A STUDY OF MELLIN - BARNES TYPE INTEGRALS, SINGLE AND MULTIVARIABLE POLYNOMIALS AND FRACTIONAL INTEGRALS

Ph.D. Thesis

RAKESH KUMAR BOHRA

ID: 2011RMA7118

DEPARTMENT OF MATHEMATICS
MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY, JAIPUR

JUNE, 2018

A STUDY OF MELLIN - BARNES TYPE INTEGRALS, SINGLE AND MULTIVARIABLE POLYNOMIALS AND FRACTIONAL INTEGRALS

Submitted in

fulfillment of the requirements for the degree of

Doctor of Philosophy

by

RAKESH KUMAR BOHRA

ID: 2011RMA7118

Under the Supervision of

Dr. SANJAY BHATTER

DEPARTMENT OF MATHEMATICS
MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY, JAIPUR

JUNE, 2018

DECLARATION

I, **Rakesh Kumar Bohra,** declare that this thesis titled **"A STUDY OF MELLIN-BARNES TYPE INTEGRALS, SINGLE AND MULTIVARIABLE POLYNOMIALS AND FRACTIONAL INTEGRALS"** and the work presented in it, are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this university.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself, jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Date:

RAKESH KUMAR BOHRA
ID: 2011RMA7118

CERTIFICATE

This is to certify that the thesis entitled **“A STUDY OF MELLIN-BARNES TYPE INTEGRALS, SINGLE AND MULTIVARIABLE POLYNOMIALS AND FRACTIONAL INTEGRALS”** being submitted by **Rakesh Kumar Bohra (ID: 2011RMA7118)** is a bonafide research work carried out under my supervision and guidance in fulfillment of the requirement for the award of degree of **Doctor of Philosophy** in the Department of Mathematics**,** Malaviya National Institute of Technology, Jaipur, India. The matter embodied in this thesis is original and has not been submitted to any other University or Institute for the award of any other degree.

Dr. Sanjay Bhatter
Assistant Professor
Department of Mathematics
MNIT Jaipur

Place: Jaipur
Date:

ACKNOWLEDGEMENTS

I take this opportunity to express my sense of gratitude to my supervisor Dr. Sanjay Bhatter, Assistant Professor, Department of Mathematics, Malaviya National Institute of Technology, Jaipur , for his continuous guidance not to leave any stone unturned. He has successfully helped me to overcome my problems.

I acknowledge my sincere gratitude to Professor Uday kumar R. Yaragatti, Director and Prof. I.K. Bhatt, Former Director, Malaviya National Institute of Technology, Jaipur, for their support and facilities to produce this research work. I am also thankful to, Dr. Santosh Choudhary, Head of the Department of Mathematics, for her kind care and help during the entire period of research.

I am thankful to prof. (Mrs.) Rashmi Jain, Convener of DPGC. I express the deep and profound sense of gratitude to my DREC member's Dr. Vatsala Mathur and Dr. Ritu Agrawal, Department of Mathematics, MNIT, Jaipur, for their persistent encouragement and suggestions and also thanks to clerical staff and peon staff of the Department of Mathematics.

I am indebted to my beloved parents Shri. Hanuman Prasad Bohra and Smt. Urmila Bohra, for encouraging me for my future career and goals of my life. They have supported me on every step of my life.

I would like to thank my wife Jinal Bohra and daughter Aaradhya Bohra for appreciating my work and guiding me through fruitful suggestions.

Last but not least, I extend our acknowledgement to my colleagues & friends for their sustained support during this work.

(RAKESH KUMAR BOHRA)

ABSTRACT

In **Chapter 1**, we introduce a pathway fractional integral operator associated with the pathway model and Pathway density. We also establish three theorems in this chapter. In first theorem we find pathway fractional operator whose kernel involves the product of Aleph-function, Multivariable's general class of polynomial and H-function. In second theorem we find pathway fractional operator whose kernel involves the product of Aleph-function, Multivariable's general class of polynomial, H-function and Mittag-Leffler function. At last, we establish third theorem on pathway fractional operator whose kernel involves the product of two Aleph-functions. Also we obtain new and known special cases of our all three theorems.

Chapter-2 deals with the study of a pair of multidimensional fractional integral operators whose kernels involve the product of a multivariable polynomial $S_V^{U_1,\ldots,U_k}$ and Aleph function. First we define the operators of our study and give conditions of existence of these operators. Then we obtain some images of certain useful functions under these operators. Next, we establish two theorems giving the multidimensional generalized Stieltjes transform of fractional integral operators and conversely. Then we present Mellin transform, Mellin convolutions and inversion formulae for these operators. Finally, we derive three new and interesting composition formulae of our multidimensional fractional integral operators.

In addition we have also evaluated a double integral of a very general nature with the help of our first composition formula. A special case of the same is also given.

Chapter-3 is divided into two parts. The main integral evaluated in **part-A**, we establish the main integral whose integrand involve the product of a general class of polynomials, a general sequence of function and Aleph-function with general arguments. The integral is sufficiently general in nature and a large number of known and new integrals follow as its special cases. We have obtained eight special cases of our main result, which are also new and known results.

In **part-B**, we evaluate three finite integrals whose integrand involving the product of generalized Legendre associated function $P_{\gamma}^{\alpha,\beta}(x)$, general sequence of function $S_{n}^{\mu,\delta,0}$ and Aleph ($\aleph$) - function. Next we establish three theorems as an application of our main findings and using three results of Orr and Bailey recorded in well-known text by Slater. Further, we evaluate certain new integrals by applications of these Theorems, which are of interest by themselves and sufficiently general in nature.

In **Chapter 4**, we establish three theorems. First two theorems whose integrand involve the product of Aleph-function and multivariable's polynomial $S_{V}^{U_1,\ldots,U_k}$ using Riemann-Liouville fractional operator and last theorem involving the product of multivariable Aleph-function and multivariable's polynomial $S_{V}^{U_1,\ldots,U_k}$ using Generalized Saigo derivative operators. Next, we give six corollaries involving useful special functions specially first class of multivariable hypergeometric polynomial, multivariable Jacobi polynomial, multivariable Bessel polynomial. Finally, theorems 1 and 2 are then employed to establish two multiplication formulae for multivariable Aleph-function from these multiplication formulas we can obtained a number of known and unknown multiplication formulae as their special cases.

In **chapter 5**, we establish two certain new double integrals. In first integral whose integrand involves the product of multivariable's polynomials $S_{L}^{h_1,\ldots,h_\nu}$ and Aleph-function and in second integral whose integrand involves the product of multivariable's general class of polynomials $S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}$ and Aleph function . Also we obtain New and known integrals as their special cases.

In chapter 6, At first we derive Riemann-Liouvile fractional integral transformation of the E-function, Multivariable polynomial and Aleph function then we obtain various known special cases. Finally establish Erdelyi- Kober fractional integral transformation and generalized fractional integral transformation of the E-function, Multivariable polynomial and Aleph function respectively then we get many known special cases.

CONTENTS

0 INTRODUCTION TO THE TOPIC OF STUDY 1

0.1 THE GAUSSIAN HYPERGEOMETRIC FUNCTION ………………………. 2

0.2 THE FOX H-FUNCTION…………………………………………………… 4

0.3 THE I-FUNCTION ………………………………………………………. 5

0.4 ALEPH- FUNCTION ……………………………………………………. 6

0.4.1 ALEPH-FUNCTION OF TWO VARIABLES …………………………. 7

0.5 MULTIVARIABLE ALEPH-FUNCTION …………………………………… 8

0.6 GENERAL CLASS OF POLYNOMIALS …………………………………… 11

0.6.1 THE MULTIVARIABLE GENERALIZATION OF THE S_V^U POLYNOMIAL **11**

0.6.2 THE GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS… 12

0.6.3 THE GENERALIZED SEQUATIONCE OF FUNCTIONS ……….…... 12

0.7 FRACTIONAL CALCULUS ……………………………………………….. 13

0.8 MITTAG-LEFFLER TYPE FUNCTIONS …………………………………. 15

0.9 INTEGRAL TRANSFORM ………………………………………………… 17

0.9.1 MELLIN TRANSFORM …………………………………………… 17

0.9.2 GENERALIZED STIELTJES TRANSFORM ……………………….... 17

0.10 PATHWAY FRACTIONAL INTEGRATION OPERATORS ……………. 18

1 **PATHWAY FRACTIONAL INTEGRAL OPERATOR ASSOCIATED WITH THE ALEPH-FUNCTION** 21

1.1 INTRODUCTION ……………………………………………………. 22

1.1.1 PATHWAY FRCATIONAL INTEGRATION OPERATOR…… 22

1.1.2 ALEPH-FUNCTION…………………………………………….. 24

1.1.3 ALEPH-FUNCTION TWO VARIABLES……………………… 25

1.1.4 THE GENERAL CLASS OF POLYNOMIAL …………………. 26

1.1.5 FOX H-FUNCTION ……………………………………………. 26

1.1.6 MITTAG-LEFFLER TYPE FUNCTION………………………. 27

1.1.7 BETA AND GAMMA FUNCTION…………………………… 27

1.2 THEOREMS …………………………………………………………..... 27

1.3 SPECIAL CASES …………………………………………………………… 34

2 A STUDY OF MULTIVARIABLE FRACTIONAL INTEGRAL OPERATORS INVOLVING MULTIVARIABLE POLYNOMIAL AND THE ALEPH-FUNCTION 41
2.1 DEFINITIONS ………………………………………………………….. 42
2.1.1 THE ALEPH-FUNCTION IN SERIES …………………………… 42
2.1.2 I AND J-INTEGRAL OPERATORS………………………………. 43
2.2 IMAGES OF THE OPERATORS…………………………………………. 45
2.3 THE MULTIDIMENSIONAL GENERALIZED STIELTJES TRANSFORM WITH I AND J- OPERATORS……………………………………………...… 47
2.4 MELLIN TRANSFORMS………………………………………………… 52
2.4.1 INVERSION FORMULAS……………………………………….. 55
2.4.2 MELLIN CONVOLUTIONS……………………………………… 56
2.5 COMPOSTIONS FORMULAE……………………………………….. 58
2.6 SPECIAL CASES OF COMPOSTIONS FORMULAE………………….. 67
3 A STUDY OF GENERALIZED EULERIAN INTEGRAL AND UNIFIED FINITE INTEGRALS INVOLVING $S_n^{\mu,\delta}$, $P_\gamma^{\alpha,\beta}$ AND THE ALEPH- FUNCTION 71
3.1 INTRODUCTION AND DEFINITIONS
3.1.1 THE GENERAL CLASS OF POLYNOMIALS S_V^U …………… 72
3.1.2 THE GENERALIZED SEQUENCE OF FUNCTIONS…………… 72
3.2 MAIN INTEGRAL……………………………………………………… 74
3.3 APPLICATION AND SPECIAL CASES ………………………………… 77
3.4 INTRODUCTION AND REQURIED RESULTS ……………………... 87
3.4.1 THE LEGENDRE ASSOCIATED FUNCTION………………….. 87
3.5 MAIN RESULTS ………………………………………………………. 88
3.5.1 FIRST INTEGRAL …………………………………………….. 88
3.5.2 SECOND INTEGRAL………………………………………….. 91
3.5.3 THIRD INTEGRAL……………………………………….…..... 92
3.6 THEOREMS……………………………………………………........ 94
3.7 APPLICATIONS AND SPECIAL CASES………………………………… 98

4 FRACTIONAL DIFFERENTIAL CALCULUS OF THE ALEPH-FUNCTION 107
4.1 DEFINTIONS
4.1.1 FRACTIONAL DIFFERNTIAL OPERATOR……………………... 108
4.1.2 RESULTS REQURIED………………………………………….. 109
4.2 MAIN INTEGRALS…………………………………………………….. 110
4.3 SPECIAL CASES OF THEOREMS……………………………………. 116
4.4 APPLICATIONS OF THEOREMS 1 AND 2…………………………... 123
4.5 SPECIAL CASES……………………………………………………….. 125
5 INTEGRAL PROPERTIES OF THE ALEPH-FUNCTION, MULTIVARIABLE'S GENERAL CLASS OF POLYNOMIALS ASSOCIATED WITH THE FEYNMAN INTEGRALS 129
5.1 RESULTS REQURIED…………………………………………………. 130
5.2 FIRST MAIN RESULT…………………………………………………. 130
5.3 SECOND MAIN RESULT……………………………………………… 137
5.4 SPECIAL CASES………………………………………………………. 141
6 **FRACTIONAL INTEGRAL TRANSFORMATIONS OF THE ALEPH-FUNCTION** 147
6.1 DEFINITIONS…………………………………………………………... 148
6.1.1 RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL TRANSFORMATION…………………………………………… 148
6.1.2 ERDELYI-KOBER FRACTIONAL INTEGRAL TRANSFORMATION 148
6.1.3 THE GENERAL MULTIVARIABLE POLYNOMIAL……………. 148
6.1.4 ALEPH-FUNCTION IN SERIES…………………………………... 149
6.1.5 MITTAG-LEFFLER TYPE E-FUNCTION………………………... 149
6.1.6 RESULTS REQUIRED……………………………………………. 150
6.2 THE IMAGE OF ALEPH -FUNCTION UNDER THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL TRANSFORMATION ……………………… 150
6.2.1 SPECIAL CASES…………………………………………………... 152
6.3 THE IMAGE OF ALEPH -FUNCTION UNDER THE ERDELTI-KOBER FRACTIONAL INTEGRAL TRANSFORMATION……………………… 155

6.3.1 SPECIAL CASES………………………………………………………... 157

6.4 THE IMAGE OF ALEPH -FUNCTION UNDER THE GENERALIZED INTEGRAL OPERATOR ………………………………………………………………. 159

6.4.1 SPECIAL CASES……………………………………………………….. 162

APPENDIX –A 167

BIBLOGRAPHY 185

BIOGRAPHICAL PROFILE OF RESEARCHER 197

LIST OF RESEARCH PAPER 199

CHAPTER-0

INTRODUCTION TO THE TOPIC OF STUDY

In this chapter, we give brief historical development of the study and contributions made by some of the earlier workers in this field.

0.1 THE GAUSSIAN HYPERGEOMETRIC FUNCTION

The theory of hypergeometric function is fundamental in the field of mathematics and mathematical physics. Most of the functions that occur in analysis are special cases of the hypergeometric functions. In 1812, C.F. Gauss defined his famous infinite series as follows

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n = 1 + \frac{a.b}{1.c} z + \frac{a.(a+1).b.(b+1)}{1.c.(c+1)} z^2 + \ldots \qquad (0.1.1)$$

where

$$(a)_n = \prod_{k=1}^{n} (a+k-1) = a(a+1) \ldots (a+n-1) \text{ for } n > 0; (a)_0 = 1; c \neq 0, -1, -2, \ldots$$

The above series is called Gauss series or the ordinary hypergeometric series. It is usually represented by the symbol ${}_2F_1(a, b; c; z)$, the well-known Gauss hypergeometric function. Here a, b, c and z may be real or complex. The series on the right hand side of (0.1.1) is not defined if c is zero or negative integer and terminates if either a, b is zero or negative integer. Again, the series given by (0.1.1) is convergent when $|z| < 1$ and when $z = 1$, provided that $R(c-a-b) > 0$ and also when $z = -1$, provided that $R(c-a-b) > -1$.

If in (0.1.1), we replace z by $\frac{z}{b}$ and let $b \to \infty$, then

$$\frac{(b)_n}{b^n} z^n \to z^n$$

and we gain the following well known Kummer's series

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n \, n!} z^n = 1 + \frac{a}{1.c} z + \frac{a(a+1)}{1.c(c+1)} z^2 + \ldots \qquad (0.1.2)$$

The above series is convergent for all values of a, c, and z are real or complex excluding c=0,-1,-2,… and is represented by the symbol ${}_1F_1(a:c:z)$,the well-known confluent hypergeometric function.

A generalization of ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$, which is defined in the following series

$${}_pF_q\left[\begin{matrix} a_1,\dots,a_p\,; \\ b_1,\dots,b_q\,; \end{matrix} z\right] = {}_pF_q\left[a_1,\dots,a_p\,;b_1,\dots,b_q\,;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n,\dots,(a_p)_n}{(b_1)_n,\dots,(b_q)_n} \frac{z^n}{n!}, \quad (0.1.3)$$

where p and q are either positive integer or zero and an empty product is interpreted as unity, the variable z and all the parameters $a_1,\dots,a_p\,;b_1,\dots,b_q$ are real or complex numbers such that no denominator parameter is zero or negative integer.

The conditions of convergence of the function ${}_pF_q$ are given in the following manner

i. When $p = q + 1$, the series is convergent if $|z| < 1$ and divergent when $|z| > 1$ and on the circle $|z| = 1$, the series is
   (a) Absolutely Convergent if $\mathrm{Re}(w) > 0$;
   (b) Conditional convergence if $-1 < \mathrm{Re}(w) < 0$ for $z \neq 1$;
   (c) Divergent if $\mathrm{Re}(w) \leq -1$. where $w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$
ii. When $p \leq q$, the series on the right hand side of (0.1.3) is convergent.
iii. If $p > q + 1$, the series never converges except when z=0 and the function is only defined when the series terminates.

   A comprehensive account of the functions ${}_2F_1$, ${}_1F_1$ and ${}_pF_q$ can be found in the works of Slater [100], Luke [55], Rainville [84] and Exton [23] and their applications can be found in Mathai and Saxena [64].

0.2 THE FOX H-FUNCTION

In an attempt to give meaning to the series (0.1.3) when $p > q+1$, TM Mac Robert [56] introduced and studied in detail a special function which is known in the literature as the E-function. C.S. Meijer [67] introduced the G- function in terms of the Mellin-Barnes type contour integral representation resulted in the rapid growth and development of the special functions. G-function is available in familiar Higher Transcendental Function Erdelyi Vol. I [22] and the book by Luke [55]. Mathai and Saxena [64] and Marichev [60].

The year 1961, Charles Fox [24] introduced a more general function, which thereafter became well known in literature as Fox H-function. Lot of research work has been done during the last four decades and can be referred in the book by Mathai and Saxena [65] and Srivastava [109].

The Fox H-function is defined and represented in terms of Mellin-Barnes type contour integral [34, 65 and 109]

$$H[z] = H_{P,Q}^{M,N}\left[z \middle| \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi i}\int_L \theta(\xi)\, z^{\xi} d\xi\,, \tag{0.2.1}$$

for all $z \neq 0$ where $i = \sqrt{(-1)}$ and

$$\theta(\xi) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j - B_j\xi\right)\prod_{j=1}^{N}\Gamma\left(1 - a_j + A_j\xi\right)}{\prod_{j=M+1}^{Q}\Gamma\left(1 - b_j + B_j\xi\right)\prod_{j=N+1}^{P}\Gamma\left(a_j - A_j\xi\right)}\,, \tag{0.2.2}$$

here M, N, P and Q are non-negative integers satisfying $0 \le N \le P, 1 \le M \le Q$; $A_j, (j = 1,\ldots,P)$ and $B_j, (j = 1,\ldots,Q)$ are assumed to be positive quantities for standardization purpose. For the convergence conditions existence of various contour L and other properties see Mathai and Saxena [65], Srivastava , Gupta and Goyal [109], Kilbas, Srivastava and Trujillo [46] and Kilbas and Saigo [45].

0.3 THE I-FUNCTION

I-function was introduced and investigated by V.P. Saxena [93, 94]. It is represented in the following manner

$$I[z] = I^{M,N}_{P_i,Q_i;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,N}, \left[(a_{ji},A_{ji})\right]_{N+1,P_i} \\ (b_j,B_j)_{1,M}, \left[(b_{ji},B_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i}\int_L \theta(\xi)\ z^{\xi} d\xi\ , \tag{0.3.1}$$

for all $z \neq 0$ where $i = \sqrt{(-1)}$ and

$$\theta(\xi) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j - B_j\xi\right)\prod_{j=1}^{N}\Gamma\left(1 - a_j + A_j\xi\right)}{\sum_{i=1}^{r}\prod_{j=M+1}^{Q_i}\Gamma\left(1 - b_{ji} + B_{ji}\xi\right)\prod_{j=N+1}^{P_i}\Gamma\left(a_{ji} - A_{ji}\xi\right)}\ , \tag{0.3.2}$$

where $P_i\ (i = 1,...,r), Q_i\ (i = 1,...,r); M, N$ are integers integration $0 \le N \le P_i, 1 \le M \le Q_i$ for $i = 1,...,r$, r is finite , $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$.

The conditions of existence of the I-function have been given by Saxena [94]:

$$\left.\begin{aligned} &\text{(i) } \phi_l > 0, \left|\arg(z)\right| < \frac{\pi}{2}\phi_l\ , l = 1,2,...r \\ &\text{(ii) } \phi_l \ge 0, \left|\arg(z)\right| < \frac{\pi}{2}\phi_l\ , R(\xi_l) + 1 < 0, \\ &\text{where } \phi_l = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \left(\sum_{j=N+1}^{P_l} A_{jl} + \sum_{j=M+1}^{Q_l} B_{jl}\right) \\ &\xi_l = \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j + \left(\sum_{j=M+1}^{Q_l} b_{jl} - \sum_{j=N+1}^{P_l} a_{jl}\right) + \frac{1}{2}(P_l - Q_l), \forall l = 1,2,...,r \end{aligned}\right\} \tag{0.3.3}$$

0.4 ALEPH (ℵ)-FUNCTION

The Aleph (ℵ)-function introduced by Sudland [125] and is presented here in the following manner in terms on the Mellin- Barnes type integral

$$\aleph[z] = \aleph^{M,N}_{P_i,Q_i;\tau_i;r}\left[z \middle| \begin{matrix} (a_j,A_j)_{1,N}, \left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i} \\ (b_j,B_j)_{1,M}, \left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i} \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i;\tau_i;r}(s) z^{-s} ds\ , \qquad (0.4.1)$$

for all $z \neq 0$ where $i = \sqrt{(-1)}$ and

$$\Omega^{M,N}_{P_i,Q_i;\tau_i;r}(s) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j + B_j s\right)\prod_{j=1}^{N}\Gamma\left(1 - a_j - A_j s\right)}{\sum_{i=1}^{r}\tau_i \prod_{j=M+1}^{Q_i}\Gamma\left(1 - b_{ji} - B_{ji} s\right)\prod_{j=N+1}^{P_i}\Gamma\left(a_{ji} + A_{ji} s\right)}, \qquad (0.4.2)$$

the integration path $L = L_{i\gamma\infty}, \gamma \in R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1 - a_j - A_j s)$, $j = i,...,N$ do not coincide with the pole of $\Gamma(b_j + B_j s)$, $j = i,...,M$ the parameter P_i, Q_i are non-negative integers satisfying: $0 \le N \le P_i, 0 \le M \le Q_i, \tau_i > 0$ for $i = 1,...,r$. The $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The empty product in (0.4.2) is interpreted as unity. The existence conditions for the defining integral (0.4.1) are as following

$$\left.\begin{aligned} &\text{(i) } \phi_l > 0,\ |\arg(z)| < \frac{\pi}{2}\phi_l\ , l = 1,2,...r \\ &\text{(ii) } \phi_l \ge 0, |\arg(z)| < \frac{\pi}{2}\phi_l\ , R(\xi_l) + 1 < 0, \\ &\text{where } \phi_l = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \tau_l\left(\sum_{j=n+1}^{P_l} A_{jl} + \sum_{j=m+1}^{Q_l} B_{jl}\right) \\ &\xi_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l\left(\sum_{j=N+1}^{Q_l} b_{jl} - \sum_{j=M+1}^{P_l} a_{jl}\right) + \frac{1}{2}(P_l - Q_l), \forall l = 1,2,...,r \end{aligned}\right\} \qquad (0.4.3)$$

For detailed account of Aleph (ℵ)-function see [125] and [126].

