f_1''(t) > 0 \ \forall \ t \in (0, \infty)$ and $f_1(1) = 0$ respectively. Further $f_1(t)$ is strictly increasing in $(0, \infty)$ as $f_1'(t) > 0$ in $(0, \infty)$.

For this exponential function, we have

$$C_{f_1}(P, Q) = G_{\exp}(P, Q) = \sum_{i=1}^{n} e^{\frac{p_i}{q_i}}(p_i - q_i). \tag{3.2.7}$$

In view of properties of $C_f(P, Q)$, we see that $G_{\exp}(P, Q)$ is positive and convex for the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$ and equal to zero (Non-degeneracy) or attains its minimum value when $p_i = q_i$. We can also see that $G_{\exp}(P, Q)$ is non- symmetric divergence w.r.t. P and Q because $G_{\exp}(P, Q) \neq G_{\exp}(Q, P)$.

3.2.2 Bounds of new exponential divergence measure

To estimate the new exponential divergence $G_{\exp}(P,Q)$, it would be very interesting to establish some upper and lower bounds. So in this subsection, we obtain bounds of the exponential divergence measure (3.2.7) in terms of other symmetric and non- symmetric divergence measures by using inequalities (3.2.2) and (3.2.4) respectively.

Proposition 3.2.1. *Let $P,Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, then we have*

$$\frac{e^{\alpha}(1+\alpha)^4}{8}\Delta(P,Q) \leq G_{\exp}(P,Q) \leq \frac{e^{\beta}(1+\beta)^4}{8}\Delta(P,Q), \tag{3.2.8}$$

where $\Delta(P,Q)$ is defined by (1.2.5).

Proof: Let us consider

$$f_2(t) = \frac{(t-1)^2}{t+1}, t \in (0,\infty) \tag{3.2.9}$$

and

$$f_2'(t) = \frac{(t-1)(t+3)}{(t+1)^2},$$

$$f_2''(t) = \frac{8}{(t+1)^3}. \tag{3.2.10}$$

Since $f_2''(t) > 0 \; \forall \; t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. For $f_2(t)$, we obtain

$$C_{f_2}(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} = \Delta(P,Q). \tag{3.2.11}$$

Now, let

$$g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{e^t(1+t)^4}{8},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (3.2.6) and (3.2.10) respectively and

$$g'(t) = \frac{e^t (1+t)^3 (5+t)}{8}.$$

It is clear that $g'(t) > 0$ for $t > 0$, therefore $g(t)$ is strictly increasing function in interval $(0, \infty)$. So

$$m = \inf_{t \in (\alpha,\beta)} g(t) = g(\alpha) = \frac{e^{\alpha} (1+\alpha)^4}{8} \tag{3.2.12}$$

and

$$M = \sup_{t \in (\alpha,\beta)} g(t) = g(\beta) = \frac{e^{\beta} (1+\beta)^4}{8}. \tag{3.2.13}$$

The result (3.2.8) is obtained by using (3.2.7), (3.2.11), (3.2.12), and (3.2.13) in inequalities (3.2.2).

Proposition 3.2.2. *Let $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$, then we have*

$$4e^{\alpha} (1+\alpha)\, \alpha^{\frac{3}{2}} h(P,Q) \leq G_{\exp}(P,Q) \leq 4e^{\beta} (1+\beta)\, \beta^{\frac{3}{2}} h(P,Q), \tag{3.2.14}$$

where $h(P,Q)$ is defined by (1.2.6).

Proof: Let us consider

$$f_2(t) = \frac{\left(1-\sqrt{t}\right)^2}{2}, t \in (0,\infty) \tag{3.2.15}$$

and

$$f_2'(t) = -\frac{\left(1-\sqrt{t}\right)}{2\sqrt{t}},$$

$$f_2''(t) = \frac{1}{4t^{\frac{3}{2}}}. \tag{3.2.16}$$

Since $f_2''(t) > 0 \ \forall \ t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. For $f_2(t)$, we have

$$C_{f_2}(P,Q) = \sum_{i=1}^{n} \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2} = h(P,Q). \tag{3.2.17}$$

Now, let

$$g(t) = \frac{f_1''(t)}{f_2''(t)} = 4e^t (1+t)\, t^{\frac{3}{2}},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (3.2.6) and (3.2.16) respectively and

$$g'(t) = 2e^t \sqrt{t}\,(3+t)(2t+1).$$

It is clear that $g'(t) > 0$ for $t > 0$, therefore $g(t)$ is strictly increasing function in interval $(0, \infty)$. So

$$m = \inf_{t \in (\alpha,\beta)} g(t) = g(\alpha) = 4e^{\alpha}(1+\alpha)\,\alpha^{\frac{3}{2}} \tag{3.2.18}$$

and

$$M = \sup_{t \in (\alpha,\beta)} g(t) = g(\beta) = 4e^{\beta}(1+\beta)\,\beta^{\frac{3}{2}}. \tag{3.2.19}$$

The result (3.2.14) is obtained by using (3.2.7), (3.2.17), (3.2.18), and (3.2.19) in inequalities (3.2.2).

In a similar procedure, we obtain the bounds of $G_{\exp}(P,Q)$ in terms of the other well known divergence measures. The results are as follows.

(a) If $f_2(t) = \frac{t}{2}\log t + \left(\frac{t+1}{2}\right)\log\frac{2}{t+1}$ and $I(P,Q)$ is (1.2.12), then we get

$$2e^{\alpha}\alpha(1+\alpha)^2 I(P,Q) \le G_{\exp}(P,Q) \le 2e^{\beta}\beta(1+\beta)^2 I(P,Q). \tag{3.2.20}$$

(b) If $f_2(t) = (t-1)\log t$ and $J(P,Q)$ is (1.2.10), then we get

$$e^{\alpha}\alpha^2 J(P,Q) \le G_{\exp}(P,Q) \le e^{\beta}\beta^2 J(P,Q). \tag{3.2.21}$$

(c) If $f_2(t) = \frac{(t-1)^2}{\sqrt{t}}$ and $E^*(P,Q)$ is (1.2.8), then we get

$$\frac{4e^{\alpha}(1+\alpha)\alpha^{\frac{5}{2}}}{3\alpha^2+2\alpha+3}E^*(P,Q) \le G_{\exp}(P,Q) \le \frac{4e^{\beta}(1+\beta)\beta^{\frac{5}{2}}}{3\beta^2+2\beta+3}E^*(P,Q). \qquad (3.2.22)$$

(d) If $f_2(t) = \frac{(t-1)^2(t+1)}{t}$ and $\psi(P,Q)$ is (1.2.9), then we get

$$\frac{e^{\alpha}\alpha^3}{2(\alpha^2-\alpha+1)}\psi(P,Q) \le G_{\exp}(P,Q) \le \frac{e^{\beta}\beta^3}{2(\beta^2-\beta+1)}\psi(P,Q). \qquad (3.2.23)$$

(e) If $f_2(t) = \left(\frac{t+1}{2}\right)\log\frac{t+1}{2\sqrt{t}}$ and $T(P,Q)$ is (1.2.11), then we get

$$\frac{4e^{\alpha}\alpha^2(1+\alpha)^2}{1+\alpha^2}T(P,Q) \le G_{\exp}(P,Q) \le \frac{4e^{\beta}\beta^2(1+\beta)^2}{1+\beta^2}T(P,Q). \qquad (3.2.24)$$

(f) If $f_2(t) = \frac{(t^2-1)^2}{2t^{\frac{3}{2}}}$ and $\psi_M(P,Q)$ is (1.2.15), then we get

$$\frac{8e^{\alpha}\alpha^{\frac{7}{2}}(1+\alpha)}{15\alpha^4+2\alpha^2+15}\psi_M(P,Q) \le G_{\exp}(P,Q) \le \frac{8e^{\beta}\beta^{\frac{7}{2}}(1+\beta)}{15\beta^4+2\beta^2+15}\psi_M(P,Q). \qquad (3.2.25)$$

(g) If $f_2(t) = (t-1)\log\frac{t+1}{2}$ and $J_R(P,Q)$ is (1.2.22), then we get

$$\frac{e^{\alpha}(1+\alpha)^3}{\alpha+3}J_R(P,Q) \le G_{\exp}(P,Q) \le \frac{e^{\beta}(1+\beta)^3}{\beta+3}J_R(P,Q). \qquad (3.2.26)$$

(h) If $f_2(t) = t\log t$ and $K(P,Q)$ is (1.2.18), then we get

$$\alpha(1+\alpha)e^{\alpha}K(P,Q) \le G_{\exp}(P,Q) \le \beta(1+\beta)e^{\beta}K(P,Q). \qquad (3.2.27)$$

(i) If $f_2(t) = \left(\frac{t+1}{2}\right)\log\frac{t+1}{2t}$ and $G(P,Q)$ is (1.2.20), then we get

$$2\alpha^2(1+\alpha)^2e^{\alpha}G(P,Q) \le G_{\exp}(P,Q) \le 2\beta^2(1+\beta)^2e^{\beta}G(P,Q). \qquad (3.2.28)$$

(j) If $f_2(t) = (t-1)^2$ and $\chi^2(P,Q)$ is (1.2.19), then we get

$$\frac{e^{\alpha}(1+\alpha)}{2}\chi^2(P,Q) \le G_{\exp}(P,Q) \le \frac{e^{\beta}(1+\beta)}{2}\chi^2(P,Q). \qquad (3.2.29)$$

(k) If $f_2(t) = t\log\frac{2t}{t+1}$ and $F(P,Q)$ is (1.2.21), then we get

$$\alpha e^{\alpha}(1+\alpha)^3 F(P,Q) \le G_{\exp}(P,Q) \le \alpha e^{\beta}(1+\beta)^3 F(P,Q). \tag{3.2.30}$$

Now, the following proposition gives the bounds of absolute value of $G_{\exp}(P,Q)$ in terms of the Variational distance by helping the inequalities (3.2.4).

Proposition 3.2.3. *Let $P,Q \in \Gamma_n$ and $0 < \alpha \le 1 \le \beta < \infty, \alpha \ne \beta$, then we have*

$$\alpha e^{\alpha} V(P,Q) \le |G_{\exp}|(P,Q) \le \beta e^{\beta} V(P,Q), \tag{3.2.31}$$

where $V(P,Q)$ is defined by (1.2.7).

Proof: Let us consider

$$f_1(t) = e^t(t-1), f_2(t) = |t-1| \ \forall\, t \in (0,\infty),$$

$$f_1'(t) = te^t, f_2'(t) = \begin{cases} -1 & \text{if } 0 < t < 1 \\ 1 & \text{if } 1 < t < \infty \end{cases}$$

and

$$f_1''(t) = e^t(t+1), f_2''(t) = 0.$$

We can see that both functions $f_1(t), f_2(t)$ are convex and normalized because $f_1''(t) \ge 0 \ \forall\, t > 0$, $f_2''(t) \ge 0 \ \forall\, t > 0$ but not at $t = 1$ and $f_1(1) = 0 = f_2(1)$ respectively.

Now for $f_1(t), f_2(t)$, we obtain the followings

$$C_{|f_1|}(P,Q) = \sum_{i=1}^{n} e^{\frac{p_i}{q_i}} |p_i - q_i| = |G_{\exp}|(P,Q) \tag{3.2.32}$$

and

$$C_{|f_2|}(P,Q) = \sum_{i=1}^{n} |p_i - q_i| = V(P,Q) \tag{3.2.33}$$

respectively.

Now, let $g(t) = \frac{|f_1'(t)|}{|f_2'(t)|} = \left|\frac{f_1'(t)}{f_2'(t)}\right| = |te^t| = te^t$, where $|f_2'(t)| = 1$ and $g'(t) = e^t(t+1) > 0$.

It is clear that $g(t)$ is strictly increasing in $(0, \infty)$, so

$$m = \inf_{t \in (\alpha,\beta)} g(t) = g(\alpha) = \alpha e^{\alpha}. \tag{3.2.34}$$

$$M = \sup_{t \in (\alpha,\beta)} g(t) = g(\beta) = \beta e^{\beta}. \tag{3.2.35}$$

The result (3.2.31) is obtained by using (3.2.32), (3.2.33), (3.2.34), and (3.2.35) in (3.2.4).

3.2.3 Numerical verification of obtained bounds

In this subsection, we take an example for calculating the divergences $\Delta(P,Q)$, $h(P,Q)$, $G(P,Q)$, $V(P,Q)$, $G_{exp}(P,Q)$, and $|G_{exp}|(P,Q)$ and then verify numerically the results (3.2.8), (3.2.14), (3.2.28), and (3.2.31) or verify the bounds of $G_{exp}(P,Q)$ and $|G_{exp}|(P,Q)$.

Example 3.2.1. *Let P be the binomial probability distribution with parameters $(n = 10, p = 0.7)$ and Q its approximated Poisson probability distribution with parameter $(\lambda = np = 7)$ for the random variable X , then we obtain*

x_i	0	1	2	3	4	5	6	7	8	9	10
$p_i \approx$	.0000059	.000137	.00144	.009	.036	.102	.200	.266	.233	.121	.0282
$q_i \approx$	.000911	.00638	.022	.052	.091	.177	.199	.149	.130	.101	.0709
$\frac{p_i}{q_i} \approx$	.00647	.0214	.0654	.173	.395	.871	1.005	1.785	1.792	1.198	.397

Table 3.1: Evaluation of Binomial and Poisson probability distributions

$$\alpha(= .00647) \leq \frac{p_i}{q_i} \leq \beta(= 1.792). \tag{3.2.36}$$

$$\Delta(P,Q)=\sum_{i=1}^{11}\frac{(p_i-q_i)^2}{p_i+q_i}\approx .1812. \tag{3.2.37}$$

$$h(P,Q)=\sum_{i=1}^{11}\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2}\approx .0502. \tag{3.2.38}$$

$$G(P,Q)=\sum_{i=1}^{11}\frac{p_i+q_i}{2}\log\left(\frac{p_i+q_i}{2p_i}\right)\approx .0746. \tag{3.2.39}$$

$$V(P,Q)=\sum_{i=1}^{11}|p_i-q_i|\approx 0.4844. \tag{3.2.40}$$

$$G_{\exp}(P,Q)=\sum_{i=1}^{11}e^{\frac{p_i}{q_i}}(p_i-q_i)\approx .97971. \tag{3.2.41}$$

$$|G_{\exp}|(P,Q)=\sum_{i=1}^{11}e^{\frac{p_i}{q_i}}|p_i-q_i|\approx 1.78872. \tag{3.2.42}$$

Put the approximated values from (3.2.36) to (3.2.42) in results (3.2.8), (3.2.14), (3.2.28), and (3.2.31) respectively and get the following results

$.0233\leq .97971\,(=G_{\exp}(P,Q))\leq 8.260, 1.508\times 10^{-4}\leq .97971\,(=G_{\exp}(P,Q))\leq 8.071,$

$6.367\times 10^{-6}\leq .97971\,(=G_{\exp}(P,Q))\leq 22.414,$ and

$3.154\times 10^{-3}\leq 1.78872\,(=|G_{\exp}|(P,Q))\leq 5.2095$

respectively.

Hence verified the bounds of $G_{\exp}(P,Q)$ and $|G_{\exp}|(P,Q)$ in terms of $\Delta(P,Q)$, $h(P,Q)$, $G(P,Q)$, and $V(P,Q)$ for $p=0.7$, where all divergence measures have their usual meanings.

Similarly, we can verify the bounds of $G_{\exp}(P,Q)$ in terms of other divergences or can verify the other inequalities for different values of p and q and for other discrete probability distributions as well, like; Negative binomial, Geometric, uniform etc.

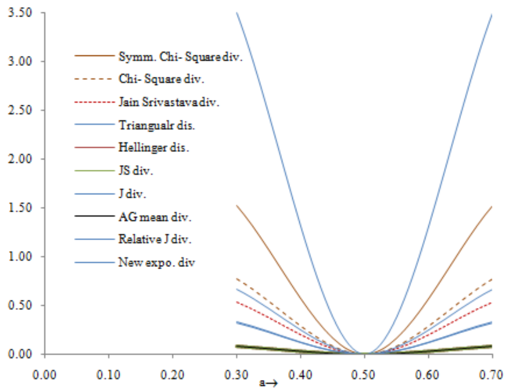

Figure 3.2: Comparison of the well known divergences with $G_{\exp}(P,Q)$

In Figure 3.2, we have considered $p_i = (a, 1-a), q_i = (1-a, a)$, where $a \in (0,1)$. It is clear from Figure that the new exponential divergence $G_{\exp}(P,Q)$ has a steeper slope than $\psi(P,Q)$, $\chi^2(P,Q)$, $E^*(P,Q)$, $\Delta(P,Q)$, $h(P,Q)$, $I(P,Q)$, $J(P,Q)$, $T(P,Q)$, and $J_R(P,Q)$.

3.3 New Divergence Measure and Bounds

In this section, a new information divergence measure of Csiszar's class is proposed and obtain the bounds of this new divergence in terms of the other well known divergences $\chi^2(P,Q)$, $K(P,Q)$, $h(P,Q)$, and $R_a(P,Q)$ (as special cases of Relative information of type 's', i.e., $K_s(P,Q)$ (1.2.23)) by using information inequalities given by theorem 3.3.1 (Taneja and Kumar [104]). The results are on the similar lines to the results presented by Taneja and Kumar [104].

Theorem 3.3.1. *Let $f : (\alpha, \beta) \subset (0, \infty) \to R$ be a mapping which is normalized, i.e., $f(1) = 0$ and suppose that*
(i). f is twice differentiable on (α, β), $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$.
(ii). There exist real constants m, M such that $m < M$ and $m \leq t^{2-s} f''(t) \leq$

M $\forall t \in (\alpha, \beta)$, $s \in R$.
If $P, Q \in \Gamma_n$ is such that $\alpha \leq \frac{p_i}{q_i} \leq \beta$ $\forall$ $i = 1, 2, 3..., n$, then we have the following inequalities

$$mK_s(P,Q) \leq C_f(P,Q) \leq MK_s(P,Q) \tag{3.3.1}$$

and

$$\begin{aligned} &m\left[\eta_s(P,Q) - K_s(P,Q)\right] \leq E_{C_{f'}}(P,Q) - C_f(P,Q) \\ &\leq M\left[\eta_s(P,Q) - K_s(P,Q)\right], \end{aligned} \tag{3.3.2}$$

where $C_f(P,Q)$, $E_{C_{f'}}(P,Q)$ have their usual meanings respectively, earlier mentioned and

$$\begin{aligned} K_s(P,Q) &= \left[s(s-1)\right]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right] \\ &= \textit{the Relative information of type } 's' (1.2.23), \end{aligned} \tag{3.3.3}$$

where $s \neq 0, 1$ and $s \in R$. Particularly

$$\lim_{s\to 1} K_s(P,Q) = K(P,Q), \lim_{s\to 0} K_s(P,Q) = K(Q,P), \tag{3.3.4}$$

where $K(P,Q)$ is well known Relative information.

$$\begin{aligned} \eta_s(P,Q) &= C_{K_s'}\left(\frac{P^2}{Q}, P\right) - C_{K_s'}(P,Q) \\ &= (s-1)^{-1}\sum_{i=1}^{n}(p_i - q_i)\left(\frac{p_i}{q_i}\right)^{s-1}, s \neq 1. \end{aligned} \tag{3.3.5}$$

3.3.1 New divergence measure

In this subsection, we introduce a new divergence measure corresponding to new convex function, and will study the properties.