0.4.1 ALEPH $(\aleph)$-FUNCTION OF TWO VARIABLES

Saxena [90] defined the Aleph $(\aleph)$ - function of two variables

$$\aleph[x,y]=\aleph^{0,n;m_1,n_1;m_2,n_2}_{p,q;p_i,q_i,\tau_i;p_i',q_i',\tau_i';r}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}A^*\\B^*\end{matrix}\right]$$

(0.4.4)

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\phi(s,\xi)\,\theta_1(s)\,\theta_2(\xi)\,x^{-s}\,y^{-\xi}\,ds\,d\xi\,,$$

$$A^*=(a_j,\alpha_j,A_j)_{1,p},(c_j,C_j)_{1,n_1},...,\left[\tau_i(c_{ji},C_{ji})\right]_{n_1+1,p_i},(e_j,E_j)_{1,n_2},...,\left[\tau_i'(e_{ji},E_{ji})\right]_{n_2+1,p_i'}$$

$$B^*=(b_j,\beta_j,B_j)_{1,q},(d_j,D_j)_{1,m_1},...,\left[\tau_i(d_{ji},D_{ji})\right]_{m_1+1,q_i},(f_j,F_j)_{1,m_2},...,\left[\tau_i'(f_{ji},F_{ji})\right]_{m_2+1,q_i'}$$

$$\phi(s,\xi)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s-A_j\xi)}{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s+A_j\xi)\prod_{j=1}^{q}\Gamma(1-b_j-\beta_j s-B_j\xi)}, \tag{0.4.5}$$

$$\theta_1(s)=\Omega^{m_1,n_1}_{p_i,q_i,\tau_i;r}(s)=\frac{\prod_{j=1}^{m_1}\Gamma(d_j+D_j s)\prod_{j=1}^{n_1}\Gamma(1-c_j-C_j s)}{\sum_{i=1}^{r}\tau_i\prod_{j=m_1+1}^{q_i}\Gamma(1-d_{ji}-D_{ji}\xi)\prod_{j=n_1+1}^{p_i}\Gamma(c_{ji}+C_{ji}\xi)}, \tag{0.4.6}$$

$$\theta_2(\xi)=\Omega^{m_2,n_2}_{p_i',q_i',\tau_i';r}(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(f_j+F_j\xi)\prod_{j=1}^{n_2}\Gamma(1-e_j-F_j\xi)}{\sum_{i=1}^{r}\tau_i'\prod_{j=m_2+1}^{q_i'}\Gamma(1-f_{ji}-F_{ji}\xi)\prod_{j=n_2+1}^{p_i'}\Gamma(e_{ji}+E_{ji}\xi)}. \tag{0.4.7}$$

0.5 MULTIVARIABLE ALEPH-FUNCTION

The Aleph - function of several variables is the generalization of the multivariable I-function recently studied by Sharma and Ahmad [96] and Ayant [5] , This function is also a generalization of G and H-function of multiple variables. The multiple Mellin-Barnes type integral occurring in this thesis will be referred as the multivariable's Aleph-function throughout our present study and it is represented in the following manner

$$\aleph[z_1,\ldots,z_r] = \aleph^{0,n;m_1,n_1;\ldots;m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)}:\ldots:p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} A^* \\ B^* \end{matrix}\right.\right]$$

$$A^* = \left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}\right],\left[\tau_i\left(a_{ji},\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{n+1,p_i};\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],$$

$$\left[\tau_{i(1)}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,p_i^{(1)}};\ldots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i(r)}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,p_i^{(r)}}$$

$$B^* = \cdots\cdots\cdots\cdots\cdots\cdots,\left[\tau_i\left(b_{ji},\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right]_{m+1,q_i};\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_1}\right],$$

$$\left[\tau_{i(1)}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}};\ldots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i(r)}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta(s_k)\, z_k^{s_k}\, ds_1\ldots ds_r\,, \qquad (0.5.1)$$

where $\omega = \sqrt{(-1)}$

$$\psi(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\cdots\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k\right)\right]},$$

(0.5.2)

$$\theta(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)\prod_{j=1}^{n_k}\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k\right)\right]},$$

(0.5.3)

where $j=1,\ldots,r$ and $k=1,\ldots,r$

Suppose, as, that the parameters

$a_j, j=1,\ldots,p;\ b_j, j=1,\ldots,q;$

$c_j^{(k)}, j=1,\ldots,n_{(k)};\ c_{ji^{(k)}}^{(k)}, j=n_{(k)}+1,\ldots,p_{i^{(k)}};$

$d_j^{(k)}, j=1,\ldots,m_{(k)};\ d_{ji^{(k)}}^{(k)}, j=m_{(k)}+1,\ldots,q_{i^{(k)}};$

With $k=1,\ldots,r,\ i=1,\ldots,R,\ i^{(k)}=1,\ldots,R^{(k)}$

are complex numbers, and $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)}=\sum_{j=1}^{n}\alpha_j^{(k)}+\tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}$$
$$-\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\le 0$$

(0.5.4)

The real numbers τ_i are positive for i=1 to R, $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\ldots,R^{(k)}$.

The contour L_k is in the s_k land and rune from $\sigma-i\infty$ to $\sigma-i\infty$ where σ is a real number with loop, if necessary, ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1,\dots,m_k$ are separated from those of $\Gamma(1 - a_j + \sum_{j=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1,\dots,n$ and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1,\dots,n_k$ to the left of the contour L_k

$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_{i^{(k)}} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \tag{0.5.5}$$

with $k = 1,\dots,r$, $i = 1,\dots,R$, $i^{(k)} = 1,\dots,R^{(k)}$.

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form

$$\aleph(z_1,\dots,z_r) = 0\left(|z_1|^{\alpha_1} \dots |z_r|^{\alpha_r}\right), \max\left[|z_1| \dots |z_r|\right] \to 0,$$

$$\aleph(z_1,\dots,z_r) = 0\left(|z_1|^{\beta_1} \dots |z_r|^{\beta_r}\right), \max\left[|z_1| \dots |z_r|\right] \to \infty,$$

where, $k = 1,\dots,r$: $\alpha_{(k)} = \min\left[\operatorname{Re}\left(\frac{d_j^{(k)}}{\delta_j^{(k)}}\right)\right]$, $j = 1,\dots,m_k$ and

$$\beta_{(k)} = \max\left[\operatorname{Re}\left(\frac{c_j^{(k)} - 1}{\gamma_j^{(k)}}\right)\right], j = 1,\dots,n_k$$

Chaurasia [15] give series representation of the Aleph function

$$\aleph^{M,N}_{P_i,Q_i,\tau_i,r}[z]=\sum_{\nu=1}^{M}\sum_{g=0}^{\infty}\theta(S_{\nu,g})\, z^{-S_{\nu,g}}, \qquad (0.5.6)$$

where

$$\theta(S_{\nu,g})=\frac{\prod_{j=1,j\neq\nu}^{M}\Gamma(b_j+B_jS_{\nu,g})\prod_{j=1}^{N}\Gamma(1-a_j-A_jS_{\nu,g})\,(-1)^g}{\sum_{i=1}^{r}\tau_i\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}S_{\nu,g})\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}S_{\nu,g})\; g!\,B_\nu},\; S_{\nu,g}=\frac{b_\nu+g}{B_\nu}. \qquad (0.5.7)$$

0.6 GENERAL CLASS OF POLYNOMIALS

The classes of hypergeometric polynomials such as Brafman polynomials, extended Jacobi polynomials and the classical orthogonal polynomials such as Hermite , Jacobi, Laguerre , Konhauser orthogonal polynomials and several other polynomials play vital role in the study of mathematics and applied physics.

All the above polynomials are the special cases of the general class of polynomials introduced by Srivastava [104]

$$S_V^U[x]=\sum_{R=0}^{[V/U]}(-V)_{UR}\,A_{V,R}\,\frac{x^R}{R!},V=0,1,2,... \qquad (0.6.1)$$

where the coefficients $A_{V,R}\,(V,R\geq 0)$ are arbitrary constants, real or complex and U is an arbitrary positive integer.

Detail information of some of the special cases of the above mentioned class of polynomials has been given in the Appendix –A at the end of the thesis.

0.6.1 THE MULTIVARIABLE GENERALIZATIONS OF THE S_V^U POLYNOMIAL

The following generalization was introduced and defined by Srivastava and Garg [107, p.686, Eq. (1.4)] as follows

$$S_V^{U_1,\dots,U_k}(x_1,\dots,x_k)=\sum_{\substack{R_1,\dots,R_k=0\\ \sum_{i=1}^{k}U_iR_i\le V}} (-V)_{\sum_{i=1}^{k}U_iR_i}\, A(V,R_1,\dots,R_k)\frac{x_i^{R_i}}{R_i!},\qquad (0.6.2)$$

Where $U_1,\dots,U_k$ are arbitrary positive integers, V=0,1,2,… and the coefficients $A(V,R_1,\dots,R_k)$ are arbitrary constants, real or complex.

0.6.2 THE GENERAL CLASS OF MULTIVARIABLE POLYNOMIALS

The general class of polynomials defined by Srivastava [103], is given in the following manner

$$S_{n_1,\dots,n_R}^{m_1,\dots,m_R}\left[x_1,\dots,x_R\right]=\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\prod_{i=1}^{R}\frac{(-n_i)_{m_is_i}}{\Gamma(s_i+1)}A_{n_i,s_i}\,x^{s_i}$$

(0.6.3)

Where $m_1,\dots,m_R$ are arbitrary positive integers, $n_1,\dots,n_R=0,1,2,\dots$ and the coefficients $A_{n_i,s_i}\,(n_i,s_i\ge 0)$ are arbitrary constant, real or complex

0.6.3 THE GENERALIZED SEQUENCE OF FUNCTIONS

A number of persons notably Srivastava and Panda [112], Patil and Thakare [76] Joshi and Prajapat [41],Dhillon [16] (See also Srivastava and Manocha [110]) have studied several unified polynomial sets defined by means of Rodrigues formula.

Agrawal and Chaubey [1, 2] were motivated by some of these works defined and studied the following sequence of functions

$$R_n^{(\alpha,\beta)}[x]=R_n^{(\alpha,\beta)}[x;A,B,C,D;G,H;\gamma,\delta;\vartheta,k;\omega(x)]$$

$$=\frac{(Ax^G+B)^{-\alpha}(Cx^H+D)^{-\beta}}{K_n\,\omega(x)}T_{\vartheta,k}^{n}\left\{(Ax^G+B)^{\alpha+\gamma n}(Cx^H+D)^{\beta+\delta n}\omega(x)\right\},\qquad (0.6.4)$$

where

$$T^{n}_{\vartheta,k} = \left[x^{k} (\vartheta + x D_{x}) \right]^{n}, \; D_{x} = \frac{d}{dx} \tag{0.6.5}$$

$\{K_n\}_{n=0}^{\infty}$ is sequence of constants, and $\omega(x)$ is independent of n and differentiable any number of times.

Raijada [80] , introduced a generalized sequence of function $S_n^{\alpha,\beta,\tau}[x]$ as follows

$$S_n^{\alpha,\beta,\tau}[x;r,s,q,A,B,m,k,l]$$
$$= (Ax+B)^{-\alpha}(1-\tau x^{r})^{-\beta/\tau}\, T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}\left(1-\tau x^{r}\right)^{\beta/\tau+sn}\right] \tag{0.6.6}$$

The above generalized sequence of functions $S_n^{\alpha,\beta,\tau}[x]$ can also be derived from the general sequence of functions defined by (0.6.6), it is of interest by itself. It unifies and extends a number of well known classical polynomials studied by several research workers such as Krall and Frink and Frink, Gould and Hopper, Singh and Srivastava , Chatterjea, Singh , Dhillon , etc.

A detailed information of important special cases of the generalized sequence of functions (0.6.6) can be refer to in Appendix – A at the end of this thesis.

0.7 FRACTIONAL CALCULUS

The term fractional calculus has its origin to the letter written by Maiquis de L' hospital in 1695 to Gottoried Leibniz, wherein he enquired whether a meaning could be ascribed to $\frac{d^{n} f(x)}{dx^{n}}$ if n was a fraction. Later it is well established that n^{th} derivative of f(x) means all values of n i.e. rational, positive, negative, real or complex. The real journey of progress of fractional calculus started in 1974, when the first article on fractional calculus was published [75].

The fractional calculus finds use in many fields of science and engineering, including the fluid flow, electrical networks electro chemistry, rhelogy, quantitative biology, statistical probability theory, chemical physics, and several other branches of mathematical analysis, like integral and differential equations, operational calculus and univalent function theory. The work of Oldham and spanier [75], Samko, Kilbas and Marichev [88], Miller and Ross [71], Podlubny [77],Caputo [12],Gorenflo and Vessella [28], Kiryakova [47],McBride [66], Nishimoto [74] provide a comprehensive account of the development and applications in the field of fractional calculus.

The theory of fractional calculus is mainly based upon the study of the well-known Fractional integral operator ${}_aD_z^{\alpha}$ defined by (Lovoie [54] and Ross [85])

$$ {}_aD_z^{\alpha}f(z) = \frac{1}{\Gamma(-\alpha)} \int_a^z (z-y)^{-\alpha-1} f(y)dy, \quad \mathrm{Re}(\alpha) < 0, \tag{0.7.1} $$

$$ = \frac{d^m}{dx^m}\, {}_aD_z^{\alpha-m} f(z), \quad \mathrm{Re}(\alpha) \geq 0, \tag{0.7.2} $$

where m is a positive integer greater than $\mathrm{Re}(\alpha)$ and the integral involved exists.

When a=0, the fractional integral operator given by (0.7.1) reduces to the classical Reimann Lioville fractional integral operator of order $(-\alpha)$ and when $a \to \infty$, it may be identified with the definition of the familiar Weyl fractional integral operator of order $(-\alpha)$.

On account of the important role played by Reimann Lioville and Weyl integral operators in several problems of mathematical Physics and applied mathematics, various generalization of the fractional integral operators have been studied from time to time by several research workers notably Sneddon [102] , Kalla [42],,Kalla and Saxena [44], Gupta and Soni [38], Saxena and Kumbhat [92], Manocha [57], Koul [52], Rain and Kiryakova [83], Garg [26], Garg and Purohit [25], Gupta [35], Saigo [86,87], Gupta and Jain [36], Kober [50], Erdelyi [19,20], Gupta ,Jain and Agrawal [37].

A detail of various Fractional integral operators studied by researchers has been given by Srivastava and Saxena [124].

In chapter 4 of this thesis, we have used the fractional integral operator ${}_aD_z^{\alpha}$ defined above (0.7.1) to obtain certain multiplication formula and its generalization of order λ (defined below) to find out some new images.

Let $0 \le \alpha < 1, \beta, \eta, x \in R, m \in N$ then generalized modified fractional derivative operator due to Saigo [86] is defined as

$$D_{0,x,m}^{\alpha,\beta,\eta} f(x) = \frac{d}{dx}\left(\frac{x^{m(\beta-\eta)}}{\Gamma(1-\alpha)} \int_0^x \left(x^m - t^m\right)^{-\alpha} {}_2F_1\left[\begin{matrix}\beta-\alpha;1-\eta;\\ 1-\alpha\end{matrix} 1-\frac{t^m}{x^m}\right] f(t)dt^m \right), \quad (0.7.3)$$

the multiplicity of $(x^m - t^m)^{-\alpha}$ in equation (0.7.3) is removed by requiring $\log\left(x^m - t^m\right)$ to be real when $(x^m - t^m) > 0$, and is assumed to be well defined in the unit disk. When m=1 then the above operator reduces to Saigo derivative operator $D_{0,x}^{\alpha,\beta,\eta}$ (cf. Srivastava , Saigo and Owa [123] ; See also Srivastava and Saxena [124]).

0.8 MITTAG-LEFFLER TYPE FUNCTIONS

Gosta Mittag-Leffler [72], introduced the function $E_\alpha(z)$, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)} z^n , \quad (0.8.1)$$

where $z, \alpha \in C; R(\alpha) \ge 0$ and $|z| > 0.$

Wiman [130] extended (0.8.1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} z^n , \quad (0.8.2)$$

where $z, \alpha, \beta \in C; R(\alpha) \ge 0, R(\beta) \ge 0.$

Shukla and Prajapati [97], defined and investigated the function $E_{\beta,\rho}^{\gamma,q}(z)$ as

$$E_{\beta,\rho}^{\gamma,q}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[-z \,\middle|\, \begin{matrix}(1-\gamma,q)\\ (0,1),(1-\rho,\beta)\end{matrix}\right]$$

$$= \frac{1}{2\pi i \Gamma(\gamma)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\gamma - qs)}{\Gamma(\rho - \beta s)} (-z)^{-s} ds , \quad (0.8.3)$$

with $\beta, \gamma, \rho \in C, R(\beta) > 0, q \in (0,1) \cup N$

Kiryakova [47] , has studied about " Multi index M-L function" defined by

$$\left[E_{(1/\rho_i),(\mu_i)}(z)\right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu_1 + n/\rho_1) \ldots \Gamma(\mu_m + n/\rho_m)} z^n , \quad (0.8.4)$$

where m>1, is an integer, $\rho_1,...,\rho_m > 0$ and $\mu_1,...,\mu_m$ are arbitrary real numbers.

Saxena and Nishimoto [95], studied an extension of M-L type function as

$$E_{\gamma,k}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_{nk}}{\prod_{j=1}^{m}(\alpha_j n+\beta_j)}\frac{z^n}{n!}, \qquad (0.8.5)$$

where $z,\alpha_j,\beta_j,\gamma\in C;\ \sum_{j=1}^{m}R(\alpha)>R(k)-1,\ j=1,\ldots,m$ and $R(k)>0.$

Kalla , Haidey and Virchenko [43] , showed Multi parameter M-L type function in the following form

$$\left[HE_{\mu_1,\ldots,\mu_\nu}^{\lambda_1,\ldots,\lambda_\nu}(z)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{\left\{\prod_{j=1}^{\nu}\Gamma\left(1+\mu_j+\lambda_j n\right)\right\}}\left(\frac{z}{\Lambda}\right)^{\Lambda n+M}, \qquad (0.8.6)$$

where $\mu_j\in C,\lambda_j>0,\ j=1,2,\ldots,\nu;\ \sum_{j=1}^{\nu}\mu_j=M$ and $\sum_{j=1}^{\nu}\lambda_j=\Lambda.$

Bhatter and Faisal [7], defined a M-L type E-function as follows

$$_{\tau}E_k^h(z)={}_{\tau}E_k^h\left[z\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{matrix}\right]={}_{\tau}E_k^h\left[z\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i),\ldots,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k)\end{matrix}\right]$$

$$=\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\cdots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\cdots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)}, \qquad (0.8.7)$$

where $z,\alpha,\beta,\gamma_i,\delta_j\in C,R(\alpha)\geq 0,R(\beta)>0,R(\gamma_i)>0,R(\delta_j)>0,q_i\geq 0$

$$p_j\geq 0;s_i\geq 0,r_j\geq 0;a,\tau\in R;\rho\in\{0,1\},\left(\sum_{i=0}^{h}q_i s_i<\sum_{j=1}^{k}p_j r_j+R(\alpha)\right)\ \text{or}$$

$$\left(\sum_{i=0}^{h}q_i s_i=\sum_{j=1}^{k}p_j r_j+R(\alpha)\ \text{when}\ \prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}\left(p_j\right)^{p_j r_j}\right]^{-1}|z^{\alpha}|<1\right), \qquad (0.8.8)$$

here $i=1,2,\ldots,h;\ j=1,2,\ldots,k.$

0.9 INTEGRAL TRANSFORM

If $f(x)$ denote a function of a prescribed class of functions defined on a given interval [a, b] and K(x, s) denotes a define function of x in that interval for each value of s, a parameter whose domain is prescribed, then the liner integral transforms T [f(x); s] of the function f(x) is define in the following manner

$$T[f(x);s]=\int_a^b K(x,s)\, f(x)\, dx \tag{0.9.1}$$

Wherein the class of functions and the domain of parameter s are so prescribed that the above integral exists. In (0.9.1), K(x, s) is known as the kernel of the transform, T [f(x); s] is the image f(x) in the said transform and f(x) is the original of T [f(x); s].

If an integral equation can be so determined that

$$f(x)=\int_\alpha^\beta \phi(s,x)\, T[f(x);s]\, ds \tag{0.9.2}$$

Then (0.9.2) is termed as the inversion formula of (0.9.1)

0.9.1 MELLIN TRANSFORM

The well known Mellin Transform is defined by

$$M[f(x);s]=\int_0^\infty x^{s-1}\, f(x)\, dx \tag{0.9.3}$$

and the inversion formula is given by

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\, M[f(x);s]\, ds \tag{0.9.4}$$

provided that the above integral exist.

0.9.2 GENERALIZED STIELTJES TRANSFORM

The generalized stieltjes transform, under appropriate conditions, is defined as follows

$$S_h[f(x);s]=\int_0^{\infty}(x+s)^{-s}\,f(x)\,dx \tag{0.9.5}$$

We have studied the multidimensional stieltjes transform in chapter 2 of the present work.

0.10 PATHWAY FRACTIONAL INTEGRATION OPERATORS

Nair [73] introduced the Pathway fractional integration operator and state as follows

Let $f(x)\in L(a,b);\eta\in C, R(\eta)>0, a>0$ and let us take a "pathway parameter" $\alpha<1$, then the pathway fractional integration operator is

$$\left[P_{0+}^{(\eta,\alpha)}f\right](x)=x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}f(t)dt\,, \tag{0.10.1}$$

the pathway model is introduced by Mathai [61] and discussed further by Mathai and Haubold [62, 63].For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.)

$$f(x)=c|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}}, \tag{0.10.2}$$

provided that $-\infty<x<\infty;\delta>0;\beta\geq 0,\gamma>0;\left[1-a(1-\alpha)|x|^{\delta}\right]>0$, where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as below

$$c=\frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)},\ \text{for}\,\alpha<1\,, \tag{0.10.3}$$

$$c = \frac{1}{2}\frac{\delta[a(\alpha-1)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta}\right)}, \text{ for } \frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0, \alpha>1, \qquad (0.10.4)$$

$$c = \frac{1}{2}\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \text{ for } \alpha \to 1. \qquad (0.10.5)$$

See that for $\alpha < 1$ it is a finite range density with $\left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and (0.10.2) remains in the extended generalized type-1 beta family. The pathway density in (0.10.2), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 1$, writing $1-\alpha = -(\alpha-1)$ we have

$$f(x) = c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{\frac{-\beta}{\alpha-1}}. \qquad (0.10.6)$$

Provided that $-\infty < x < \infty; \delta > 0; \beta > 0, \alpha > 1$, which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the *F*-density, the Student-*t* density, the Cauchy density, etc.

Here we take only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (0.10.2) and (0.10.6) take the exponential form, since

$$\lim_{\alpha \to 1} c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} = \lim_{\alpha \to 1} c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{\frac{-\beta}{\alpha-1}}$$

$$= c|x|^{\gamma-1} e^{-a\eta|x|^{\delta}}. \qquad (0.10.7)$$

In this the generalized gamma, the Weibull, the Chi-square, the Laplace, Maxwell-Boltzmann and other related densities are included.

.

CHAPTER 1

PATHWAY FRACTIONAL INTEGRAL OPERATOR ASSOCIATED WITH THE ALEPH-FUNCTION

Publications

1. New Pathway fractional integral operator associated with Aleph-function, Multivariable's general class of polynomial with H-function, (IJMTT), and Vol. 14, 2014, 84-87.

2. Pathway fractional integral operator associated with Aleph-function, Multivariable's general class of polynomial, Mittag-Leffler function with H-function, International J. of Math. Sci. & Engg. Appls. (IJMSEA), 10(1) 2016, 15-21.

3. Pathway fractional integral operator of the product of two Aleph-functions, (IOSR-JM), 13(2), 2017, 27-30.

The aim of the present chapter is to study of a pathway fractional integral operator associated with the pathway model and Pathway density for Aleph-function. We establish three theorems. In first theorem we find pathway fractional operator whose kernel involves the product of Aleph-function, Multivariable's general class of polynomial and H-function. In second theorem we find pathway fractional operator whose kernel involves the product of Aleph-function, Multivariable's general class of polynomial, H-function and Mittag-Leffler function. At last, we establish third theorem on pathway fractional operator whose kernel involves the product of two Aleph-functions. Also we obtain new and known special cases of our all three theorems.

Various definitions of the operators of the classical and generalized fractional calculus (FC) are formerly well known and broadly used in the applications to mathematical models of fractional order. Two well known classical and generalized fractional calculus operators are Riemann-Liouville fractional integral given by Samko-Kilbas-Marichev[88]) and Erd∂elyi-Kober operators are given (cf. [88], Kober [50], Erd∂elyi [20], Kiryakova [48]). Further Saigo [86] was introduced and examined generalized fractional calculus as the hypergeometric integral operators. Kiryakova [47] find generalized fractional calculus whose integrand involves G and H-functions respectively. We bring out a fractional integration operator, which can be considered as an extension of the left-sided Riemann-Liouville fractional integral operator. We come up with some results for this operator, including the images of the Aleph-function, Mittag-Leffler function, and their appropriate cases.

In this chapter we obtain a result, which create a connection to broad classes of statistical distributions. The pathway parameter α establishes a path for one distribution to another and to different classes of distributions. If α goes to 1,we get generalized gamma type function.

1.1 INTRODUCTION

1.1.1 PATHWAY FRACTIONAL INTEGRATION

Nair [73], introduced the Pathway fractional integration operator and state as follows

Take $\alpha < 1$ and let $f(x) \in L(a,b)$, $R(\eta) > 0, a > 0, \eta \in C$, then the pathway fractional integration operator is

$$\left[P_{0+}^{(\eta,\alpha)} f\right](x) = x^{\eta} \int_{0}^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} f(t)\,dt. \tag{1.1.1}$$

Mathai [61] introduced the pathway model and further discussed by Mathai and Haubold [62, 63]. For real scalar α, the following probability density function (p. d. f.) is the pathway model for scalar random variables

$$f(x) = c|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}}, \tag{1.1.2}$$

provided that $-\infty < x < \infty; \delta > 0; \beta \geq 0, \gamma > 0; \left[1-a(1-\alpha)|x|^{\delta}\right] > 0$, where c is the normalizing constant. For real α, the normalizing constant c is given as follows

$$c = \frac{1}{2}\frac{\delta[a(\alpha-1)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta}\right)},\ \text{for}\ \frac{1}{\alpha-1}-\frac{\gamma}{\delta} > 0, \alpha > 1, \tag{1.1.3}$$

$$c = \frac{1}{2}\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)},\ \text{for}\ \alpha \to 1, \tag{1.1.4}$$

$$c = \frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)},\ \text{for}\ \alpha < 1. \tag{1.1.5}$$

Provided that for $\alpha < 1$ it is a finite range density with $\left[1-a(1-\alpha)|x|^{\delta}\right] > 0$ and (1.1.2) remains in the extended generalized type-1 beta family. The pathway density in (1.1.2), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other probability density function.

For $\alpha > 1$, taking $1-\alpha = -(\alpha-1)$ we get

$$f(x) = c\,|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}}, \tag{1.1.6}$$

provided that $-\infty < x < \infty; \delta > 0; \beta > 0, \alpha > 1$, which is the extended generalized type-2 beta model for real x.

Here we take only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (1.1.2) and (1.1.6) take the exponential form, since

$$\lim_{\alpha \to 1} c\,|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} = \lim_{\alpha \to 1} c\,|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{\frac{-\beta}{\alpha-1}}$$
$$= c\,|x|^{\gamma-1} e^{-a\eta|x|^{\delta}}. \tag{1.1.7}$$

This includes the generalized gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities.

For more details on the pathway model, the reader is referred to go through the recent papers of Mathai and Haubold [62], [63].

1.1.2 ALEPH (ℵ) - FUNCTION

Sudland [125] introduced the Aleph (ℵ) - function, however the notation and complete definition is presented here in the following manner in terms and the Mellin- Barnes type integrals of Aleph-function is

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n}, \left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i} \\ (b_j,B_j)_{1,m}, \left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i} \end{matrix} \right. \right]$$
$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)\, z^{-s} ds, \tag{1.1.8}$$

for all $z \neq 0$ where $\omega = \sqrt{(-1)}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma\left(b_j + B_j s\right)\prod_{j=1}^{n}\Gamma\left(1 - a_j - A_j s\right)}{\sum_{i=1}^{r}\tau_i \prod_{j=m+1}^{q_i}\Gamma\left(1 - b_{ji} - B_{ji} s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji} + A_{ji} s\right)}. \tag{1.1.9}$$

The integration path $L = L_{i\gamma\infty}, \gamma \in R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$ and is such that the poles, assumed to be simple of $\Gamma(1 - a_j - A_j s)$, j=1,...,n do not coincide with the pole of $\Gamma(b_j + B_j s)$, $j = i,..., m$ the parameter p_i, q_i are non-negative integers satisfying $0 \le n \le p_i, 0 \le m \le q_i, \tau_i > 0$, $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$ for $i = 1,..., r$ ·The empty product in (1.1.9) is interpreted as unity. The existence conditions for the defining integral (1.1.8) are as following

$$\left.\begin{array}{l} \text{(i) } \phi_l > 0, \ |\arg(z)| < \frac{\pi}{2}\phi_l \ , l = 1,2,...r \\ \text{(ii) } \phi_l \ge 0, |\arg(z)| < \frac{\pi}{2}\phi_l \ , R(\xi_l) + 1 < 0, \\ \text{where } \phi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \\ \xi_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=n+1}^{q_l} b_{jl} - \sum_{j=m+1}^{p_l} a_{jl} \right) + \frac{1}{2}(p_l - q_l), \forall l = 1,2,...,r \end{array}\right\} \quad (1.1.10)$$

For detailed introduction of Aleph (ℵ) function given in [125] and [126].

1.1.3 ALEPH (ℵ) - FUNCTION OF TWO VARIABLES

Saxena [90], defined the Aleph (ℵ) - function of two variables

$$\aleph[x, y] = \aleph^{0,n;m_1,n_1;m_2,n_2}_{p,q;p_i,q_i,\tau_i;p'_i,q'_i,\tau'_i;r}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} A^* \\ B^* \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \phi(s,\xi)\theta_1(s)\theta_2(\xi) x^{-s} y^{-\xi} ds\, d\xi \, , \quad (1.1.11)$$

where

$$A^* = (a_j, \alpha_j, A_j)_{1,p}, (c_j, C_j)_{1,n_1}, ..., \left[\tau_i(c_{ji}, C_{ji})\right]_{n_1+1,p_i}, (e_j, E_j)_{1,n_2}, ..., \left[\tau'_i(e_{ji}, E_{ji})\right]_{n_2+1,p'_i},$$

$$B^* = (b_j, \beta_j, B_j)_{1,q}, (d_j, D_j)_{1,m_1}, ..., \left[\tau_i(d_{ji}, D_{ji})\right]_{m_1+1,q_i}, (f_j, F_j)_{1,m_2}, ..., \left[\tau'_i(f_{ji}, F_{ji})\right]_{m_2+1,q'_i},$$

$$\phi(s,\xi)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s-A_j\xi)}{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s+A_j\xi)\prod_{j=1}^{q}\Gamma(1-b_j-\beta_j s-B_j\xi)}, \tag{1.1.12}$$

$$\theta_1(s)=\Omega_{p_i,q_i,\tau_i;r}^{m_1,n_1}(s)=\frac{\prod_{j=1}^{m_1}\Gamma(d_j+D_j s)\prod_{j=1}^{n_1}\Gamma(1-c_j-C_j s)}{\sum_{i=1}^{r}\tau_i\prod_{j=m_1+1}^{q_i}\Gamma(1-d_{ji}-D_{ji}\xi)\prod_{j=n_1+1}^{p_i}\Gamma(c_{ji}+C_{ji}\xi)}, \tag{1.1.13}$$

and

$$\theta_2(\xi)=\Omega_{p_i',q_i',\tau_i';r}^{m_2,n_2}(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(f_j+F_j\xi)\prod_{j=1}^{n_2}\Gamma(1-e_j-F_j\xi)}{\sum_{i=1}^{r}\tau_i'\prod_{j=m_2+1}^{q_i'}\Gamma(1-f_{ji}-F_{ji}\xi)\prod_{j=n_2+1}^{p_i'}\Gamma(e_{ji}+E_{ji}\xi)}. \tag{1.1.14}$$

1.1.4 THE GENERAL MULTIVARIABLE POLYNOMIAL

The general class of polynomials defined by Srivastava [103], is presented in the following manner

$$S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left[x_1,\ldots,x_R\right]=\sum_{s_1=0}^{n_1/m_1}\cdots\sum_{s_R=0}^{n_R/m_R}\prod_{i=1}^{R}\frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)}A_{n_i,s_i}\,x_i^{s_i}, \tag{1.1.15}$$

where $n_1,\ldots,n_R=0,1,2,\ldots$, the coefficients $A_{n_i,s_i}\ (n_i,s_i\geq 0)$ are arbitrary constant, real or complex and $m_1,\ldots,m_R$ are arbitrary positive integers,

1.1.5 FOX H- FUNCTION

Mathai and Saxena [65] given the series representation for Fox H- function as follows

$$H_{P,Q}^{M,N}[z]=H_{P,Q}^{M,N}\left[z\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]=\sum_{h=1}^{N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h}\left(\frac{1}{z}\right)^{\xi}, \tag{1.1.16}$$

where $\xi=\dfrac{(e_h-1-h)}{E_h}$, $h=1,2,\ldots,N$

and

$$\chi(\xi)=\frac{\prod_{j=1}^{M}\Gamma\left(f_j+F_j\xi\right)\prod_{j=1}^{N}\Gamma\left(1-e_j-E_j\xi\right)}{\prod_{j=M+1}^{Q}\Gamma\left(1-f_j-F_j\xi\right)\prod_{j=N+1}^{P}\Gamma\left(e_j+E_j\xi\right)}.$$

For convergence conditions and other details of the above function see Mathai and Saxena [65].

Mathai and Saxena [65, Page 11, equation (1.7.81)] has given the relation of Wright's function ${}_p\psi_q$ and H-function, as follows

$${}_p\psi_q\left[\begin{matrix}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{matrix}\middle| z\right]=H^{1,p}_{p,q+1}\left[-z\middle|\begin{matrix}(1-a_1,A_1),...,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),...,(1-b_q,B_q)\end{matrix}\right]. \quad (1.1.17)$$

1.1.6 MITTAG-LEFFLER TYPE FUNCTION

Shukla and Prajapati [97], defined and investigated the function $E^{\gamma,q}_{\beta,\rho}(z)$, as follows

$$\begin{aligned}E^{\gamma,q}_{\beta,\rho}(z)&=\frac{1}{\Gamma(\gamma)}H^{1,1}_{1,2}\left[-z\middle|\begin{matrix}(1-\gamma,q)\\(0,1),(1-\rho,\beta)\end{matrix}\right]\\&=\frac{1}{2\pi i\Gamma(\gamma)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\rho-\beta s)}(-z)^{-s}ds,\end{aligned} \quad (1.1.18)$$

with $\beta,\gamma,\rho\in C, R(\beta)>0, q\in(0,1)\cup N$

1.1.7 BETA AND GAMMA FUNCTION RELATION

$$\int_0^1 (z)^{m-1}(1-z)^{n-1}dz=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \quad (1.1.19)$$

1.2 THEOREMS

Theorem1: let $\eta,u,u_1,u_R\in C$, $R(\delta)>, R\left(1+\frac{\eta}{(1-\alpha)}\right)>0, R(\eta,u,u_1,u_R)>0, \lambda>0$, $\beta>0, R(\beta)>0$, and m_i is an arbitrary integer and coefficients $A_{n_i,s_i}(n_i,s_i\geq 0)$ are arbitrary constants, real or complex and set of sufficient conditions (1.1.10) hold

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\dots,n_R}^{m_1,\dots,m_R}(x^{u_1},\dots,x^{u_R}) \aleph_{p_i,q_i;\tau_i;r}^{m,n}\left\{ d\,x^{\delta} \left| \begin{matrix} (a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i} \\ (b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i} \end{matrix} \right.\right\}\right.$$

$$\left.\times H_{P,Q}^{M,N}\left\{ c\,x^{\lambda} \left| \begin{matrix} (e_P,E_P) \\ (f_Q,F_Q) \end{matrix} \right.\right\}\right] = \frac{x^{\eta+u+u_1s_1+\dots+u_Rs_R}}{[a(1-\alpha)]^{u+u_1s_1+\dots+u_Rs_R}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)$$

$$\times \sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_is_i}}{\Gamma(s_i+1)}\right\}A_{n_i,s_i}\; H_{P,Q}^{M,N}\left[\frac{c\,x^{\lambda}}{[a(1-\alpha)]^{\lambda}}\left| \begin{matrix} (e_P,E_P) \\ (f_Q,F_Q) \end{matrix} \right.\right]$$

$$\times \aleph_{p_i+1,q_i+1;\tau_i;r}^{m,n+1}\left[\frac{d\,x^{\delta}}{[a(1-\alpha)]^{\delta}}\left| \begin{matrix} (1-u-u_1s_1-\dots-u_Rs_R+\lambda\xi,\delta),\,(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i} \\ (b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i},(-\frac{\eta}{1-\alpha}-u-u_1s_1-\dots-u_Rs_R+\lambda\xi,\delta) \end{matrix} \right.\right].$$

(1.2.1)

Proof: In the left hand side of (1.2.1), we use these definitions (1.1.1), (1.1.8), (1.1.15) and (1.1.16) then we get

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_is_i}}{\Gamma(s_i+1)}x^{u_is_i}\right\}A_{n_i,s_i}\right.$$

$$\left.\times\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i;\tau_i;r}^{m,n}(s)\,(d\,x^{\delta})^{-s}\,ds\sum_{h=1}^{N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h}\left\{\left(c\,x^{\lambda}\right)^{-1}\right\}^{\xi}\right]$$

$$= x^{\eta}\left[\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left\{t^{u-1+u_1s_1+\dots+u_Rs_R-\delta s-\lambda\xi}\left(1-\frac{a(1-\alpha)t}{x}\right)^{1+\frac{\eta}{1-\alpha}-1}dt\right\}\right.$$

$$\times \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} A_{n_i,s_i}$$

$$\times \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(d)^{-s} ds \sum_{h=1}^{N} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h} \left(\frac{1}{c}\right)^{\xi} \Bigg], \qquad (1.2.2)$$

In equation (1.2.2) , if we take $\frac{a(1-\alpha)t}{x} = v_1$ and interchange the order of integration, we get

$$= \left[\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} \frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi}}{[a(1-\alpha)]^{u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi}} A_{n_i,s_i} \right.$$

$$\times \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(d)^{-s} ds \sum_{h=1}^{N} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h} \left(\frac{1}{c}\right)^{\xi} \Bigg]$$

$$\times \int_0^1 \left\{ v_1^{u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi-1} (1-v_1)^{1+\frac{\eta}{1-\alpha}-1} dv_1 \right\}, \qquad (1.2.3)$$

now we evaluate the inner most integral with the help of known result (1.1.19), we get

$$= \left[\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} \frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi}}{[a(1-\alpha)]^{u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi}} A_{n_i,s_i} \right.$$

$$\times \frac{\Gamma(u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi)\,\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi+1+\frac{\eta}{1-\alpha})} \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)\,(d)^{-s} ds$$

$$\times \sum_{h=1}^{N} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h} \left(\frac{1}{c}\right)^{\xi} \Bigg],$$

after a little simplification ,we get required result (1.2.1).

Theorem2: let $\eta, u, u_1, u_R \in C, R(\delta) >, R\left(1+\frac{\eta}{(1-\alpha)}\right) > 0, R(\eta, u, u_1, u_R) > 0, \lambda > 0, \beta, \gamma \in C,$

$R(\beta), R(\eta),\ R(\gamma), R(\rho) > 0,\ \lambda \in (0,1) \cup N$ and m_i is an arbitrary integer and coefficients $A_{n_i,s_i}\ (n_i, s_i \geq 0)$ are arbitrary constants, real or complex and set of sufficient conditions (1.1.10) hold

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}(x^{u_1},\ldots,x^{u_R})\ \aleph_{p_i,q_i;\tau_i;r}^{m,n}\left\{d x^{\delta}\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right\}\right.$$

$$\left.\times H_{P,Q}^{M,N}\left\{c x^{\lambda}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right\} E_{\beta,\rho}^{\gamma,q}\left(b x^{\beta}\right)\right] = \frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R}}{[a(1-\alpha)]^{u+u_1 s_1+\ldots+u_R s_R}}$$

$$\times\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\frac{1}{\Gamma(\gamma)}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)}\right\}A_{n_i,s_i}$$

$$\times H_{P,Q}^{M,N}\left[\frac{c x^{\lambda}}{[a(1-\alpha)]^{\lambda}}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[\frac{d x^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]$$

$$\times{}_2\Psi_2\left[\begin{matrix}(\gamma,q),(u+u_1 s_1+\ldots+u_R s_R-\lambda\xi,\beta)\\(\rho,\beta),(1+\frac{\eta}{1-\alpha}+u+u_1 s_1+\ldots+u_R s_R-\lambda\xi,\beta)\end{matrix}\left|\frac{b x^{\beta}}{[a(1-\alpha)]^{\beta}}\right.\right] \tag{1.2.4}$$

Proof:-

In the left hand side of (1.2.4), we use these definitions (1.1.1), (1.1.8), (1.1.15), (1.1.16) and (1.1.18) then we get

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)}x^{u_i s_i}\right\}A_{n_i,s_i}\right.$$

$$\times \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(d x^{\delta})^{-s} ds \sum_{h=1}^{N}\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h}\left\{\frac{1}{(c x^{\lambda})}\right\}^{\xi}$$

$$\left.\times \frac{1}{2\pi i\Gamma(\gamma)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\zeta)\Gamma(\gamma-q\zeta)}{\Gamma(\rho-\beta\zeta)}\left(-b t^{\beta}\right)^{-\zeta} d\zeta\right]$$

$$= x^{\eta}\left[\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left\{t^{u-1+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi-\beta\zeta}\left(1-\frac{a(1-\alpha)t}{x}\right)^{(\frac{\eta}{1-\alpha}+1)-1} dt\right\}\right.$$

$$\times \sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)}\right\}A_{n_i,s_i}$$

$$\times \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(d)^{-s} ds \sum_{h=1}^{N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h}\left(\frac{1}{c}\right)^{\xi}$$

$$\left.\times \frac{1}{2\pi i\Gamma(\gamma)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\zeta)\Gamma(\gamma-q\zeta)}{\Gamma(\rho-\beta\zeta)}(-b)^{-\zeta} d\zeta\right], \tag{1.2.5}$$

In equation (1.2.5), if we put $\frac{a(1-\alpha)t}{x}=v_1$ and interchanging the order of integration, we get

$$= \sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)}\right\}\frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi-\beta\zeta}}{[a(1-\alpha)]^{u+u_1 s_1+\ldots+u_R s_R-\delta s-\lambda\xi-\beta\zeta}} A_{n_i,s_i}$$

$$\times \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(d)^{-s} ds \sum_{h=1}^{N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\chi(\xi)}{\Gamma(\nu+1)E_h}\left(\frac{1}{c}\right)^{\xi}$$

$$\left.\times \frac{1}{2\pi i\Gamma(\gamma)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\zeta)\Gamma(\gamma-q\zeta)}{\Gamma(\rho-\beta\zeta)}(-b)^{-\zeta} d\zeta\right]$$

$$\times\int_0^1\left\{v_1^{\,u+u_1s_1+\ldots+u_Rs_R-\delta s-\lambda\xi-\beta\zeta-1}(1-v_1)^{(\frac{\eta}{1-\alpha}+1)-1}dv_1\right\}, \qquad (1.2.6)$$

now we evaluate the inner most integral with the help of known result (1.1.19), we get

$$\frac{x^{\eta+u+u_1s_1+\ldots+u_Rs_R}}{[a(1-\alpha)]^{u+u_1s_1+\ldots+u_Rs_R}}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_is_i}}{\Gamma(s_i+1)}x^{s_i}\right\}A_{n_i,s_i}$$

$$\times H^{M,N}_{P,Q}\left[\frac{c\,x^{\lambda}}{[a(1-\alpha)]^{\lambda}}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]\aleph^{m,n}_{p_i,q_i;\tau_i;r}\left[\frac{d\,x^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]$$

$$\times\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\frac{1}{\Gamma(\gamma)}H^{1,2}_{2,3}\left[-\frac{b\,x^{\beta}}{[a(1-\alpha)]^{\beta}}\left|\begin{matrix}(1-\gamma,\lambda),(1-u-u_1s_1-\ldots-u_Rs_R+\lambda\xi,\beta)\\(0,1),(1-\rho,\beta),(-\frac{\eta}{1-\alpha}-u-u_1s_1-\ldots-u_Rs_R+\lambda\xi,\beta)\end{matrix}\right.\right],$$

(1.2.7)

In equation (1.2.7) with the help of known result (1.1.17) and after a little simplification, we get the required result (1.2.4).

Theorem 3. Let $\eta,\beta,\rho,\gamma\in C, R(\eta), R(\beta), R(\rho), R(\gamma)>0, R\left(1+\frac{\eta}{(1-\alpha)}\right)>0.$

and set of sufficient conditions (1.1.10) hold

$$P^{(\eta,\alpha)}_{0+}\left[x^{\rho-1}\aleph^{m_1,n_1}_{p_i,q_i;\tau_i;r}\left\{\lambda x^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},[\tau_i(a_{ji},A_{ji})]_{n_1+1,p_i}\\(b_j,B_j)_{1,m_1},[\tau_i(b_{ji},B_{ji})]_{m_1+1,q_i}\end{matrix}\right.\right\}\right.$$

$$\left.\times\aleph^{m_2,n_2}_{p_i',q_i';\tau_i';r}\left\{w\,x^{\nu}\left|\begin{matrix}(c_j,C_j)_{1,n_2},[\tau_i'(c_{ji},C_{ji})]_{n_2+1,p_i'}\\(d_j,D_j)_{1,m_2},[\tau_i'(d_{ji},D_{ji})]_{m_2+1,q_i'}\end{matrix}\right.\right\}\right]$$

$$=\frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\aleph^{0,1;m_1,n_1;m_2,n_2}_{1,1;p_i,q_i,\tau_i;p'_i,q'_i;\tau'_i;r}\left[\begin{array}{c|c}\frac{\lambda x^{\mu}}{[a(1-\alpha)]^{\mu}} & C^* \\ \frac{w x^{\nu}}{[a(1-\alpha)]^{\nu}} & D^*\end{array}\right], \qquad (1.2.8)$$

where

$$\left.\begin{array}{l} C^*=(1-\sigma,\mu,\nu),(a_j,A_j)_{1,n_1},\left[\tau_i(a_{ji},A_{ji})\right]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},\left[\tau'_i(c_{ji},C_{ji})\right]_{n_2+1,p'_i} \\ \text{and} \\ D^*=(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu),(b_j,B_j)_{1,m_1},\left[\tau_i(b_{ji},B_{ji})\right]_{m_1+1,q_i};(d_j,D_j)_{1,m_2},\left[\tau'_i(d_{ji},D_{ji})\right]_{m_2+1,q'_i}\end{array}\right\}$$

Proof: - In the left hand side of (1.2.8), we use these definitions (1.1.1) and (1.1.8), we get

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i;\tau_i;r}(s)\int_{L_2}\Omega^{m_2,n_2}_{p'_i,q'_i;\tau'_i;r}(\xi)\,\lambda^{-s}w^{-\xi}\left\{P^{(\eta,\alpha)}_{0+}(x^{\rho-\mu s-\nu\xi-1})\right\}(x)ds\,d\xi$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i;\tau_i;r}(s)\int_{L_2}\Omega^{m_2,n_2}_{p'_i,q'_i;\tau'_i;r}(\xi)\,\lambda^{-s}w^{-\xi}(x)$$

$$\times x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left\{t^{\rho-\mu s-\nu\xi-1}\left(1-\frac{a(1-\alpha)t}{x}\right)^{\frac{\eta}{1-\alpha}}dt\right\}ds\,d\xi, \qquad (1.2.9)$$

In equation (1.2.9).if we put $\frac{a(1-\alpha)t}{x}=v_1$ and interchanging the order of integration then we get the following form

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i;\tau_i;r}(s)\int_{L_2}\Omega^{m_2,n_2}_{p'_i,q'_i;\tau'_i;r}(\xi)\,\lambda^{-s}w^{-\xi}(x)$$

$$\times x^{\eta}\int_0^1\left\{\left(\frac{xv_1}{a(1-\alpha)}\right)^{\rho-\mu s-\nu\xi-1}(1-v_1)^{\frac{\eta}{1-\alpha}}\frac{x}{a(1-\alpha)}dv_1\right\}ds\,d\xi$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i;\tau_i;r}(s)\int_{L_2}\Omega^{m_2,n_2}_{p_i,q_i;\tau_i;r}(\xi)\,\lambda^{-s}w^{-\xi}(x)$$

$$\times\frac{x^{\eta+\rho-\mu s-\nu\xi}}{\{a(1-\alpha)\}^{\rho-\mu s-\nu\xi}}\int_0^1\left\{(v_1)^{\rho-\mu s-\nu\xi-1}(1-v_1)^{(\frac{\eta}{1-\alpha}+1)-1}dv_1\right\}ds\,d\xi \qquad (1.2.10)$$

now we evaluate the inner most integral with the help of known result (1.1.19), we get

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i;\tau_i;r}(s)\int_{L_2}\Omega^{m_2,n_2}_{p_i,q_i;\tau_i;r}(\xi)\,\lambda^{-s}w^{-\xi}(x)ds\,d\xi$$

$$\times\frac{x^{\eta+\rho-\mu s-\nu\xi}}{\{a(1-\alpha)\}^{\rho-\mu s-\nu\xi}}\frac{\Gamma(\frac{\eta}{1-\alpha}+1)\Gamma(\rho-\mu s-\nu\xi)}{\Gamma(\frac{\eta}{1-\alpha}+\rho-\mu s-\nu\xi+1)}=\frac{x^{\eta+\rho-\mu s-\nu\xi}}{\{a(1-\alpha)\}^{\rho-\mu s-\nu\xi}}\Gamma(\frac{\eta}{1-\alpha}+1)$$

$$\times\left[\frac{1}{(2\pi i)^2}\int_{L_1}\Omega^{m_1,n_1}_{p_i,q_i;\tau_i;r}(s)\int_{L_2}\Omega^{m_2,n_2}_{p_i,q_i;\tau_i;r}(\xi)\,\lambda^{-s}w^{-\xi}(x)\frac{\Gamma(\rho-\mu s-\nu\xi)}{\Gamma(\frac{\eta}{1-\alpha}+\rho-\mu s-\nu\xi+1)}\right]ds\,d\xi.$$

(1.2.11)

Now we use definition (1.1.11) in (1.2.11) then we get required result (1.2.8), after a little simplification.

1.3. SPECIAL CASES

1.3.1 BY THEOREM 1

(i). If we put $\tau_i=1$ in (1.2.1) then we get following result in term of I-function

$$P^{(\eta,\alpha)}_{0+}\left[x^{u-1}S^{m_1,\ldots,m_R}_{n_1,\ldots,n_R}(x^{u_1},\ldots,x^{u_R})I^{m,n}_{p_i,q_i;r}\left\{dx^{\delta}\left|\begin{matrix}(a_j,A_j)_{1,n},\left[(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[(b_{ji},B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right\}\right.$$

$$\left.\times H^{M,N}_{P,Q}\left\{cx^{\lambda}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right\}\right]=\frac{x^{\eta+u+u_1s_1+\ldots+u_Rs_R}}{[a(1-\alpha)]^{u+u_1s_1+\ldots+u_Rs_R}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)$$

$$\times \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} A_{n_i, s_i} \; H_{P,Q}^{M,N}\left[\frac{c x^{\lambda}}{[a(1-\alpha)]^{\lambda}} \middle| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\times I_{p_i+1, q_i+1; r}^{m, n+1}\left[\frac{d x^{\delta}}{[a(1-\alpha)]^{\delta}} \middle| \begin{matrix} (1-u-u_1 s_1 - \cdots - u_R s_R - \lambda\xi, \delta), (a_j, A_j)_{1,n}, \left[(a_{ji}, A_{ji})\right]_{n+1, p_i} \\ (b_j, B_j)_{1,m}, \left[(b_{ji}, B_{ji})\right]_{m+1, q_i}, (-\frac{\eta}{1-\alpha} - u - u_1 s_1 - \cdots - u_R s_R + \lambda\xi, \delta) \end{matrix} \right]$$

(1.3.1)

(ii). If we choose $\tau_i = 1$ and $r = 1$ in (1.2.1) then we get following result in term of H-function

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\dots,n_R}^{m_1,\dots,m_R}(x^{u_1},\dots,x^{u_R}) H_{p,q}^{m,n}\left\{ d x^{\delta} \middle| \begin{matrix} (a_j, A_j)_{1,n}, \left[(a_j, A_j)\right]_{n+1,p} \\ (b_j, B_j)_{1,m}, \left[(b_j, B_j)\right]_{m+1,q} \end{matrix} \right\} \right.$$

$$\left. \times H_{P,Q}^{M,N}\left\{ c x^{\lambda} \middle| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right\} \right] = \frac{x^{\eta+u+u_1 s_1 + \cdots + u_R s_R}}{[a(1-\alpha)]^{u+u_1 s_1 + \cdots + u_R s_R}} \Gamma\left(1 + \frac{\eta}{1-\alpha}\right)$$

$$\times \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} A_{n_i, s_i} \; H_{P,Q}^{M,N}\left[\frac{c x^{\lambda}}{[a(1-\alpha)]^{\lambda}} \middle| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right]$$

$$\times H_{p+1, q+1}^{m, n+1}\left[\frac{d x^{\delta}}{[a(1-\alpha)]^{\delta}} \middle| \begin{matrix} (1-u-u_1 s_1 - \cdots - u_R s_R + \lambda\xi, \delta), (a_j, A_j)_{1,n}, \left[(a_j, A_j)\right]_{n+1, p} \\ (b_j, B_j)_{1,m}, \left[(b_j, B_j)\right]_{m+1, q}, (-\frac{\eta}{1-\alpha} - u - u_1 s_1 - \cdots - u_R s_R + \lambda\xi, \delta) \end{matrix} \right]$$

(1.3.2)

(iii). If we choose $M = N = P = Q = 1$ and $d = 1$ in (1.2.1) then we get following result due to Jain and Arekar [40]

$$P_{0+}^{(\eta,\alpha)}\left[x^{u-1} S_{n_1,\dots,n_R}^{m_1,\dots,m_R}\left\{ x^{u_1},\dots,x^{u_R} \right\} \aleph_{p_i, q_i; \tau_i; r}^{m,n}\left\{ x^{\delta} \middle| \begin{matrix} (a_j, A_j)_{1,n}, \left[\tau_i (a_{ji}, A_{ji})\right]_{n+1, p_i} \\ (b_j, B_j)_{1,m}, \left[\tau_i (b_{ji}, B_{ji})\right]_{m+1, q_i} \end{matrix} \right\} \right]$$

$$= \frac{x^{\eta+u+u_1 s_1+\cdots+u_R s_R}}{[a(1-\alpha)]^{u+u_1 s_1+\cdots+u_R s_R}} \Gamma\left(1+\frac{\eta}{1-\alpha}\right) \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} A_{n_i, s_i}$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1;\tau_i;r} \left[\frac{d x^{\delta}}{[a(1-\alpha)]^{\delta}} \middle| \begin{matrix} (1-u-u_1 s_1-\cdots-u_R s_R, \delta), (a_j, A_j)_{1,n}, [\tau_i (a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, [\tau_i (b_{ji}, B_{ji})]_{m+1,q_i}, (-\frac{\eta}{1-\alpha}-u-u_1 s_1-\cdots-u_R s_R, \delta) \end{matrix} \right]$$

(1.3.3)

(iv) If we substitute $\tau_i = 1$, $r = 1$, $M = N = P = Q = 1$ and general class of polynomials is unity in equation (1.2.1) then we obtain known result due to Nair [73, equation (15)].