Let $f : (0, \infty) \to R$ be a mapping, defined as

$$f(t) = \frac{(t-1)^4}{t}, \forall\, t \in (0, \infty), \tag{3.3.6}$$

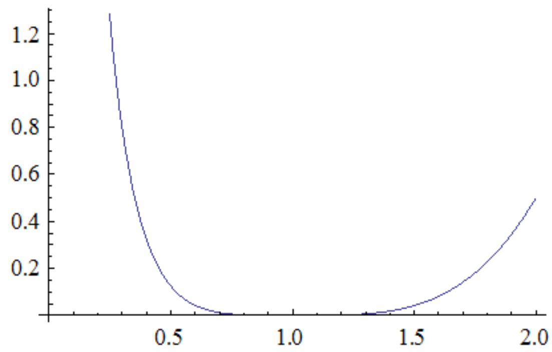

Figure 3.3: Convex function $f(t)$

and

$$f'(t) = \frac{(t-1)^3(3t+1)}{t^2},$$

$$f''(t) = \frac{2(t-1)^2(3t^2+2t+1)}{t^3}. \tag{3.3.7}$$

Now for the function $f(t)$, we obtain the following new divergence measure

$$C_f(P,Q) = V^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i-q_i)^4}{p_i q_i^2}. \tag{3.3.8}$$

Since $f''(t) \geq 0$ and $f(1) = 0$, therefore $f(t)$ is convex and normalized respectively. We can also see that $f'(t) < 0$ at $(0,1)$ and > 0 at $(1,\infty)$, i.e., $f(t)$ is strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, and $f'(1) = 0$. Moreover by the properties of $C_f(P,Q)$, we see that $V^*(P,Q) > 0$ and convex in the pair of probability distribution $(P,Q) \in \Gamma_n \times \Gamma_n$ and $V^*(P,Q) = 0$ (Non- degeneracy) if $P = Q$ or attains its minimum value when $p_i = q_i$. We can also see that $V^*(P,Q)$ is non- symmetric divergence w.r.t. P and Q as $V^*(P,Q) \neq V^*(Q,P)$.

3.3.2 Bounds of new divergence measure

In this subsection, we derive bounds of $V^*(P,Q)$ by using the inequalities (3.3.1) and (3.3.2). Actually, these all propositions are the special cases on Relative information of type s , i.e., $K_s(P,Q)$ at $s=2, s=1, s=\frac{1}{2}, s=0$,and $s=-1$.

Proposition 3.3.1. *For $P,Q \in \Gamma_n$ and $0<\alpha\le 1\le \beta<\infty$ with $\alpha \neq \beta$, we have*
(i) If $0<\alpha<1$, then

$$0 \le V^*(P,Q) \le max\left[\frac{(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^3}, \frac{(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^3}\right]\chi^2(P,Q). \tag{3.3.9}$$

$$\begin{aligned} 0 &\le V^*_\rho(P,Q) - V^*(P,Q) \\ &\le max\left[\frac{(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^3}, \frac{(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^3}\right]\chi^2(P,Q). \end{aligned} \tag{3.3.10}$$

(ii) If $\alpha=1$, then

$$0 \le V^*(P,Q) \le \frac{(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^3}\chi^2(P,Q). \tag{3.3.11}$$

$$0 \le V^*_\rho(P,Q) - V^*(P,Q) \le \frac{(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^3}\chi^2(P,Q), \tag{3.3.12}$$

where $\chi^2(P,Q)$ is defined by (1.2.19).

Proof: Firstly put $s=2$ in (3.3.3) and (3.3.5), we get the followings respectively

$$\begin{aligned} K_2(P,Q) &= \frac{1}{2}\sum_{i=1}^{n}\frac{p_i^2}{q_i} - 1 = \frac{1}{2}\sum_{i=1}^{n}\left[\frac{p_i^2}{q_i} - 2p_i + q_i\right] \\ &= \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i} = \frac{1}{2}\chi^2(P,Q). \end{aligned} \tag{3.3.13}$$

$$\begin{aligned}\eta_2(P,Q) &= \sum_{i=1}^{n} \frac{(p_i - q_i)\,p_i}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 = \sum_{i=1}^{n} \left[\frac{p_i^2}{q_i} - 2p_i + q_i\right] \\ &= \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \chi^2(P,Q).\end{aligned} \tag{3.3.14}$$

Now for $f'(t)$, we obtain

$$E_{C_{f'}}(P,Q) = V_\rho^*(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^4 (3p_i + q_i)}{(p_i q_i)^2}. \tag{3.3.15}$$

Now, let $g(t) = f''(t) = \frac{2(t-1)^2\left(3t^2+2t+1\right)}{t^3}$ (after putting $s = 2$ in $t^{2-s}f''(t)$) and

$$g'(t) = \frac{6\,(t^4 - 1)}{t^4}, g''(t) = \frac{24}{t^5}.$$

If $g'(t) = 0 \Rightarrow t^4 - 1 = 0 \Rightarrow t = 1, -1$ and $g''(1) = 24 > 0$.

It is clear that $g(t)$ is strictly decreasing on $(0,1)$ and increasing on $[1,\infty)$ and $g(t)$ has minimum value at $t = 1$, so

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = 0, \tag{3.3.16}$$

and

(i) If $0 < \alpha < 1$, then

$$\begin{aligned}M &= \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] \\ &= max\left[\frac{2(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^3}, \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^3}\right].\end{aligned} \tag{3.3.17}$$

(ii) If $\alpha = 1$, then

$$M = \sup_{t\in[1,\beta)} g(t) = g(\beta) = \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^3}. \tag{3.3.18}$$

The results (3.3.9), (3.3.10), (3.3.11), and (3.3.12) are obtained by using (3.3.8), (3.3.13), (3.3.14), (3.3.15), (3.3.16), (3.3.17) and (3.3.18) in (3.3.1) and (3.3.2).

Proposition 3.3.2. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*
(i) If $0 < \alpha < 1$, then

$$0 \leq V^* (P, Q) \leq max \left[\frac{2 (\alpha - 1)^2 (3\alpha^2 + 2\alpha + 1)}{\alpha^2}, \frac{2 (\beta - 1)^2 (3\beta^2 + 2\beta + 1)}{\beta^2} \right] K (P, Q) . \tag{3.3.19}$$

$$\begin{aligned} 0 &\leq V^*_\rho (P, Q) - V^* (P, Q) \\ &\leq max \left[\frac{2 (\alpha - 1)^2 (3\alpha^2 + 2\alpha + 1)}{\alpha^2}, \frac{2 (\beta - 1)^2 (3\beta^2 + 2\beta + 1)}{\beta^2} \right] K (Q, P) . \end{aligned} \tag{3.3.20}$$

(ii) If $\alpha = 1$, then

$$0 \leq V^* (P, Q) \leq \frac{2 (\beta - 1)^2 (3\beta^2 + 2\beta + 1)}{\beta^2} K (P, Q) . \tag{3.3.21}$$

$$0 \leq V^*_\rho (P, Q) - V^* (P, Q) \leq \frac{2 (\beta - 1)^2 (3\beta^2 + 2\beta + 1)}{\beta^2} K (Q, P) , \tag{3.3.22}$$

where $K (P, Q)$ is defined by (1.2.18).

Proof: Firstly put $s = 1$ in (3.3.3) and (3.3.5), we get the followings respectively

$$\lim_{s \to 1} K_s (P, Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} = K (P, Q) . \tag{3.3.23}$$

$$\lim_{s \to 1} \eta_s (P, Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} + \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i} = K (P, Q) + K (Q, P) . \tag{3.3.24}$$

Now, let $g (t) = t f'' (t) = \frac{2(t-1)^2 \left(3t^2+2t+1\right)}{t^2}$ (after putting $s = 1$ in $t^{2-s} f'' (t)$) and

$$g' (t) = \frac{4 (t - 1) (3t^3 + t^2 + t + 1)}{t^3}, g'' (t) = \frac{12 (t^4 + 1)}{t^4}.$$

If $g' (t) = 0 \Rightarrow (t - 1) (3t^3 + t^2 + t + 1) = 0 \Rightarrow t = 1, -0.63$ and $g'' (1) = 24 > 0$.

It is clear that $g (t)$ is strictly decreasing on $(0, 1)$ and increasing on $[1, \infty)$ and

$g(t)$ has minimum value at $t = 1$, so

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = 0, \tag{3.3.25}$$

and

(i) If $0 < \alpha < 1$, then

$$\begin{aligned} M &= \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] \\ &= max\left[\frac{2(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^2}, \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^2}\right]. \end{aligned} \tag{3.3.26}$$

(ii) If $\alpha = 1$, then

$$M = \sup_{t\in[1,\beta)} g(t) = g(\beta) = \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^2}. \tag{3.3.27}$$

The results (3.3.19), (3.3.20), (3.3.21), and (3.3.22) are obtained by using (3.3.8), (3.3.15), (3.3.23), (3.3.24), (3.3.25), (3.3.26) and (3.3.27) in (3.3.1) and (3.3.2).

Proposition 3.3.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*
(i) If $0 < \alpha < 1$, then

$$\begin{aligned} &0 \leq V^*(P,Q) \\ &\leq max\left[\frac{8(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^{\frac{3}{2}}}, \frac{8(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^{\frac{3}{2}}}\right] h(P,Q). \end{aligned} \tag{3.3.28}$$

$$\begin{aligned} &0 \leq V_\rho^*(P,Q) - V^*(P,Q) \\ &\leq max\left[\frac{8(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^{\frac{3}{2}}}, \frac{8(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^{\frac{3}{2}}}\right] \\ &\left[\frac{1}{2}\Big(R_{\frac{3}{2}}(Q,P) - B(P,Q)\Big) - h(P,Q)\right]. \end{aligned} \tag{3.3.29}$$

(ii) If $\alpha = 1$, then

$$0 \le V^*(P,Q) \le \frac{8(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^{\frac{3}{2}}} h(P,Q). \tag{3.3.30}$$

$$\begin{aligned} &0 \le V^*_\rho(P,Q) - V^*(P,Q) \\ &\le \frac{8(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^{\frac{3}{2}}} \left[\frac{1}{2}\Big(R_{\frac{3}{2}}(Q,P) - B(P,Q)\Big) - h(P,Q)\right], \end{aligned} \tag{3.3.31}$$

where $h(P,Q)$, $R_a(P,Q)$ are defined by (1.2.6), (1.2.27) respectively, and $B(P,Q) = \sum_{i=1}^{n}\sqrt{p_i q_i}$ is the well known Bhattacharya distance [10].

Proof: Firstly put $s = \frac{1}{2}$ in (3.3.3) and (3.3.5), we get the followings respectively

$$\begin{aligned} K_{\frac{1}{2}}(P,Q) &= 4\left[1 - \sum_{i=1}^{n}\sqrt{p_i q_i}\right] = 2\left[2 - 2\sum_{i=1}^{n}\sqrt{p_i q_i}\right] = 2\sum_{i=1}^{n}\left[p_i + q_i - 2\sqrt{p_i q_i}\right] \\ &= 4\sum_{i=1}^{n}\frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2} = 4h(P,Q). \end{aligned} \tag{3.3.32}$$

$$\begin{aligned} \eta_{\frac{1}{2}}(P,Q) &= 2\sum_{i=1}^{n}(q_i - p_i)\sqrt{\frac{q_i}{p_i}} = 2\sum_{i=1}^{n}\left(\frac{q_i^{\frac{3}{2}}}{p_i^{\frac{1}{2}}} - \sqrt{p_i q_i}\right) \\ &= 2\left[R_{\frac{3}{2}}(Q,P) - B(P,Q)\right]. \end{aligned} \tag{3.3.33}$$

Now, let $g(t) = t^{\frac{3}{2}} f''(t) = \frac{2(t-1)^2(3t^2+2t+1)}{t^{\frac{3}{2}}}$ (after putting $s = \frac{1}{2}$ in $t^{2-s} f''(t)$) and

$$g'(t) = \frac{3(t-1)(5t^3+t^2+t+1)}{t^{\frac{5}{2}}}, g''(t) = \frac{3(15t^4 - 4t^3 + 5)}{2t^{\frac{7}{2}}}.$$

If $g'(t) = 0 \Rightarrow (t-1)(5t^3+t^2+t+1) = 0 \Rightarrow t = 1, -0.53$ and $g''(1) = 24 > 0$. It is clear that $g(t)$ is strictly decreasing on $(0,1)$ and increasing on $[1,\infty)$ and $g(t)$ has minimum value at $t = 1$, so

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = 0, \tag{3.3.34}$$

and

(i) If $0 < \alpha < 1$, then

$$\begin{aligned} M &= \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] \\ &= max\left[\frac{2(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha^{\frac{3}{2}}}, \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^{\frac{3}{2}}}\right]. \end{aligned} \tag{3.3.35}$$

(ii) If $\alpha = 1$, then

$$M = \sup_{t\in[1,\beta)} g(t) = g(\beta) = \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta^{\frac{3}{2}}}. \tag{3.3.36}$$

The results (3.3.28), (3.3.29), (3.3.30), and (3.3.31) are obtained by using (3.3.8), (3.3.15), (3.3.32), (3.3.33), (3.3.34), (3.3.35) and (3.3.36) in (3.3.1) and (3.3.2).

Similarly, for $s = 0$, we obtain

(i) If $0 < \alpha < 1$, then

$$\begin{aligned} &0 \le V^*(P,Q) \\ &\le max\left[\frac{2(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha}, \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta}\right] K(Q,P). \end{aligned} \tag{3.3.37}$$

$$\begin{aligned} &0 \le V_\rho^*(P,Q) - V^*(P,Q) \\ &\le max\left[\frac{2(\alpha-1)^2(3\alpha^2+2\alpha+1)}{\alpha}, \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta}\right] \\ &\left[\chi^2(Q,P) - K(Q,P)\right]. \end{aligned} \tag{3.3.38}$$

(ii) If $\alpha = 1$, then

$$0 \le V^*(P,Q) \le \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta} K(Q,P). \tag{3.3.39}$$

$$\begin{aligned} &0 \le V_\rho^*(P,Q) - V^*(P,Q) \\ &\le \frac{2(\beta-1)^2(3\beta^2+2\beta+1)}{\beta}\left[\chi^2(Q,P) - K(Q,P)\right]. \end{aligned} \tag{3.3.40}$$

Now for $s = -1$, we obtain

(i) If $0 < \alpha < 1$, then

$$0 \leq V^*(P,Q) \leq max\left[(\alpha-1)^2\left(3\alpha^2+2\alpha+1\right),(\beta-1)^2\left(3\beta^2+2\beta+1\right)\right]\chi^2(Q,P). \tag{3.3.41}$$

$$\begin{aligned}0 &\leq V_\rho^*(P,Q) - V^*(P,Q) \\ &\leq max\left[(\alpha-1)^2\left(3\alpha^2+2\alpha+1\right),(\beta-1)^2\left(3\beta^2+2\beta+1\right)\right] \\ &\left[R_3(Q,P) - R_2(Q,P) - \chi^2(Q,P)\right].\end{aligned} \tag{3.3.42}$$

(ii) If $\alpha = 1$, then

$$0 \leq V^*(P,Q) \leq (\beta-1)^2\left(3\beta^2+2\beta+1\right)\chi^2(Q,P). \tag{3.3.43}$$

$$\begin{aligned}0 &\leq V_\rho^*(P,Q) - V^*(P,Q) \\ &\leq (\beta-1)^2\left(3\beta^2+2\beta+1\right)\left[R_3(Q,P) - R_2(Q,P) - \chi^2(Q,P)\right].\end{aligned} \tag{3.3.44}$$

In Figure 3.4, we have considered $p_i = (a, 1-a), q_i = (1-a, a)$, where $a \in (0,1)$.

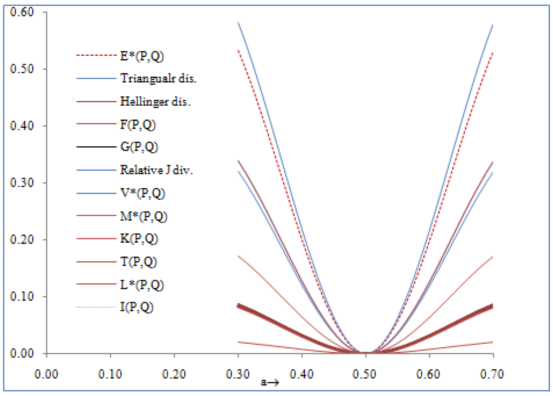

Figure 3.4: Comparison of the well known divergences with $V^*(P,Q)$

It is clear from Figure that the new divergence measure $V^*(P,Q)$ has a steeper slope than $E^*(P,Q)$, $\Delta(P,Q)$, $h(P,Q)$, $F(P,Q)$, $G(P,Q)$, $J_R(P,Q)$, $M^*(P,Q)$, $K(P,Q)$, $T(P,Q)$, $L^*(P,Q)$, and $I(P,Q)$.

3.4 Conclusion

In this chapter, we introduced two different non- symmetric divergence measures of Csiszar's class. One of them is by exponential convex function and other is by algebraic convex function. Further evaluated bounds of these divergence measures separately by using well known information inequalities on $C_f(P, Q)$. Numerical verification by taking Binomial and Poisson distributions, has been done as well. By comparison graph, we compared new divergence measure with well known divergence measures.

4

SERIES OF NEW DIVERGENCE MEASURES AND APPLICATIONS

4.1 Introduction

In chapter 3, new divergence measures of Csiszar's class have been introduced by suitably defined the convex functions and did a detail study of these divergences. In this chapter, different series of divergence measures of Csiszar's class are proposed.

This chapter is organized as follows: This chapter contains 2 sections excluding Introduction and Conclusion. Since there are many series of divergences and their studies, therefore we make two sections with the same title differing by Roman number I and II respectively. Sections 4.2 and 4.3 introduce different new series of convex functions and corresponding new series of divergence measures of Csiszar's class, further obtain the bounds of a particular divergence of a series separately by using well known information inequalities together with numerical verifications. Intra relations among new series of divergences obtain separately.

4.2 Series of New Divergence Measures and Applications-I

In this section, we introduce new series of divergence measures as a family of Csiszar's generalized divergence, characterize the properties of convex functions and divergences, compare several divergences, and derive important and interesting intra relation among divergences of these new series. Also get the bounds of a particular member of that series together with numerical verification of obtained bounds.

4.2.1 Series of convex functions and properties- I

In this subsection, we develop some new series of convex functions and study their properties. For this, firstly let $f : (0, \infty) \to R$ be a mapping defined as

$$f_m(t) = \frac{(t^2 - 1)^{2m}}{t^{2m-1}}, m = 1, 2, 3... \tag{4.2.1}$$

and

$$f'_m(t) = \frac{(t^2 - 1)^{2m-1}\left[t^2(2m+1) + 2m - 1\right]}{t^{2m}}, \tag{4.2.2}$$

$$f''_m(t) = \frac{2m(t^2 - 1)^{2m-2}}{t^{2m+1}}\left[t^4(2m+1) + 4t^2(m-1) + 2m - 1\right]. \tag{4.2.3}$$

Since $f''_m(t) \geq 0$ for $t > 0$ and $m = 1, 2, ...$, therefore $f_m(t)$ are convex functions for each m.

Now, from (4.2.1), we get the following new convex functions at $m = 1, 2, 3...$ respectively.

$$f_1(t) = \frac{\left(t^2-1\right)^2}{t}, f_2(t) = \frac{\left(t^2-1\right)^4}{t^3}, f_3(t) = \frac{\left(t^2-1\right)^6}{t^5}... \tag{4.2.4}$$

Since, we know that the linear combination of convex functions is also a convex function, i.e., $a_1 f_1(t) + a_2 f_2(t) + a_3 f_3(t) + ...$ is a convex function as well, where $a_1, a_2, a_3, ...$ are positive constants. Therefore, we have following two cases to obtain new series of convex functions.