(vii) If we take $\tau_i = 1$, $r = 1$ and general class of polynomials is unity in equation (1.2.1) then we obtain known result due to Chaurasia and Gill [14, equation (12)].

1.3.2 BY THEOREM 2

(i). If we put $\tau_i = 1$ in equation (1.2.4) then we obtain the following result in term of I-function

$$P^{(\eta,\alpha)}_{0+} \left[x^{u-1} S^{m_1,\ldots,m_R}_{n_1,\ldots,n_R}(x^{u_1},\ldots,x^{u_R}) I^{m,n}_{p_i,q_i;r} \left\{ d x^{\delta} \middle| \begin{matrix} (a_j, A_j)_{1,n}, [(a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, [(b_{ji}, B_{ji})]_{m+1,q_i} \end{matrix} \right\} \right.$$

$$\left. \times H^{M,N}_{P,Q} \left\{ c x^{\lambda} \middle| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right\} E^{\gamma,q}_{\beta,\rho}\left(b x^{\beta}\right) \right] = \frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R}}{[a(1-\alpha)]^{u+u_1 s_1+\ldots+u_R s_R}}$$

$$\times \Gamma\left(1+\frac{\eta}{1-\alpha}\right) \frac{1}{\Gamma(\gamma)} \sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_R=0}^{[n_R/m_R]} \left\{ \prod_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} \right\} A_{n_i, s_i}$$

$$\times H^{M,N}_{P,Q} \left[\frac{c x^{\lambda}}{[a(1-\alpha)]^{\lambda}} \middle| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right] I^{m,n}_{p_i,q_i;r} \left[\frac{d x^{\delta}}{[a(1-\alpha)]^{\delta}} \middle| \begin{matrix} (a_j, A_j)_{1,n}, [(a_{ji}, A_{ji})]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, [(b_{ji}, B_{ji})]_{m+1,q_i} \end{matrix} \right]$$

$$
{}_2\Psi_2\left[\begin{matrix}(\gamma,q),(u+u_1s_1+\ldots+u_Rs_R-\delta s-\delta\xi,\beta)\\(\rho,\beta),(1+\frac{\eta}{1-\alpha}+u+u_1s_1+\ldots+u_Rs_R-\delta s-\delta\xi,\beta)\end{matrix}\middle|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right] \tag{1.3.4}
$$

(ii). If we choose $\tau_i = 1$ and $r = 1$ in (1.2.4) then we get the following result in term of H-function

$$
P_{0+}^{(\eta,\alpha)}\left[x^{u-1}S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}(x^{u_1},\ldots,x^{u_R})H_{p,q}^{m,n}\left\{dx^{\delta}\middle|\begin{matrix}(a_j,A_j)_{1,n},\left[(a_j,A_j)\right]_{n+1,p}\\(b_j,B_j)_{1,m},\left[(b_j,B_j)\right]_{m+1,q}\end{matrix}\right\}\right.
$$

$$
\left.\times H_{P,Q}^{M,N}\left\{cx^{\lambda}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right\}E_{\beta,\rho}^{\gamma,\lambda}\left(bx^{\beta}\right)\right]=\frac{x^{\eta+u+u_1s_1+\ldots+u_Rs_R}}{[a(1-\alpha)]^{u+u_1s_1+\ldots+u_Rs_R}}
$$

$$
\times\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\frac{1}{\Gamma(\gamma)}\sum_{s_1=0}^{[n_1/m_1]}\cdots\sum_{s_R=0}^{[n_R/m_R]}\left\{\prod_{i=1}^{R}\frac{(-n_i)_{m_is_i}}{\Gamma(s_i+1)}\right\}A_{n_i,s_i}
$$

$$
\times H_{P,Q}^{M,N}\left[\frac{cx^{\lambda}}{[a(1-\alpha)]^{\lambda}}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]H_{p,q}^{m,n}\left[\frac{dx^{\delta}}{[a(1-\alpha)]^{\delta}}\middle|\begin{matrix}(a_j,A_j)_{1,n},\left[(a_j,A_j)\right]_{n+1,p}\\(b_j,B_j)_{1,m},\left[(b_j,B_j)\right]_{m+1,q}\end{matrix}\right]
$$

$$
{}_2\Psi_2\left[\begin{matrix}(\gamma,\lambda),(u+u_1s_1+\ldots+u_Rs_R-\delta s-\lambda\xi,\beta)\\(\rho,\beta),(1+\frac{\eta}{1-\alpha}+u+u_1s_1+\ldots+u_Rs_R-\delta s-\lambda\xi,\beta)\end{matrix}\middle|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right] \tag{1.3.5}
$$

(iii). If we choose General polynomials is unity, Aleph function unity and $q = 1$ in (1.2.4) then we get the following result due to Chaurasia and Gill [14, equation (21)]

$$
P_{0+}^{(\eta,\alpha)}\left[x^{u-1}H_{P,Q}^{M,N}\left\{cx^{\lambda}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right\}E_{\beta,\rho}^{\gamma,1}\left(bx^{\beta}\right)\right]=\frac{x^{\eta+u}}{[a(1-\alpha)]^{u}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\frac{1}{\Gamma(\gamma)}
$$

$$
\times H_{P,Q}^{M,N}\left[\frac{cx^{\lambda}}{[a(1-\alpha)]^{\lambda}}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]{}_2\Psi_2\left[\begin{matrix}(\gamma,1),(u-\lambda\xi,\beta)\\(\rho,\beta),(1+\frac{\eta}{1-\alpha}+u-\lambda\xi,\beta)\end{matrix}\middle|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]
$$

(1.3.6)

(iv) If we put $M = N = P = Q = 1$ and Wright's function ${}_2\psi_2$ convert in terms of H-function then we get the known result due to Nair [73, equation (24)].

1.3.3 BY THEOREM 3

(i). If we put $\tau_i = 1, \tau_i' = 1$ in (1.2.8) then we obtain the following results in term of I -function

$$P_{0+}^{(\eta,\alpha)}\left[x^{\rho-1} I_{p_i,q_i;r}^{m_1,n_1}\left\{\lambda x^{\mu} \,\middle|\, \begin{matrix} (a_j,A_j)_{1,n_1},\left[(a_{ji},A_{ji})\right]_{n_1+1,p_i} \\ (b_j,B_j)_{1,m_1},\left[(b_{ji},B_{ji})\right]_{m_1+1,q_i} \end{matrix}\right\}\right.$$

$$\left.\times I_{p_i',q_i';r}^{m_2,n_2}\left\{w x^{\nu} \,\middle|\, \begin{matrix} (c_j,C_j)_{1,n_2},\left[(c_{ji},C_{ji})\right]_{n_2+1,p_i'} \\ (d_j,D_j)_{1,m_2},\left[(d_{ji},D_{ji})\right]_{m_2+1,q_i'} \end{matrix}\right\}\right]$$

$$= \frac{x^{\eta+\rho}}{\left[a(1-\alpha)\right]^{\rho}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right) I_{1,1;p_i,q_i;p_i',q_i';r}^{0,1;m_1,n_1;m_2,n_2}\left[\begin{matrix} \dfrac{\lambda x^{\mu}}{\left[a(1\text{-}\alpha)\right]^{\mu}} \\ \dfrac{w x^{\nu}}{\left[a(1\text{-}\alpha)\right]^{\nu}} \end{matrix} \,\middle|\, \begin{matrix} E^* \\ F^* \end{matrix}\right], \tag{1.3.7}$$

where

$$\left.\begin{aligned} E^* &= (1-\sigma,\mu,\nu),(a_j,A_j)_{1,n_1},\left[(a_{ji},A_{ji})\right]_{n_1+1,p_i};(c_j,C_j)_{1,n_2},\left[(c_{ji},C_{ji})\right]_{n_2+1,p_i'} \\ F^* &= (-\tfrac{\eta}{1-\alpha}-\sigma,\mu,\nu),(b_j,B_j)_{1,m_1},\left[(b_{ji},B_{ji})\right]_{m_1+1,q_i};(d_j,D_j)_{1,m_2},\left[(d_{ji},D_{ji})\right]_{m_2+1,q_i'} \end{aligned}\right\}$$

2. If we put $\tau_i = 1, \tau_i' = 1$ and $r = 1, i = 1,2,...,r$ in (1.2.8) then we get the following result in term of H- function

$$P_{0+}^{(\eta,\alpha)}\left[x^{\rho-1}H_{p,q}^{m_1,n_1}\left\{\lambda x^{\mu}\left|\begin{matrix}(a_j,A_j)_{1,n_1},\left[(a_j,A_j)\right]_{n_1+1,p}\\(b_j,B_j)_{1,m_1},\left[(b_j,B_j)\right]_{m_1+1,q}\end{matrix}\right.\right\}\right.$$

$$\left.\times H_{p',q'}^{m_2,n_2}\left\{w x^{\nu}\left|\begin{matrix}(c_j,C_j)_{1,n_2},\left[(c_j,C_j)\right]_{n_2+1,p'}\\(d_j,D_j)_{1,m_2},\left[(d_j,D_j)\right]_{m_2+1,q'}\end{matrix}\right.\right\}\right]$$

$$=\frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)H_{1,1;p,q;p',q'}^{0,1;m_1,n_1;m_2,n_2}\left[\begin{matrix}\frac{\lambda x^{\mu}}{[a(1\text{-}\alpha)]^{\mu}}\\\frac{w x^{\nu}}{[a(1\text{-}\alpha)]^{\nu}}\end{matrix}\left|\begin{matrix}G^*\\H^*\end{matrix}\right.\right], \tag{1.3.8}$$

where

$$\left.\begin{matrix}G^*=(1-\sigma,\mu,\nu),(a_j,A_j)_{1,n_1},\left[(a_j,A_j)\right]_{n_1+1,p};(c_j,C_j)_{1,n_2},\left[(c_j,C_j)\right]_{n_2+1,p'}\\H^*=(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu),(b_j,B_j)_{1,m_1},\left[(b_j,B_j)\right]_{m_1+1,q};(d_j,D_j)_{1,m_2},\left[(d_j,D_j)\right]_{m_2+1,q'}\end{matrix}\right\}$$

3. If we use the relation (1.1.18) in (1.3.8) then we arrive at the following result

$$P_{0+}^{(\eta,\alpha)}\left[x^{\rho-1}E_{\xi_1,\eta_1}^{\delta_1}\left(\lambda x^{\mu}\right)E_{\xi_2,\eta_2}^{\delta_2}\left(\lambda x^{\nu}\right)\right]$$

$$=\frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(\frac{\eta}{1-\alpha}\right)H_{1,1;1,2;1,2}^{0,1;1,1;1,1}\left[\begin{matrix}\frac{\lambda x^{\mu}}{[a(1\text{-}\alpha)]^{\mu}}\\\frac{w x^{\nu}}{[a(1\text{-}\alpha)]^{\nu}}\end{matrix}\left|\begin{matrix}(1-\sigma,\mu,\nu),(1-\delta_1,\xi_1),(1-\delta_2,\xi_2)\\(-\frac{\eta}{1-\alpha}-\sigma,\mu,\nu),(0,1);(1-\eta_1,\xi_1);(0,1);(1-\eta_2,\xi_2)\end{matrix}\right.\right]. \tag{1.3.9}$$

.

CHAPTER 2

A STUDY OF MULTIVARIABLE FRACTIONAL INTEGRAL OPERATORS INVOLVING MULTIVARIABLE POLYNOMIAL AND THE ALEPH (ℵ)-FUNCTION

Publications:

1. Multidimensional fractional integral operators involving general class of Polynomial and Aleph (ℵ) function, International Journal of Mathematics and its Applications, Vol. 5, Number 2 (B) (2017), pp. 293-300.

2. On some composition formulae for multidimensional fractional integral operators associated aleph (ℵ)- function and general class of polynomial, International journal of computational and applied mathematics , 12(2) , (2017), pp. 255-272.

In this chapter, we develop and present multidimensional fractional integral operators whose kernels involve the product of a multivariable polynomial (0.6.2) and the Aleph $(\aleph)$-function (0.4.1). First we define the operators of our study and give conditions of existence of these operators. Further, we obtain some images of certain useful functions under these operators. Next, we establish two theorems giving the multidimensional generalized Stieltjes transform of fractional integral operators and conversely. Then we present Mellin transform, Mellin convolutions and inversion formulae for these operators. Finally, we derive three new and interesting composition formulae of our multidimensional fractional integral operators.

In addition we have also evaluated a double integral of a very general nature with the help of our first composition formula. A special case of the same is also given. The results obtained by Goyal and Jain [30], Erdelyi [21], Goyal, Jain and Gaur [31], Raina [81] and many others follow as special cases of our composition formulae.

2.1 DEFINITIONS

2.1.1 THE ALEPH $(\aleph)$ – FUNCTION IN SERIES

Chaurasia [15] give series representation of the Aleph function

$$\aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\right]=\sum_{\nu=1}^{M}\sum_{g=0}^{\infty}\theta(S_{\nu,g})\,z^{-S_{\nu,g}}, \tag{2.1.1}$$

where

$$\theta(S_{\nu,g})=\frac{\prod\limits_{j=1,\,j\neq\nu}^{M}\Gamma(b_j+B_jS_{\nu,g})\prod\limits_{j=1}^{N}\Gamma(1-a_j-A_jS_{\nu,g})\,(-1)^g}{\sum\limits_{i=1}^{r}\tau_i\prod\limits_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}S_{\nu,g})\prod\limits_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}S_{\nu,g})\;g!\;B_\nu},\quad S_{\nu,g}=\frac{b_\nu+g}{B_\nu}, \tag{2.1.2}$$

The behaviors of the Aleph function for small and large of z is given by Chaurasia [15] in the following manner

$$\aleph_{P_i,Q_i,\tau_i,r}^{M,N}[z] = O\left[|z|^{a}\right] \text{For small z, where } a = \min_{1\le j\le M}\left[\operatorname{Re}\left(\frac{b_j}{B_j}\right)\right], \tag{2.1.3}$$

$$\aleph_{P_i,Q_i,\tau_i,r}^{M,N}[z] = O\left[|z|^{b}\right] \text{For small z, where } b = \min_{1\le j\le N}\left[\operatorname{Re}\left(\frac{a_j - 1}{A_j}\right)\right] \tag{2.1.4}$$

and conditions (0.4.3) are also satisfied.

2.1.2 I AND J- INTEGRAL OPERATORS

Throughout in this chapter we assume that A denotes the class of function for which

for which $\int\cdots\int_{\Lambda_s} \left|f(t_1,...,t_s)\right| dt_1,...,dt_s < \infty$

for every bounded s- dimensional region Λ_s ,excluding the origin and

$$f(t_1,...,t_s) = \begin{cases} O\prod_{j=1}^{s}\left(\left|t_j\right|^{U_j}\right) & \max\left\{\left|t_j\right|\right\}\to 0 \\ O\prod_{j=1}^{s}\left(\left|t_j\right|^{-V_j} e^{-W_j\left|t_j\right|}\right) & \min\left\{\left|t_j\right|\right\}\to\infty \end{cases} \quad ; \quad j = i,...,s. \tag{2.1.5}$$

Now, we defined and represented two multidimensional fractional integral operators with kernels involving Aleph-function and multivariable polynomial having general arguments as follows

$$I_x\left[f(t_1,...,t_s)\right] = I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left[f(t_1,...,t_s);x_1,...,x_s\right]$$

$$= \left(\prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left[\prod_{j=1}^{s} t_j^{\rho_j}(x_j - t_j)^{\sigma_j - 1}\right]$$

$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},...,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j}\left|\begin{matrix}(a_j,A_j)_{1,N},\left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i;r}\\(b_j,B_j)_{1,M},\left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i;r}\end{matrix}\right.\right] f(t_1,\dots,t_s)\,dt_1,\dots,dt_s\,,$$

(2.1.6)

where

$$\left.\begin{array}{l}\text{(i)}\quad \min \operatorname{Re}\ (e_j,f_j,\eta_j,\lambda_j)\ge 0\text{, not all zero simultaneously;}\\ \text{(ii)}\quad \min\limits_{1\le k\le M}\ \operatorname{Re}\left[1+\rho_j+U_j+\eta_j\frac{b_k}{B_k}\right]>0,\ \min\limits_{1\le k\le M}\ \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0\\ \qquad\qquad\text{where } j=1,\dots,s.\end{array}\right\}. \qquad (2.1.7)$$

$$J_x\left[f(t_1,\dots,t_s)\right]=J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f(t_1,\dots,t_s);x_1,\dots,x_s\right]$$

$$=\left(\prod_{j=1}^{s}x_j^{\rho_j}\right)\int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty}\left[\prod_{j=1}^{s}t_j^{-\rho_j-\sigma_j}(t_j-x_j)^{\sigma_j-1}\right]$$

$$\times S_V^{U_1,\dots,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{e_1}\left(1-\frac{x_1}{t_1}\right)^{f_1},\dots,E_s\left(\frac{x_s}{t_s}\right)^{e_s}\left(1-\frac{x_s}{t_s}\right)^{f_s}\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{\eta_j}\left(1-\frac{x_j}{t_j}\right)^{\lambda_j}\left|\begin{matrix}(a_j,A_j)_{1,N},\left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i;r}\\(b_j,B_j)_{1,M},\left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i;r}\end{matrix}\right.\right] f(t_1,\dots,t_s)\,dt_1,\dots,dt_s\,,$$

(2.1.8)

where

$$\left.\begin{array}{l}\text{(i)}\quad \min\ \operatorname{Re}\ (e_j,f_j,\eta_j,\lambda_j)\ge 0\text{, not all zero simultaneously;}\\ \text{(ii)}\ \operatorname{Re}\ (W_j)=0,\ \min\limits_{1\le k\le M}\ \operatorname{Re}\left[1+\rho_j+V_j+\eta_j\frac{b_k}{B_k}\right]>0,\ \min\limits_{1\le k\le M}\ \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0\\ \quad\text{or }\operatorname{Re}\ (W_j)>0,\ \min\limits_{1\le k\le M}\ \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0\ \text{, where}\quad j=1,\dots,s.\end{array}\right\}.$$

(2.1.9)

2.2. SOME IMPORTANT IMAGES OF THE INTEGRAL OPERATORS

In this section we will obtain the images of some functions $\prod_{j=1}^{s} t_j^{\gamma_j}(h_j+t_j)^{-\delta_j}$ under the operators given by (2.1.6) and (2.1.8) as follows

$$I_x\left[\prod_{j=1}^{s} t_j^{\gamma_j}(h_j+t_j)^{-\delta_j}\right]=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(-\frac{x_j}{h_j}\right)^{n}\left(\frac{x_j^{\gamma_j}}{h_j^{\delta_j}}\right)\left(1+\frac{x_j}{h_j}\right)^{\sigma_j+f_jR_j-\delta_j}$$

$$\times\sum_{\substack{R_1,\ldots,R_s=0\\ \sum_{i=1}^{s}U_iR_i\le V}}(-V)_{\sum_{i=1}^{s}U_iR_i}\,A(V,R_1,\ldots,R_s)\frac{E_i^{R_i}}{R_i!}\,\aleph^{M,N+3s}_{P_i+3s,Q_i+2s,\tau_i,r}\left[z\prod_{i=1}^{s}\left(1+\frac{x_j}{h_j}\right)^{\lambda_j}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right],$$

(2.2.1)

where

$$A^*=\left(a_j,A_j\right)_{1,N},\left(-\rho_j-\gamma_j-e_jR_j;\eta_j\right)_{1,s},\left(1-\sigma_j-f_jR_j-n;\lambda_j\right)_{1,s},$$

$$\left(-\sigma_j-\rho_j-\gamma_j-\left(e_j+f_j\right)R_j+\delta_j-n;\lambda_j+\eta_j\right)_{1,s},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r}$$

and

$$B^*=\left(b_j,B_j\right)_{1,M},\left(-\sigma_j-\rho_j-\gamma_j-\left(e_j+f_j\right)R_j+\delta_j;\lambda_j+\eta_j\right)_{1,s},$$

$$\left(-\sigma_j-\rho_j-\gamma_j-\left(e_j+f_j\right)R_j-n;\lambda_j+\eta_j\right)_{1,s},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r}\quad.$$

Provided that $\min\mathrm{Re}\,(e_j,f_j,\eta_j,\lambda_j)\ge 0$ not all zero simultaneously,

$$\min_{1\le k\le M}\mathrm{Re}\left[1+\rho_j+\gamma_j+\eta_j\frac{b_k}{B_k}\right]>0\text{ and }\min_{1\le k\le M}\mathrm{Re}\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0,(j=1,\ldots,s).$$

Also, for J-Operator

$$J_x\left[\prod_{j=1}^{s} t_j^{\gamma_j}\left(h_j+t_j\right)^{-\delta_j}\right]=\sum_{R_1,\ldots,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}(-V)_{\sum_{i=1}^{s}U_iR_i}\,A(V,R_1,\ldots,R_s)\frac{E_i^{R_i}}{R_i!}$$

$$\times\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(-\frac{h_j}{x_j}\right)^n\left(x_j^{\gamma_j-\delta_j}\right)\left(1+\frac{h_j}{x_j}\right)^{\sigma_j+f_jR_j-\delta_j}$$

$$\times\ \aleph^{M,N+3s}_{P_i+3s,Q_i+2s,\tau_i,r}\left[z\prod_{i=1}^{s}\left(1+\frac{h_j}{x_j}\right)^{\lambda_j}\middle|\begin{matrix}C^*\\D^*\end{matrix}\right],\qquad(2.2.2)$$

where

$$C^*=\left(a_j,A_j\right)_{1,N},\left(1-\rho_j+\gamma_j-e_jR_j-\delta_j;\eta_j\right)_{1,s},\left(1-\sigma_j-f_jR_j-n;\lambda_j\right)_{1,s},$$
$$\left(1-\sigma_j-\rho_j+\gamma_j-\left(e_j+f_j\right)R_j-n;\lambda_j+\eta_j\right)_{1,s},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r}$$

and

$$D^*=\left(b_j,B_j\right)_{1,M},\left(1-\sigma_j-\rho_j+\gamma_j-\left(e_j+f_j\right)R_j;\lambda_j+\eta_j\right)_{1,s},$$
$$\left(1-\sigma_j-\rho_j+\gamma_j-\left(e_j+f_j\right)R_j-n;\lambda_j+\eta_j\right)_{1,s},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r}\ .$$

Provided that $\min\operatorname{Re}(e_j,f_j,\eta_j,\lambda_j)\ge 0$, $(j=1,\ldots,s)$ not all zero simultaneously,

$$\min_{1\le k\le M}\operatorname{Re}\left[\rho_j-\gamma_j+\delta_j+\eta_j\frac{b_k}{B_k}\right]>0\ \text{ and }\ \min_{1\le k\le M}\operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0,(j=1,\ldots,s).$$

Proof: To prove (2.2.1), at first we express I-operator in the integral form with the help of equation (2.1.6). Next, we write generalized multivariable polynomial in the series by using (0.6.2). Then, we change the order of t_j-integral and the series and express the Aleph function in term of Mellin Barnes type contour integrals with the help of (0.4.1) additionally we change the order of $t_j\,(j=1,2,\ldots,s)$ and $\xi-$integrals (which is permissible under the given conditions). Solving the t_j-integrals with the help of known result Gradshteyn [33, p.287, Eq. 3.197 (8)], we get the following expression

$$I_x\left[\prod_{j=1}^{s} t_j^{\gamma_j}(h_j+t_j)^{-\delta_j}\right]=\sum_{\substack{R_1,\ldots,R_s=0\\ \sum_{i=1}^{s}U_iR_i\le V}}(-V)_{\sum_{i=1}^{s}U_iR_i}\,A(V,R_1,\ldots,R_s)\frac{E_i^{R_i}}{R_i!}$$

$$\prod_{j=1}^{s}\left(x_j^{\gamma_j-\delta_j}\right)\left(\frac{h_j}{x_j}\right)^{-\delta_j}\frac{1}{2\pi i}\int_L \phi(\xi)\,z^{-\xi}\,B\left(\sigma_j+f_jR_j-\lambda_j\xi,\rho_j+\gamma_j+e_jR_j-\eta_j\xi+1\right)$$

$${}_2F_1\left[\begin{matrix}\delta_j,\rho_j+\gamma_j+e_jR_j-\eta_j\xi+1\\ \sigma_j+\rho_j+\gamma_j+(e_j+f_j)R_j-(\lambda_j+\eta_j)\xi+1\end{matrix};-\frac{x_j}{h_j}\right]d\xi\,, \tag{2.2.3}$$

where

$$\left|\arg\left(\frac{x_j}{h_j}\right)\right|<\pi\,,\mathrm{Re}\left(\sigma_j+f_jR_j-\lambda_j\xi\right)>0,\mathrm{Re}\left(\rho_j+\gamma_j+e_jR_j-\eta_j\xi+1\right)>0\,,(j=1,\ldots,s).$$

By using the transformation formula Rainville [84, p.60, Eq. (5)] and rearranging the above result, we easily reach at the required result after a little simplification.

Again the proof of result (2.2.2) can be easily obtained on the similar lines of the above result.

2.3. THE MULTIDIMENSIONAL GENERALIZED STIELTJES TRANSFORM WITH I AND J-INTEGRAL OPERATORS

The multidimensional generalized Stieltjes transform of a function $\phi(t_1,\ldots,t_s)$ is expressed as

$$S_{w_1,\ldots,w_s}(\phi)\,(h_1,\ldots,h_s)=\int_0^\infty\cdots\int_0^\infty \phi(t_1,\ldots,t_s)\prod_{j=1}^{s}(t_j+h_j)^{-w_j}\,dt_1\ldots dt_s\,, \tag{2.3.1}$$

provided that the integral is valid.

The multidimensional generalized Stieltjes transform of the I and J-integral operators can be given by the following theorems.

Theorem 1 Let $\phi(t_1,...,t_s) \in A$, $\min \operatorname{Re}(e_j, f_j, \eta_j, \lambda_j) \geq 0$, $(j=1,...,s)$, are not all zero simultaneously, $\min \operatorname{Re}\left[\rho_j + \eta_j + w_j\right] > 0$ and $\min \operatorname{Re}\left[\sigma_j + \lambda_j\right] > 0$. then

(i) $S_{w_1,...,w_s}(I_t\phi)(h_1,...,h_s)$

$$= \int_0^\infty \cdots \int_0^\infty \phi(x_1,...,x_s)\, \psi_1(x_1,...,x_s; h_1,...,h_s)\, dx_1...dx_s, \tag{2.3.2}$$

$$= \sum_{\substack{R_1,...,R_s = 0 \\ \sum_{i=1}^{s} U_i R_i \leq V}} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V, R_1,..., R_s) \frac{E_i^{R_i}}{R_i!} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)} \prod_{j=1}^{s} \left(-\frac{h_j}{x_j}\right)^n$$

$$\times \left(x_j^{-w_j}\right)\left(1+\frac{h_j}{x_j}\right)^{\sigma_j + f_j R_j - w_j} \aleph^{M,N+3s}_{P_i+3s, Q_i+2s, \tau_i; r}\left[z \prod_{i=1}^{s}\left(1+\frac{h_j}{x_j}\right)^{\lambda_j} \middle| \begin{matrix} E^* \\ F^* \end{matrix}\right], \tag{2.3.3}$$

where

$$\psi_1(x_1,...,x_s; h_1,...,h_s) = J_x\left[\prod_{j=1}^{s} (h_j + t_j)^{-w_j}\right],$$

$$E^* = \left(a_j, A_j\right)_{1,N}, \left(1-\rho_j - e_j R_j - w_j; \eta_j\right)_{1,s}, \left(1-\sigma_j - f_j R_j - n; \lambda_j\right)_{1,s},$$
$$\left(1-\sigma_j - \rho_j - \left(e_j + f_j\right) R_j - n; (\lambda_j + \eta_j)\right)_{1,s}, \left[\tau_i\left(a_{ji}, A_{ji}\right)\right]_{N+1, P_i; r}$$

and

$$F^* = \left(b_j, B_j\right)_{1,M}, \left(1-\sigma_j - \rho_j - \left(e_j + f_j\right) R_j; (\lambda_j + \eta_j)\right)_{1,s},$$
$$\left(1+w_j - \sigma_j - \rho_j - \left(e_j + f_j\right) R_j - n; \lambda_j + \eta_j\right)_{1,s}, \left[\tau_i\left(b_{ji}, B_{ji}\right)\right]_{M+1, Q_i; r}$$

(ii) For $\min \operatorname{Re}(e_j, f_j, \eta_j, \lambda_j) \geq 0$, $(j=1,\ldots,s)$, not all zero simultaneously, $\min \operatorname{Re}\left[1+\rho_j+\eta_j\right]>0$ and $\min \operatorname{Re}\left[\sigma_j+\lambda_j\right]>0$ .then

$$S_{w_1,\ldots,w_s}(J_t\phi)(h_1,\ldots,h_s) = \int_0^\infty \cdots \int_0^\infty \phi(x_1,\ldots,x_s)\,\psi_2(x_1,\ldots,x_s;h_1,\ldots,h_s)\,dx_1\ldots dx_s\,, \tag{2.3.4}$$

$$= \sum_{\substack{R_1,\ldots,R_s=0 \\ \sum_{i=1}^{s} U_i R_i \leq V}} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V,R_1,\ldots,R_s) \frac{E_i^{R_i}}{R_i!} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)} \prod_{j=1}^{s} \left(-\frac{x_j}{h_j}\right)^n$$

$$\times \left(h_j^{-w_j}\right)\left(1+\frac{x_j}{h_j}\right)^{\sigma_j+f_jR_j-w_j} \aleph_{P_i+3s,Q_i+2s,\tau_i;r}^{M,N+3s}\left[z\prod_{i=1}^{s}\left(1+\frac{x_j}{h_j}\right)^{\lambda_j}\middle|\begin{matrix}G^*\\H^*\end{matrix}\right], \tag{2.3.5}$$

where

$$\psi_2(x_1,\ldots,x_s;h_1,\ldots,h_s) = I_x\left[\prod_{j=1}^{s}(h_j+t_j)^{-w_j}\right],$$

$$G^* = \left(a_j,A_j\right)_{1,N}, \left(-\rho_j-e_jR_j;\eta_j\right)_{1,s}, \left(1-\sigma_j-f_jR_j-n;\lambda_j\right)_{1,s},$$
$$\left(-\sigma_j-\rho_j-\left(e_j+f_j\right)R_j+w_j-n;\lambda_j+\eta_j\right)_{1,s}, \left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r}$$

and

$$H^* = \left(b_j,B_j\right)_{1,M}, \left(-\sigma_j-\rho_j-\left(e_j+f_j\right)R_j+w_j;\lambda_j+\eta_j\right)_{1,s},$$
$$\left(-\sigma_j-\rho_j-\left(e_j+f_j\right)R_j-n;\lambda_j+\eta_j\right)_{1,s}, \left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r}$$

It is pretended that the integrals on the right hand side of equation (2.3.3) and (2.3.5) exist.

Proof: To prove first of theorem 1, we express the left hand side of (2.3.2) with the help of (2.1.6) and (2.3.2) as follows

$$= \int_0^\infty \cdots \int_0^\infty \left[\left(\prod_{j=1}^{s} t_j^{-\rho_j - \sigma_j} \right) \int_0^{t_1} \cdots \int_0^{t_s} \left[\prod_{j=1}^{s} x_j^{\rho_j} (t_j - x_j)^{\sigma_j - 1} \right] \right.$$

$$\times S_V^{U_1,\ldots,U_s} \left[E_1 \left(\frac{x_1}{t_1} \right)^{e_1} \left(1 - \frac{x_1}{t_1} \right)^{f_1}, \ldots, E_s \left(\frac{x_s}{t_s} \right)^{e_s} \left(1 - \frac{x_s}{t_s} \right)^{f_s} \right]$$

$$\times \aleph_{P_i,Q_i,\tau_i,r}^{M,N} \left[z \prod_{j=1}^{s} \left(\frac{x_j}{t_j} \right)^{\eta_j} \left(1 - \frac{x_j}{t_j} \right)^{\lambda_j} \left| \begin{matrix} (a_j, A_j)_{1,N}, \left[\tau_i (a_{ji}, A_{ji}) \right]_{N+1, P_i; r} \\ (b_j, B_j)_{1,M}, \left[\tau_i (b_{ji}, B_{ji}) \right]_{M+1, Q_i; r} \end{matrix} \right. \right]$$

$$\left. \times \phi(x_1, \ldots, x_s)\, dx_1, \ldots, dx_s \right] \prod_{j=1}^{s} \left\{ (h_j + t_j)^{-w_j} \right\} dt_1, \ldots, dt_s, \tag{2.3.6}$$

Now we interchange the order of x_j and t_j -integrals (which is the Permissible under the conditions stated with the theorem), we get

$$= \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=1}^{s} (x_j^{\rho_j})\ \phi(x_1, \ldots, x_s) \int_{x_1}^\infty \cdots \int_{x_s}^\infty \left[\prod_{j=1}^{s} (t_j^{-\rho_j - \sigma_j})\, (t_j - x_j)^{\sigma_j - 1} \right] \right.$$

$$\times S_V^{U_1,\ldots,U_s} \left[E_1 \left(\frac{x_1}{t_1} \right)^{e_1} \left(1 - \frac{x_1}{t_1} \right)^{f_1}, \ldots, E_s \left(\frac{x_s}{t_s} \right)^{e_s} \left(1 - \frac{x_s}{t_s} \right)^{f_s} \right]$$

$$\times \aleph_{P_i,Q_i,\tau_i,r}^{M,N} \left[z \prod_{j=1}^{s} \left(\frac{x_j}{t_j} \right)^{\eta_j} \left(1 - \frac{x_j}{t_j} \right)^{\lambda_j} \left| \begin{matrix} (a_j, A_j)_{1,N}, \left[\tau_i (a_{ji}, A_{ji}) \right]_{N+1, P_i; r} \\ (b_j, B_j)_{1,M}, \left[\tau_i (b_{ji}, B_{ji}) \right]_{M+1, Q_i; r} \end{matrix} \right. \right]$$

$$\left. \times \prod_{j=1}^{s} \left\{ (h_j + t_j)^{-w_j} \right\} dt_1 \ldots dt_s \right] dx_1 \ldots dx_s. \tag{2.3.7}$$

Now by interpreting the t_j-integrals in terms of the operators defined by (2.1.8), the above result can be taken in the following manner

$$= \int_0^\infty \cdots \int_0^\infty \phi(x_1,...,x_s)\; J_x\left[\prod_{j=1}^{s} (h_j + t_j)^{-w_j}\right] dx_1...dx_s \,.$$

(2.3.8)

Now using the result (2.2.2) and take $\gamma_j = 0$, to find the value of $J_x\left[\prod_{j=1}^{s} (h_j + t_j)^{-w_j}\right]$, after a little simplification ,we reach at required result (2.3.2).

The proof of result (2.3.4) can be developed by proceeding on similar lines to the above result.

The following theorem gives the I and J-integral operator of Multidimensional Generalized Stieltjes transform.

Theorem 2. Let $\phi(t_1,...,t_s) \in A$, $\min \text{Re}\,(e_j, f_j, \eta_j, \lambda_j) \geq 0$,$(j=1,...,s)$ not all zero simultaneously, then

i. For $\min\ \text{Re}\left[1+\rho_j+\eta_j\right] > 0$, $(j=1,...,s)$

$$I_y\left[S_{w_1,...,w_s}\,\phi(t_1,...,t_s);(x_1,...,x_s)\right]$$

$$= \int_0^\infty \cdots \int_0^\infty \phi(t_1,...,t_s)\;\psi_2(t_1,...,t_s;x_1,...,x_s)\,dt_1...dt_s\,, \tag{2.3.9}$$

ii. For $\min\ \text{Re}\left[\rho_j+\eta_j+w_j\right] > 0$, $(j=1,...,s)$

$$J_y\left[S_{w_1,...,w_s}\,\phi(t_1,...,t_s);(x_1,...,x_s)\right]$$

$$= \int_0^\infty \cdots \int_0^\infty \phi(t_1,...,t_s)\;\psi_1(t_1,...,t_s;x_1,...,x_s)\,dt_1...dt_s\,, \tag{2.3.10}$$

where $\psi_1(t_1,...,t_s;x_1,...,x_s)$ and $\psi_2(t_1,...,t_s;x_1,...,x_s)$ are as given in (2.3.3) and (2.3.5) respectively, provided that the integrals in the right hand side of the equations (2.3.9) and (2.3.10) exists.

Proof: The proof of theorem 2 for results (2.3.9) and (2.3.10) , can be obtained by the similar proof of Theorem 1.

2.4. MELLIN TRANSFORMS, INVERSION FORMULAS AND MELLIN CONVOLUTIONS

The multidimensional generalized Mellin transform of the function $f(t_1,...,t_s) \in A$ is defined by the Srivastava and Panda [115, part I, p.125, Eq. (3.5)] as follows

$$M\left[f(t_1,...,t_s);\theta_1,...,\theta_s\right]=\int_0^\infty\cdots\int_0^\infty \prod_{j=1}^{s} t^{\theta_j -1} f(t_1,...,t_s)\,dt_1...dt_s\,, \tag{2.4.1}$$

provided that the integral exists.

Now we obtain the following results which give the multidimensional Mellin transforms of I and J- fractional integral operators (2.1.6) and (2.1.8) respectively, we also find the inversion formulae and mellin convolutions of these operators.

Result 1

If $M\left[I_x\{f(t_1,...,t_s);\theta_1,...,\theta_s\}\right]$ and the conditions of the existence of the operator $I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f(t_1,...,t_s)\right]$ exists, then

$$M\left[I_x\{f(t_1,...,t_s);\theta_1,...,\theta_s\}\right]=M\left[f(t_1,...,t_s);\theta_1,...,\theta_s\right]\Lambda(\theta_1,...,\theta_s)\,, \tag{2.4.2}$$

where

$$\Lambda(\theta_1,...,\theta_s)=\sum_{\substack{R_1,...,R_s=0 \\ \sum_{i=1}^{s} U_i R_i \le V}} (-V)_{\sum_{i=1}^{s} U_i R_i}\, A(V,R_1,...,R_s)\,\frac{E_i^{R_i}}{R_i!} \tag{2.4.3}$$

$$\times\aleph^{M,N+2s}_{P_i+2s,Q_i+s,\tau_i;r}\left[z\,\Big|\,\begin{matrix} I^* \\ J^* \end{matrix}\right],$$

here

$$I^{*}=\left(a_j,A_j\right)_{1,N},\left(-\rho_j+\theta_j-e_jR_j;\eta_j\right)_{1,s},\left(1-\sigma_j-f_jR_j;\lambda_j\right)_{1,s},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r}$$

and

$$J^{*}=\left(b_j,B_j\right)_{1,M},\left(-\sigma_j-\rho_j+\theta_j-\left(e_j+f_j\right)R_j;(\lambda_j+\eta_j)\right)_{1,s},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r}$$

Result 2

If $M\left[J_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]$ and the conditions of the existence of the operator

$J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f(t_1,...,t_s)\right]$ exists, then

$$M\left[J_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]=M\left[f(t_1,...,t_s);\theta_1,...,\theta_s\right]\Lambda\,(1-\theta_1,...,1-\theta_s)\,, \tag{2.4.4}$$

where $\Lambda\,(1-\theta_1,...,1-\theta_s)$ can be obtained by replacing θ_s by $1-\theta_s$ in (2.4.3).

Proof: To prove the result 1, first of all with the help of the equation (2.4.1), we write the multidimensional Mellin transform of the I-operator defined by (2.1.6), we have

$$M\left[I_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]$$

$$=\int_0^\infty\cdots\int_0^\infty\left(\prod_{j=1}^{s}x_j^{\theta_j-1}\right)\left[\prod_{j=1}^{s}\left(x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left[\prod_{j=1}^{s}t_j^{\rho_j}(x_j-t_j)^{\sigma_j-1}\right]\right.$$

$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},...,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right]$$

$$\times\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j}\left|\begin{matrix}\left(a_j,A_j\right)_{1,N},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r}\\\left(b_j,B_j\right)_{1,M},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r}\end{matrix}\right.\right]$$

$$\left.\times f(t_1,...,t_s)dt_1,...,dt_s\right]dx_1,...,dx_s. \tag{2.4.5}$$

Now, we interchange the order of x_j and t_j-integrals (which is the Permissible under the conditions stated with the theorem) and we obtain the right hand side of (2.4.5) as follows:

$$= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{s} (t_j^{\rho_j})\, f(t_1,\ldots,t_s) \left[\int_{t_1}^\infty \cdots \int_{t_s}^\infty \prod_{j=1}^{s} (x_j^{\theta_j - \rho_j - \sigma_j - 1})\, (x_j - t_j)^{\sigma_j - 1} \right.$$

$$\times S_V^{U_1,\ldots,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},\ldots, E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s} \right]$$

$$\times \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left[z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j} \middle| \begin{matrix} \left(a_j,A_j\right)_{1,N}, \left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i;r} \\ \left(b_j,B_j\right)_{1,M}, \left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i;r} \end{matrix} \right]$$

$$\left. \times\ dx_1 \ldots dx_s \right] dt_1 \ldots dt_s .$$

(2.4.6)

The above result can be reduced by using the definition of (2.1.8)

$$\int_0^\infty \cdots \int_0^\infty f(t_1,\ldots,t_s)\, J_t\left[\prod_{j=1}^{s} x_j^{\theta_j - 1} \right] dt_1 \ldots dt_s . \tag{2.4.7}$$

Further by using the below result

$$I_x\left[\prod_{j=1}^{s} t_j^{\nu_j} (x_j - t_j)^{\delta_j} \right] = \left(\prod_{j=1}^{s} x_j^{2\nu_j + \delta_j + 1} \right) J_x\left[\prod_{j=1}^{s} t_j^{-(1+\nu_j+\delta_j)} (t_j - x_j)^{\delta_j} \right], \tag{2.4.8}$$

the integral (2.4.7) reduce to

$$\int_0^\infty \cdots \int_0^\infty f(t_1,\ldots,t_s) \prod_{j=1}^{s} (t_j^{2\theta_j - 1})\, I_t\left(\prod_{j=1}^{s} x_j^{-\theta_j} \right) dt_1 \ldots dt_s , \tag{2.4.9}$$

finally, by using the result (2.2.1), we find $I_t\left[\prod_{j=1}^{s} x_j^{-\theta_j}\right]$ occurring in the above equation and we easily reached at desired result (2.4.2) after a little simplification.

Similarly, the proof of result (2.4.4) can be obtained in the similar lines of the above result.

2.4.1 INVERSION FORMULAS

Srivastava and Panda [115, part I p.125, Lemma 2], give the inversion theorem for the multidimensional Mellin transform (2.4.1),with the help of this theorem we can easily get the following inversion formulae for the fractional integral operators defined by (2.1.6) and (2.1.8).

Result 3

$$f(t_1,\ldots,t_s) = \frac{1}{(2\pi i)^s}$$

$$\times \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_s-i\infty}^{c_s+i\infty} \frac{\prod_{j=1}^{s} t_j^{-\theta_j}}{\Lambda(\theta_1,\ldots,\theta_s)} \, M\left[I_x\left\{f(t_1,\ldots,t_s);\theta_1,\ldots,\theta_s\right\}\right] d\theta_1 \ldots d\theta_s \,, \tag{2.4.10}$$

where $\Lambda(\theta_1,\ldots,\theta_s)$ is given by (2.4.3).

Result 4

$$f(t_1,\ldots,t_s) = \frac{1}{(2\pi i)^s}$$

$$\times \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_s-i\infty}^{c_s+i\infty} \frac{\prod_{j=1}^{s} t_j^{-\theta_j}}{\Lambda(1-\theta_1,\ldots,1-\theta_s)} \, M\left[J_x\left\{f(t_1,\ldots,t_s);\theta_1,\ldots,\theta_s\right\}\right] d\theta_1 \ldots d\theta_s \,, \tag{2.4.11}$$

where $\Lambda(1-\theta_1,\ldots,1-\theta_s)$ can be obtaining by replacing θ_s, by $1-\theta_s$ in (2.4.3).

The validity conditions for the inversion formula (2.4.10) and (2.4.11) can be deduced from the existence condition of the fractional integral operators defined by (2.1.6) and (2.1.8) and their multidimensional Mellin transform started earlier.

2.4.2 MELLIN CONVOLUTIONS

The Multidimensional Mellin convolutions of two functions $f(t_1,...,t_s)$ and $g(t_1,...,t_s)$ will be defined by

$$\left(f*g\right)(t_1,...,t_s)=\left(g*f\right)(t_1,...,t_s)$$

$$=\int_0^\infty\cdots\int_0^\infty\left(\prod_{j=1}^{s}x_j^{-1}\right)f\left(\frac{t_1}{x_1},...,\frac{t_s}{x_s}\right)g(x_1,...,x_s)\,dx_1\,...\,dx_s\,, \qquad (2.4.12)$$

provided that the multiple integral in right hand side exists.

If $f(t_1,...,t_s)\in A$, then the fractional integral operators expressed by (2.1.6) and (2.1.8) respectively can be written as multidimensional Mellin convolutions in the following way

Result 5

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\,g(t_1,...,t_s)$$

$$=\left(I_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}*g\right)(x_1,...,x_s)\,, \qquad (2.4.13)$$

where

$$I_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho_j-\sigma_j}(x_j-1)^{\sigma_j-1}U(x_j-1)\right)$$

$$\times\, S_V^{U_1,...,U_s}\left[\prod_{j=1}^{s}E_j\,(x_j)^{-e_j-f_j}(x_j-1)^{f_j}\right.$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\prod_{j=1}^{s}(x_j)^{-\eta_j-\lambda_j}(x_j-1)^{\lambda_j} \middle| \begin{matrix} (a_j,A_j)_{1,N},\left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i;r} \\ (b_j,B_j)_{1,M},\left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i;r} \end{matrix} \right],$$

(2.4.14)

here $U(x)$ being the Heaviside's unit function.

Result 6

$$J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\, g(t_1,...,t_s)$$

$$=\left(J_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z} * g\right)(x_1,..., x_s), \qquad (2.4.15)$$

where

$$J_{\rho,\sigma;e,f;\eta,\lambda,x:U,V;Z}$$

$$=\left(\prod_{j=1}^{s} x_j^{\rho_j}\,(1-x_j)^{\sigma_j-1}\,U(1-x_j)\right)$$

$$\times\, S_V^{U_1,...,U_s}\left[\prod_{j=1}^{s} E_j (x_j)^{e_j}(1-x_j)^{f_j}\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\prod_{j=1}^{s}(x_j)^{\eta_j}(1-x_j)^{\lambda_j} \middle| \begin{matrix} (a_j,A_j)_{1,N},\left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i;r} \\ (b_j,B_j)_{1,M},\left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i;r} \end{matrix} \right],$$

(2.4.16)

here $U(x)$ being the Heaviside's unit function.

Proof- To Prove result 5, first we express the I-operator defined by (2.1.6) in the following form with the help of Heaviside's unit function.

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\, g(t_1,\ldots,t_s)$$

$$= \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^{s} t_j^{-1}\right)\left[\prod_{j=1}^{s}\left\{\left(\frac{x_j}{t_j}\right)^{-\rho_j-\sigma_j}\left(\frac{x_j}{t_j}-1\right)^{\sigma_j-1} U\left(\frac{x_j}{t_j}-1\right)\right\}\right]$$

$$\times S_V^{U_1,\ldots,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{-e_1-f_1}\left(\frac{x_1}{t_1}-1\right)^{f_1},\ldots,E_s\left(\frac{x_s}{t_s}\right)^{-e_s-f_s}\left(\frac{x_s}{t_s}-1\right)^{f_s}\right]$$

$$\times \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left[z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{-\eta_j-\lambda_j}\left(\frac{x_j}{t_j}-1\right)^{\lambda_j}\left|\begin{matrix}(a_j,A_j)_{1,N},\left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i;r}\\(b_j,B_j)_{1,M},\left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i;r}\end{matrix}\right.\right] f(t_1,\ldots,t_s)\,dt_1,\ldots,dt_s\,.$$

(2.4.17)

By using the definition of the Mellin convolutions given by (2.4.12) and equation (2.4.14) in the above equation (2.4.17), we easily arrive at the required result.

The proof of the result 6 can be developed by proceeding on the similar lines of the above result.