(i) If we take $a_1 = a_2 = 1, a_3 = a_4 = a_5 = ... = 0$, then we have

$$f_{1,2}(t) = f_1(t) + f_2(t) = \frac{\left(t^2-1\right)^2}{t} + \frac{\left(t^2-1\right)^4}{t^3} = \frac{\left(t^2-1\right)^2\left(t^4-t^2+1\right)}{t^3}. \tag{4.2.5}$$

Similarly, if we take $a_2 = a_3 = 1, a_1 = a_4 = a_5 = ... = 0$, then we have

$$f_{2,3}(t) = f_2(t) + f_3(t) = \frac{\left(t^2-1\right)^4}{t^3} + \frac{\left(t^2-1\right)^6}{t^5} = \frac{\left(t^2-1\right)^4\left(t^4-t^2+1\right)}{t^5}. \tag{4.2.6}$$

In this way, we can write for $m = 1, 2, 3...$

$$\begin{aligned} f_{m,m+1}(t) &= f_m(t) + f_{m+1}(t) \\ &= \frac{\left(t^2-1\right)^{2m}}{t^{2m-1}} + \frac{\left(t^2-1\right)^{2m+2}}{t^{2m+1}} = \frac{\left(t^2-1\right)^{2m}\left(t^4-t^2+1\right)}{t^{2m+1}}. \end{aligned} \tag{4.2.7}$$

(ii) If we take $a_1 = 1, a_2 = \log_e b, a_3 = \frac{(\log_e b)^2}{2!}, a_4 = \frac{(\log_e b)^3}{3!}, ..., b > 1$, then we have

$$\begin{aligned} g_1(t) &= f_1(t) + (\log_e b) f_2(t) + \frac{(\log_e b)^2}{2!} f_3(t) + ... \\ &= \frac{\left(t^2-1\right)^2}{t} + (\log_e b)\frac{\left(t^2-1\right)^4}{t^3} + \frac{(\log_e b)^2}{2!}\frac{\left(t^2-1\right)^6}{t^5} + ... \\ &= \frac{\left(t^2-1\right)^2}{t}\left[1 + (\log_e b)\frac{\left(t^2-1\right)^2}{t^2} + \frac{(\log_e b)^2}{2!}\frac{\left(t^2-1\right)^4}{t^4} + ...\right] \\ &= \frac{\left(t^2-1\right)^2}{t} b^{\frac{\left(t^2-1\right)^2}{t^2}}, b > 1. \end{aligned} \tag{4.2.8}$$

Similarly, if we take $a_1 = 0, a_2 = 1, a_3 = \log_e b, a_4 = \frac{(\log_e b)^2}{2!}, a_5 = \frac{(\log_e b)^3}{3!}, ..., b > 1,$

then we have

$$\begin{aligned} g_2(t) &= \frac{(t^2-1)^4}{t^3} + (\log_e b)\frac{(t^2-1)^6}{t^5} + \frac{(\log_e b)^2}{2!}\frac{(t^2-1)^8}{t^7} + ..., b > 1 \\ &= \frac{(t^2-1)^4}{t^3}\left[1 + (\log_e b)\frac{(t^2-1)^2}{t^2} + \frac{(\log_e b)^2}{2!}\frac{(t^2-1)^4}{t^4} + ...\right] \\ &= \frac{(t^2-1)^4}{t^3} b^{\frac{(t^2-1)^2}{t^2}}, b > 1. \end{aligned} \tag{4.2.9}$$

In this way, we can write

$$g_m(t) = \frac{(t^2-1)^{2m}}{t^{2m-1}} b^{\frac{(t^2-1)^2}{t^2}}, b > 1, m = 1, 2, 3, ... \tag{4.2.10}$$

Remark 4.2.1. *If we take $b = e \approx 2.71828$ then from (4.2.10), we obtain the following series.*

$$g_m(t) = \frac{(t^2-1)^{2m}}{t^{2m-1}} e^{\frac{(t^2-1)^2}{t^2}} = \frac{(t^2-1)^{2m}}{t^{2m-1}} \exp\frac{(t^2-1)^2}{t^2}, m = 1, 2, 3, ... \tag{4.2.11}$$

Properties of convex functions defined by (4.2.1), (4.2.7) and (4.2.11), are as follows.

- Since $f_m(1) = 0 = f_{m,m+1}(1) = g_m(1) \Rightarrow f_m(t), f_{m,m+1}(t)$ and $g_m(t)$ are normalized functions for each m.
- Since $f'_m(t) < 0$ at $(0,1)$ and > 0 at $(1,\infty) \Rightarrow f_m(t)$ are strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, for each value of m and $f'_m(1) = 0$.

Figures 4.1, 4.2, and 4.3 shows the behavior of convex functions and shows that $f_m(t), f_{m,m+1}(t)$, and $g_m(t)$ has a stepper slope for increasing values of m respectively.

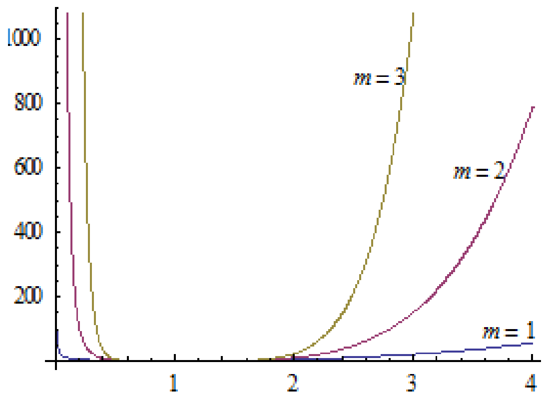

Figure 4.1: Behavior of convex functions $f_m(t)$

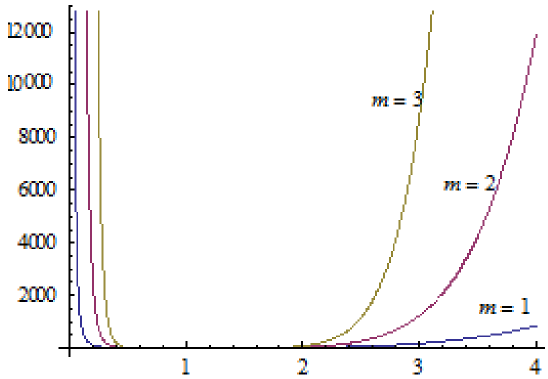

Figure 4.2: Behavior of convex functions $f_{m,m+1}(t)$

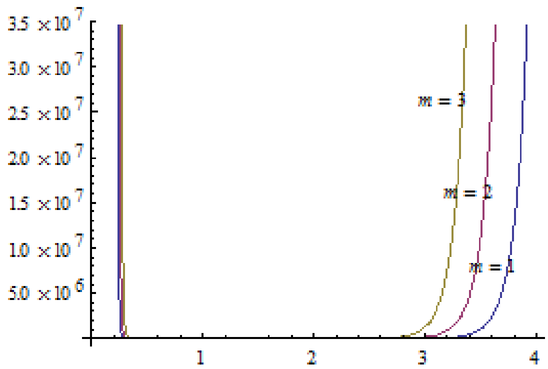

Figure 4.3: Behavior of convex functions $g_m(t)$

4.2.2 New series of divergence measures- I

In this subsection, we obtain new series of divergence measures of Csiszar's class corresponding to series of convex functions defined in subsection 4.2.1 and study their properties. Now for convex functions (4.2.1), we get the following new series of divergences.

$$C_f(P,Q) = \gamma_m(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2m}}{p_i^{2m-1} q_i^{2m}}, m = 1,2,3... \tag{4.2.12}$$

$$\gamma_1(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^2}{p_i q_i^2}, \gamma_2(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^4}{p_i^3 q_i^4}, ... \tag{4.2.13}$$

where $C_f(P,Q)$ is well known Csiszar's generalized divergence.

Similarly for (4.2.7), we get the following new series of divergences.

$$C_f(P,Q) = \eta_m(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2m} \left(p_i^4 - p_i^2 q_i^2 + q_i^4\right)}{p_i^{2m+1} q_i^{2m+2}}, m = 1,2,... \tag{4.2.14}$$

$$\eta_1(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^2 \left(p_i^4 - p_i^2 q_i^2 + q_i^4\right)}{p_i^3 q_i^4}, \tag{4.2.15}$$

$$\eta_2(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^4 \left(p_i^4 - p_i^2 q_i^2 + q_i^4\right)}{p_i^5 q_i^6}, ... \tag{4.2.16}$$

Similarly for (4.2.11), we get the following new series of divergences.

$$C_f(P,Q) = \rho_m(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2m}}{p_i^{2m-1} q_i^{2m}} \exp \frac{\left(p_i^2 - q_i^2\right)^2}{\left(p_i q_i\right)^2}, m = 1,2,... \tag{4.2.17}$$

$$\rho_1(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^2}{p_i q_i^2} \exp \frac{\left(p_i^2 - q_i^2\right)^2}{\left(p_i q_i\right)^2}, \tag{4.2.18}$$

$$\rho_2(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^4}{p_i^3 q_i^4} \exp \frac{\left(p_i^2 - q_i^2\right)^2}{\left(p_i q_i\right)^2}, ... \tag{4.2.19}$$

Properties of divergences defined by (4.2.12), (4.2.14) and (4.2.17), are as follows.

- In view of properties of $C_f(P,Q)$, we can say that $\gamma_m(P,Q), \eta_m(P,Q), \rho_m(P,Q) >$

0 and are convex in the pair of probability distribution $P, Q \in \Gamma_n$.

• $\gamma_m(P,Q) = 0 = \eta_m(P,Q) = \rho_m(P,Q)$ if $P = Q$ or $p_i = q_i$ (attains its minimum value).

• Since $\gamma_m(P,Q) \neq \gamma_m(Q,P), \eta_m(P,Q) \neq \eta_m(Q,P), \rho_m(P,Q) \neq \rho_m(Q,P) \Rightarrow$ $\gamma_m(P,Q), \eta_m(P,Q)$, $\rho_m(P,Q)$ are non- symmetric divergence measures.

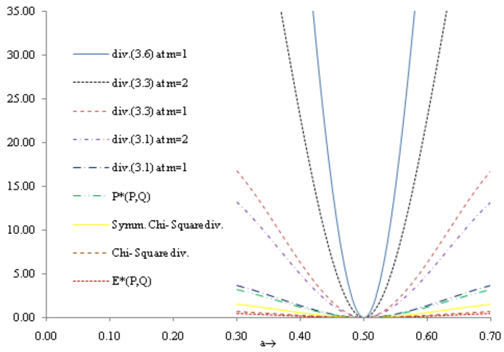

Figure 4.4: Comparison of the well known divergences with new series of divergences

Figure 4.4 shows the behavior of $\gamma_1(P,Q)$, $\gamma_2(P,Q)$, $\eta_1(P,Q)$, $\eta_2(P,Q)$, $\rho_1(P,Q)$, $P^*(P,Q)$, $\psi(P,Q)$, $\chi^2(P,Q)$, and $E^*(P,Q)$. We have considered $p_i = (a, 1-a), q_i = (1-a, a)$, where $a \in (0,1)$. It is clear from Figure that the new divergences $\gamma_1(P,Q)$, $\gamma_2(P,Q)$, $\eta_1(P,Q)$, $\eta_2(P,Q)$, and $\rho_1(P,Q)$ has a steeper slope than others.

4.2.3 Intra relation and bounds- I

First, we derive an intra relation among new series of divergence measures (4.2.12), (4.2.14), and (4.2.17), as follows.

Proposition 4.2.1. *Let $P, Q \in \Gamma_n$, then we have the following new intra relation.*

$$\gamma_m(P,Q) \leq \eta_m(P,Q) \leq \rho_m(P,Q), \tag{4.2.20}$$

where $m = 1, 2, 3...$ and $\gamma_m(P,Q)$, $\eta_m(P,Q)$, and $\rho_m(P,Q)$ are given by (4.2.12), (4.2.14), and (4.2.17) respectively.

Proof: Since

$$\frac{(t^2-1)^{2m}(t^4-t^2+1)}{t^{2m+1}} = \frac{(t^2-1)^{2m}}{t^{2m-1}} + \frac{(t^2-1)^{2m+2}}{t^{2m+1}}$$

and

$$\frac{(t^2-1)^{2m}}{t^{2m-1}} \exp\frac{(t^2-1)^2}{t^2} = \frac{(t^2-1)^{2m}}{t^{2m-1}}\left[1+\frac{(t^2-1)^2}{t^2}+\frac{(t^2-1)^4}{2!t^4}+...\right].$$

Therefore, for $m = 1, 2, 3...$ and $t > 0$, we have the following inequalities.

$$\begin{aligned}\frac{(t^2-1)^{2m}}{t^{2m-1}} &\le \frac{(t^2-1)^{2m}}{t^{2m-1}} + \frac{(t^2-1)^{2m+2}}{t^{2m+1}}\\ &\le \frac{(t^2-1)^{2m}}{t^{2m-1}}\left[1+\frac{(t^2-1)^2}{t^2}+\frac{(t^2-1)^4}{2!t^4}+...\right].\end{aligned} \tag{4.2.21}$$

Now put $t = \frac{p_i}{q_i}, i = 1, 2, 3..., n$ in (4.2.21), multiply by q_i and then sum over all $i = 1, 2, 3..., n$, we get the relation (4.2.20).

Particularly from (4.2.20), we will obtain the followings.

$$\gamma_1(P,Q) \le \eta_1(P,Q) \le \rho_1(P,Q), \gamma_2(P,Q) \le \eta_2(P,Q) \le \rho_2(P,Q), ... \tag{4.2.22}$$

Now, bounds of a particular member $\gamma_1(P,Q)$ of one of the series of divergences, are obtained in terms of the well known divergences $h(P,Q)$(1.2.6), $\psi(P,Q)$(1.2.9), $J(P,Q)$(1.2.10), $T(P,Q)$(1.2.11), $I(P,Q)$(1.2.12), $K(P,Q)$(1.2.18), $\chi^2(P,Q)$(1.2.19), $G(P,Q)$(1.2.20), and $F(P,Q)$(1.2.21) by using information inequalities (3.2.2) on $C_f(P,Q)$ given by Taneja [95]. The results are on the similar lines to the results presented by Taneja [95]. Firstly, let us consider

$$f_1(t) = \frac{(t^2-1)^2}{t}, t > 0, f_1(1) = 0, f_1'(t) = \frac{(t^2-1)(3t^2+1)}{t^2}$$

and

$$f_1''(t) = \frac{2(3t^4+1)}{t^3}. \tag{4.2.23}$$

For $f_1(t)$, we have

$$C_{f_1}(P,Q) = \sum_{i=1}^{n} \frac{(p_i^2 - q_i^2)^2}{p_i q_i^2} = \gamma_1(P,Q). \tag{4.2.24}$$

Now the following propositions give the bounds of $\gamma_1(P,Q)$.

Proposition 4.2.2. *For $P,Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*
(i) If $0 < \alpha \leq .67$, then

$$23.4h(P,Q) \leq \gamma_1(P,Q) \leq 8max\left[\frac{3\alpha^4+1}{\alpha^{\frac{3}{2}}}, \frac{3\beta^4+1}{\beta^{\frac{3}{2}}}\right] h(P,Q). \tag{4.2.25}$$

(ii) If $.67 < \alpha \leq 1$, then

$$\frac{8(3\alpha^4+1)}{\alpha^{\frac{3}{2}}} h(P,Q) \leq \gamma_1(P,Q) \leq \frac{8(3\beta^4+1)}{\beta^{\frac{3}{2}}} h(P,Q). \tag{4.2.26}$$

Proof: Let us consider

$$f_2(t) = \frac{1}{2}\left(1-\sqrt{t}\right)^2, t \in (0,\infty), f_2(1) = 0, f_2'(t) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{t}}\right) \text{ and}$$

$$f_2''(t) = \frac{1}{4t^{\frac{3}{2}}}. \tag{4.2.27}$$

Since $f_2''(t) > 0 \; \forall \; t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we get

$$C_{f_2}(P,Q) = \sum_{i=1}^{n} \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2} = h(P,Q). \tag{4.2.28}$$

Now, let $g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{8\left(3t^4+1\right)}{t^{\frac{3}{2}}}$ and

$$g'(t) = \frac{4(15t^4-3)}{t^{\frac{5}{2}}}, g''(t) = 30\left(3\sqrt{t} + \frac{1}{t^{\frac{7}{2}}}\right),$$

where $f_1''(t)$ and $f_2''(t)$ are given by (4.2.23) and (4.2.27) respectively.

If $g'(t) = 0 \Rightarrow t = .6687403 \approx .67$. It is clear that $g'(t) < 0$ in $(0, .67)$ and > 0 in $(.67, \infty)$ with $g''(.67) = 195.5276 \approx 195.5 > 0$, i.e., $g(t)$ is strictly decreasing in $(0, .67)$ and strictly increasing in $(.67, \infty)$. So $g(t)$ has a minimum value at $t = .67$. Therefore

(i) If $0 < \alpha \leq .67$, then

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(.67) = 23.405968 \approx 23.4. \tag{4.2.29}$$

$$\begin{aligned} M &= \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] \\ &= max\left[\frac{8(3\alpha^4+1)}{\alpha^{\frac{3}{2}}}, \frac{8(3\beta^4+1)}{\beta^{\frac{3}{2}}}\right]. \end{aligned} \tag{4.2.30}$$

(ii) If $.67 < \alpha \leq 1$, then

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{8(3\alpha^4+1)}{\alpha^{\frac{3}{2}}}. \tag{4.2.31}$$

$$M = \sup_{t\in(\alpha,\beta)} g(t) = g(\beta) = \frac{8(3\beta^4+1)}{\beta^{\frac{3}{2}}}. \tag{4.2.32}$$

The results (4.2.25) and (4.2.26) are obtained by using (4.2.24), (4.2.28), (4.2.29), (4.2.30), (4.2.31), and (4.2.32) in (3.2.2).

Proposition 4.2.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*

(i) If $0 < \alpha \leq .51$, then

$$14.24G(P,Q) \leq \gamma_1(P,Q) \leq 4max\left[\frac{(\alpha+1)(3\alpha^4+1)}{\alpha}, \frac{(\beta+1)(3\beta^4+1)}{\beta}\right]G(P,Q). \tag{4.2.33}$$

(ii) If $.51 < \alpha \leq 1$, then

$$\frac{4(\alpha+1)(3\alpha^4+1)}{\alpha}G(P,Q) \leq \gamma_1(P,Q) \leq \frac{4(\beta+1)(3\beta^4+1)}{\beta}G(P,Q). \tag{4.2.34}$$

Proof: Let us consider

$$f_2(t) = \left(\frac{t+1}{2}\right)\log\frac{t+1}{2t}, t \in (0,\infty), f_2(1) = 0, f_2'(t) = \frac{1}{2}\left[\log\frac{t+1}{2t} - \frac{1}{t}\right] \text{ and}$$

$$f_2''(t) = \frac{1}{2t^2(t+1)}. \tag{4.2.35}$$

Since $f_2''(t) > 0 \; \forall \; t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we get

$$C_{f_2}(P,Q) = \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\frac{p_i+q_i}{2p_i} = G(P,Q). \tag{4.2.36}$$

Now, let $g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{4(t+1)\left(3t^4+1\right)}{t}$ and

$$g'(t) = \frac{4\left(12t^5+9t^4-1\right)}{t^2}, g''(t) = 8\left(18t^2+9t+\frac{1}{t^3}\right),$$

where $f_1''(t)$ and $f_2''(t)$ are given by (4.2.23) and (4.2.35) respectively.

If $g'(t) = 0 \Rightarrow t = .507385 \approx .51$. It is clear that $g'(t) < 0$ in $(0, .51)$ and > 0 in $(.51, \infty)$ with $g''(.51) = 134.4830294 \approx 134.45 > 0$, i.e., $g(t)$ is strictly decreasing in $(0, .51)$ and strictly increasing in $(.51, \infty)$. So $g(t)$ has a minimum value at $t = .51$. Therefore

(i) If $0 < \alpha \leq .51$, then

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(.51) = 14.24677337 \approx 14.24. \tag{4.2.37}$$

$$\begin{aligned} M &= \sup_{t\in(\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] \\ &= max\left[\frac{4(\alpha+1)\left(3\alpha^4+1\right)}{\alpha}, \frac{4(\beta+1)\left(3\beta^4+1\right)}{\beta}\right]. \end{aligned} \tag{4.2.38}$$

(ii) If $.51 < \alpha \leq 1$, then

$$m = \inf_{t\in(\alpha,\beta)} g(t) = g(\alpha) = \frac{4(\alpha+1)\left(3\alpha^4+1\right)}{\alpha}. \tag{4.2.39}$$

$$M = \sup_{t \in (\alpha,\beta)} g(t) = g(\beta) = \frac{4(\beta+1)(3\beta^4+1)}{\beta}. \tag{4.2.40}$$

The results (4.2.33) and (4.2.34) are obtained by using (4.2.24), (4.2.36), (4.2.37), (4.2.38), (4.2.39), and (4.2.40) in (3.2.2).

By using the similar procedure, we obtain the bounds of $\gamma_1(P,Q)$ in terms of other standard divergences. These inequalities (results) are as follows, omitting the details.

(a) If we take $f_2(t) = (t-1)^2$, then we have

(i) If $0 < \alpha < 1$, then

$$4\chi^2(P,Q) \le \gamma_1(P,Q) \le max\left[\frac{3\alpha^4+1}{\alpha^3}, \frac{3\beta^4+1}{\beta^3}\right]\chi^2(P,Q). \tag{4.2.41}$$

(ii) If $\alpha = 1$, then

$$4\chi^2(P,Q) \le \gamma_1(P,Q) \le \frac{3\beta^4+1}{\beta^3}\chi^2(P,Q). \tag{4.2.42}$$

(b) If we take $f_2(t) = t\log t$, then we have

(i) If $0 < \alpha \le .76$, then

$$6.9K(P,Q) \le \gamma_1(P,Q) \le 2max\left[\frac{3\alpha^4+1}{\alpha^2}, \frac{3\beta^4+1}{\beta^2}\right]K(P,Q). \tag{4.2.43}$$

(ii) If $.76 < \alpha \le 1$, then

$$\frac{2(3\alpha^4+1)}{\alpha^2}K(P,Q) \le \gamma_1(P,Q) \le \frac{2(3\beta^4+1)}{\beta^2}K(P,Q). \tag{4.2.44}$$

(c) If we take $f_2(t) = t\log\frac{2t}{t+1}$, then we have

(i) If $0 < \alpha \le .62$, then

$$19.7F(P,Q) \le \gamma_1(P,Q) \le 2max\left[\frac{(\alpha+1)^2(3\alpha^4+1)}{\alpha^2}, \frac{(\beta+1)^2(3\beta^4+1)}{\beta^2}\right]F(P,Q). \tag{4.2.45}$$

(ii) If $.62 < \alpha \leq 1$, then

$$\frac{2\left(\alpha+1\right)^{2}\left(3\alpha^{4}+1\right)}{\alpha^{2}}F\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq \frac{2\left(\beta+1\right)^{2}\left(3\beta^{4}+1\right)}{\beta^{2}}F\left(P,Q\right). \tag{4.2.46}$$

(d) If we take $f_2\left(t\right) = \left(t-1\right)\log t$, then we have

(i) If $0 < \alpha \leq .65$, then

$$2.87J\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq 2max\left[\frac{3\alpha^{4}+1}{\alpha\left(\alpha+1\right)}, \frac{3\beta^{4}+1}{\beta\left(\beta+1\right)}\right]J\left(P,Q\right). \tag{4.2.47}$$

(ii) If $.65 < \alpha \leq 1$, then

$$\frac{2\left(3\alpha^{4}+1\right)}{\alpha\left(\alpha+1\right)}J\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq \frac{2\left(3\beta^{4}+1\right)}{\beta\left(\beta+1\right)}J\left(P,Q\right). \tag{4.2.48}$$

(e) If we take $f_2\left(t\right) = \frac{t+1}{2}\log\frac{t+1}{2\sqrt{t}}$, then we have

(i) If $0 < \alpha \leq .62$, then

$$21.8T\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq 8max\left[\frac{\left(3\alpha^{4}+1\right)\left(\alpha+1\right)}{\alpha\left(\alpha^{2}+1\right)}, \frac{\left(3\beta^{4}+1\right)\left(\beta+1\right)}{\beta\left(\beta^{2}+1\right)}\right]T\left(P,Q\right). \tag{4.2.49}$$

(ii) If $.62 < \alpha \leq 1$, then

$$\frac{8\left(3\alpha^{4}+1\right)\left(\alpha+1\right)}{\alpha\left(\alpha^{2}+1\right)}T\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq \frac{8\left(3\beta^{4}+1\right)\left(\beta+1\right)}{\beta\left(\beta^{2}+1\right)}T\left(P,Q\right). \tag{4.2.50}$$

(f) If we take $f_2\left(t\right) = \frac{(t-1)^2(t+1)}{t}$, then we have

(i) If $0 < \alpha \leq .25$, then

$$\psi\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq max\left[\frac{3\alpha^{4}+1}{\alpha^{3}+1}, \frac{3\beta^{4}+1}{\beta^{3}+1}\right]\psi\left(P,Q\right). \tag{4.2.51}$$

(ii) If $.25 < \alpha \leq 1$, then

$$\frac{3\alpha^{4}+1}{\alpha^{3}+1}\psi\left(P,Q\right) \leq \gamma_{1}\left(P,Q\right) \leq \frac{3\beta^{4}+1}{\beta^{3}+1}\psi\left(P,Q\right). \tag{4.2.52}$$

(g) If we take $f_2(t) = \frac{t}{2}\log t + \frac{t+1}{2}\log\frac{2}{t+1}$, then we have

(i) If $0 < \alpha \leq .69$, then

$$23.86 I(P,Q) \leq \gamma_1(P,Q) \leq 4max\left[\frac{(\alpha+1)(3\alpha^4+1)}{\alpha^2}, \frac{(\beta+1)(3\beta^4+1)}{\beta^2}\right] I(P,Q). \tag{4.2.53}$$

(ii) If $.69 < \alpha \leq 1$, then

$$\frac{4(\alpha+1)(3\alpha^4+1)}{\alpha^2} I(P,Q) \leq \gamma_1(P,Q) \leq \frac{4(\beta+1)(3\beta^4+1)}{\beta^2} I(P,Q). \tag{4.2.54}$$

4.2.4 Numerical verification of obtained bounds- I

In this subsection, we take an example for calculating the divergences $h(P,Q)$, $G(P,Q)$, and $\gamma_1(P,Q)$ and then verify numerically the results (4.2.25) and (4.2.33) or verify the bounds of $\gamma_1(P,Q)$ in terms of $h(P,Q)$ and $G(P,Q)$.

Example 4.2.1. *We are taking the example same as example 3.2.1 (subsection-3.2.3) for $p = 0.7$ and $q = 0.3$ by considering two discrete probability distributions Binomial and Poisson, so the values of α, β, $h(P,Q)$, $G(P,Q)$ are same already defined in that example, given by (3.2.36), (3.2.38), (3.2.39) and $\gamma_1(P,Q)$ is defined as follows.*

$$\gamma_1(P,Q) = \sum_{i=1}^{11} \frac{(p_i^2 - q_i^2)^2}{p_i q_i^2} \approx 2.25065. \tag{4.2.55}$$

Now, put the approximated numerical values from (3.2.36), (3.2.38), (3.2.39), and (4.2.55) in (4.2.25) and (4.2.33), we get the followings respectively

$$1.17468 \leq 2.25065 \,(= \gamma_1(P,Q)) \leq 771.68$$

and

$$1.062304 \leq 2.25065 \,(= \gamma_1(P,Q)) \leq 46.4161.$$

Hence verify the inequalities (4.2.25) and (4.2.33) for $p = 0.7$.

Similarly, we can verify the other obtained inequalities numerically for different values of p and q by taking other discrete probability distributions, like: Geometric, Negative Binomial, Uniform etc.

4.3 Series of New Divergence Measures and Applications- II

In this section, we again introduce new series of divergence measures as a family of Csiszar's generalized divergence, characterize the properties of convex functions and divergences, compare several divergences, and derive important and interesting intra relation among divergences of these new series. Also get the bounds of a particular member of that series together with numerical verification of obtained bounds.

4.3.1 Series of convex functions and properties- II

In this subsection, we develop some series of convex functions and will study their properties. For this, Let $f : (0, \infty) \to R$ be a real valued mapping, defined as

$$f_m(t) = \frac{\left(t^2 - 1\right)^{2m}}{t^{\frac{2m-1}{2}}}, m = 1, 2, 3..., \tag{4.3.1}$$

and

$$f'_m(t) = \frac{\left(t^2 - 1\right)^{2m-1}\left[t^2(6m+1) + 2m - 1\right]}{2t^{\frac{2m+1}{2}}}, \tag{4.3.2}$$

$$f''_m(t) = \frac{\left(t^2 - 1\right)^{2m-2}}{4t^{\frac{2m+3}{2}}}\left[t^4\left(36m^2 - 1\right) + 2t^2\left(12m^2 - 16m + 1\right) + 4m^2 - 1\right]. \tag{4.3.3}$$

Since $f''_m(t) \geq 0$ for $t > 0$ and $m = 1, 2, ...$, therefore $f_m(t)$ are convex functions for each m.

Now, from (4.3.1), we get the following new convex functions at $m = 1, 2, 3...$ respectively.

$$f_1(t) = \frac{\left(t^2-1\right)^2}{t^{\frac{1}{2}}},\ f_2(t) = \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}},\ f_3(t) = \frac{\left(t^2-1\right)^6}{t^{\frac{5}{2}}}... \tag{4.3.4}$$

Now by using (4.3.4), we get the following series of convex functions as well.

$$f_{1,2}(t) = f_1(t) + f_2(t) = \frac{\left(t^2-1\right)^2}{t^{\frac{1}{2}}} + \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}} = \frac{\left(t^2-1\right)^2\left(t^4-2t^2+t+1\right)}{t^{\frac{3}{2}}}. \tag{4.3.5}$$

$$f_{2,3}(t) = f_2(t) + f_3(t) = \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}} + \frac{\left(t^2-1\right)^6}{t^{\frac{5}{2}}} = \frac{\left(t^2-1\right)^4\left(t^4-2t^2+t+1\right)}{t^{\frac{5}{2}}}. \tag{4.3.6}$$

In this way, we can write for $m = 1, 2, 3...$

$$\begin{aligned} f_{m,m+1}(t) = f_m(t) + f_{m+1}(t) &= \frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} + \frac{\left(t^2-1\right)^{2m+2}}{t^{\frac{2m+1}{2}}} \\ &= \frac{\left(t^2-1\right)^{2m}\left(t^4-2t^2+t+1\right)}{t^{\frac{2m+1}{2}}}. \end{aligned} \tag{4.3.7}$$

Since, we know that the linear combination of convex functions is also a convex function, i.e., $a_1 f_1(t) + a_2 f_2(t) + a_3 f_3(t) + ...$ is a convex function as well, where $a_1, a_2, a_3, ...$ are positive constants. So, we get another series of convex functions by using (4.3.4), defined as follows.

(i) If we take $a_1 = 1, a_2 = \log_e b, a_3 = \frac{(\log_e b)^2}{2!}, ..., b > 1$, then we have

$$\begin{aligned} g_1(t) &= f_1(t) + (\log_e b) f_2(t) + \frac{(\log_e b)^2}{2!} f_3(t) + ... \\ &= \frac{\left(t^2-1\right)^2}{t^{\frac{1}{2}}} + (\log_e b) \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}} + \frac{(\log_e b)^2}{2!} \frac{\left(t^2-1\right)^6}{t^{\frac{5}{2}}} + ... \\ &= \frac{\left(t^2-1\right)^2}{t^{\frac{1}{2}}} \left[1 + (\log_e b) \frac{\left(t^2-1\right)^2}{t} + \frac{(\log_e b)^2}{2!} \frac{\left(t^2-1\right)^4}{t^2} + ...\right] \\ &= \frac{\left(t^2-1\right)^2}{t^{\frac{1}{2}}} b^{\frac{\left(t^2-1\right)^2}{t}}, b > 1. \end{aligned} \tag{4.3.8}$$

(ii) If we take $a_1 = 0, a_2 = 1, a_3 = \log_e b, a_4 = \frac{(\log_e b)^2}{2!}, ..., b > 1$, then we have

$$\begin{aligned} g_2(t) &= \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}} + (\log_e b) \frac{\left(t^2-1\right)^6}{t^{\frac{5}{2}}} + \frac{(\log_e b)^2}{2!} \frac{\left(t^2-1\right)^8}{t^{\frac{7}{2}}} + ..., b > 1 \\ &= \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}} \left[1 + (\log_e b) \frac{\left(t^2-1\right)^2}{t} + \frac{(\log_e b)^2}{2!} \frac{\left(t^2-1\right)^4}{t^2} + ...\right] \\ &= \frac{\left(t^2-1\right)^4}{t^{\frac{3}{2}}} b^{\frac{\left(t^2-1\right)^2}{t}}, b > 1. \end{aligned} \tag{4.3.9}$$

In this way, we can write

$$g_m(t) = \frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} b^{\frac{\left(t^2-1\right)^2}{t}}, b > 1, m = 1, 2, 3, ... \tag{4.3.10}$$

Remark 4.3.1. *If we take $b = e \approx 2.71828$ then from (4.3.10), we obtain the following series*

$$g_m(t) = \frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} e^{\frac{\left(t^2-1\right)^2}{t}} = \frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} \exp \frac{\left(t^2-1\right)^2}{t}, m = 1, 2, 3, ... \tag{4.3.11}$$

Properties of convex functions defined by (4.3.1), (4.3.7) and (4.3.11), are as follows.

- Since $f_m(1) = 0 = f_{m,m+1}(1) = g_m(1) \Rightarrow f_m(t), f_{m,m+1}(t)$ and $g_m(t)$ are normalized functions for each m.

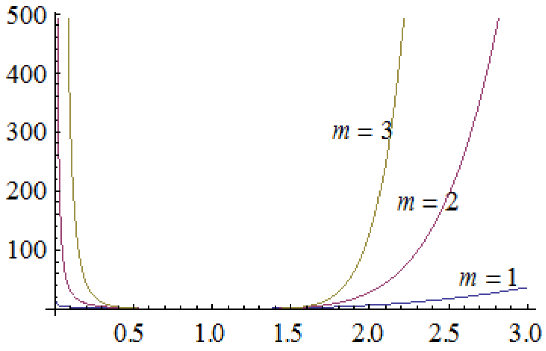

Figure 4.5: Behavior of convex functions $f_m(t)$

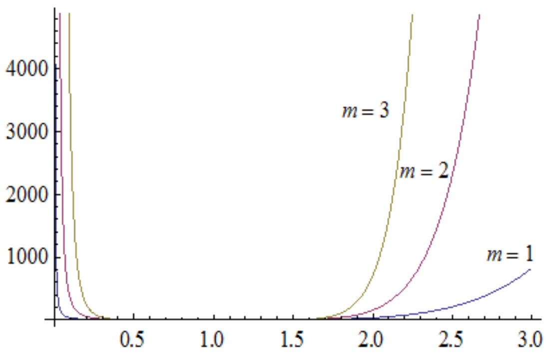

Figure 4.6: Behavior of convex functions $f_{m,m+1}(t)$

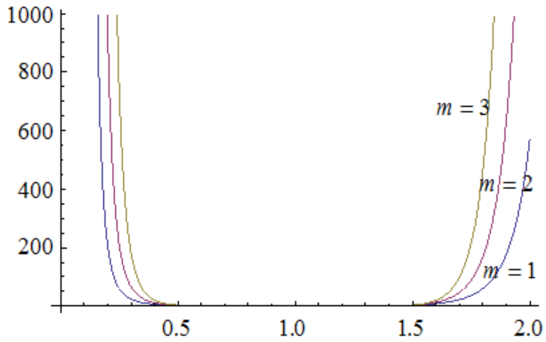

Figure 4.7: Behavior of convex functions $g_m(t)$

• Since $f'_m(t) < 0$ at $(0,1)$ and > 0 at $(1,\infty) \Rightarrow f_m(t)$ are strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, for each value of m and $f'_m(1) = 0$.

Figures 4.5, 4.6, and 4.7 show that $f_m(t)\,, f_{m,m+1}(t)$ and $g_m(t)$ have a stepper slope for increasing values of m respectively.

4.3.2 New series of divergence measures- II

In this subsection, we obtain new series of divergence measures of Csiszar's class corresponding to convex functions defined in subsection 4.3.1, and study their properties in detail.

Now for convex functions (4.3.1), we get the following new series of divergences of Csiszar's class.

$$C_f(P,Q) = \xi_m(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2m}}{(p_i q_i)^{\frac{2m-1}{2}} q_i^{2m}}, m = 1,2,3... \tag{4.3.12}$$

$$\xi_1(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2}}{(p_i q_i)^{\frac{1}{2}} q_i^{2}}, \xi_2(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{4}}{(p_i q_i)^{\frac{3}{2}} q_i^{4}}, ... \tag{4.3.13}$$

Similarly for (4.3.7), we obtain the following new series.

$$C_f(P,Q) = \zeta_m(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2m}\left(p_i^4 - 2p_i^2 q_i^2 + p_i q_i^3 + q_i^4\right)}{(p_i q_i)^{\frac{2m+1}{2}} q_i^{2m+2}}, m = 1,2,... \tag{4.3.14}$$

$$\zeta_1(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2}\left(p_i^4 - 2p_i^2 q_i^2 + p_i q_i^3 + q_i^4\right)}{(p_i q_i)^{\frac{3}{2}} q_i^{4}}, \tag{4.3.15}$$

$$\zeta_2(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{4}\left(p_i^4 - 2p_i^2 q_i^2 + p_i q_i^3 + q_i^4\right)}{(p_i q_i)^{\frac{5}{2}} q_i^{6}}, ... \tag{4.3.16}$$

Similarly for convex functions (4.3.11), we have the following new series of divergences.

$$C_f(P,Q) = \omega_m(P,Q) = \sum_{i=1}^{n} \frac{\left(p_i^2 - q_i^2\right)^{2m}}{(p_i q_i)^{\frac{2m-1}{2}} q_i^{2m}} \exp\frac{\left(p_i^2 - q_i^2\right)^{2}}{p_i q_i^3}, m = 1,2,... \tag{4.3.17}$$

$$\omega_1(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^2}{\left(p_iq_i\right)^{\frac{1}{2}}q_i^2}\exp\frac{\left(p_i^2-q_i^2\right)^2}{p_iq_i^3}, \tag{4.3.18}$$

$$\omega_2(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^4}{\left(p_iq_i\right)^{\frac{3}{2}}q_i^4}\exp\frac{\left(p_i^2-q_i^2\right)^2}{p_iq_i^3},\ldots \tag{4.3.19}$$

Properties of divergences defined by (4.3.12), (4.3.14) and (4.3.17), are as follows.

- In view of properties of $C_f(P,Q)$, we can say that $\xi_m(P,Q),\zeta_m(P,Q),\omega_m(P,Q)>0$ and are convex in the pair of probability distribution $P,Q\in\Gamma_n$.
- $\xi_m(P,Q)=0=\zeta_m(P,Q)=\omega_m(P,Q)$ if $P=Q$ or $p_i=q_i$ (attains its minimum value).
- Since $\xi_m(P,Q)\neq\xi_m(Q,P),\zeta_m(P,Q)\neq\zeta_m(Q,P),\omega_m(P,Q)\neq\omega_m(Q,P)\Rightarrow$ $\xi_m(P,Q),\zeta_m(P,Q)$, $\omega_m(P,Q)$ are non- symmetric divergence measures.

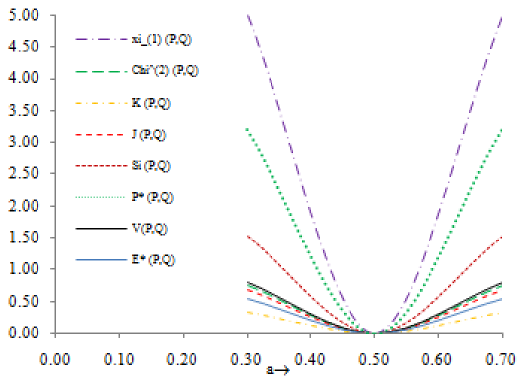

Figure 4.8: Comparison of the well known divergences with new series of divergences

Figure 4.8 shows the behavior of $\xi_1(P,Q)$, $P^*(P,Q)$, $\psi(P,Q)$, $V(P,Q)$, $\chi^2(P,Q)$, $J(P,Q)$, $E^*(P,Q)$, and $K(P,Q)$. We have considered $p_i=(a,1-a),q_i=(1-a,a)$, where $a\in(0,1)$. It is clear from the Figure that the new divergence $\xi_1(P,Q)$ has a stepper slope than remaining divergences.

4.3.3 Intra relation and bounds- II

Firstly we derive an intra relation among new series of divergence measures (4.3.12), (4.3.14), and (4.3.17), which is the following proposition.

Proposition 4.3.1. *Let $P, Q \in \Gamma_n$, then we have the following new intra relation.*

$$\xi_m(P,Q) \le \zeta_m(P,Q) \le \omega_m(P,Q), \tag{4.3.20}$$

where $m = 1, 2, ...$ and $\xi_m(P,Q)$, $\zeta_m(P,Q)$, and $\omega_m(P,Q)$ are given by (4.3.12), (4.3.14), and (4.3.17) respectively.