2.5 COMPOSTIONS FORMULAE FOR THE MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS DEFINED BY EQUATIONS (2.1.6) AND (2.1.8)

Result 7

$$I_{y:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{I_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f(t_1,\ldots,t_s)\right]\right\}=\left(\prod_{j=1}^{s} x_j^{-\rho'_j-1}\right)$$

$$\times\int_0^{x_1}\cdots\int_0^{x_s}\left(\prod_{j=1}^{s} t_j^{\rho'_j}\right) G\left(\frac{t_1}{x_1},\ldots,\frac{t_s}{x_s}\right) f(t_1,\ldots,t_s)\,dt_1\ldots dt_s\,, \qquad (2.5.1)$$

where $G(t_1,...,t_s) = \sum_{\substack{R_1,...,R_s=0 \\ \sum_{i=1}^{s} U_i R_i \le V}} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V,R_1,...,R_s) \frac{E_i^{R_i}}{R_i!}$

$$\times \sum_{\substack{R'_1,...,R'_s=0 \\ \sum_{i=1}^{s} U'_i R'_i \le V'}} (-V')_{\sum_{i=1}^{s} U'_i R'_i} A(V',R'_1,...,R'_s) \frac{E_i'^{R'_i}}{R'_i!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta(S_{\nu,g})\, z'^{-S_{\nu,g}}\, \Gamma\left(\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g}\right)$$

$$\times \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)} \left(1-t_j\right)^{\sigma_j+\sigma'_j+f_j R_j+f'_j R'_j-\lambda'_j S_{\nu,g}+n-1}\, t_j^{e'_j R'_j-\eta'_j S_{\nu,g}}$$

$$\times \aleph^{M,N+2s}_{P_i+2s,Q_i+2s,\tau_i,r}\left[z \prod_{j=1}^{s} \left(1-t_j\right)^{\lambda_j} \middle| \begin{matrix} A^{**} \\ B^{**} \end{matrix} \right], \qquad (2.5.2)$$

$$A^{**} = \left(a_j, A_j\right)_{1,N}, \left[\tau_i\left(a_{ji}, A_{ji}\right)\right]_{N+1,P_i}, \left(1-n-\sigma_j - f_j R_j; \lambda_j\right)_{1,s},$$
$$\left(-\rho_j+\rho'_j+\sigma'_j - e_j R_j + \left(e'_j + f'_j\right) R'_j - \left(\eta'_j + \lambda'_j\right) S_{\nu,g}; \eta_j\right)_{1,s}$$

and

$$B^{**} = \left(b_j, B_j\right)_{1,M}, \left(-\rho_j+\rho'_j+\sigma'_j + n - e_j R_j + \left(e'_j + f'_j\right) R'_j - \left(\eta'_j + \lambda'_j\right) S_{\nu,g}; \eta_j\right)_{1,s},$$
$$\left(1-n-\sigma_j - \sigma'_j - f_j R_j - f'_j R'_j + \lambda'_j S_{\nu,g}; \lambda_j\right)_{1,s}, \left[\tau_i\left(b_{ji}, B_{ji}\right)\right]_{M+1,Q_i}$$

and $\theta(S_{\nu,g})$ and $S_{\nu,g}$ are given by (2.1.2). also the following conditions are satisfied:

$$\left.\begin{array}{l}\min\limits_{1\le k\le M}\operatorname{Re}\left[1+\rho_j+\eta_j\frac{b_k}{jB_k}\right]>0,\ \min\limits_{1\le k\le M}\operatorname{Re}\left[1+\rho'_j+U_j+\eta'_j\frac{b_k}{jB_k}\right]>0\ ,\\ \min\limits_{1\le k\le M}\operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{B_k}\right]>0\ ,\ \min\limits_{1\le k\le M}\operatorname{Re}\left[\sigma'_j+\lambda'_j\frac{b'_k}{B'_k}\right]>0\ ,j=1,2,...s.\end{array}\right\} \qquad (2.5.3)$$

Result 8

$$J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left\{J^{\rho',\sigma';e',f';\eta',\lambda'}_{y:U',V';Z'}\left[f(t_1,...,t_s)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{\rho'_j}\right)\int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{-\rho'_j-1}\right)G\left(\frac{x_1}{t_1},...,\frac{x_s}{t_s}\right)f(t_1,...,t_s)\,dt_1...dt_s\ , \qquad (2.5.4)$$

where $f(t_1,...,t_s)\in A$ and $G(t_1,...,t_s)$ is given by (2.5.2), also the integral occurring in the left hand side of (2.5.4) is absolutely convergent and the following conditions are satisfied:

$$\left.\begin{array}{l}\min\limits_{1\le k\le M}\operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{jB_k}\right]>0,\ \min\limits_{1\le k\le M}\operatorname{Re}\left[\sigma'_j+\lambda'_j\frac{b_k}{jB_k}\right]>0\ ,\\ \operatorname{Re}(W_j)=0\ ,\ \operatorname{Re}(W_j)>0,\ \min\limits_{1\le k\le M}\operatorname{Re}\left[1+\rho_j++\sigma_j+V_j+\eta_j\frac{b_k}{jB_k}\right]>0,j=1,2,...s.\end{array}\right\}$$

Result 9

$$I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left\{J^{\rho',\sigma';e',f';\eta',\lambda'}_{y:U',V';Z'}\left[f(t_1,...,t_s)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho_j-1}\right)\int_{0}^{x_1}\cdots\int_{0}^{x_s}\left(\prod_{j=1}^{s}t_j^{\rho_j}\right)G\left(\frac{t_1}{x_1},...,\frac{t_s}{x_s}\right)f(t_1,...,t_s)\,dt_1...dt_s$$
$$+\left(\prod_{j=1}^{s}x_j^{\rho'_j}\right)\int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{-\rho'_j-1}\right)G'\left(\frac{x_1}{t_1},...,\frac{x_s}{t_s}\right)f(t_1,...,t_s)\,dt_1...dt_s\ , \qquad (2.5.5)$$

where

$$G(t_1,\dots,t_s)=\sum_{R_1,\dots,R_s=0}^{\sum_{i=1}^{s}U_iR_i\le V}(-V)_{\sum_{i=1}^{s}U_iR_i}\,A(V,R_1,\dots,R_s)\frac{E_i^{R_i}}{R_i!}$$

$$\times\sum_{R_1',\dots,R_s'=0}^{\sum_{i=1}^{s}U_i'R_i'\le V'}(-V')_{\sum_{i=1}^{s}U_i'R_i'}\,A(V',R_1',\dots,R_s')\frac{E_i'^{R_i'}}{R_i'!}$$

$$\times\sum_{\nu=1}^{M'}\sum_{g=0}^{\infty}\theta\left(S_{\nu,g}\right)z'^{-S_{\nu,g}}\left(1-t_j\right)^{\sigma_j+\sigma_j'+f_jR_j+f_j'R_j'-\lambda_j'S_{\nu,g}-1}$$

$$\times\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}t_j^{e_jR_j+n}\,\Gamma\left(\sigma_j'+f_j'R_j'-\lambda_j'S_{\nu,g}+n\right)$$

$$\times\aleph^{M,N+2s}_{P_i+2s,Q_i+2s,\tau_i,r}\left[z\prod_{j=1}^{s}\left(t_j\right)^{\eta_j}\left(1-t_j\right)^{\lambda_j}\middle|\begin{matrix}C^{**}\\D^{**}\end{matrix}\right],\qquad(2.5.6)$$

here

$$C^{**}=\left(a_j,A_j\right)_{1,N},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i},\left(-\rho_j-\rho_j'-e_jR_j-e_j'R_j'-\eta_j'S_{\nu,g};\eta_j\right)_{1,s},$$
$$\left(1-\sigma_j-\rho_j-\rho_j'-\sigma_j'-\left(e_j+f_j\right)R_j-\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{\nu,g}-n;\left(\eta_j+\lambda_j\right)\right)_{1,s}$$

and

$$D^{**}=\left(b_j,B_j\right)_{1,M},\left(-\rho_j'-n-\rho_j-\sigma_j'-e_jR_j-\left(e_j'+f_j'\right)R_j'-\left(\lambda_j'+\eta_j'\right)S_{\nu,g};\eta_j\right)_{1,s},$$
$$\left(1-\rho_j-\rho_j'-\sigma_j-\sigma_j'-\left(e_j+f_j\right)R_j-\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{\nu,g};\left(\lambda_j+\eta_j\right)\right)_{1,s},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i}$$

also $\theta(S_{\nu,g})$ and $S_{\nu,g}$ are given by (2.1.2) and the following conditions are satisfied

$$\min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j+\eta_j\frac{b_k}{B_k}\right]>0\ ,\ \min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j'+U_j+\eta_j'\frac{b_k}{B_k}\right]>0$$

$$\operatorname{Re}(W_j)>0\ \text{or}\ \operatorname{Re}(W_j)=0\ ,\ \min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j++\sigma_j+V_j+\eta_j\frac{b_k}{B_k}\right]>0\ j=1,2,\dots s.$$

Proof- To Prove results 7, we first define the I-operators involved in its left hand side of (2.5.1) in the integral form, we have

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{I_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f(t_1,...,t_s)\right]\right\}$$

$$=\left(\prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left(\prod_{j=1}^{s} t_j^{\rho_j'}\right) f(t_1,...,t_s)\,\Delta\, dt_1...dt_s\,, \tag{2.5.7}$$

where

$$\Delta=\int_{t_1}^{x_1}\cdots\int_{t_s}^{x_s}\left[\prod_{j=1}^{s} y_j^{\rho_j-\rho_j'-\sigma_j'}\left(x_j-y_j\right)^{\sigma_j-1}\left(y_j-t_j\right)^{\sigma_j'-1}\right]$$

$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},...,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]$$

$$\times S_V^{U_1',...,U_s'}\left[E_1'\left(\frac{t_1}{y_1}\right)^{e_1'}\left(1-\frac{t_1}{y_1}\right)^{f_1'},...,E_s\left(\frac{t_s}{y_s}\right)^{e_s'}\left(1-\frac{t_s}{y_s}\right)^{f_s'}\right]$$

$$\times\aleph\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\eta_j}\left(1-\frac{y_j}{x_j}\right)^{\lambda_j}\right]\aleph'\left[z'\prod_{j=1}^{s}\left(\frac{t_j}{y_j}\right)^{\eta_j'}\left(1-\frac{t_j}{y_j}\right)^{\lambda_j'}\right]dy_1,...,dy_s. \tag{2.5.8}$$

To determine Δ, we first replace both the multivariable polynomials in terms of their respective series with the help of (0.6.2), and interchange the order of these series and y_j -integrals (j=1,…, s), Next, we express one Aleph $\aleph'$ -function in terms of series (2.1.1) and second Aleph $\aleph$-function in terms of Mellin –Barnes type contour integral (0.4.1), and interchange the order of ξ_j and γ_j- integrals (which is the permissible under the conditions stated). We arrive at the following expression after a little simplification.

$$\Delta = \sum_{R_1,\ldots,R_s=0}^{\sum_{i=1}^{s} U_i R_i \le V} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V,R_1,\ldots,R_s) \frac{E_i^{R_i}}{R_i!}$$

$$\times \sum_{R'_1,\ldots,R'_s=0}^{\sum_{i=1}^{s} U'_i R'_i \le V'} (-V')_{\sum_{i=1}^{s} U'_i R'_i} A(V',R'_1,\ldots,R'_s) \frac{E_i'^{R'_i}}{R'_i!} \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta\left(S_{\nu,g}\right) z'^{-S_{\nu,g}}$$

$$\times \frac{1}{2\pi i} \int_L \varphi(\xi) z^{-\xi} \prod_{j=1}^{s} x^{-e_j R_j - f_j R_j + (\eta_j + \lambda_j)\xi} \, t^{e'_j R'_j - \eta'_j S_{\nu,g}}$$

$$\times \int_{t_1}^{x_1} \cdots \int_{t_s}^{x_s} \left[\prod_{j=1}^{s} y_j^{\rho_j - \rho'_j - \sigma'_j + e_j R_j - (e'_j + f'_j) R'_j + (\eta'_j + \lambda'_j) S_{\nu,g} - \eta_j \xi} \right.$$

$$\left. \times \left(x_j - y_j\right)^{\sigma_j + f_j R_j - \lambda_j \xi - 1} \left(y_j - t_j\right)^{\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g} - 1} dy_1 \ldots dy_s \right] d\xi. \tag{2.5.9}$$

On setting $u_j = \dfrac{(x_j - y_j)}{(x_j - t_j)}$ involved in the above equation (2.5.9) and calculate the u_j-integrals with help of the following result Gradshteyn [33, p.287.eq.3.197 (8)]

$$\int_0^1 x^{\delta-1} (x+a)^{\lambda} (1-x)^{\mu-1} dx = a^{\lambda} \beta(\delta,\mu) \, {}_2F_1\left[-\lambda, \delta; \delta+\mu; -\frac{1}{a}\right]. \tag{2.5.10}$$

On substituting the value of Δ in the equation (2.5.7), we get the desired result (2.5.1) after a little simplification.

The proof of result 8 can be developed by proceeding on the similar line to result 7.

Proof the results 9 - we express the I-operator and J- operator involve in the left hand side in the integral form with the help of equations (2.1.6) and (2.1.8) respectively, then we get the integral given below

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{J_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f(t_1,...,t_s)\right]\right\}$$

$$=\left(\prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left[\prod_{j=1}^{s} y_j^{\rho_j}(x_j-y_j)^{\sigma_j-1}\right]$$

$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},...,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]$$

$$\times \aleph\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\eta_j}\left(1-\frac{y_j}{x_j}\right)^{\lambda_j}\right]\left(\prod_{j=1}^{s} y_j^{\rho'_j}\right)\int_{y_1}^{\infty}\cdots\int_{y_s}^{\infty}\left[\prod_{j=1}^{s} t_j^{-\rho'_j-\sigma'_j}\left(t_j-y_j\right)^{\sigma'_j-1}\right]$$

$$\times S_V^{U'_1,...,U'_s}\left[E'_1\left(\frac{y_1}{t_1}\right)^{e'_1}\left(1-\frac{y_1}{t_1}\right)^{f'_1},...,E_s\left(\frac{y_s}{t_s}\right)^{e'_s}\left(1-\frac{y_s}{t_s}\right)^{f'_s}\right]$$

$$\times \aleph'\left[z'\prod_{j=1}^{s}\left(\frac{y_j}{t_j}\right)^{\eta'_j}\left(1-\frac{y_j}{t_j}\right)^{\lambda'_j}\right] f(t_1,...,t_s)\ dt_1,...,dt_s dy_1,...,dy_s\ . \qquad (2.5.11)$$

Now we change the order of y_j and t_j - integrals (which is permissible under the conditions stated), we get the following expression after a little simplification

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{J_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f(t_1,...,t_s)\right]\right\}$$

$$= \int_0^{x_1} \cdots \int_0^{x_s} \left\{ \int_0^{t_1} \cdots \int_0^{t_s} \Lambda \, dy_1 \ldots dy_s \right\} f(t_1,\ldots,t_s)\, dt_1 \ldots dt_s$$

$$+ \int_{x_1}^{\infty} \cdots \int_{x_s}^{\infty} \left\{ \int_0^{x_1} \cdots \int_0^{x_s} \Lambda \, dy_1 \ldots dy_s \right\} f(t_1,\ldots,t_s)\, dt_1 \ldots dt_s$$

$$= \int_0^{x_1} \cdots \int_0^{x_s} I_1 \, f(t_1,\ldots,t_s)\, dt_1 \ldots dt_s + \int_{x_1}^{\infty} \cdots \int_{x_s}^{\infty} I_2 \, f(t_1,\ldots,t_s)\, dt_1 \ldots dt_s \,, \qquad (2.5.12)$$

where

$$\Lambda = \left(\prod_{j=1}^{s} x_j^{-\rho_j - \sigma_j} \, t_j^{-\rho_j' - \sigma_j'} \, y_j^{\rho_j + \rho_j'} (x_j - y_j)^{\sigma_j - 1} (t_j - y_j)^{\sigma_j' - 1} \right)$$

$$S_V^{U_1,\ldots,U_s} \left[E_1 \left(\frac{y_1}{x_1} \right)^{e_1} \left(1 - \frac{y_1}{x_1} \right)^{f_1}, \ldots, E_s \left(\frac{y_s}{x_s} \right)^{e_s} \left(1 - \frac{y_s}{x_s} \right)^{f_s} \right]$$

$$\times S_V^{U_1',\ldots,U_s'} \left[E_1' \left(\frac{y_1}{t_1} \right)^{e_1'} \left(1 - \frac{y_1}{t_1} \right)^{f_1'}, \ldots, E_s \left(\frac{y_s}{t_s} \right)^{e_s'} \left(1 - \frac{y_s}{t_s} \right)^{f_s'} \right]$$

$$\times \aleph \left[z \prod_{j=1}^{s} \left(\frac{y_j}{x_j} \right)^{\eta_j} \left(1 - \frac{y_j}{x_j} \right)^{\lambda_j} \right] \aleph' \left[z' \prod_{j=1}^{s} \left(\frac{y_j}{t_j} \right)^{\eta_j'} \left(1 - \frac{y_j}{t_j} \right)^{\lambda_j'} \right] dy_1,\ldots,dy_s \qquad (2.5.13)$$

and $$I_1 = \left\{ \int_0^{t_1} \cdots \int_0^{t_s} \Lambda \, dy_1 \ldots dy_s \right\}, I_2 = \left\{ \int_0^{x_1} \cdots \int_0^{x_s} \Lambda \, dy_1 \ldots dy_s \right\}. \qquad (2.5.14)$$

To evaluate I_1, involved in the integral on the right hand side of (2.5.12), we express the multivariable polynomials $S_V^{U_1,\ldots,U_s}$, $S_V^{U_1',\ldots,U_s'}$ and Aleph $\aleph'$ -function expressed in terms of their respective series with the help of equation (0.6.2) and (2.1.1) respectively and also write the Aleph $\aleph$-function in terms of Mellin –Barnes type contour integral by using (0.4.1). Now change the order of summations and Mellin-Barnes contour integral with y_j -integrals and solving the y_j -integrals, we have

$$I_1 = \int_0^{t_1} \cdots \int_0^{t_s} \Lambda \, dy_1 \ldots dy_s = \sum_{\substack{R_1,\ldots,R_s = 0 \\ \sum_{i=1}^{s} U_i R_i \le V}} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V, R_1, \ldots, R_s) \frac{E_i^{R_i}}{R_i!}$$

$$\times \sum_{\substack{R'_1,\ldots,R'_s = 0 \\ \sum_{i=1}^{s} U'_i R'_i \le V'}} (-V')_{\sum_{i=1}^{s} U'_i R'_i} A(V', R'_1, \ldots, R'_s) \frac{E_i'^{R'_i}}{R'_i!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta(S_{\nu,g}) \, z'^{-S_{\nu,g}} \frac{1}{2\pi i} \int_L \varphi(\xi) \, z^{-\xi} \, d\xi$$

$$\times \left(\prod_{j=1}^{s} x^{-\rho_j - \sigma_j - e_j R_j - f_j R_j + (\eta_j + \lambda_j)\xi} \, t_j^{-\rho'_j - \sigma'_j - (e'_j + f'_j) R'_j + (\eta'_j + \lambda'_j) S_{\nu,g}} \right)$$

$$\times \int_0^{t_1} \cdots \int_0^{t_s} \left[\prod_{j=1}^{s} y_j^{\rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi} \right.$$

$$\left. \times (x_j - y_j)^{\sigma_j + f_j R_j - \lambda_j \xi - 1} \, (t_j - y_j)^{\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g} - 1} \right] dy_1 \ldots dy_s \, . \tag{2.5.15}$$

Now we substitute $y_j = t_j u_j$ in (2.5.15) and integrate it with help of the known result Srivastava and Garg [107, p.47, Th. 1.6], we get the following equation

$$I_1 = \int_0^{t_1} \cdots \int_0^{t_s} \Lambda \, dy_1 \ldots dy_s = \sum_{\substack{R_1,\ldots,R_s = 0 \\ \sum_{i=1}^{s} U_i R_i \le V}} (-V)_{\sum_{i=1}^{s} U_i R_i} A(V, R_1, \ldots, R_s) \frac{E_i^{R_i}}{R_i!}$$

$$\times \sum_{\substack{R'_1,\ldots,R'_s=0 \\ \sum_{i=1}^{s} U'_i R'_i \le V'}} (-V')_{\sum_{i=1}^{s} U'_i R'_i} \; A(V',R'_1,\ldots,R'_s) \frac{E_i'^{R'_i}}{R'_i!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta(S_{\nu,g}) \, z'^{-S_{\nu,g}} \frac{1}{2\pi i} \int_L \varphi(\xi) z^{-\xi} d\xi$$

$$\times \left(\prod_{j=1}^{s} x_j^{-\rho_j - e_j R_j + \eta_j \xi - 1} \; t_j^{\rho_j + e_j R_j - \eta_j \xi} \right)$$

$$\times \frac{\Gamma(\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g}) \, \Gamma(\rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi + 1)}{\Gamma(\rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi + 1 + \sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g})}$$

$$\times {}_2F_1\left[\begin{matrix} -\left(\sigma_j + f_j R_j - \lambda_j \xi - 1\right); 1 + \rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi \\ 1 + \rho_j + \rho'_j + \sigma'_j + e_j R_j + \left(e'_j + f'_j\right) R'_j - \left(\eta'_j + \lambda'_j\right) S_{\nu,g} - \eta_j \xi \end{matrix}; \left(\frac{t_j}{x_j}\right)\right]. \quad (2.5.16)$$

By using the following result Rainville [84, p.60, Eq. (5)], on right hand side of (2.5.16),

$${}_2F_1[a,b;c;z] = (1-z)^{c-a-b} \, {}_2F_1[c-a, c-b; c; z]; \quad |z| < 1$$

and expressing the ${}_2F_1$ in the series form and re-arranging the result in terms of Aleph function we arrive at the solution of I_1.

The proof of $I_2 = \int_0^{x_1} \ldots \int_0^{x_s} \Lambda \, dy_1 \ldots dy_s$, can be developed by proceeding on the similar line as I_1, the only difference that we put $y_j = x_j u_j$ in the equation (2.5.15). On substituting the values of I_1 and I_2 in (2.5.12), we get the required result (2.5.5).

2.6 SPECIAL CASE OF COMPOSITION FORMULAE

We now find a two dimensional analogue of our first composition formula. By substituting $s=2$ and taking the general class of polynomials as unity, we get

$$I_{x,y;Z}^{\rho,m,\sigma,n\,;\eta,\delta,\lambda,\mu}\left\{I_{s,t;Z'}^{\rho',m',\sigma',n'\,;\eta',\delta',\lambda',\mu'}\left[f(u,v)\right]\right\}$$

$$=I_{x,y;Z}^{\rho,m,\sigma,n\,;\eta,\delta,\lambda,\mu}\left\{s^{-\rho'-\sigma'}t^{-m'-n'}\int_0^s\int_0^t u^{\rho'}v^{m'}(s-u)^{\sigma'-1}(t-v)^{n'-1}\right.$$

$$\left.\times\aleph'\left[z'\left(\frac{u}{s}\right)^{\eta'}\left(\frac{v}{t}\right)^{\delta'}\left(1-\frac{u}{s}\right)^{\lambda'}\left(1-\frac{v}{t}\right)^{\mu'}\right]f(u,v)\,du\,dv\right\}$$

$$=\sum_{\nu=1}^{M'}\sum_{g=0}^{\infty}\theta\left(S_{\nu,g}\right)z'^{-S_{\nu,g}}\,\Gamma(\sigma'-\lambda'S_{\nu,g})\,\Gamma(n'-\mu'S_{\nu,g})$$

$$\times\sum_{k'=0}^{\infty}\frac{x^{-\sigma-\rho'-\sigma'-k'+(\eta'_j+\lambda'_j)S_{\nu,g}}}{\Gamma(k'+1)}\;y^{-n-m'-n'-k'+(\delta'_j+\mu'_j)S_{\nu,g}}$$

$$\times\int_0^x\int_0^y\left[(x-u)^{\sigma+\sigma'+k'-\lambda'S_{\nu,g}-1}(y-v)^{n+n'+k'-\mu'_jS_{\nu,g}-1}u^{\rho'-\eta'S_{\nu,g}}v^{m'-\delta'S_{\nu,g}}\right]$$

$$\times\aleph_{P_i+4,Q_i+4,\tau_i,r}^{M+2,N+2}\left[z\left(1-\frac{u}{x}\right)^{\lambda}\left(1-\frac{v}{y}\right)^{\mu}\middle|\begin{matrix}E^{**}\\F^{**}\end{matrix}\right]f(u,v)\,du\,dv\,, \tag{2.6.1}$$

where

$$E^{**}=\left(a_j,A_j\right)_{1,N},\left(1-k'-\sigma;\lambda\right),\left(1-k'-n;\mu'\right),\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i}, \tag{2.6.2}$$

$$\left(\rho'-\rho+\sigma'-\left(\eta'+\lambda'\right)S_{\nu,g};\eta\right),\left(-m+m'+n'-\left(\delta'+\mu'\right)S_{\nu,g};\delta\right)$$

and

$$F^{**} = \left(b_j, B_j\right)_{1,M}, \left(\rho' - \rho + \sigma' - \left(\eta' + \lambda'\right) S_{v,g} + k'; \eta\right), \left(m' + n' - m - \left(\delta' + \mu'\right) S_{v,g} + k'; \delta\right),$$

$$\left[\tau_i\left(b_{ji}, B_{ji}\right)\right]_{M+1,Q_i}, \left(1 - \sigma - \sigma' - k' + \lambda' S_{v,g}; \lambda\right), \left(1 - n - n' - k' + \mu' S_{v,g}; \mu\right) \tag{2.6.3}$$

Also $\theta(S_{v,g})$ and $S_{v,g}$ are given by (2.1.2) and following conditions are satisfied:

$$\min_{1\le k\le M} \operatorname{Re}\left[1 + \rho + m + (\eta + \delta)\left(\frac{b_k}{B_k}\right)\right] > 0 \ , \ \min_{1\le k\le M} \operatorname{Re}\left[\sigma + \eta + (\lambda + \mu)\left(\frac{b_k}{B_k}\right)\right] > 0$$

$$\min_{1\le k\le M} \operatorname{Re}\left[1 + \rho' + m' + \left(\eta' + \delta'\right)\left(\frac{b'_k}{B'_k}\right)\right] > 0 \ , \ \min_{1\le k\le M} \operatorname{Re}\left[\sigma' + \eta' + \left(\lambda' + \mu'\right)\left(\frac{b'_k}{B'_k}\right)\right] > 0$$

In the same way two dimensional formulae can be obtained easily from result 2 and 3. These formulae can be further reduced into the results given by Raina [81] by taking $\tau_i = 1$ and I-function to unity.