Proof: Since

$$\frac{(t^2-1)^{2m}(t^4-2t^2+t+1)}{t^{\frac{2m+1}{2}}} = \frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} + \frac{(t^2-1)^{2m+2}}{t^{\frac{2m+1}{2}}}$$

and

$$\frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{(t^2-1)^2}{t} = \frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}}\left[1 + \frac{(t^2-1)^2}{t} + \frac{(t^2-1)^4}{2!t^2} + ...\right].$$

Therefore, for $m = 1, 2, 3...$ and $t > 0$, we have the following inequalities.

$$\begin{aligned} \frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} &\le \frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} + \frac{(t^2-1)^{2m+2}}{t^{\frac{2m+1}{2}}} \\ &\le \frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}}\left[1 + \frac{(t^2-1)^2}{t} + \frac{(t^2-1)^4}{2!t^2} + ...\right]. \end{aligned} \tag{4.3.21}$$

Now put $t = \frac{p_i}{q_i}, i = 1, 2, 3..., n$ in (4.3.21), multiply by q_i and then sum over all $i = 1, 2, 3..., n$, we obtain the relation (4.3.20).

Particularly from (4.3.20), we have the followings as well.

$$\xi_1(P,Q) \le \zeta_1(P,Q) \le \omega_1(P,Q), \xi_2(P,Q) \le \zeta_2(P,Q) \le \omega_2(P,Q), ... \tag{4.3.22}$$

Now, bounds of a particular member $\xi_1(P,Q)$ of one of the series of divergences, are obtained in terms of the well known divergences $K(P,Q)$(1.2.18) and $\chi^2(P,Q)$(1.2.19) by using information inequalities (3.2.2) on $C_f(P,Q)$ given by Taneja [95]. The results are on the similar lines to the results presented by Taneja [95].

Firstly, let us consider

$$f_1(t)=\frac{\left(t^2-1\right)^2}{\sqrt{t}}, t>0, f_1(1)=0, f_1'(t)=\frac{\left(t^2-1\right)\left(7t^2+1\right)}{2t^{\frac{3}{2}}}$$

and

$$f_1''(t)=\frac{\left(35t^4-6t^2+3\right)}{4t^{\frac{5}{2}}}. \tag{4.3.23}$$

For $f_1(t)$, we obtain

$$C_{f_1}(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i^2-q_i^2\right)^2}{\left(p_iq_i\right)^{\frac{1}{2}}q_i^2}=\xi_1(P,Q). \tag{4.3.24}$$

Now, the following two propositions give the upper and lower bounds of new divergence $\xi_1(P,Q)$.

Proposition 4.3.2. *For $P,Q\in\Gamma_n$ and $0<\alpha\le 1\le\beta<\infty$ with $\alpha\neq\beta$, we have*

$$\frac{35\alpha^{\frac{9}{2}}-6\alpha^{\frac{5}{2}}+3\alpha^{\frac{1}{2}}}{8}\chi^2(Q,P)\le\xi_1(P,Q)\le\frac{35\beta^{\frac{9}{2}}-6\beta^{\frac{5}{2}}+3\beta^{\frac{1}{2}}}{8}\chi^2(Q,P). \tag{4.3.25}$$

Proof: Let us consider

$$f_2(t)=\frac{(t-1)^2}{t}, t\in(0,\infty), f_2(1)=0, f_2'(t)=\frac{t^2-1}{t^2} \text{ and}$$

$$f_2''(t)=\frac{2}{t^3}. \tag{4.3.26}$$

Since $f_2''(t)>0\ \forall\ t>0$ and $f_2(1)=0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we get

$$C_{f_2}(P,Q)=\sum_{i=1}^{n}\frac{\left(p_i-q_i\right)^2}{p_i}=\chi^2(Q,P). \tag{4.3.27}$$

Now, let

$$g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{35t^{\frac{9}{2}} - 6t^{\frac{5}{2}} + 3t^{\frac{1}{2}}}{8}, g'(t) = \frac{3\left(105t^4 - 10t^2 + 1\right)}{16t^{\frac{1}{2}}},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (4.3.23) and (4.3.26) respectively.

It is clear that $g'(t) > 0 \; \forall \; t > 0$ or $g(t)$ is always strictly increasing in $(0, \infty)$, so

$$m = \inf_{t \in (\alpha, \beta)} g(t) = g(\alpha) = \frac{35\alpha^{\frac{9}{2}} - 6\alpha^{\frac{5}{2}} + 3\alpha^{\frac{1}{2}}}{8}. \tag{4.3.28}$$

$$M = \sup_{t \in (\alpha, \beta)} g(t) = g(\beta) = \frac{35\beta^{\frac{9}{2}} - 6\beta^{\frac{5}{2}} + 3\beta^{\frac{1}{2}}}{8}. \tag{4.3.29}$$

The result (4.3.25) is obtained by using (4.3.24), (4.3.27), (4.3.28) and (4.3.29) in (3.2.2).

Proposition 4.3.3. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*
(i) If $0 < \alpha \leq .3916$, then

$$\begin{aligned} &1.158K(Q,P) \leq \xi_1(P,Q) \\ &\leq max\left[\frac{35\alpha^{\frac{7}{2}} - 6\alpha^{\frac{3}{2}} + 3\alpha^{\frac{-1}{2}}}{4}, \frac{35\beta^{\frac{7}{2}} - 6\beta^{\frac{3}{2}} + 3\beta^{\frac{-1}{2}}}{4}\right] K(Q,P). \end{aligned} \tag{4.3.30}$$

(ii) If $.3916 < \alpha \leq 1$, then

$$\frac{35\alpha^{\frac{7}{2}} - 6\alpha^{\frac{3}{2}} + 3\alpha^{\frac{-1}{2}}}{4} K(Q,P) \leq \xi_1(P,Q) \leq \frac{35\beta^{\frac{7}{2}} - 6\beta^{\frac{3}{2}} + 3\beta^{\frac{-1}{2}}}{4} K(Q,P). \tag{4.3.31}$$

Proof: Let us consider

$$f_2(t) = -\log t, t \in (0, \infty), f_2(1) = 0, f_2'(t) = -\frac{1}{t} \text{ and}$$

$$f_2''(t) = \frac{1}{t^2}. \tag{4.3.32}$$

Since $f_2''(t) > 0 \; \forall \, t > 0$ and $f_2(1) = 0$, so $f_2(t)$ is strictly convex and normalized function respectively. Now for $f_2(t)$, we get

$$C_{f_2}(P,Q) = \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i} = K(Q,P). \tag{4.3.33}$$

Now, let

$$g(t) = \frac{f_1''(t)}{f_2''(t)} = \frac{35t^{\frac{7}{2}} - 6t^{\frac{3}{2}} + 3t^{\frac{-1}{2}}}{4}, g'(t) = \frac{245t^4 - 18t^2 - 3}{8t^{\frac{3}{2}}}$$

and

$$g''(t) = \frac{1225t^4 - 18t^2 + 9}{16t^{\frac{5}{2}}},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (4.3.23) and (4.3.32) respectively.

It is clear that $g'(t) < 0$ in $(0, .3916)$ and $g'(t) \geq 0$ in $[.3916, \infty)$ with $g''(.3916) > 0$, i.e., $g(t)$ is strictly decreasing in $(0, .3916)$ and increasing in $[.3916, \infty)$. So $g(t)$ has a minimum value at $t = .3916$. Therefore

(i) If $0 < \alpha \leq .3916$, then

$$m = \inf_{t \in (\alpha,\beta)} g(t) = g(.3916) = 1.158. \tag{4.3.34}$$

$$\begin{aligned} M &= \sup_{t \in (\alpha,\beta)} g(t) = max\left[g(\alpha), g(\beta)\right] \\ &= max\left[\frac{35\alpha^{\frac{7}{2}} - 6\alpha^{\frac{3}{2}} + 3\alpha^{\frac{-1}{2}}}{4}, \frac{35\beta^{\frac{7}{2}} - 6\beta^{\frac{3}{2}} + 3\beta^{\frac{-1}{2}}}{4}\right]. \end{aligned} \tag{4.3.35}$$

(ii) If $.3916 < \alpha \leq 1$, then

$$m = \inf_{t \in (\alpha,\beta)} g(t) = g(\alpha) = \frac{35\alpha^{\frac{7}{2}} - 6\alpha^{\frac{3}{2}} + 3\alpha^{\frac{-1}{2}}}{4}. \tag{4.3.36}$$

$$M = \sup_{t \in (\alpha,\beta)} g(t) = g(\beta) = \frac{35\beta^{\frac{7}{2}} - 6\beta^{\frac{3}{2}} + 3\beta^{\frac{-1}{2}}}{4}. \tag{4.3.37}$$

The results (4.3.30) and (4.3.31) are obtained by using (4.3.24), (4.3.33), (4.3.34), (4.3.35), (4.3.36) and (4.3.37) in (3.2.2).

4.3.4 Numerical verification of obtained bounds- II

In this subsection, we take an example for calculating the divergences $\chi^2(Q,P)$, $K(Q,P)$, and $\xi_1(P,Q)$ and then verify numerically the results (4.3.25) and (4.3.30) or verify the bounds of $\xi_1(P,Q)$ in terms of $\chi^2(P,Q)$ and $K(P,Q)$.

Example 4.3.1. *We are taking the example same as example 3.2.1 (subsection-3.2.3) for $p = 0.7$ and $q = 0.3$ by considering two discrete probability distributions Binomial and Poisson, so the values of α, β are same already defined in that example, given by (3.2.36) and the values of $\chi^2(Q,P)$, $K(Q,P)$, and $\xi_1(P,Q)$ are defined as follows.*

$$\chi^2(Q,P) = \sum_{i=1}^{11} \frac{(p_i - q_i)^2}{p_i} \approx 1.2259. \tag{4.3.38}$$

$$K(Q,P) = \sum_{i=1}^{11} q_i \log \frac{q_i}{p_i} \approx .2467. \tag{4.3.39}$$

$$\xi_1(P,Q) = \sum_{i=1}^{11} \frac{(p_i^2 - q_i^2)^2}{(p_i q_i)^{\frac{1}{2}} q_i^2} \approx 1.5703. \tag{4.3.40}$$

Now, put the approximated numerical values from (3.2.36) and (4.3.38) to (4.3.40) in (4.3.25) and (4.3.30), we get the followings respectively

$$.03697 \le \xi_1(P,Q) = 1.5703 \le 70.700, .2849 \le \xi_1(P,Q) = 1.5703 \le 15.8406.$$

Hence verify the inequalities (4.3.25) and (4.3.30) for $p = 0.7$.

Similarly, we can verify the same inequalities numerically for different values of p and q by taking other discrete probability distributions, like: Geometric, Negative Binomial, Uniform etc.

4.4 Conclusion

In this chapter, we introduced parametric series of divergence measures of Csiszar's class for series of convex functions of algebraic type. Nature of all convex functions with respect to the value of their parameter, has shown graphically. Comparison of new members of Csiszar's class with old standard members, has also been presented. We have evaluated bounds of a member of the new series of divergences, together with numerical verification.

5

NEW RELATIONS AMONG SEVERAL DIVERGENCES

5.1 Introduction

Since we know that divergence measures are very useful in information theory and practical problems, therefore it is very important to derive some relations among them.

In this chapter, we derive many new important and interesting relations among several divergences by helping some algebraic, exponential, and logarithmic inequalities. This chapter contains only one section excluding introduction and Conclusion.

5.2 Inequalities, Relations Among Divergences

This section contains several algebraic, exponential, and logarithmic inequalities and then further we establish many new relations among new divergence measures and well known old divergence measures by using these inequalities. Several interesting and important new relations among divergences, also obtain by helping old relations.

5.2.1 Algebraic, exponential, and logarithmic inequalities

The following inequalities (5.2.1) and (5.2.2) are famous in literature of pure and applied mathematics, which are important tools to prove many interesting and important results in information theory.

$$1 + t \leq e^t \leq 1 + te^t,\ t > 0. \tag{5.2.1}$$

$$\frac{t}{1+t} \leq \log\left(1+t\right) \leq t,\ t > 0. \tag{5.2.2}$$

Besides above inequalities, we are introducing the following algebraic and exponential inequalities as well together with their proofs.

Proposition 5.2.1. *Let $t \in (0, \infty)$ and $m = 1, 2, ...,$ then we have the following inequalities.*

$$\frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} > \frac{\left(t-1\right)^{2m}}{t^{\frac{2m-1}{2}}}, \tag{5.2.3}$$

$$\frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} > \frac{\left(t-1\right)^{2m}}{\left(t+1\right)^{2m-1}}, \tag{5.2.4}$$

$$\frac{\left(t^2-1\right)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{\left(t^2-1\right)^2}{t} > \frac{\left(t-1\right)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{\left(t-1\right)^2}{t}, \tag{5.2.5}$$

$$\frac{\left(t^2-1\right)^2}{\sqrt{t}} > \left(t-1\right)^2, \tag{5.2.6}$$

and

$$\frac{\left(t^2-1\right)^2}{\sqrt{t}} \geq 1 - \sqrt{t}. \tag{5.2.7}$$

All functions involve in (5.2.3) to (5.2.7) are convex and normalized, since $f''(t) \geq 0\ \forall\, t > 0$ and $f(1) = 0$ respectively.

Proof: From (5.2.3), we have to prove that

$$\frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} > \frac{(t-1)^{2m}}{t^{\frac{2m-1}{2}}} \Rightarrow (t+1)^{2m} - 1 > 0,$$

which is true (obvious) for $t > 0$. Hence proved (5.2.3) for each m.

Further, from (5.2.4), we have to prove that

$$\frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} > \frac{(t-1)^{2m}}{(t+1)^{2m-1}} \Rightarrow (t+1)^{4m-1} - t^{\frac{2m-1}{2}} > 0,$$

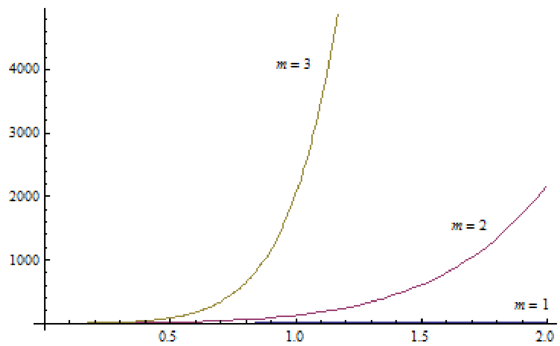

Figure 5.1: Graph of $(t+1)^{4m-1} - t^{\frac{2m-1}{2}}$ for $m = 1, 2, ...$

which is true (Figure 5.1) for $t > 0$. Hence proved (5.2.4) for each m.

Similarly, from (5.2.5), we have to prove that

$$\frac{(t^2-1)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{(t^2-1)^2}{t} > \frac{(t-1)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{(t-1)^2}{t}$$

$$\Rightarrow (t+1)^{2m} \exp\frac{(t^2-1)^2}{t} > \exp\frac{(t-1)^2}{t}$$

$$\Rightarrow (t+1)^{2m} \exp\left[\frac{(t^2-1)^2}{t} - \frac{(t-1)^2}{t}\right] > 1$$

$$\Rightarrow (t+1)^{2m} \exp\left[(t+2)(t-1)^2\right] - 1 > 0,$$

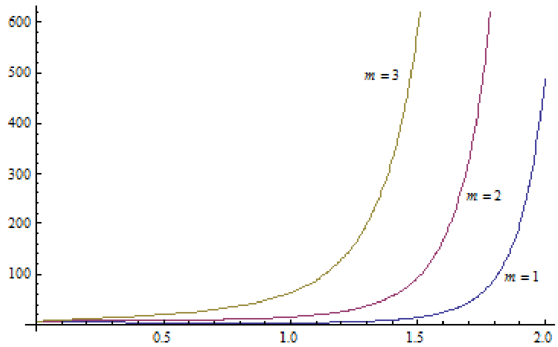

Figure 5.2: Graph of $(t+1)^{2m}\exp\left[(t+2)(t-1)^2\right]-1$ for $m=1,2,...$

which is true (Figure 5.2) for $t>0$. Hence proved (5.2.5) for each m.

Second lastly, from (5.2.6), we have to prove that

$$\frac{(t^2-1)^2}{\sqrt{t}} > (t-1)^2 \Rightarrow (t+1)^2-\sqrt{t}>0,$$

which is true (obvious) for $t>0$. Hence proved (5.2.6).

Lastly, from (5.2.7), we have to prove that

$$\frac{(t^2-1)^2}{\sqrt{t}} \geq 1-\sqrt{t} \Rightarrow \frac{(t^2-1)^2}{\sqrt{t}}+\sqrt{t}\geq 1$$
$$\Rightarrow \left(t^2-1\right)^2+t-\sqrt{t}\geq 0,$$

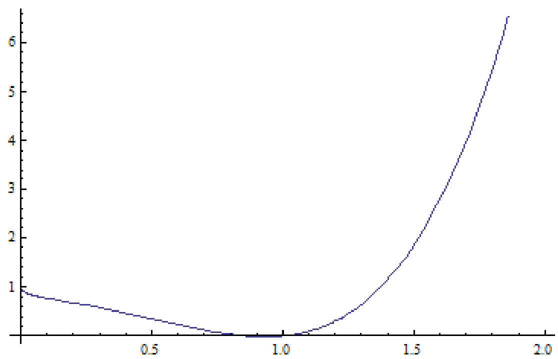

Figure 5.3: Graph of $\left(t^2-1\right)^2+t-\sqrt{t}$

which is true (Figure 5.3) for $t>0$. Hence proved (5.2.7).

Proposition 5.2.2. *Let $t \in (0, \infty)$ and $m = 1, 2, 3...$ then we have the following inequalities.*

$$\frac{\left(t^2-1\right)^{2m}}{t^{2m-1}} > \frac{\left(t-1\right)^{2m}}{t^{\frac{2m-1}{2}}}, \tag{5.2.8}$$

$$\frac{\left(t^2-1\right)^{2m}}{t^{2m-1}} > \frac{\left(t-1\right)^{2m}}{\left(t+1\right)^{2m-1}}, \tag{5.2.9}$$

$$\frac{\left(t^2-1\right)^{2m}}{t^{2m-1}} > \left(t-1\right)^{2m}, \tag{5.2.10}$$

and

$$\frac{\left(t^2-1\right)^{2m}}{t^{2m-1}} \exp\frac{\left(t^2-1\right)^2}{t^2} > \frac{\left(t-1\right)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{\left(t-1\right)^2}{t}. \tag{5.2.11}$$

All functions involve in (5.2.8) to (5.2.11) are convex and normalized, since $f''(t) \geq 0 \; \forall \, t > 0$ and $f(1) = 0$ respectively.

Proof:From (5.2.8), we have to prove that

$$\frac{\left(t^2-1\right)^{2m}}{t^{2m-1}} > \frac{\left(t-1\right)^{2m}}{t^{\frac{2m-1}{2}}} \Rightarrow \left(t+1\right)^{2m} > t^{m-\frac{1}{2}}$$

$$\Rightarrow \sqrt{t}\left(t+1\right)^{2m} - t^m > 0,$$

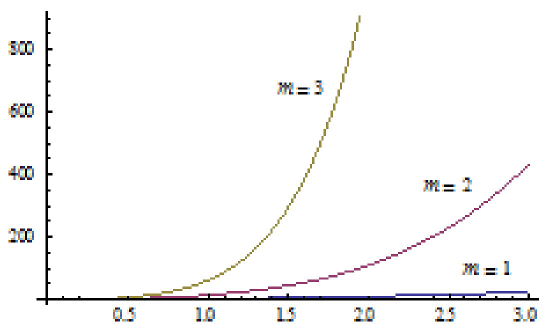

Figure 5.4: Graph of $\sqrt{t}\left(t+1\right)^{2m} - t^m$ for $m = 1, 2, ...$

which is true (Figure 5.4) for $t > 0, m = 1, 2, 3....$ Hence proved the result (5.2.8).

Now from (5.2.9), we have to prove that

$$\frac{(t^2-1)^{2m}}{t^{2m-1}} > \frac{(t-1)^{2m}}{(t+1)^{2m-1}} \Rightarrow (t+1)^{4m-1} > t^{2m-1}$$

$$\Rightarrow (t+1)^{4m-1} - t^{2m-1} > 0,$$

which is true for $t > 0, m = 1, 2, 3....$ Hence proved the result (5.2.9).

Similarly from (5.2.10), we have to prove that

$$\frac{(t^2-1)^{2m}}{t^{2m-1}} > (t-1)^{2m} \Rightarrow (t+1)^{2m} - t^{2m-1} > 0,$$

which is true (obvious) for $t > 0, m = 1, 2, 3....$ Hence proved the result (5.2.10).