If in these composition formulas we reduce both the generalized class of polynomials to unity, we reach at the multidimensional analogue of the simpler results given by Erdelyi [21], Again reducing the generalized class of polynomials to unity and the Aleph-function to the generalized hypergeometric function, we arrive at the result due to Goyal and Jain [30, p.253, eq. (2.4), p. 254, eq. (2.7); p.255, eq. (2.12)] after a little simplification. Further , if we reduce generalized class of polynomials to S_V^U polynomials we arrive at the result which are in core the same as those obtained by Goyal , Jain and Gaur [31, p.404-405,eq. (2.1);p.406,eq. (2.7); p. 407-408,eq.(2.12)].

.

.

CHAPTER 3

A STUDY OF GENERALIZED EULERIAN INTEGRAL AND UNIFIED FINITE INTEGRALS INVOLVING $S_n^{\mu,\delta,0}$, $P_\gamma^{\alpha,\beta}$ AND THE ALEPH- FUNCTION

Publications:

1. A Study of unified integrals involving the generalized polynomial set, generalized Legendre's associated function and Aleph$(\aleph)$- function With Applications, South East Asian J. of Math.& Math. Sci, Vol.13, No.1 2017, pp. 27-46.

2. The general Eulerian integral with Aleph $(\aleph)$-Function, South East Asian J. of Math.& Math. Sci, Vol.12, No.1 2016, pp. 57-66.

This chapter is divided in two parts A and B.

Part-A

We establish the main integral whose integrand involves the product of a general class of polynomials, a general sequence of function and Aleph-function with general arguments. The integral is sufficiently general in nature and a large number of known and new integrals follow as its special cases. We have obtained eight special cases of our main result, which are also new and known results.

3.1 INTRODUCTION AND DEFINITIONS

3.1.1 THE GENERAL CLASS OF POLYNOMIALS S_V^U

A general class of polynomials introduced by Srivastava [104] is represented as

$$S_V^U[x]=\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}x^R,\quad V=0,1,2,\ldots\quad,\tag{3.1.1}$$

where U is an arbitrary positive integral and the coefficients $A_{V,R}\,(V,R\geq 0)$ are arbitrary constants, real or complex. A number of well known polynomials follow as special cases of S_V^U are referred in Appendix A.

3.1.2 THE GENERALIZED SEQUENCE OF FUNCTIONS

Raijada [80, p.64, Eq. (2.18)], defined and represented the following Rodrigues type formula of generalized polynomial as

$$S_n^{\mu,\delta,\zeta}[x;w,s,q,A,B,m,\xi,l]$$

$$=(Ax+B)^{-\mu}\left(1-\zeta x^w\right)^{\frac{-\delta}{\zeta}}T_{\xi,l}^{m+n}\left[(Ax+B)^{\mu+qn}\left(1-\zeta x^w\right)^{\frac{\delta}{\zeta}+sn}\right],\tag{3.1.2}$$

with the differential operator

$$T_{k,l}=x^l\left[k+x\frac{d}{dx}\right].$$

The generalized sequence of functions in the explicit series form is given by Raijada [80, p.71, Eq. (2.3.4)]

$$S_n^{\mu,\delta,\zeta}[x;w,s,q,A,B,m,\xi,l]=B^{qn}\,x^{l(m+n)}(1-\zeta x^{l})^{sn}$$

$$\times l^{m+n}\sum_{\sigma=0}^{m+n}\sum_{r=0}^{\sigma}\sum_{j=0}^{m+n}\sum_{i=0}^{j}\frac{(-1)^{j}(-j)_i\,(\mu)_i\,(-\sigma)_i\,(-\mu-qn)_i}{j!\,\sigma!\,r!\,i!\,(1-\mu-j)_i}$$

$$\times\left(-\frac{\delta}{\zeta}-sn\right)_{\sigma}\left(\frac{i+\xi+wr}{l}\right)_{m+n}\left(\frac{-\zeta x^{w}}{1-\zeta x^{w}}\right)\left(\frac{Ax}{B}\right)^{i}. \tag{3.1.3}$$

Raijada [80] has given some special cases of (3.1.3). We shall use the following special case in our finding.

If we put $A=1, B=0$ in (3.1.3) and let $\zeta\to 0$ and using the well known results

$$Lt_{\zeta\to 0}\left(1-\zeta x^{w}\right)^{\frac{\delta}{\zeta}}=\exp\left(-\delta x^{r}\right),\ Lt_{|b|\to\infty}(b)_n\left(\frac{z}{b}\right)^{n}=z^{n},$$

then we reach at the following important polynomial set

$$S_n^{\mu,\delta,0}[x]=S_n^{\mu,\delta,0}[x;w,q,1,0,m,\xi,l]=x^{qn+l(m+n)}\,l^{m+n} \tag{3.1.4}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{r=0}^{\eta}\frac{(-\eta)_r\left(\frac{\mu+qn+\xi+rw}{l}\right)_{m+n}}{\eta!\,r!}\left(\delta x^{w}\right)^{\eta}.$$

The following results also required to prove our main integral.

The binomial expression

$$(ut+v)^{\gamma}=(au+v)^{\gamma}\sum_{l=0}^{\infty}\frac{(-\gamma)_l}{l!}\left\{-\frac{(t-a)u}{au+v}\right\}^{l}, \tag{3.1.5}$$

$$\left[|(t-a)u|<|au+v|;t\in(a,b)\right].$$

The Eulerian beta function

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}dt=(b-a)^{\alpha+\beta-1}B(\alpha,\beta), \tag{3.1.6}$$

where $[\mathrm{Re}(\alpha)>0;\mathrm{Re}(\beta)>0;a\neq b]$, using (3.1.5) and (3.1.6) we readily find that Gradshteyn and Ryzbik [33, p.287, Eq.3.198; 7, p.301, Eq.2.2.6.1]

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+\nu)^{\gamma}\,dt$$
$$= (b-a)^{\alpha+\beta-1}(au+\nu)^{\gamma} B(\alpha,\beta)\; {}_2F_1\left[\alpha,-\gamma;\alpha+\beta;-\frac{(b-a)u}{au+\nu}\right], \tag{3.1.7}$$

where

$$\left[\operatorname{Re}(\alpha)>0:\operatorname{Re}(\beta)>0;\left|\arg\left(\frac{bu+\nu}{au+\nu}\right)\right|\le \pi-\varepsilon\,(0<\varepsilon<\pi);a\ne b\right],$$

again if we put $\gamma=-\alpha-\beta$, $u=\lambda-\mu$ and $\nu=(1+\mu)b-(1+\lambda)a$ then the special case of (3.1.7) given by Prudnikov [79, p.301, Eq.2.2.6.1].

$$\int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{[b-a+\lambda(t-a)+\mu(b-t)]^{\alpha+\beta}}dt=\frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{(b-a)}B(\alpha,\beta), \tag{3.1.8}$$

$[\operatorname{Re}(\alpha)>0;\operatorname{Re}(\beta)>0;b-a+\lambda(t-a)+\mu(b-t)\ne 0,t\in[a,b];a\ne b]$.

3.2 MAIN INTEGRAL

$$\int_l^w \frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}} \aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},[\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i}\\(b_j,\beta_j)_{1,M},[\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\times S_V^U\left[y\{g(t)\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\{g(t)\}^{\vartheta}\right]dt$$

$$=(w-l)^{-1}(1+\rho)^{-\lambda-\gamma\Lambda-1}(1+\sigma)^{-\mu-\Lambda\delta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}}{R!}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}h^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!\,e!}\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta(1+\rho)}\right\}^{k}$$

$$\times \aleph^{M,N+3}_{P_i+3,Q_i+2,\tau_i;r}\left[z\left\{\frac{(1+\rho)^{\gamma}}{\beta\,(1+\sigma)^{-\delta}}\right\}^{-\nu} \middle| \begin{matrix} A^*,\left(a_j,\alpha_j\right)_{1,N},\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1.P_i} \\ \left(b_j,\beta_j\right)_{1,M},\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1.Q_i},B^* \end{matrix}\right], \tag{3.2.1}$$

where

$$\left.\begin{aligned} A^* &= (1-k-\Lambda,\nu),(-k-\lambda-\gamma\Lambda,\gamma\nu),(-\mu-\delta\Lambda,\delta\nu) \\ B^* &= (1-\Lambda,\nu),[-k-\lambda-\mu-(\gamma+\delta)\Lambda-1,\nu(\gamma+\delta)] \end{aligned}\right\} \tag{3.2.2}$$

$$R' = qn + h(m+n), \Lambda = \rho' R + \vartheta R' + d\eta,$$

$$f(t) = w - l + \rho(t-l) + \sigma(w-t), \tag{3.2.3}$$

and

$$g(t) = \frac{(t-l)^{\gamma}(w-t)^{\delta}\{f(t)\}^{1-\gamma-\delta}}{\beta(w-l)+(\beta\rho+\alpha-\beta)(t-l)+\beta\sigma(w-t)}. \tag{3.2.4}$$

The integral (3.2.1), hold true when

(a) $V > 0\,;\gamma > 0;\delta > 0\,;\beta \neq 0;w \neq 1\,;\,\rho,\sigma \neq -1$
and $\{w - l + \rho(t-l) + \sigma(w-t)\} \neq 0, t \in [l,w]$,

(b) $\operatorname{Re}\left\{1+\lambda+\gamma\nu\left(\frac{b_j}{B_j}\right)\right\} > 0$ and $\operatorname{Re}\left\{1+\mu+\nu\delta\left(\frac{b_j}{B_j}\right)\right\} > 0;\,(j=1,\ldots,M)$ Where M is an arbitrary positive integer,

(c) M, N, P, Q are positive integers constrained by $1 \leq M \leq Q, 0 \leq N \leq P$,

(d) $|\arg(z)| < \frac{\pi}{2}\phi_l\,, l = 1,\ldots,r$,

(e) $|(\beta-\alpha)(t-l)| < |\beta[w-l+\rho(t-l)+\sigma(w-t)]|, t \in [a,b]$,

(f) The coefficients $A_{V,R}$ $(V,R \geq 0)$ are arbitrary constants real or complex and U is an arbitrary positive integer and

(g) The series on the right hand side of (3.2.1) converges absolutely.

Proof: We first put the value of S_V^U and $S_n^{\alpha',\beta',0}$ by the using of equation (3.1.1) and (3.1.4) respectively and expressing the Aleph function in terms of Mellin-Barnes integral with the help of (0.4.1) for evaluating the above integral (3.2.1). Changing the order of summation and s and t integral (which is permissible under the conditions stated), the left hand side of (3.2.1) convert into the following form (say $\Re$):

$$\Re = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} A_{V,R}}{R!} (y)^R (x)^{R'} h^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{\eta} \frac{(-\eta)_e \left(\frac{\alpha' + qn + \xi + de}{h} \right)_{m+n}}{\eta! e!}$$

$$\times \left(\beta' x^d\right)^{\eta} \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) z^{-s} ds \left[\int_l^w \frac{(t-l)^{\lambda} (w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}} \{g(t)\}^{-\nu s + \rho' R + \vartheta R' + d\eta} dt \right], \tag{3.2.5}$$

where $\Omega(s), f(t)$ and $g(t)$ are given by (0.4.2), (3.2.3) and (3.2.4) respectively.

Now we put the value of $g(t)$ in (3.2.5) and after a little simplification we get the following integral

$$\Re = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} A_{V,R}}{R!} (y)^R (x)^{R'} h^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{\eta} \frac{(-\eta)_e \left(\frac{\alpha' + qn + \xi + de}{h} \right)_{m+n}}{\eta! e!}$$

$$\times \left(\beta' (x)^d \right)^{\eta} \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) \left(\frac{z}{\beta^{\nu}} \right)^{-s} ds$$

$$\times \frac{1}{\beta^{\Lambda}} \int_l^w \frac{(t-l)^{\lambda - \nu\gamma s + \Lambda\gamma} (w-t)^{\mu - \delta\nu s + \Lambda\delta}}{\{f(t)\}^{\lambda+\mu+(\gamma+\delta)(-\nu s+\Lambda)+2}} \left\{ 1 - \frac{(\beta-\alpha)(t-l)}{\beta f(t)} \right\}^{\nu s - \Lambda} dt \tag{3.2.6}$$

By using binomial expansion, the integral (3.2.6) reduces to

$$\Re = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} A_{V,R}}{R!} \left(\frac{y}{\beta^{\rho'}} \right)^R \left(\frac{x}{\beta^{\vartheta}} \right)^{R'} h^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{\eta} \frac{(-\eta)_e \left(\frac{\alpha' + qn + \xi + de}{h} \right)_{m+n}}{\eta! e!}$$

$$\times \left(\beta' \left(\frac{x}{\beta} \right)^d \right)^{\eta} \sum_{k=0}^{\infty} \left(\frac{\beta - \alpha}{\beta} \right)^k \frac{1}{k!} \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) \frac{\Gamma(-\nu s + \Lambda + k)}{\Gamma(-\nu s + \Lambda)} \left(\frac{z}{\beta^{\nu}} \right)^{-s} ds$$

$$\times\left\{\int_{l}^{w}\frac{(t-l)^{(\lambda+k-\nu\gamma s+\Lambda\gamma+1)-1}(w-t)^{(\mu-\delta\nu s+\Lambda\delta+1)-1}}{\{f(t)\}^{\lambda+\mu+k+(\gamma+\delta)(-\gamma s+\Lambda)+2}}\right\}dt\ ,$$

(3.2.7)

then the inner most integral in (3.2.7) can be evaluated by using the known integral (3.1.8) and rearranging the expression in terms of Aleph **(ℵ) -** function, after a little simplification we get the required result (3.2.1).

3.3 APPLICATIONS AND SPECIAL CASES

In this part we specifically show how the general integral formula (3.2.1) can be applied (and suitably maneuvered) to derive various interesting (and potentially useful) results including those given by Wille [128].

(i) If we put $\rho=\sigma=0$ and $z=(w-l)^{(\gamma+\delta-1)\nu}$ in (3.2.1) then we have

$$\int_{l}^{w}\frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[(w-l)^{(\gamma+\delta-1)\nu}\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},[\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i}\\(b_j,\beta_j)_{1,M},[\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\times S^{U}_{V}\left[y\{g(t)\}^{\rho'}\right]S^{\alpha',\beta',0}_{n}\left[x\{g(t)\}^{\vartheta}\right]dt$$

$$=(w-l)^{-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}}{R!}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}h^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta}\right\}^{k}$$

$$\times\aleph^{M,N+3}_{P_i+3,Q_i+2,\tau_i;r}\left[\left\{\frac{(w-l)^{\gamma+\delta-1}}{\beta}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},[\tau_i(a_{ji},\alpha_{ji})]_{N+1.P_i}\\(b_j,\beta_j)_{1,M},[\tau_i(b_{ji},\beta_{ji})]_{M+1.Q_i},B^*\end{matrix}\right.\right],$$

(3.3.1)

where A^* and B^* represented by (3.2.2) and provided that the conditions of (3.2.1) are satisfied.

Next if we substitute $\beta=\alpha=\frac{1}{\varepsilon}$ in (3.3.1), then we have

$$\int_l^w \frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}} S_V^U\left[y\{g(t)\}^{\rho'}\right] S_n^{\alpha',\beta',0}\left[x\{g(t)\}^{\vartheta}\right]$$

$$\times \aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[(w-l)^{(\gamma+\delta-1)\nu}\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i}\\(b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i}\end{matrix}\right.\right]dt$$

$$=(w-l)^{-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}}{R!}\left(y\varepsilon^{\rho'}\right)^{R}\left(x\varepsilon^{\vartheta}\right)^{R'}h^{m+n}$$

$$\times\left(\beta'(x\varepsilon)^{d}\right)^{\eta}\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}$$

$$\times \aleph_{P_i+3,Q_i+2,\tau_i;r}^{M,N+3}\left[\left\{\varepsilon(w-l)^{\gamma+\delta-1}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1.P_i}\\(b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1.Q_i},B^*\end{matrix}\right.\right], \tag{3.3.2}$$

where A^* and B^* represented by (3.2.2).

Further if we set $\gamma=\delta=1,\lambda=\mu=\frac{-1}{2},\alpha\to\alpha^2$ and $\beta\to\beta^2$ in the integral formula (3.3.1) and sum the resulting series with the help of a known formula Erdelyi [22, p.101.Eq.2.8 (6)], then after a little simplification, we obtain the following integral

$$\int_l^w (t-l)^{-1/2}(w-t)^{-1/2} S_V^U\left[y\left\{\frac{(t-l)(w-t)(w-l)^{-1}}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\rho'}\right] S_n^{\alpha',\beta',0}\left[x\left\{\frac{(t-l)(w-t)(w-l)^{-1}}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\vartheta}\right]$$

$$\times \aleph_{P_i,Q_i\tau_i,r}^{M,N}\left[(w-l)^{(\gamma+\delta-1)\nu}\left\{\frac{(t-l)(w-t)}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i}\\(b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i}\end{matrix}\right.\right]dt$$

$$=\sqrt{\pi}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left(\frac{y}{(\alpha+\beta)^{2\rho'}}\right)^{R}\left(\frac{x}{(\alpha+\beta)^{2\vartheta}}\right)^{R'}h^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}\left(\beta'\left(\frac{x}{(\alpha+\beta)^2}\right)^{d}\right)^{\eta}$$

$$\times\aleph^{M,N+1}_{P_i+1,Q_i+1,\tau_i;r}\left[\left\{\frac{(w-l)}{(\alpha+\beta)^2}\right\}^{-\nu}\left|\begin{matrix}\left(\frac{1}{2}-\Lambda,\nu\right),\left(a_j,\alpha_j\right)_{1,N},\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1.P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1.Q_i},(-\Lambda,\nu)\end{matrix}\right.\right],$$

(3.3.3)

again if we substitute $\gamma=\delta=\frac{1}{2},\mu=-\lambda-2\,\text{and}\,\nu\to2\nu$ in (3.3.1) , we obtain following interesting integral after a little simplification

$$\int_l^w(t-l)^{\lambda}(w-t)^{-\lambda-2}S_V^U\left[y\left\{\frac{(t-l)^{1/2}(w-t)^{1/2}}{\alpha(t-l)+\beta(w-t)}\right\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\left\{\frac{(t-l)^{1/2}(w-t)^{1/2}}{\alpha(t-l)+\beta(w-t)}\right\}^{\vartheta}\right]$$

$$\times\aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[\left\{\frac{[(t-l)(w-t)]^{\nu}}{[\alpha(t-l)+\beta(w-t)]^{2\nu}}\right\}\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,N},\ldots,\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i}\\ \left(b_j,\beta_j\right)_{1,M},\ldots,\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i}\end{matrix}\right.\right]dt$$

$$=\sqrt{\pi}(w-l)^{-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}2^{1-\Lambda}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}\left(\frac{\beta}{\alpha}\right)^{1+\lambda+\frac{\Lambda}{2}}$$

$$\times h^{m+n}\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+2,\tau_i;r}\left[\{4\alpha\beta\}^{-\nu}\left|\begin{matrix}\left(-\lambda-\frac{1}{2}\Lambda,\nu\right),\left(\lambda+2-\frac{1}{2}\Lambda,\nu\right),\left(a_j,\alpha_j\right)_{1,N},\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1.P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1.Q_i},\left(1-\frac{1}{2}\Lambda,\nu\right),\left\{\frac{1}{2}-\frac{1}{2}\Lambda,\nu\right\}\end{matrix}\right.\right].$$

(3.3.4)

(ii) If the general class of polynomial S_V^U reduce to Hermite polynomials (A.2) and $S_n^{\alpha',\beta',0}[x]$ reduce to unity in equation (3.2.1), then we get the following result

$$\int_l^w \frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}\aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\{g(t)\}^{\nu}\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,N},\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\times\left[y\{g(t)\}^{\rho'}\right]^{\frac{V}{2}} H_V\left[\frac{1}{2\sqrt{y\{g(t)\}^{\rho'}}}\right]dt=(w-l)^{-1}(1+\rho)^{-\lambda-\gamma\rho' R-1}$$

$$\times(1+\sigma)^{-\mu-\delta\rho' R-1}\sum_{R=0}^{[V/2]}\frac{(-V)_{2R}\ (-1)^R}{R\,!}\left(-\frac{y}{\beta^{\rho'}}\right)^R\sum_{k=0}^{\infty}\frac{1}{k\,!}\left\{\frac{(\beta-\alpha)}{\beta(1+\rho)}\right\}^k$$

$$\times \aleph^{M,N+3}_{P_i+3,Q_i+2,\tau_i;r}\left[z\left\{\frac{(1+\rho)^{\gamma}}{\beta\ (1+\sigma)^{-\delta}}\right\}^{-\nu}\left|\begin{matrix}C^*,\left(a_j,\alpha_j\right)_{1,N},\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1.P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1.Q_i},D^*\end{matrix}\right.\right],$$

(3.3.5)

where

$$\left.\begin{aligned}C^*&=\left(1-k-\rho' R,\nu\right),\left(-k-\lambda-\gamma\rho' R,\gamma\nu\right),\left(-\mu-\delta\rho' R,\delta\nu\right)\\ D^*&=\left(1-\rho' R,\nu\right),\left[-k-\lambda-\mu-(\gamma+\delta)\rho' R-1,\nu(\gamma+\delta)\right]\end{aligned}\right\}\qquad(3.3.6)$$

and above integral holds true under the same conditions as those required for (3.2.1).

(iii) If we reduce the general class of polynomials to Laguerre polynomial (A.8) and $S_n^{\alpha',\beta',0}[x]$ to Gould and Hopper polynomial (A.10) in integral (3.2.1), then we get

$$\int_{l}^{w}\frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}\aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i}\\(b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\times L_{n'}^{(k')}\left[y\{g(t)\}^{\rho'}\right]H_n^{(h)}\left[y\{g(t)\}^{\vartheta},\alpha',\beta'\right]dt$$

$$=(w-l)^{-1}(1+\rho)^{-\lambda-\gamma\Lambda-1}(1+\sigma)^{-\mu-\Lambda\delta-1}\sum_{R=0}^{[V]}\frac{(-V)_R}{R!}\binom{n'+k'}{n'}\frac{1}{(k'+1)_R}$$

$$\times\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}(-1)^{R}\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}}{\eta!e!}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta(1+\rho)}\right\}^{k}$$

$$\times\aleph_{P_i+3,Q_i+2,\tau_i;r}^{M,N+3}\left[z\left\{\frac{(1+\rho)^{\gamma}}{\beta(1+\sigma)^{-\delta}}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1.P_i}\\(b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1.Q_i},B^*\end{matrix}\right.\right],$$

(3.3.7)

where A^* and B^* represented in (3.2.2).

(iv) If we put $\tau_i=1$ in (3.2.1), then Aleph -function reduce to I- function result due to Bhatter [10, equation (4.2.1)] and we get

$$\int_{l}^{w}\frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}S_V^U\left[y\{g(t)\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\{g(t)\}^{\vartheta}\right]dt$$

$$\times I_{P_i,Q_i,r}^{M,N}\left[z\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$= (w-l)^{-1}(1+\rho)^{-\lambda-\gamma\Lambda-1}(1+\sigma)^{-\mu-\Lambda\delta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}$$

$$\times h^{m+n}\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!\;e!}\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta(1+\rho)}\right\}^{k}$$

$$I^{M,N+3}_{P_i+3,Q_i+2,r}\left[z\left\{\frac{(1+\rho)^{\gamma}}{\beta(1+\sigma)^{-\delta}}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1.P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1.Q_i},B^*\end{matrix}\right.\right],$$

(3.3.8)

where A^* and B^* represented in (3.2.2).

Again if we put $\rho=\sigma=0$ and $z=(w-l)^{(\gamma+\delta-1)\nu}$ in (3.3.8), then we get the following interesting integral

$$\int_l^w \frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}\,S_V^U\left[y\{g(t)\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\{g(t)\}^{\vartheta}\right]dt$$

$$\times I^{M,N}_{P_i,Q_i,r}\left[(w-l)^{(\gamma+\delta-1)\nu}\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$= (w-l)^{-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}h^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta}\right\}^{k}$$

$$\times I^{M,N+3}_{P_i+3,Q_i+2;r}\left[\left\{\frac{(w-l)^{\gamma+\delta-1}}{\beta}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1.P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1.Q_i},B^*\end{matrix}\right.\right],$$

(3.3.9)

where A^* and B^* represented in (3.2.2) and provided that the conditions of (3.3.1) are satisfied.

Again setting $\beta=\alpha=\frac{1}{\varepsilon}$ in (3.3.9), we obtain

$$\int_l^w \frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}} S_V^U\left[y\{g(t)\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\{g(t)\}^{\vartheta}\right]$$

$$\times I^{M,N}_{P_i,Q_i,r}\left[(w-l)^{(\gamma+\delta-1)\nu}\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]dt$$

$$=(w-l)^{-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left(y\,\varepsilon^{\rho'}\right)^{R}\left(x\,\varepsilon^{\vartheta}\right)^{R'}h^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}\left(\beta'(x\varepsilon)^d\right)^{\eta}}{\eta!e!}$$

$$\times I^{M,N+3}_{P_i+3,Q_i+2;r}\left[\left\{\varepsilon(w-l)^{\gamma+\delta-1}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1.P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1.Q_i},B^*\end{matrix}\right.\right],\quad (3.3.10)$$

where A^* and B^* represented in (3.2.2).