Similarly from (5.2.11), we have to prove that

$$\frac{(t^2-1)^{2m}}{t^{2m-1}} \exp\frac{(t^2-1)^2}{t^2} > \frac{(t-1)^{2m}}{t^{\frac{2m-1}{2}}} \exp\frac{(t-1)^2}{t}$$

$$\Rightarrow \frac{(t+1)^{2m} e^{\frac{(t-1)^2\left(t^2+t+1\right)}{t^2}}}{t^{m-\frac{1}{2}}} > 1 \Rightarrow (t+1)^{2m} e^{\frac{(t-1)^2\left(t^2+t+1\right)}{t^2}} - t^{m-\frac{1}{2}} > 0,$$

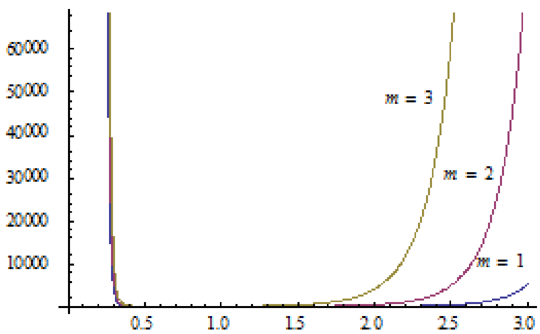

Figure 5.5: Graph of $(t+1)^{2m} e^{\frac{(t-1)^2\left(t^2+t+1\right)}{t^2}} - t^{m-\frac{1}{2}}$ for $m = 1, 2, ...$

which is true (Figure 5.5) for $t > 0, m = 1, 2, 3....$ Hence proved the result (5.2.11).

5.2.2 Several new relations among divergences

In subsection 5.2.1, many inequalities (algebraic, exponential, and logarithmic) have been introduced. Now, in this part of the section, we establish several relations among many divergence measures together with standard means. We have already defined all the divergence measures and means (using in upcoming propositions) in introduction chapter. So we are not repeating that all divergences. We start with the following proposition.

Proposition 5.2.3. *Let $P, Q \in \Gamma_n$ and $m = 1, 2, 3...$, then we have the following new relations*

$$N_m^*(P,Q) - N_{m+1}^*(P,Q) \leq \Delta_m(P,Q) \tag{5.2.12}$$

and

$$\Delta_{m+1}(P,Q) \leq N_{m+1}^*(P,Q), \tag{5.2.13}$$

where $\Delta_m(P,Q)$ and $N_m^*(P,Q)$ are defined by (1.2.30) and (1.2.48) respectively.

Proof: Put $t = \frac{(p_i-q_i)^2}{(p_i+q_i)^2}$ in inequalities (5.2.1), we get

$$1 + \frac{(p_i-q_i)^2}{(p_i+q_i)^2} \leq \exp\frac{(p_i-q_i)^2}{(p_i+q_i)^2} \leq 1 + \frac{(p_i-q_i)^2}{(p_i+q_i)^2}\exp\frac{(p_i-q_i)^2}{(p_i+q_i)^2}.$$

Now multiply the above expression by $\frac{(p_i-q_i)^{2m}}{(p_i+q_i)^{2m-1}}, m = 1, 2, 3...$ and sum over all $i = 1, 2, 3..., n$, we get

$$\sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_i+q_i)^{2m-1}} + \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m+2}}{(p_i+q_i)^{2m+1}} \leq \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_i+q_i)^{2m-1}}\exp\frac{(p_i-q_i)^2}{(p_i+q_i)^2}$$
$$\leq \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_i+q_i)^{2m-1}} + \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m+2}}{(p_i+q_i)^{2m+1}}\exp\frac{(p_i-q_i)^2}{(p_i+q_i)^2}\text{ , i.e.,}$$

$$\Delta_m(P,Q) + \Delta_{m+1}(P,Q) \leq N_m^*(P,Q) \leq \Delta_m(P,Q) + N_{m+1}^*(P,Q). \tag{5.2.14}$$

From second and third part of (5.2.14), we obtain inequality (5.2.12) and from first and third part, we obtain (5.2.13). Particularly

at $m=1$:

$$N_1^*(P,Q) - N_2^*(P,Q) \leq \Delta_1(P,Q) = \Delta(P,Q), \Delta_2(P,Q) \leq N_2^*(P,Q). \quad (5.2.15)$$

at $m=2$:

$$N_2^*(P,Q) - N_3^*(P,Q) \leq \Delta_2(P,Q), \Delta_3(P,Q) \leq N_3^*(P,Q), \quad (5.2.16)$$

and so on.

Proposition 5.2.4. *Let $P,Q \in \Gamma_n$ and $m = 1,2,...$, then we have the following new relations*

$$J_m^*(P,Q) - J_{m+1}^*(P,Q) \leq E_m^*(P,Q) \quad (5.2.17)$$

and

$$E_{m+1}^*(P,Q) \leq J_{m+1}^*(P,Q), \quad (5.2.18)$$

where $E_m^*(P,Q)$ and $J_m^*(P,Q)$ are defined by (1.2.28) and (1.2.29) respectively.

Proof: Put $t = \frac{(p_i-q_i)^2}{p_iq_i}$ in inequalities (5.2.1), we get

$$1 + \frac{(p_i-q_i)^2}{p_iq_i} \leq \exp\frac{(p_i-q_i)^2}{p_iq_i} \leq 1 + \frac{(p_i-q_i)^2}{p_iq_i}\exp\frac{(p_i-q_i)^2}{p_iq_i}.$$

Now multiply the above expression by $\frac{(p_i-q_i)^{2m}}{(p_iq_i)^{\frac{2m-1}{2}}}, m = 1,2,...$ and sum over all $i = 1,2,3...,n$, we get

$$\sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_iq_i)^{\frac{2m-1}{2}}} + \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m+2}}{(p_iq_i)^{\frac{2m+1}{2}}} \leq \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_iq_i)^{\frac{2m-1}{2}}}\exp\frac{(p_i-q_i)^2}{p_iq_i}$$
$$\leq \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m}}{(p_iq_i)^{\frac{2m-1}{2}}} + \sum_{i=1}^{n}\frac{(p_i-q_i)^{2m+2}}{(p_iq_i)^{\frac{2m+1}{2}}}\exp\frac{(p_i-q_i)^2}{p_iq_i}, \text{ i.e.,}$$

$$E_m^*(P,Q) + E_{m+1}^*(P,Q) \leq J_m^*(P,Q) \leq E_m^*(P,Q) + J_{m+1}^*(P,Q). \quad (5.2.19)$$

From second and third part of (5.2.19), we get inequality (5.2.17) and from first and third part, we get (5.2.18). Particularly

at $m = 1$:

$$J_1^*(P,Q) - J_2^*(P,Q) \le E_1^*(P,Q) = E^*(P,Q), E_2^*(P,Q) \le J_2^*(P,Q). \quad (5.2.20)$$

at $m = 2$:

$$J_2^*(P,Q) - J_3^*(P,Q) \le E_2^*(P,Q), E_3^*(P,Q) \le J_3^*(P,Q), \quad (5.2.21)$$

and so on.

Except the above results, from first and second part of the inequalities (5.2.19), we can easily see that at $m = 1$

$$E_1^*(P,Q) \le J_1^*(P,Q). \quad (5.2.22)$$

Proposition 5.2.5. *Let $P, Q \in \Gamma_n$, then we have the following new relations*

$$\psi(P,Q) - 2E_1^*(P,Q) \le S^*(P,Q) \quad (5.2.23)$$

and

$$S^*(P,Q) + \psi(P,Q) \le \psi_M(P,Q), \quad (5.2.24)$$

where $E_1^*(P,Q)$, $\psi(P,Q)$, $S^*(P,Q)$, and $\psi_M(P,Q)$ are defined by (1.2.8), (1.2.9), (1.2.13), and (1.2.15) respectively.

Proof: Put $t = \frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}$ in inequalities (5.2.2), we get

$$\frac{\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}}{1+\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}} \le \log\left[1+\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}\right] \le \frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_i q_i}}, \text{ i.e.,}$$

$$\frac{p_i+q_i-2\sqrt{p_i q_i}}{p_i+q_i} \le \log\frac{p_i+q_i}{2\sqrt{p_i q_i}} \le \frac{p_i+q_i-2\sqrt{p_i q_i}}{2\sqrt{p_i q_i}}.$$

Now multiply the above expression by $\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}$ and sum over all $i = 1, 2, 3..., n$, we get

$$\sum_{i=1}^{n}\left[\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}\right]\left[\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i}\right] \le \sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}\log\frac{p_i+q_i}{2\sqrt{p_iq_i}}$$
$$\le \sum_{i=1}^{n}\left[\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i}\right]\left[\frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}\right] \text{, i.e.,}$$

$$\sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i} - 2\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{\sqrt{p_iq_i}} \le S^*(P,Q)$$
$$\le \sum_{i=1}^{n}\frac{(p_i^2-q_i^2)^2}{2(p_iq_i)^{\frac{3}{2}}} - \sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_iq_i} \text{, i.e.,}$$
$$\psi(P,Q) - 2E_1^*(P,Q) \le S^*(P,Q) \le \psi_M(P,Q) - \psi(P,Q). \tag{5.2.25}$$

From first and second part of (5.2.25), we get inequality (5.2.23) and from second and third part, we get (5.2.24).

Except these, if we add (5.2.23) and (5.2.24), we get

$$2\psi(P,Q) \le \psi_M(P,Q) + 2E_1^*(P,Q). \tag{5.2.26}$$

From second and third part of the inequalities (5.2.25), we can easily see that

$$S^*(P,Q) \le \psi_M(P,Q). \tag{5.2.27}$$

By taking both (5.2.23) and (5.2.27), we can write

$$\psi(P,Q) - 2E_1^*(P,Q) \le S^*(P,Q) \le \psi_M(P,Q). \tag{5.2.28}$$

Proposition 5.2.6. *Let $P, Q \in \Gamma_n$, then we have the following new relations*

$$L(P,Q) + \Delta(P,Q) \le \frac{1}{2}E_1^*(P,Q) \tag{5.2.29}$$

and

$$\Delta(P,Q) \leq L(P,Q) + 2\sum_{i=1}^{n} \frac{(p_i - q_i)^2 \sqrt{p_i q_i}}{(p_i + q_i)^2}, \tag{5.2.30}$$

where $\Delta(P,Q)$, $E_1^*(P,Q)$, *and* $L(P,Q)$ *are defined by (1.2.5), (1.2.8), and (1.2.14) respectively.*

Proof: Put $t = \frac{(\sqrt{p_i}-\sqrt{q_i})^2}{2\sqrt{p_i q_i}}$ in inequalities (5.2.2), we get

$$\frac{p_i + q_i - 2\sqrt{p_i q_i}}{p_i + q_i} \leq \log \frac{p_i + q_i}{2\sqrt{p_i q_i}} \leq \frac{p_i + q_i - 2\sqrt{p_i q_i}}{2\sqrt{p_i q_i}}.$$

Now multiply the above expression by $\frac{(p_i - q_i)^2}{p_i + q_i}$ and sum over all $i = 1, 2, 3..., n$, we get

$$\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} \left[\frac{p_i + q_i - 2\sqrt{p_i q_i}}{p_i + q_i}\right] \leq \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} \log \frac{p_i + q_i}{2\sqrt{p_i q_i}}$$
$$\leq \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} \left[\frac{p_i + q_i - 2\sqrt{p_i q_i}}{2\sqrt{p_i q_i}}\right], \text{ i.e.,}$$

$$\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} - 2\sum_{i=1}^{n} \frac{(p_i - q_i)^2 \sqrt{p_i q_i}}{(p_i + q_i)^2} \leq L(P,Q) \leq \frac{1}{2}\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{\sqrt{p_i q_i}} - \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}, \text{ i.e.,}$$

$$\Delta(P,Q) - 2\sum_{i=1}^{n} \frac{(p_i - q_i)^2 \sqrt{p_i q_i}}{(p_i + q_i)^2} \leq L(P,Q) \leq \frac{1}{2}E_1^*(P,Q) - \Delta(P,Q). \tag{5.2.31}$$

From second and third part of (5.2.31), we get inequality (5.2.29) and from first and second part, we get (5.2.30).

From inequality (5.2.29), we can easily see that

$$\Delta(P,Q) \leq \frac{1}{2}E_1^*(P,Q). \tag{5.2.32}$$

Proposition 5.2.7. *Let* $P, Q \in \Gamma_n$ *and* $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, *then we have the following new relations*

$$A(P,Q) \leq h(P,Q) \leq T(P,Q), \tag{5.2.33}$$

$$A(P,Q) + h(P,Q) \le \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i+q_i)^2}{\sqrt{p_iq_i}}, \tag{5.2.34}$$

and

$$A(P,Q) + T(P,Q) \le \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i+q_i)^2}{\sqrt{p_iq_i}}, \tag{5.2.35}$$

where $h(P,Q)$ and $T(P,Q)$ are defined by (1.2.6) and (1.2.11) respectively, and $A(P,Q) = \sum_{i=1}^{n}\frac{p_i+q_i}{2} = 1$.

Proof: Put $t = \frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2\sqrt{p_iq_i}}$ in inequalities (5.2.2), we get

$$\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i} \le \log\frac{p_i+q_i}{2\sqrt{p_iq_i}} \le \frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}.$$

Now multiply the above expression by $\frac{p_i+q_i}{2}$ and sum over all $i = 1,2,3...,n$, we get

$$\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\left(\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i}\right) \le \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\frac{p_i+q_i}{2\sqrt{p_iq_i}}$$
$$\le \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\left(\frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}\right) \text{ , i.e.,}$$

$$\sum_{i=1}^{n}\frac{p_i+q_i-2\sqrt{p_iq_i}}{2} \le T(P,Q) \le \sum_{i=1}^{n}\frac{(p_i+q_i)^2}{4\sqrt{p_iq_i}} - 1 \text{ , i.e.,}$$

$$\sum_{i=1}^{n}\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2} \le T(P,Q) \le \sum_{i=1}^{n}\frac{(p_i+q_i)^2}{4\sqrt{p_iq_i}} - 1 \text{ , i.e.,}$$

$$h(P,Q) \le T(P,Q) \le \sum_{i=1}^{n}\frac{(p_i+q_i)^2}{4\sqrt{p_iq_i}} - 1. \tag{5.2.36}$$

From first and third part of (5.2.36), we get inequality (5.2.34) and from second and third part, we get (5.2.35).

Except these, from (5.2.34) and (5.2.36), we can easily see the followings

$$A(P,Q) \le \sum_{i=1}^{n}\frac{(p_i+q_i)^2}{4\sqrt{p_iq_i}}, \tag{5.2.37}$$

$$h(P,Q) \leq \sum_{i=1}^{n} \frac{(p_i+q_i)^2}{4\sqrt{p_i q_i}}, \tag{5.2.38}$$

and

$$h(P,Q) \leq T(P,Q). \tag{5.2.39}$$

Now do (5.2.37)-(5.2.38), we get

$$A(P,Q) \leq h(P,Q). \tag{5.2.40}$$

By taking both (5.2.39) and (5.2.40), we get the inequalities (5.2.33).

Proposition 5.2.8. *Let $P,Q \in \Gamma_n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, then we have the following new relations*

$$G(Q,P) \geq \frac{1}{2} - \log 2 \tag{5.2.41}$$

and

$$\log 2 + G(Q,P) \leq \frac{1}{2}\left[R_2(P,Q)+1\right], \tag{5.2.42}$$

where $G(P,Q)$ and $R_2(P,Q)$ are defined by (1.2.20) and (1.2.27) respectively.

Proof: Put $t = \frac{p_i}{q_i}$ in inequalities (5.2.2), we get

$$\frac{p_i}{p_i+q_i} \leq \log\frac{p_i+q_i}{q_i} \leq \frac{p_i}{q_i}.$$

Now multiply the above expression by $\frac{p_i+q_i}{2}$ and sum over all $i = 1,2,3...,n$, we get

$$\sum_{i=1}^{n} \frac{p_i+q_i}{2}\frac{p_i}{p_i+q_i} \leq \sum_{i=1}^{n} \frac{p_i+q_i}{2}\log\frac{2(p_i+q_i)}{2q_i} \leq \sum_{i=1}^{n} \frac{p_i+q_i}{2}\frac{p_i}{q_i}\text{ , i.e.,}$$

$$\sum_{i=1}^{n}\frac{p_i}{2} \leq \log 2\sum_{i=1}^{n}\frac{p_i+q_i}{2} + \sum_{i=1}^{n}\frac{p_i+q_i}{2}\log\frac{p_i+q_i}{2q_i} \leq \sum_{i=1}^{n}\frac{p_i^2}{2q_i} + \sum_{i=1}^{n}\frac{p_i}{2}\text{ , i.e.,}$$

$$\frac{1}{2} \leq \log 2 + G(Q,P) \leq \frac{1}{2}\left[R_2(P,Q)+1\right]. \tag{5.2.43}$$

From first and second part of (5.2.43), we get inequality (5.2.41) and from second and third part, we get (5.2.42).

Proposition 5.2.9. *Let $P, Q \in \Gamma_n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, then we have the following new relations*

$$\log 2 - F(P,Q) \leq A(P,Q) \tag{5.2.44}$$

and

$$\frac{1}{2} H(P,Q) + F(P,Q) \leq \log 2, \tag{5.2.45}$$

where $F(P,Q)$ is defined by (1.2.21) and $H(P,Q) = \sum_{i=1}^{n} \frac{2p_i q_i}{p_i+q_i}$.

Proof: Put $t = \frac{p_i}{q_i}$ in inequalities (5.2.2), we get

$$\frac{p_i}{p_i+q_i} \leq \log \frac{p_i+q_i}{q_i} \leq \frac{p_i}{q_i}.$$

Now multiply the above expression by $2q_i$ and sum over all $i = 1, 2, 3..., n$, we get

$$\sum_{i=1}^{n} 2q_i \frac{p_i}{p_i+q_i} \leq \sum_{i=1}^{n} 2q_i \log \frac{2(p_i+q_i)}{2q_i} \leq \sum_{i=1}^{n} 2q_i \frac{p_i}{q_i} \text{ , i.e.,}$$

$$H(P,Q) \leq 2\log 2 \sum_{i=1}^{n} q_i - 2\sum_{i=1}^{n} q_i \log \frac{2q_i}{p_i+q_i} \leq 2\sum_{i=1}^{n} p_i \text{ , i.e.,}$$

$$H(P,Q) \leq 2\log 2 - 2F(Q,P) \leq 2.$$

After interchanging P and Q, we obtain the following

$$H(P,Q) \leq 2\log 2 - 2F(P,Q) \leq 2. \tag{5.2.46}$$

From second and third part of (5.2.46), we get inequality (5.2.44) and from first and second part, we get (5.2.45).

Some more new relations:

The following inequalities (5.2.47) is a famous relation from literature (Taneja [101]). These expressions have been already defined in introduction of this thesis.

$$\begin{aligned} H(P,Q) &\leq B(P,Q) \leq N_3(P,Q) \leq A(P,Q) \\ &\leq R(P,Q) \leq S(P,Q) \leq C(P,Q). \end{aligned} \tag{5.2.47}$$

Now we can obtain some other important relations among various divergences with the help of above inequalities, these are as follows.

From (5.2.33) and (5.2.47), we obtain

$$H\left(P,Q\right) \le B\left(P,Q\right) \le N_3\left(P,Q\right) \le A\left(P,Q\right) \le h\left(P,Q\right) \le T\left(P,Q\right). \quad (5.2.48)$$

From (5.2.44) and (5.2.47), we obtain

$$\log 2 - F\left(P,Q\right) \le A\left(P,Q\right) \le R\left(P,Q\right) \le S\left(P,Q\right) \le C\left(P,Q\right). \quad (5.2.49)$$

From (5.2.33) and (5.2.44), we obtain

$$\log 2 - F\left(P,Q\right) \le A\left(P,Q\right) \le h\left(P,Q\right) \le T\left(P,Q\right). \quad (5.2.50)$$

Do (5.2.42) - (5.2.44), we get

$$G\left(Q,P\right) + F\left(Q,P\right) \le \frac{1}{2}\left[R_2\left(P,Q\right) + 1\right] - A\left(P,Q\right) \text{ , i.e.,}$$

$$2A\left(P,Q\right) + 2\left[G\left(Q,P\right) + F\left(Q,P\right)\right] \le R_2\left(P,Q\right) + 1 \text{ , i.e.,}$$

$$2A\left(P,Q\right) + J_R\left(P,Q\right) \le R_2\left(P,Q\right) + 1. \quad (5.2.51)$$

From (5.2.15), (5.2.22) and (5.2.32), we obtain

$$N_1^*\left(P,Q\right) - N_2^*\left(P,Q\right) \le \Delta\left(P,Q\right) \le \frac{1}{2}E_1^*\left(P,Q\right) \le \frac{1}{2}J_1^*\left(P,Q\right). \quad (5.2.52)$$

From (5.2.15) and (5.2.29), we obtain

$$N_1^*\left(P,Q\right) - N_2^*\left(P,Q\right) \le \Delta\left(P,Q\right) \le \frac{1}{2}E_1^*\left(P,Q\right) - L\left(P,Q\right). \quad (5.2.53)$$

Proposition 5.2.10. *Let $P, Q \in \Gamma_n$, then we have the followings new inter relations.*

$$\xi_m(P,Q) > E_m^*(P,Q). \tag{5.2.54}$$

$$\xi_m(P,Q) > \Delta_m(P,Q). \tag{5.2.55}$$

$$\omega_m(P,Q) > J_m^*(P,Q). \tag{5.2.56}$$

$$\xi_1(P,Q) > \chi^2(P,Q). \tag{5.2.57}$$

$$\xi_1(P,Q) \geq h(P,Q), \tag{5.2.58}$$

where $\xi_m(P,Q)$ and $\omega_m(P,Q)$ are defined by (4.3.12) and (4.3.17) respectively.

Proof: If we put $t = \frac{p_i}{q_i}, i = 1,2,3...,n$ in (5.2.3) to (5.2.7) and multiply by q_i, and then sum over all $i = 1,2,3...,n$, we get the desired relations (5.2.54) to (5.2.58) respectively.

Now we can easily say from (5.2.54), (5.2.55) and (5.2.56) that

$$\xi_1(P,Q) > E_1^*(P,Q), \xi_2(P,Q) > E_2^*(P,Q), ..., \tag{5.2.59}$$

$$\xi_1(P,Q) > \Delta_1(P,Q) = \Delta(P,Q), \xi_2(P,Q) > \Delta_2(P,Q), ..., \tag{5.2.60}$$

and

$$\omega_1(P,Q) > J_1^*(P,Q), \omega_2(P,Q) > J_2^*(P,Q), ... \tag{5.2.61}$$

respectively.

Proposition 5.2.11. *Let $P, Q \in \Gamma_n$, then we have the followings new inter relations.*

$$\begin{aligned} &2\left[N_1^*(P,Q) - N_2^*(P,Q)\right] \leq 2\Delta(P,Q) \leq 8I(P,Q) \leq 8h(P,Q) \leq J(P,Q) \\ &\leq 8T(P,Q) \leq E_1^*(P,Q) \leq \xi_1(P,Q) \leq \left[\zeta_1(P,Q), \omega_1(P,Q)\right]. \end{aligned} \tag{5.2.62}$$

$$\begin{aligned} &H(P,Q) \le B(P,Q) \le L_*(P,Q) \le N_1(P,Q) \le N_3(P,Q) \le N_2(P,Q) \\ &\le A(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \end{aligned} \tag{5.2.63}$$

$$\frac{1}{4}J_R(P,Q) \le K(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.64}$$

$$2\Delta(P,Q) - \frac{1}{2}\psi(P,Q) \le \chi^2(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.65}$$

$$\frac{1}{2}\left[\psi_M(P,Q) - \frac{1}{2}J_1^*(P,Q)\right] \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.66}$$

$$[J_1^*(P,Q) - J_2^*(P,Q)] \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.67}$$

$$2[\Delta(P,Q) + L(P,Q)] \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.68}$$

$$4M_{SA}(P,Q) \le \frac{4}{3}M_{SH}(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.69}$$

$$\frac{1}{2}M_{SB}(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.70}$$

$$32d(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.71}$$

$$2F(P,Q) \le \xi_1(P,Q) \le [\zeta_1(P,Q), \omega_1(P,Q)]. \tag{5.2.72}$$

$$6D_{\psi J}(P,Q) \le 64D_{\psi T}(P,Q) \le E_2^*(P,Q) \le \xi_2(P,Q) \le [\zeta_2(P,Q), \omega_2(P,Q)], \tag{5.2.73}$$

where $M_{SA}(P,Q)$, $M_{SB}(P,Q)$, $M_{SH}(P,Q)$, $D_{\psi T}(P,Q)$, $D_{\psi J}(P,Q)$, and $\zeta_m(P,Q)$ are defined by (1.2.42), (1.2.43), (1.2.44), (1.2.45), (1.2.46), and (4.3.14) respectively.

Proof: Since we know the following relations. Relations (5.2.75), (5.2.77), (5.2.81), and (5.2.82) have taken from literature (Jain and Chhabra [43]), relations (5.2.74) and (5.2.80) are from literature (Jain and Srivastava [49]), relations (5.2.79) and (5.2.86) are from (Jain and Chhabra [44]), relations (5.2.83) and (5.2.84) have taken from (Taneja [97]), whereas relations (5.2.78), (5.2.85), (5.2.76), and (5.2.87) are from literatures (Jain and Saraswat [48]), (Taneja [100]), (Taneja [101]), and (Taneja [99]) respectively.

$$\begin{aligned} &\frac{1}{4}\Delta\left(P,Q\right) \le I\left(P,Q\right) \le h\left(P,Q\right) \le \frac{1}{8}J\left(P,Q\right) \\ &\le T\left(P,Q\right) \le \frac{1}{8}E_1^*\left(P,Q\right). \end{aligned} \tag{5.2.74}$$

$$N_1^*\left(P,Q\right) - N_2^*\left(P,Q\right) \le \Delta\left(P,Q\right). \tag{5.2.75}$$

$$\begin{aligned} &H\left(P,Q\right) \le B\left(P,Q\right) \le L_*\left(P,Q\right) \le N_1\left(P,Q\right) \\ &\le N_3\left(P,Q\right) \le N_2\left(P,Q\right) \le A\left(P,Q\right). \end{aligned} \tag{5.2.76}$$

$$A\left(P,Q\right) \le h\left(P,Q\right). \tag{5.2.77}$$

$$\frac{1}{4}J_R\left(P,Q\right) \le K\left(P,Q\right) \le J\left(P,Q\right). \tag{5.2.78}$$

$$\Delta\left(P,Q\right) \le \frac{1}{2}\left[\frac{1}{2}\psi\left(P,Q\right) + \chi^2\left(P,Q\right)\right]. \tag{5.2.79}$$

$$\frac{1}{2}\psi_M\left(P,Q\right) \le E_1^*\left(P,Q\right) + \frac{1}{4}J_1^*\left(P,Q\right). \tag{5.2.80}$$

$$J_1^*\left(P,Q\right) - J_2^*\left(P,Q\right) \le E_1^*\left(P,Q\right). \tag{5.2.81}$$

$$\Delta\left(P,Q\right) \le \frac{1}{2}E_1^*\left(P,Q\right) - L\left(P,Q\right). \tag{5.2.82}$$

$$M_{SA}\left(P,Q\right) \le \frac{1}{3}M_{SH}\left(P,Q\right) \le \frac{1}{4}\Delta\left(P,Q\right). \tag{5.2.83}$$

$$\frac{1}{2}M_{SB}\left(P,Q\right) \le h\left(P,Q\right). \tag{5.2.84}$$

$$4d\left(P,Q\right) \le \frac{1}{8}J\left(P,Q\right). \tag{5.2.85}$$

$$F\left(P,Q\right) \le \frac{1}{2}\Delta\left(P,Q\right). \tag{5.2.86}$$

$$\frac{1}{4}D_{\psi J}\left(P,Q\right) \le \frac{8}{3}D_{\psi T}\left(P,Q\right) \le \frac{1}{24}E_2^*\left(P,Q\right). \tag{5.2.87}$$

By taking (5.2.74), (5.2.75) and first part of the relations (5.2.59) and (4.3.22) together, we get the relation (5.2.62).

By taking (5.2.58), (5.2.76), (5.2.77) and first part of the relation (4.3.22), we get the relation (5.2.63).

By taking (5.2.78) and fifth, eighth, ninth elements of the proved relation (5.2.62) together, we get the relation (5.2.64).

By taking (5.2.57), (5.2.79) and first part of the relation (4.3.22) together, we get the relation (5.2.65).

By taking (5.2.80) and first part of the relations (5.2.59) and (4.3.22) together, we get the relation (5.2.66).

By taking (5.2.81) and first part of the relations (5.2.59) and (4.3.22) together, we get the relation (5.2.67).

By taking (5.2.82) and first part of the relations (5.2.59) and (4.3.22) together, we get the relation (5.2.68).

By taking (5.2.83) and first part of the relations (5.2.60) and (4.3.22) together, we get the relation (5.2.69).

By taking (5.2.58), (5.2.84) and first part of the relation (4.3.22) together, we get the relation (5.2.70).

By taking (5.2.85) and fifth, eighth, ninth elements of the proved relation (5.2.62) together, we get the relation (5.2.71).

By taking (5.2.86) and first part of the relations (5.2.60) and (4.3.22) together, we get the relation (5.2.72).

By taking (5.2.87) and second part of the relations (5.2.59) and (4.3.22) together, we get the relation (5.2.73).

Proposition 5.2.12. *Let $P, Q \in \Gamma_n$, then we have the followings new inter relations.*

$$\gamma_m(P,Q) > E_m^*(P,Q), \tag{5.2.88}$$

$$\gamma_m(P,Q) > \Delta_m(P,Q), \tag{5.2.89}$$

$$\gamma_m(P,Q) > \chi^{2m}(P,Q), \tag{5.2.90}$$

and

$$\rho_m(P,Q) > J_m^*(P,Q), \tag{5.2.91}$$

where $\Delta_m(P,Q)$, $\chi^{2m}(P,Q)$, $\gamma_m(P,Q)$, and $\rho_m(P,Q)$ are defined by (1.2.30), (1.2.31), (4.2.12), and (4.2.17) respectively..

Proof: If we put $t = \frac{p_i}{q_i}, i = 1, 2, 3..., n$ in (5.2.8) to (5.2.11), multiply by q_i and then sum over all $i = 1, 2, 3..., n$, we get the desired relations (5.2.88) to (5.2.91) respectively.

Now we can easily say from (5.2.88) to (5.2.91), that

$$\gamma_1(P,Q) > E_1^*(P,Q) = E^*(P,Q), \gamma_2(P,Q) > E_2^*(P,Q), ..., \tag{5.2.92}$$

$$\gamma_1(P,Q) > \Delta_1(P,Q) = \Delta(P,Q), \gamma_2(P,Q) > \Delta_2(P,Q), ..., \tag{5.2.93}$$

$$\gamma_1(P,Q) > \chi^2(P,Q), \gamma_2(P,Q) > \chi^4(P,Q), ..., \tag{5.2.94}$$

and

$$\rho_1(P,Q) > J_1^*(P,Q), \rho_2(P,Q) > J_2^*(P,Q), ..., \tag{5.2.95}$$

respectively.

Proposition 5.2.13. *Let $P, Q \in \Gamma_n$, then we have the followings new inter relations.*

$$\rho_m(P,Q) > J_m^*(P,Q) \geq E_m^*(P,Q), \tag{5.2.96}$$

$$\rho_1(P,Q) > 2\Delta(P,Q) \geq 2\left[N_1^*(P,Q) - N_2^*(P,Q)\right], \tag{5.2.97}$$

$$\rho_1(P,Q) > 8T(P,Q) \geq J(P,Q) \geq 8h(P,Q) \geq 8I(P,Q), \tag{5.2.98}$$

and

$$\begin{aligned}&\rho_1(P,Q) > 8A(P,Q) \geq 8N_2(P,Q) \geq 8N_3(P,Q) \geq 8N_1(P,Q)\\ &\geq 8L_*(P,Q) \geq 8B(P,Q) \geq 8H(P,Q).\end{aligned} \tag{5.2.99}$$

Proof: Since we know the followings. Relations (5.2.100), (5.2.101), and (5.2.103) are from (Jain and Chhabra [43]), whereas relations (5.2.102) and (5.2.104) are from literatures (Jain and Srivastava [49]) and (Taneja [101]) respectively.

$$J_m^*(P,Q) \geq E_m^*(P,Q), \tag{5.2.100}$$

$$\frac{1}{2}E^*(P,Q) \geq \Delta(P,Q) \geq [N_1^*(P,Q) - N_2^*(P,Q)], \tag{5.2.101}$$

$$\frac{1}{2}E^*(P,Q) \geq T(P,Q) \geq \frac{1}{8}J(P,Q) \geq h(P,Q) \geq I(P,Q), \tag{5.2.102}$$

$$T(P,Q) \geq A(P,Q), \tag{5.2.103}$$

and

$$\begin{aligned}&A(P,Q) \geq N_2(P,Q) \geq N_3(P,Q) \geq N_1(P,Q)\\ &\geq L_*(P,Q) \geq B(P,Q) \geq H(P,Q).\end{aligned} \tag{5.2.104}$$

By taking (5.2.91) and (5.2.100) together, we get the relation (5.2.96).

By taking first and third part of the proved relation (5.2.96) at $m = 1$ together with (5.2.101), we get the relation (5.2.97).

By taking first and third part of the proved relation (5.2.96) at $m = 1$ together with (5.2.102), we get the relation (5.2.98).

By taking first and second part of the proved relation (5.2.98) together with (5.2.103) and (5.2.104), we get the relation (5.2.99).

5.3 Conclusion

In this chapter, we derived many new important and interesting relations among several divergences by using some algebraic, exponential, and logarithmic inequalities.

6

NEW GENERALIZED INFORMATION DIVERGENCE FOR COMPARING FINITE PROBABILITY DISTRIBUTIONS AND APPLICATIONS

6.1 Introduction

Several generalized divergences had been introduced in information theory for comparing two probability distributions at a time, like; Csiszar's divergence ([2], [20]), Bregman's divergence ([14]), Burbea- Rao's divergence ([16]), Renyi's divergence ([79]), Jain and Saraswat's divergence ([48]) etc. This chapter introduces new generalized divergence measure for comparing finite number of discrete probability distributions.

This chapter is organized as follows: Besides introduction and conclusion sections, there are two more sections as well. In section 6.2, we introduce a new generalized divergence for comparing $2j, j = 1, 2, 3..., n$ discrete probability distributions at a time. we also obtain a sequence of intra relations among this generalized measure and the measures with $2j+2, 2j-2, 2j-4, ..., 4, 2\ \forall j = 1, 2, 3..., n$ discrete probability distributions, respectively. Relation with other generalized divergence has been obtained as well. In section 6.3, Special cases as an application, are discussed.

6.2 New Generalized Divergence, Properties and Relations

Let $\Gamma_m = \{P = (p_1, p_2, p_3, ..., p_m) : p_i > 0, \sum_{i=1}^m p_i = 1\}$, $m \geq 2$ be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3..., m$, then we have to suppose that $0f(0) = 0f\left(\frac{0}{0}\right) = 0$.

Let $P_1 = (p_{11}, ..., p_{m1}), ..., P_n = (p_{1n}, ..., p_{mn})$ and $Q_1 = (q_{11}, ..., q_{m1}), ..., Q_n = (q_{1n}, ..., q_{mn})$ be discrete probability distributions such that $P_j, Q_j \in \Gamma_m\ \forall\ j = 1, 2, ..., n$. Now we define a new generalized information divergence measure among $2n$ discrete probability distributions by

$$S_f^n(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n) = \sum_{i=1}^m \sum_{i=1}^m ... \sum_{i=1}^m q_{i1}q_{i2}...q_{in} f\left(\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}} + \frac{p_{i2}+q_{i2}}{2q_{i2}} + ... + \frac{p_{in}+q_{in}}{2q_{in}}}{n}\right), \tag{6.2.1}$$

where $f : (0, \infty) \to R$ (set of real no.) is real, continuous, and convex function. Particularly, Jain and Saraswat's generalized divergence measure (1.2.47) is a special case of this measure, which is

$$S_f^1(P_1, Q_1) = \sum_{i=1}^{m} q_{i1} f\left(\frac{p_{i1} + q_{i1}}{2q_{i1}}\right) = \sum_{i=1}^{m} q_i f\left(\frac{p_i + q_i}{2q_i}\right). \tag{6.2.2}$$

Now we define the following basic properties of measure (6.2.1).

(a). $S_f^n(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n) > 0$ and is convex in the pair of probability distribution $P, Q \in \Gamma_m$.

(b). $S_f^n(P_j, Q_j) = 0$ if $P_j = Q_j \; \forall \; j = 1, 2..., n$ (attains its minimum value).

(c). $S_f^n(P_j, Q_j)$ attains its maximum value when P_j and Q_j are perpendicular to each other for each j.

6.2.1 Intra relation among new generalized divergences

Now, we derive an important and fruitful relation among new generalized divergence measures. The sequence of these measures are basically special cases of (6.2.1) according to the number of probability distributions.

Theorem 6.2.1. *Let $f : (0, \infty) \to R$ be a differentiable, convex function, i.e., $f''(t) \geq 0 \; \forall t > 0$. For $P_j, Q_j \in \Gamma_m \; \forall \; j = 1, 2..., n$, we have*

$$\begin{aligned} &S_f^1(P_1, Q_1) \geq S_f^2(P_1, P_2, Q_1, Q_2) \geq ... \geq S_f^n(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n) \\ &\geq S_f^{n+1}(P_1, P_2, ..., P_n, P_{n+1}, Q_1, Q_2, ..., Q_n, Q_{n+1}) \geq f(1), \end{aligned} \tag{6.2.3}$$

where $S_f^n(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n)$ is given by (6.2.1).

Proof: By using Jensen inequality (1.3.3) for multiple summations for the discrete probability distributions, we get

$$\sum_{i=1}^{m} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1} q_{i2} ... q_{in} q_{i(n+1)} f\left(\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}} + \frac{p_{i2}+q_{i2}}{2q_{i2}} + ... + \frac{p_{in}+q_{in}}{2q_{in}} + \frac{p_{i(n+1)}+q_{i(n+1)}}{2q_{i(n+1)}}}{n+1}\right)$$

$$\geq f\left[\sum_{i=1}^{m}\sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}q_{i2}...q_{in}q_{i(n+1)}\left(\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}}+\frac{p_{i2}+q_{i2}}{2q_{i2}}+...+\frac{p_{in}+q_{in}}{2q_{in}}+\frac{p_{i(n+1)}+q_{i(n+1)}}{2q_{i(n+1)}}}{n+1}\right)\right]$$

$$= f\left[\frac{1}{n+1}\left(\sum_{i=1}^{m}\left(\frac{p_{i1}+q_{i1}}{2}\right)\sum_{i=1}^{m}q_{i2}...\sum_{i=1}^{m}q_{i(n+1)}+...+\sum_{i=1}^{m}\left(\frac{p_{i(n+1)}+q_{i(n+1)}}{2}\right)\sum_{i=1}^{m}q_{i1}...\sum_{i=1}^{m}q_{in}\right)\right]$$

$$= f\left[\frac{1}{n+1}(1+1+...+1)\right] = f\left(\frac{n+1}{n+1}\right) = f(1), \text{ i.e.,}$$

$$S_f^{n+1}(P_1,P_2,...,P_n,P_{n+1},Q_1,Q_2,...,Q_n,Q_{n+1}) \geq f(1). \tag{6.2.4}$$

Hence the last inequality of relation (6.2.3) is proved.

Now apply Jensen inequality (1.3.3) for $x_1, x_2, ..., x_{n+1}$, where $x_i \in (0,\infty)$ $\forall\ i = 1, 2, ..., n+1$, we obtain

$$\frac{1}{n+1}\left[f(x_1)+f(x_2)+...+f(x_n)+f(x_{n+1})\right] \geq f\left[\frac{x_1+x_2+...+x_n+x_{n+1}}{n+1}\right]. \tag{6.2.5}$$

Let

$$x_1 = \frac{z_1+z_2+...+z_n}{n}, x_2 = \frac{z_2+z_3+...+z_n+z_{n+1}}{n}, ..., x_{n+1} = \frac{z_{n+1}+z_1+...+z_{n-1}}{n},$$

where $z_i \in (0,\infty)$ $\forall\ i = 1, 2, ..., n+1$.

Then by inequality (6.2.5), we get

$$\begin{aligned}&\frac{1}{n+1}\left[f\left(\frac{z_1+z_2+...+z_n}{n}\right)+...+f\left(\frac{z_{n+1}+z_1+...+z_{n-1}}{n}\right)\right]\\ &\geq f\left[\frac{1}{n+1}\left(\frac{z_1+z_2+...+z_n}{n}+...+\frac{z_{n+1}+z_1+...+z_{n-1}}{n}\right)\right]\\ &= f\left(\frac{n(z_1+z_2+...+z_n+z_{n+1})}{n(n+1)}\right) = f\left(\frac{z_1+z_2+...+z_n+z_{n+1}}{n+1}\right).\end{aligned} \tag{6.2.6}$$

Now put $z_j = \frac{p_{ij}+q_{ij}}{2q_{ij}}$ in (6.2.6), multiply with q_{ij} $\forall\ j = 1, ..., n+1$ and for each $i = 1, ..., m$ and then summation $n+1$ times from $i = 1$ to $i = m$, we get

$$\frac{1}{n+1}\left[\sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}...q_{i(n+1)}f\left(\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}}+...+\frac{p_{in}+q_{in}}{2q_{in}}}{n}\right)\right]$$

$$+...+\frac{1}{n+1}\left[\sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}...q_{i(n+1)}f\left(\frac{\frac{p_{i(n+1)}+q_{i(n+1)}}{2q_{i(n+1)}}+\frac{p_{i1}+q_{i1}}{2q_{i1}}+...+\frac{p_{i(n-1)}+q_{i(n-1)}}{2q_{i(n-1)}}}{n}\right)\right]$$

$$\geq \sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}...q_{i(n+1)}f\left(\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}}+...+\frac{p_{i(n+1)}+q_{i(n+1)}}{2q_{i(n+1)}}}{n+1}\right), \text{ i.e.,}$$

$$S_f^n\left(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n\right) \geq S_f^{n+1}\left(P_1,P_2,...,P_{n+1},Q_1,Q_2,...,Q_{n+1}\right). \tag{6.2.7}$$

Hence the second last inequality of (6.2.3) is proved for all n, and the theorem is thus proved.

Remark 6.2.1. *If function is normalized, i.e., $f(1) = 0$, then we obtain the following sequence of new relations from (6.2.3).*

$$\begin{aligned}&S_f^1\left(P_1,Q_1\right) \geq S_f^2\left(P_1,P_2,Q_1,Q_2\right) \geq ... \geq S_f^n\left(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n\right)\\&\geq S_f^{n+1}\left(P_1,P_2,...,P_n,P_{n+1},Q_1,Q_2,...,Q_n,Q_{n+1}\right) \geq 0.\end{aligned} \tag{6.2.8}$$

6.2.2 Relation between two different generalized divergences

The following generalized divergence measure for comparing finite discrete probability distributions, is introduced by Dragomir [26], which is

$$\begin{aligned}&C_f^n\left(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n\right)\\&=\sum_{i=1}^{m}\sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}q_{i2}...q_{in}f\left(\frac{\frac{p_{i1}}{q_{i1}}+\frac{p_{i2}}{q_{i2}}+...+\frac{p_{in}}{q_{in}}}{n}\right),\end{aligned} \tag{6.2.9}$$

where $f:(0,\infty)\to R$ (set of real no.) is real, continuous, and convex function. Particularly, Ciszar's generalized divergence measure (1.2.1) is a special case of this measure, which is

$$C_f^1\left(P_1,Q_1\right) = \sum_{i=1}^{m}q_{i1}f\left(\frac{p_{i1}}{q_{i1}}\right) = \sum_{i=1}^{m}q_{i}f\left(\frac{p_{i}}{q_{i}}\right). \tag{6.2.10}$$

Now we derive a special and important relation between generalized measure (6.2.1) and (6.2.9), by the following theorem .

Theorem 6.2.2. *Let $f:(0,\infty)\to R$ be a differentiable, convex and normalized function, i.e., $f''(t)\geq 0\ \forall t>0$ and $f(1)=0$ respectively. For $P_j, Q_j \in \Gamma_m\ \forall\ j=1,2...,n$, we have*

$$S_f^n\left(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n\right)\leq \frac{1}{2}C_f^n\left(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n\right). \quad (6.2.11)$$

Proof: Apply Jensen inequality (1.3.3) for the domain $I\subset(0,\infty)$, by putting $\lambda_1=\lambda_2=\frac{1}{2}, \lambda_3=...=\lambda_n=0$, we get

$$f\left(\frac{t_1+t_2}{2}\right)\leq\frac{1}{2}\left[f\left(t_1\right)+f\left(t_2\right)\right]. \quad (6.2.12)$$

Now put $t_1=t$ and $t_2=1$ in above inequality, we obtain

$$f\left(\frac{t+1}{2}\right)\leq\frac{1}{2}f\left(t\right). \quad (6.2.13)$$

Now take $t=\frac{\sum_{j=1}^{n}\frac{p_{ij}}{q_{ij}}}{n}$ in inequality (6.2.13), multiply with $\prod_{j=1}^{n}q_{ij}$ for each i and then summation over n times from $i=1$ to $i=m$, we obtain the required relation (6.2.11).

Remark 6.2.2. *By considering two probability distributions at a time, we get the following well known result from inequality (6.2.11)*

$$\sum_{i=1}^{m}q_if\left(\frac{p_i+q_i}{2q_i}\right)\leq\frac{1}{2}\sum_{i=1}^{m}q_if\left(\frac{p_i}{q_i}\right)\Rightarrow S_f\left(P,Q\right)\leq\frac{1}{2}C_f\left(P,Q\right),$$

where $S_f\left(P,Q\right), C_f\left(P,Q\right)$ are given by (6.2.2) and (6.2.10) respectively.

6.3 Application of New Generalized Divergence

In previous section, we introduced new generalized divergence measure for comparing finite discrete probability distributions. In this section, we apply this new generalized divergence on Variational distance, Chi- square divergence and Exponential divergence respectively and obtain the interesting relations. The results are on the similar lines to the results presented by Dragomir [26].

Proposition 6.3.1. *Let $P_j, Q_j \in \Gamma_m \ \forall \ j = 1, 2, ..., n$, then we have*

$$V_n\left(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n\right) \leq \sum_{j=1}^{n} V\left(P_j, Q_j\right) \tag{6.3.1}$$

and

$$\begin{aligned}&V_1\left(P_1, Q_1\right) = V\left(P, Q\right) \geq \frac{1}{2} V_2\left(P_1, P_2, Q_1, Q_2\right) \geq ... \geq \frac{1}{n} V_n\left(P_1, ..., P_n, Q_1, ..., Q_n\right)\\ &\geq \frac{1}{n+1} V_{n+1}\left(P_1, ..., P_n, P_{n+1}, Q_1, ..., Q_n, Q_{n+1}\right) \geq 0.\end{aligned} \tag{6.3.2}$$

Proof: Let $f(t) = |t-1|, t > 0$. Here $f(t)$ is convex and normalized function because $f''(t) \geq 0 \ \forall \ t > 0$ but not at $t = 1$ and $f(1) = 0$ respectively.

Put $f(t)$ in (6.2.1) and (6.2.2), we obtain the followings respectively.

$$\begin{aligned}S_f^n\left(P_1, ..., P_n, Q_1, ..., Q_n\right) &= \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left| \frac{\frac{p_{i1}+q_{i1}}{2q_{i1}} + ... + \frac{p_{in}+q_{in}}{2q_{in}}}{n} - 1 \right|\\ &= \frac{1}{n} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left| \left(\frac{p_{i1}+q_{i1}}{2q_{i1}} - 1\right) + ... + \left(\frac{p_{in}+q_{in}}{2q_{in}} - 1\right) \right|\\ &= \frac{1}{2n} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left| \frac{p_{i1}-q_{i1}}{q_{i1}} + ... + \frac{p_{in}-q_{in}}{q_{in}} \right|\\ &= \frac{1}{2n} V_n\left(P_1, ..., P_n, Q_1, ..., Q_n\right)\end{aligned} \tag{6.3.3}$$

and

$$\frac{1}{2}\sum_{i=1}^{m}|p_i - q_i| = \frac{1}{2}V(P,Q), \tag{6.3.4}$$

where $V_n(P_1,...,P_n,Q_1,...,Q_n)$ is designated as generalized Variational distance and $V(P,Q)$ is the well known Variational distance, a special case of $V_n(P_1,...,P_n,Q_1,...,Q_n)$ for comparing two probability distributions.

Now, equation (6.3.3) can be written as

$$\frac{1}{2n}V_n(P_1,...,P_n,Q_1,...,Q_n) \le \frac{1}{2n}\sum_{i=1}^{m}...\sum_{i=1}^{m}q_{i1}...q_{in}\left[\left|\frac{p_{i1}-q_{i1}}{q_{i1}}\right| + ... + \left|\frac{p_{in}-q_{in}}{q_{in}}\right|\right]$$

$$= \frac{1}{2n}\left[\sum_{i=1}^{m}|p_{i1}-q_{i1}|\sum_{i=1}^{m}q_{i2}...\sum_{i=1}^{m}q_{in} + ... + \sum_{i=1}^{m}|p_{in}-q_{in}|\sum_{i=1}^{m}q_{i1}...\sum_{i=1}^{m}q_{i(n-1)}\right]$$

$$= \frac{1}{2n}\left[\sum_{i=1}^{m}|p_{i1}-q_{i1}| + ... + \sum_{i=1}^{m}|p_{in}-q_{in}|\right] = \frac{1}{2n}\sum_{j=1}^{n}\sum_{i=1}^{m}|p_{ij}-q_{ij}| = \frac{1}{2n}\sum_{j=1}^{n}V(P_j,Q_j).$$

$$\Rightarrow V_n(P_1,...,P_n,Q_1,...,Q_n) \le \sum_{j=1}^{n}V(P_j,Q_j).$$

Hence prove the relation (6.3.1) and sequence of inequalities (6.3.2) can be obtained by using (6.2.8) directly.

Proposition 6.3.2. *Let $P_j, Q_j \in \Gamma_m \ \forall\ j = 1,2,...,n$, then we have*

$$\chi_n^2(P_1,P_2,...,P_n,Q_1,Q_2,...,Q_n) = \sum_{j=1}^{n}\chi^2(P_j,Q_j) \tag{6.3.5}$$

and

$$\chi_1^2(P_1,Q_1) = \chi^2(P,Q) \ge \frac{1}{2^2}\chi_2^2(P_1,P_2,Q_1,Q_2) \ge ... \ge \frac{1}{n^2}\chi_n^2(P_1,...,P_n,Q_1,...,Q_n)$$
$$\ge \frac{1}{(n+1)^2}\chi_{n+1}^2(P_1,...,P_n,P_{n+1},Q_1,...,Q_n,Q_{n+1}) \ge 0. \tag{6.3.6}$$

Proof: Let $f(t) = (t-1)^2, t > 0$. Here $f(t)$ is convex and normalized function because $f''(t) \geq 0 \; \forall \; t > 0$ and $f(1) = 0$ respectively.

Put $f(t)$ in (6.2.1) and (6.2.2), we obtain the followings respectively.

$$\begin{aligned}
S_f^n(P_1, ..., P_n, Q_1, ..., Q_n) &= \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left[\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}} + ... + \frac{p_{in}+q_{in}}{2q_{in}}}{n} - 1 \right]^2 \\
&= \frac{1}{n^2} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left[\left(\frac{p_{i1}+q_{i1}}{2q_{i1}} - 1 \right) + ... + \left(\frac{p_{in}+q_{in}}{2q_{in}} - 1 \right) \right]^2 \\
&= \frac{1}{4n^2} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left[\frac{p_{i1}-q_{i1}}{q_{i1}} + ... + \frac{p_{in}-q_{in}}{q_{in}} \right]^2 \\
&= \frac{1}{4n^2} \chi_n^2 (P_1, ..., P_n, Q_1, ..., Q_n)
\end{aligned} \tag{6.3.7}$$

and

$$\frac{1}{4} \sum_{i=1}^{m} \frac{(p_i - q_i)^2}{q_i} = \frac{1}{4} \chi^2 (P, Q), \tag{6.3.8}$$

where $\chi_n^2(P_1, ..., P_n, Q_1, ..., Q_n)$ is designated as generalized Chi- square divergence and $\chi^2(P,Q)$ is the well known Chi- square divergence, a special case of $\chi_n^2(P_1, ..., P_n, Q_1, ..., Q_n)$ for comparing two probability distributions.

Now, the above equation (6.3.7) can be written as

$$\frac{1}{4n^2} \chi_n^2 (P_1, ..., P_n, Q_1, ..., Q_n)$$

$$= \frac{1}{n^2} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left[\sum_{j=1}^{n} \left(\frac{p_{ij}+q_{ij}}{2q_{ij}} - 1 \right)^2 + 2 \sum_{1 \leq j < k \leq n} \left(\frac{p_{ij}+q_{ij}}{2q_{ij}} - 1 \right) \left(\frac{p_{ik}+q_{ik}}{2q_{ik}} - 1 \right) \right]$$

$$= \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left(\frac{p_{ij}+q_{ij}}{2q_{ij}} - 1 \right)^2$$

$$+ \frac{2}{n^2} \sum_{1 \leq j < k \leq n} \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in} \left(\frac{p_{ij}+q_{ij}}{2q_{ij}} - 1 \right) \left(\frac{p_{ik}+q_{ik}}{2q_{ik}} - 1 \right)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{i1} ... \sum_{i=1}^{m} q_{ij}\left(\frac{p_{ij}+q_{ij}}{2q_{ij}}-1\right)^2 ... \sum_{i=1}^{m} q_{in}\right]$$

$$+\frac{2}{n^2}\sum_{1\le j<k\le n}\left[\sum_{i=1}^{m} q_{i1} ... \sum_{i=1}^{m} q_{ij}\left(\frac{p_{ij}+q_{ij}}{2q_{ij}}-1\right) ... \sum_{i=1}^{m} q_{ik}\left(\frac{p_{ik}+q_{ik}}{2q_{ik}}-1\right) ... \sum_{i=1}^{m} q_{in}\right]$$

$$= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{m} q_{ij}\left(\frac{p_{ij}+q_{ij}}{2q_{ij}}-1\right)^2 + \frac{2}{n^2}\sum_{1\le j<k\le n}\sum_{i=1}^{m}\left(\frac{p_{ij}-q_{ij}}{2}\right)\sum_{i=1}^{m}\left(\frac{p_{ik}-q_{ik}}{2}\right)$$

$$= \frac{1}{4n^2}\sum_{j=1}^{n}\chi^2\left(P_j, Q_j\right) + 0 = \frac{1}{4n^2}\sum_{j=1}^{n}\chi^2\left(P_j, Q_j\right).$$

$$\Rightarrow \chi_n^2\left(P_1, ..., P_n, Q_1, ..., Q_n\right) = \sum_{j=1}^{n}\chi^2\left(P_j, Q_j\right).$$

Hence prove the relation (6.3.5) and sequence of inequalities (6.3.6) can be obtained by using (6.2.8) directly.

Proposition 6.3.3. *Let $P_j, Q_j \in \Gamma_m \; \forall \; j = 1, 2, ..., n$, then we have*

$$D_{exp}^{n}\left(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n\right) = \prod_{j=1}^{n} D_{exp^{\frac{1}{n}}}\left(P_j, Q_j\right) \tag{6.3.9}$$

and

$$D_{exp}^{1}\left(P_1, Q_1\right) = D_{exp}\left(P, Q\right) \ge D_{exp}^{2}\left(P_1, P_2, Q_1, Q_2\right) \ge ... \ge D_{exp}^{n}\left(P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n\right)$$
$$\ge D_{exp}^{n+1}\left(P_1, P_2, ..., P_n, P_{n+1}, Q_1, Q_2, ..., Q_n, Q_{n+1}\right) \ge e. \tag{6.3.10}$$

Proof: Let $f(t) = e^t, t > 0$. Here $f(t)$ is convex but not normalized function because $f''(t) \ge 0 \; \forall \; t > 0$ and $f(1) \ne 0$ respectively.

Put $f(t)$ in (6.2.1) and (6.2.2), we obtain the followings respectively.

$$D_{exp}^{n}\left(P_1, ..., P_n, Q_1, ..., Q_n\right) = \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in}\exp\left(\frac{\frac{p_{i1}+q_{i1}}{2q_{i1}} + ... + \frac{p_{in}+q_{in}}{2q_{in}}}{n}\right) \tag{6.3.11}$$

and

$$\sum_{i=1}^{m} q_i e^{\frac{p_i+q_i}{2q_i}} = D_{exp}\left(P,Q\right), \tag{6.3.12}$$

where (6.3.11) is designated as generalized Exponential divergence and (6.3.12) is called Exponential divergence, a special case of (6.3.11) for comparing two probability distributions.

Now, equation (6.3.11) can be written as

$$D^n_{exp}\left(P_1,...,P_n,Q_1,...,Q_n\right) = \sum_{i=1}^{m} ... \sum_{i=1}^{m} q_{i1}...q_{in}\left(e^{\frac{p_{i1}+q_{i1}}{2nq_{i1}}}...e^{\frac{p_{in}+q_{in}}{2nq_{in}}}\right)$$

$$= \sum_{i=1}^{m} q_{i1}e^{\frac{p_{i1}+q_{i1}}{2nq_{i1}}} ... \sum_{i=1}^{m} q_{in}e^{\frac{p_{in}+q_{in}}{2nq_{in}}}$$

$$= \prod_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}e^{\frac{p_{ij}+q_{ij}}{2nq_{ij}}}\right] = \prod_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}\left(e^{\frac{p_{ij}+q_{ij}}{2q_{ij}}}\right)^{\frac{1}{n}}\right]$$

$$= \prod_{j=1}^{n} D_{exp^{\frac{1}{n}}}\left(P_j,Q_j\right).$$

Hence prove the relation (6.3.9) and sequence of inequalities (6.3.10) can be obtained by using (6.2.3) directly by considering $f\left(1\right) = e \neq 0$.

6.4 Conclusion

In this chapter, we have introduced a new generalized divergence for comparing more than two discrete probability distributions at a time. This new generalized measure is an extension of Jain and Saraswat's [48] generalized divergence measure. We also derived a relation between this new measure and another generalized measure (Dragomir [26]). Interesting relations on Variational distance (1.2.7), Chi- square divergence (1.2.19), and Exponential divergence have been evaluated as well by using new generalized divergence measure (6.2.1).

Future Scope:

With several new directions that open with the study reported here, there is scope of further work. Some of that we can suggest are the followings:

a. Study of new information inequalities in Mutual information sense (Dragomir etc. all [28]), which tell us how far the joint distribution is from its independency if distributions are independent to each other.

b. Study of Metric spaces over the set of positive real numbers by helping new symmetric divergence measures, also can be seen in literatures (Bhatia and Singh [11], Jain and Chhabra [45]). So we strongly believe that divergence measures can be extended to other significant problems of functional analysis and its applications and such investigations are actually in progress because this is also an area worth being investigated.

c. Study of divergences in fuzzy mathematics as fuzzy directed divergences and fuzzy entropies (Bajaj and Hooda [4], Hooda [39], Jha and Mishra [52]), which are very useful to find the amount of average ambiguity or difficulty in making a decision whether an element belongs to a set or not. Fuzzy information measures have recently found applications to fuzzy aircraft control, fuzzy traffic control, engineering, medicines, computer science, management and decision making etc.

d. Study of utilities of different events (Bhullar etc. all [12], Taneja and Tuteja [91]), i.e., an event is how much useful compare to other events.

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14. K.C. Jain and Praphull Chhabra, New information inequalities on absolute value of the functions and its application, **Communicated**.

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Achievements: CSIR-NET (June-2011), GATE (2011).

Teaching Experience:

1. University of Engineering and Management, Jaipur from August 01- 2012 to December 31- 2012.
2. Vivekananda Institute of Technology, Jaipur (Rajasthan) from July 09- 2011 to July 20- 2012.
3. Alankaar P.G. College, Jaipur (Rajasthan) from July 01- 2010 to March 31- 2011.

Strengths: Positive attitude, Complete dedication with work, Punctual, and Honest.

I hereby declare that the above statements are true and best of my knowledge.

Praphull Chhabra