Next, if we put $\gamma=\delta=1,\lambda=\mu=\frac{-1}{2},\alpha\to\alpha^2$ and $\beta\to\beta^2$ in the integral formula (3.3.8), then we get

$$\int_{l}^{w}(t-1)^{-1/2}(w-t)^{-1/2}S_V^U\left[y\left\{\frac{(t-l)(w-t)(w-l)^{-1}}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\left\{\frac{(t-l)(w-t)(w-l)^{-1}}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\vartheta}\right]$$

$$\times I_{P_i,Q_i,r}^{M,N}\left[(w-l)^{\gamma+\delta-1}\left\{\frac{(t-l)(w-t)}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]dt$$

$$=\sqrt{\pi}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}}{R!}\left(\frac{y}{(\alpha+\beta)^{2\rho'}}\right)^{R}\left(\frac{x}{(\alpha+\beta)^{2\vartheta}}\right)^{R'}h^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}\left(\beta'\left(\frac{x}{(\alpha+\beta)^2}\right)^{d}\right)^{\eta}$$

$$\times I_{P_i+1,Q_i+1;r}^{M,N+1}\left[\left\{\frac{(w-1)}{(\alpha+\beta)^2}\right\}^{-\nu}\left|\begin{matrix}\left(\frac{1}{2}-\Lambda,\nu\right),(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1.P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1.Q_i},(-\Lambda,\nu)\end{matrix}\right.\right]. \quad (3.3.11)$$

If in equation (3.3.8) we set $\gamma=\delta=\frac{1}{2},\mu=-\lambda-2$ and $\nu\to 2\nu$, after a little simplification, we obtain following interesting integral

$$\int_{l}^{w}(t-l)^{\lambda}(w-t)^{-\lambda-2}S_V^U\left[y\left\{\frac{(t-l)^{1/2}(w-t)^{1/2}}{\alpha(t-l)+\beta(w-t)}\right\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\left\{\frac{(t-l)^{1/2}(w-t)^{1/2}}{\alpha(t-l)+\beta(w-t)}\right\}^{\vartheta}\right]$$

$$\times I_{P_i,Q_i,r}^{M,N}\left[z\left\{\frac{[(t-l)(w-t)]^{\nu}}{[\alpha(t-l)+\beta(w-t)]^{2\nu}}\right\}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]dt$$

$$=\sqrt{\pi}(w-l)^{-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}}{R!}2^{1-\Lambda}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}\left(\frac{\beta}{\alpha}\right)^{1+\lambda+\frac{\Lambda}{2}}$$

$$\times h^{m+n}\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+de}{h}\right)_{m+n}}{\eta!e!}\left(\beta'\left(\frac{x}{\beta}\right)^d\right)^\eta$$

$$\times I^{M,N+2}_{P_i+2,Q_i+2;r}\left[\{4\alpha\beta\}^{-\nu}\left|\begin{matrix}\left(-\lambda-\frac{1}{2}\Lambda,\nu\right),\left(\lambda+2-\frac{1}{2}\Lambda,\nu\right),\left(a_j,\alpha_j\right)_{1,N},\left(a_{ji},\alpha_{ji}\right)_{N+1.P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left(b_{ji},\beta_{ji}\right)_{M+1.Q_i},\left(1-\frac{\Lambda}{2},\nu\right),\left\{\frac{1}{2}-\frac{1}{2}\Lambda,\nu\right\}\end{matrix}\right.\right].$$

(3.3.12)

(v) If we put $\tau_i=1$ in (3.3.5) then Aleph $(\aleph)$-function reduce to I- function result due to Bhatter [10, equation (4.2.11)] and we get

$$\int_l^w\frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}I^{M,N}_{P_i,Q_i,r}\left[z\{g(t)\}^{\nu}\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,N},\left(a_{ji},\alpha_{ji}\right)_{N+1,P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left(b_{ji},\beta_{ji}\right)_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\times\left[y\{g(t)\}^{\rho'}\right]^{\frac{V}{2}}H_V\left[\frac{1}{2\sqrt{y\{g(t)\}^{\rho'}}}\right]dt=(w-l)^{-1}(1+\rho)^{-\lambda-\gamma\,\rho'R-1}$$

$$\times(1+\sigma)^{-\mu-\delta\,\rho'R-1}\sum_{R=0}^{[V/2]}\frac{(-V)_{2R}A_{V,R}}{R!}\left(-\frac{y}{\beta^{\rho'}}\right)^R\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta(1+\rho)}\right\}^k$$

$$\times I^{M,N+3}_{P_i+3,Q_i+2;r}\left[z\left\{\frac{\beta(1+\rho)^{\gamma}}{(1+\sigma)^{-\delta}}\right\}^{-\nu}\left|\begin{matrix}C^*,\left(a_j,\alpha_j\right)_{1,N},\left(a_{ji},\alpha_{ji}\right)_{N+1.P_i}\\ \left(b_j,\beta_j\right)_{1,M},\left(b_{ji},\beta_{ji}\right)_{M+1.Q_i},D^*\end{matrix}\right.\right],$$

(3.3.13)

where C^* and D^* represented in (3.3.6).

(vi) If we put $\tau_i=1$ in (3.3.7) then $\aleph$-function reduces to I- function result due to Bhatter [10, equation (4.2.12)] and we get

$$\int_{l}^{w}\frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}\, I^{M,N}_{P_i,Q_i,r}\left[z\{g(t)\}^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\times L^{(k')}_{n'}\left[y\{g(t)\}^{\rho'}\right]H^{(h)}_{n}\left[y\{g(t)\}^{\vartheta},\alpha',\beta'\right]dt$$

$$=(w-l)^{-1}(1+\rho)^{-\lambda-\gamma\Lambda-1}(1+\sigma)^{-\mu-\Lambda\delta-1}\sum_{R=0}^{[V]}\frac{(-V)_R}{R!}\binom{n'+k'}{n'}\frac{1}{(k'+1)_R}$$

$$\times\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\vartheta}}\right)^{R'}(-1)^{R}\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\beta'\left(\frac{x}{\beta}\right)^{d}\right)^{\eta}}{\eta!e!}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{(\beta-\alpha)}{\beta(1+\rho)}\right\}^{k}$$

$$\times I^{M,N+3}_{P_i+3,Q_i+2;r}\left[z\left\{\frac{\beta(1+\rho)^{\gamma}}{(1+\sigma)^{-\delta}}\right\}^{-\nu}\left|\begin{matrix}A^*,(a_j,\alpha_j)_{1,N},\ldots,(a_{ji},\alpha_{ji})_{N+1.P_i}\\(b_j,\beta_j)_{1,M},\ldots,(b_{ji},\beta_{ji})_{M+1.Q_i},B^*\end{matrix}\right.\right],$$

(3.3.14)

where A^* and B^* represented in (3.2.2).

(vii) On taking $\tau_i=1, r=1$ and reducing $S^U_V[x]$ and $S^{\alpha',\beta',0}_n[x]$ to unity in (3.2.1) the result reduces to the known results due to Srivastava H. M and Raina R. K. ([117], p.693, eq.15).

(viii) By putting $\tau_i=1, r=1$ and reducing $S^U_V[x]$ and $S^{\alpha',\beta',0}_n[x]$ to unity in (3.2.1) the result reduces to another known results obtained by Srivastava H. M and Raina R. K. ([117], p.695, eq.20 or 21).

Part B

In this part of chapter, we evaluate three finite integrals whose integrand involving the product of generalized Legendre associated function $P_{\gamma}^{\alpha,\beta}(x)$, general sequence of function $S_n^{\mu,\delta,0}$ and Aleph ($\aleph$) - function. Next we establish three theorems as an application of our main findings and using three results of Orr and Bailey recorded in well-known text by Slater [100]. Further, we evaluate certain new integrals by applications of these theorems, which are of interest by themselves and sufficiently general in nature.

3.4 INTRODUCTION AND RESULTS REQUIRED

3.4.1 THE LEGENDRE ASSOCIATED FUNCTION

B. Meulenbeld [68, p.560, equation (3)], is defined and represents Generalized Legendre Associated function $P_{\gamma}^{\alpha,\beta}(x)$ as follows

$$P_{\gamma}^{\alpha,\beta}(x)=\frac{(1+x)^{\frac{\beta}{2}}}{\Gamma(1-\alpha)(1-x)^{\frac{\alpha}{2}}}\;{}_2F_1\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1\;,-\gamma-\frac{\alpha-\beta}{2}\\1-\alpha\end{matrix};\frac{1-x}{2}\right\}, \tag{3.4.1}$$

where α is negative integer and $\beta\,,\gamma$ are unrestricted.

If we substitute $\alpha=\beta$ in (3.4.1), then $P_{\gamma}^{\alpha,\beta}(x)$ becomes the associated Legendre Function $P_{\gamma}^{\alpha}(x)$ given by Glasser [27, p. 999, Eq. (8.704)] and also if we put $\alpha=\beta=0$ in (3.4.1) then $P_{\gamma}^{\alpha,\beta}(x)$ reduces to known result Rainville [84, p.166, Eq.2].

The following three results will be needed in this chapter.

(A) Meulenbeld and Robin [69,p.343,Eq. (38)],given

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_{\gamma}^{\alpha,\beta}(x)\,dx$$

$$= \frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}\,\Gamma\left(1+\rho-\frac{\alpha}{2}\right)\Gamma\left(1+\sigma+\frac{\beta}{2}\right)}{\Gamma(1-\alpha)\,\Gamma\left(2+\rho+\sigma+\frac{\beta-\alpha}{2}\right)}\;{}_3F_2\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2},1+\rho-\frac{\alpha}{2}\\ 1-\alpha\;;2+\rho+\sigma+\frac{\beta-\alpha}{2}\end{matrix};1\right\}, \tag{3.4.2}$$

where α is a negative integer and

$$\mathrm{Re}\left(1+\rho-\frac{\alpha}{2}\right)>0,\ \mathrm{Re}\left(1+\sigma+\frac{\beta}{2}\right)>0,$$

(B) Sneddon [101, p.61, Eq.2.16 (ii)], given

$$\int_0^1 x^{l-1}(1-x)^{m-1}\;{}_pF_q\left\{\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};\frac{1-x}{2}\right\}dx$$

$$= B(l,m)\;{}_{p+1}F_{q+1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,m\\ \beta_1,\ldots,\beta_q,l+m\end{matrix};\frac{1}{2}\right], \tag{3.4.3}$$

(C) Sneddon [101, p.61, Eq 2.16 (iii)],given

$$\int_0^1 \left(1-x^2\right)^{m-1}\;{}_pF_q\left\{\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};\frac{1-x}{2}\right\}dx$$

$$= B\left(\frac{1}{2},m\right)\;{}_{p+1}F_{q+1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,m\\ \beta_1,\ldots,\beta_q,2m\end{matrix};\frac{1}{2}\right]. \tag{3.4.4}$$

3.5 MAIN RESULTS

3.5.1 FIRST INTEGRAL

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\,P_\gamma^{\alpha,\beta}(x)\,S_n^{\mu,\delta,0}\left\{y\,x^u(1-x)^v\right\}$$

$$\times\aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left\{z\,x^h(1-x)^k\left|\begin{matrix}(a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i}\\ (b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i}\end{matrix}\right.\right\}dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^{\eta} \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \middle| \begin{matrix} A^{**},\ (a_j,\alpha_j)_{1,N},\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i},B^{**} \end{matrix} \right\}, \tag{3.5.1}$$

where

$$\left.\begin{aligned} A^{**} &= \left(1-\rho-uR'-uw\eta,h\right),\left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t,k\right) \\ B^{**} &= \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)(R'+w\eta)-t,(h+k)\right\} \end{aligned}\right\} \tag{3.5.2}$$

and

$R' = qn + l\,(m+n)$.

The integral is valid under following conditions

1. $\alpha > 0; u \geq 0, v \geq 0, h \geq 0, k \geq 0$ (not both zero simultaneously),

2. $\mathrm{Re}(\rho) + h \min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{B_j}\right)\right] + 1 > 0$ and

$\mathrm{Re}(\sigma - \frac{\alpha}{2}) + k \min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{B_j}\right)\right] + 1 > 0$,

3. Equation (0.4.3).

Proof:-

To prove the above integral, first of all we write the generalized polynomial set $S_n^{\mu,\delta,0}[x]$ in its series form with the help of (3.1.4), Generalized Legendre function with the help of (3.4.1) and expressing Aleph ($\aleph$) function in terms of Mellin- Barnes type integral with the help of equation (0.4.1) in the left hand side of (3.5.1) and change the order of integration and summation (which is permissible under the condition stated), then we get

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\frac{(1+x)^{\beta/2}}{(1-x)^{\alpha/2}\Gamma(1-\alpha)}\;{}_2F_1\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1\,,-\gamma-\frac{\alpha-\beta}{2}\\1-\alpha\end{matrix};\frac{1-x}{2}\right\}$$

$$\times\left(y\,x^{u}(1-x)^{v}\right)^{qn+l(m+n)}\,l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\;\eta!}$$

$$\times\left[\delta\left\{y\,x^{u}(1-x)^{v}\right\}^{w}\right]^{\eta}\frac{1}{2\pi i}\int_L\Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)\left\{z\,x^{h}(1-x)^{k}\right\}^{-s}ds\,dx$$

$$=y^{R'+w\eta}\,l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\;\eta!}\,\delta^{\eta}\,\frac{1}{\Gamma(1-\alpha)}$$

$$\times\frac{1}{2\pi i}\int_L\Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)z^{-s}ds\int_0^1 x^{\rho+uR'+uw\eta-hs-1}$$

$$\times(1-x)^{\sigma-\frac{\alpha}{2}+vR'+vw\eta-ks-1}\;{}_2F_1\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1\,,-\gamma-\frac{\alpha-\beta}{2}\\1-\alpha\end{matrix};\frac{1-x}{2}\right\}dx\,,$$

then the above integral can be evaluated with the help of the known result (3.4.3), we get

$$=y^{R'+w\eta}l^{m+n}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\;\eta!}\,\delta^{\eta}\,\frac{1}{\Gamma(1-\alpha)}$$

$$\times\frac{1}{2\pi i}\int_L\Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)z^{-s}ds\;\;B\left\{\rho+uR'+uw\eta-hs\,,\sigma-\frac{\alpha}{2}+vR'+vw\eta-ks\right\}$$

$$\times{}_3F_2\left\{\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1\,,-\gamma-\frac{\alpha-\beta}{2}\,,\sigma-\frac{\alpha}{2}+vR'+vw\eta-ks\\1-\alpha\;\;;\rho+\sigma-\frac{\alpha}{2}+(u+v)\,(R'+w\eta)-(h+k)s\end{matrix};\frac{1}{2}\right\},$$

after a little Simplification and rearranging the result , we get the required result in terms of Aleph-function.

3.5.2 SECOND INTEGRAL

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_{\gamma}^{\alpha,\beta}(x)\; S_{n}^{\mu,\delta,0}\left\{y(1-x)^{u}(1+x)^{v}\right\}$$

$$\times \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left\{ z(1-x)^{h}(1+x)^{k} \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, \left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, \left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n}\, 2^{(R'+w\eta)(u+v)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_{\gamma}\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!}\delta^{\eta}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{t!\,\Gamma(1-\alpha+t)}\, 2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}$$

$$\times \aleph_{P_i+2,Q_i+1,\tau_i;r}^{M,N+2}\left\{ z\,2^{h+k} \left| \begin{matrix} C^{**}, (a_j,\alpha_j)_{1,N}, \left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, \left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i}, D^{**} \end{matrix} \right. \right\}, \tag{3.5.3}$$

where

$$\left. \begin{aligned} C^{**} &= \left(\frac{\alpha}{2}-\rho-uR'-uw\eta-t,h\right), \left(-\sigma-\frac{\beta}{2}-vR'-vw\eta,k\right) \\ D^{**} &= \left\{-\rho-\sigma-t-\frac{(\beta-\alpha)}{2}-(u+v)(R'+w\eta)-1,(h+k)\right\} \end{aligned} \right\} \tag{3.5.4}$$

and $R' = qn + l(m+n)$.

The integral is valid under following conditions

1. $\alpha > 0; u \geq 0, v \geq 0, h \geq 0, k \geq 0$ (not both zero simultaneously),

2. $\text{Re}(1+\rho-\frac{\alpha}{2}) + h \min_{1\le j\le M}\left[\text{Re}\left(\frac{b_j}{B_j}\right)\right] > 0$, $\text{Re}(1+\sigma+\frac{\beta}{2}) + k \min_{1\le j\le M}\left[\text{Re}\left(\frac{b_j}{B_j}\right)\right] > 0$,

3. Equation (0.4.3).

Proof: To prove the above integral, first of all we write the generalized polynomial set $S_n^{\mu,\delta,0}[x]$ in its series form with the help of (3.1.4), Generalized Legendre function with the help of (3.4.1) and expressing Aleph ($\aleph$) function in terms of Mellin- Barnes type integral with the help of equation (0.4.1).Now changing the order of summation and integration (which is permissible under the condition stated) and evaluate the integral, by using (3.4.2). We get the right hand side of integral (3.5.3) after a little simplification.

3.5.3 THIRD INTEGRAL

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^2)^{\rho-1} P_\gamma^{\alpha,\beta}(x)\; S_n^{\mu,\delta,0}\left\{y(1-x^2)^k\right\}$$

$$\times \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left\{ z(1-x^2)^h \left| \begin{matrix} \left(a_j,\alpha_j\right)_{1,N}, \left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i} \\ \left(b_j,\beta_j\right)_{1,M}, \left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta} l^{m+n} \sqrt{(\pi)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{2t}\; t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph_{P_i+2,Q_i+2,\tau_i;r}^{M,N+2}\left\{ z \left| \begin{matrix} E^{**}, \left(a_j,\alpha_j\right)_{1,N}, \left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i} \\ \left(b_j,\beta_j\right)_{1,M}, \left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i}, F^{**} \end{matrix} \right. \right\}, \tag{3.5.5}$$

where

$$\left.\begin{aligned} E^{**} &= \left(1-\rho-kR'-kw\eta, h\right), \left(1-\rho-kR'-kw\eta-t, h\right) \\ F^{**} &= \left(1-\rho-kR'-kw\eta-\frac{t}{2}, h\right), \left(1-\rho-kR'-kw\eta-\frac{t}{2}-\frac{1}{2}, h\right) \end{aligned}\right\}. \tag{3.5.6}$$

and $R' = qn + l(m+n)$.

The integral is valid under the following conditions

1. $\alpha > 0; u \geq 0, v \geq 0, h \geq 0, k \geq 0$ (not both zero simultaneously),

2. $\mathrm{Re}(\rho) + h \min_{1 \leq j \leq M} \left[\mathrm{Re}\left(\frac{b_j}{B_j} \right) \right] + 1 > 0$ and

3. Equation (0.4.3).

Proof:

First of all we write the Generalized Legendre function in terms of ${}_2F_1$ with help of (3.4.1). Generalized Polynomial set $S_n^{\mu,\delta,0}[x]$ in its series form using (3.1.4) and Aleph- function in terms of Mellin Barnes contour integral with help of (0.4.1) in the left hand side of (3.5.5). By changing the order of summation and integration (which is permissible under the condition stated) we get the following form of integral (Δ)

$$\Delta = y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu + qn + \xi + \gamma w}{l} \right)_{m+n}}{\gamma!\, \eta!} \delta^\eta \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) z^{-s} ds$$

$$\times \frac{1}{\Gamma(1-\alpha)} \left\{ \int_0^1 \left(1 - x^2\right)^{\rho + kR' + kw\eta - hs - 1} {}_2F_1 \left\{ \begin{matrix} \gamma - \frac{\alpha - \beta}{2} + 1, -\gamma - \frac{\alpha - \beta}{2} \\ 1 - \alpha \end{matrix} ; \frac{1-x}{2} \right\} dx \right\}. \quad (3.5.7)$$

Applying well known result (3.4.4) and expressing the function ${}_3F_2$ in terms of series and interchanging the order of summation and integration on inner most integral (3.5.7) after a little simplification. we get following form

$$\Delta = y^{R'+w\eta} l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu + qn + \xi + \gamma w}{l} \right)_{m+n}}{\gamma!\, \eta!} \delta^\eta$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha - \beta}{2} + 1 \right)_t \left(-\gamma - \frac{\alpha - \beta}{2} \right)_t}{t!\, \Gamma(1 - \alpha + t)} \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) z^{-s} ds$$

$$\times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\rho+kR'+kw\eta-hs+t\right)\left[\Gamma\left\{2\left(\rho+kR'+kw\eta-hs\right)\right\}\right]}{\Gamma\left(\rho+kR'+kw\eta-hs+\frac{1}{2}\right)\left[\Gamma\left\{2\left(\rho+kR'+kw\eta-hs+\frac{t}{2}\right)\right\}\right]},$$

now using the well known duplication formula $\Gamma\left(\frac{1}{2}\right)\Gamma(2n)=2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$. and rearranging the resulting expression in terms of Aleph function, we reached at the desired result (3.5.5).

3.6 THEOREMS

THEOREM 1

$$\text{If } (1-x)^{a+b-c}{}_2F_1[2a,2b;2c;x]=\sum_{n'=0}^{\infty} a_{n'}\, x^{n'} \tag{3.6.1}$$

Then

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\left\{yx^u(1-x)^v\right\}$$

$$\times {}_2F_1\left[a,b;c+\frac{1}{2};x\right] {}_2F_1\left[c-a,c-b;c+\frac{1}{2};x\right]$$

$$\times \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left\{zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N},[\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},[\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix}\right.\right\}dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta \sum_{n'=0}^{\infty} \frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)_{n'} n'!} a_{n'}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph_{P_i+2,Q_i+1,\tau_i;r}^{M,N+2}\left\{ z \left| \begin{matrix} G^{**}, \left(a_j,\alpha_j\right)_{1,N}, \left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i} \\ \left(b_j,\beta_j\right)_{1,M}, \left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i}, H^{**} \end{matrix} \right. \right\}, \tag{3.6.2}$$

where

$$\left.\begin{aligned} G^{**} &= \left(1-\rho-uR'-uw\eta-n', h\right), \left(1-n'-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t, k\right) \\ H^{**} &= \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)\,(R'+w\eta)-n'-t, (h+k)\right\} \end{aligned}\right\} \tag{3.6.3}$$

The conditions of validity given in first integral (3.5.1) are satisfied.

Proof-

To prove the above theorem, we use the following result given by Slater [100, p.75, equation (2.5.2)]

$${}_2F_1\left[a,b;c+\frac{1}{2};z\right] {}_2F_1\left[c-a,c-b;c+\frac{1}{2};z\right] = \sum_{n'=0}^{\infty} \frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)_{n'} n'!} a_{n'} z^{n'}, \tag{3.6.4}$$

we let an operation (say Λ)

$$\Lambda = x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_{\gamma}^{\alpha,\beta}(x) S_n^{\mu,\delta,0}\left\{yx^u(1-x)^v\right\}$$

$$\times \aleph_{P_i,Q_i,\tau_i;r}^{M,N}\left\{ zx^h(1-x)^k \left| \begin{matrix} \left(a_j,\alpha_j\right)_{1,N}, \left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{N+1,P_i} \\ \left(b_j,\beta_j\right)_{1,M}, \left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{M+1,Q_i} \end{matrix} \right. \right\},$$

Now we multiply both side of equation (3.6.4) by Λ and integration with respect to x from $x=0$ to 1. Now interchanging the order of integration and summation in the right hand side we get the following result

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_{\gamma}^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\left\{yx^u(1-x)^v\right\}$$

$$\times {}_2F_1\left[a,b;c+\frac{1}{2};x\right] {}_2F_1\left[c-a,c-b;c+\frac{1}{2};x\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= \sum_{n'=0}^{\infty} \frac{(c)_{n'}}{\left(c+\frac{1}{2}\right)_{n'} n'!} a^{n'} \int_0^1 x^{\rho+n'-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_{\gamma}^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\}, \qquad (3.6.5)$$

finally, we easily reach at the required result by evaluating the integral on the right hand side with help of same procedure given in the integral first.

THEOREM 2

$$\text{If } {}_2F_1[a,b;c;x]\ {}_2F_1[a,b;d;x] = \sum_{n'=0}^{\infty} c_{n'} x^{n'} \qquad (3.6.6)$$

Then

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_{\gamma}^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times {}_4F_3\left[a,b,\frac{c}{2}+\frac{d}{2},\frac{c}{2}+\frac{b}{2}-\frac{1}{2};a+b,c,d;4x(1-x)\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu + qn + \xi + \gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(c+d-1)_{n'}}{(a+b)_{n'}\, n'!} (c)_{n'} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2} + 1\right)_t \left(-\gamma - \frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\, \Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2, Q_i+1, \tau_i; r}\left\{ z \middle| \begin{matrix} I^{**},\ (a_j,\alpha_j)_{1,N}, \left[\tau_i (a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, \left[\tau_i (b_{ji},\beta_{ji})\right]_{M+1,Q_i}, J^{**} \end{matrix} \right\}, \quad (3.6.7)$$

where

$$\left.\begin{aligned} I^{**} &= \left(1-\rho-uR'-uw\eta-n', h\right), \left(1-\sigma+\frac{\alpha}{2}-vR'-vw\eta-t, k\right) \\ J^{**} &= \left\{1-\rho-\sigma+\frac{\alpha}{2}-(u+v)(R'+w\eta)-n'-t, (h+k)\right\} \end{aligned}\right\}. \quad (3.6.8)$$

The conditions of validity given in Integral first are satisfied.

Proof- To prove the above theorem, we use the following results given by Slater [100, p.79,equation (2.5.27)]

$$_4F_3\left[a, b; \frac{c}{2}+\frac{d}{2}, \frac{c}{2}+\frac{b}{2}-\frac{1}{2}; a+b, c, d; 4x(1-x)\right] = \sum_{n'=0}^{\infty} \frac{(c+d-1)_{n'}}{(a+b)_{n'}\, n'!} (c)_{n'}\, x^{n'}, \quad (3.6.9)$$

where $c_{n'}$ is given by (3.6.6) . We reach at the desired result (3.6.7) by processing the same lines as given in the Theorem 1.

THEOREM 3

If $_2F_1[a, b; c; x]\ _2F_1[a, b; d; x] = \sum_{n'=0}^{\infty} c_{n'} x^{n'}$ (3.6.10)

Then

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\left\{ yx^u (1-x)^v \right\}$$

$$\times {}_4F_3\left[a, b, d, c-a; \frac{b}{2}+\frac{d}{2}, \frac{b}{2}+\frac{d}{2}+\frac{1}{2}, c; \frac{-x^2}{4(1-x)}\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, \left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, \left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(c)_{n'}(d)_{n'}}{(d+b)_{n'}\, n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma - \frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \left| \begin{matrix} I^{**},\ (a_j,\alpha_j)_{1,N}, \left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, \left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i}, J^{**} \end{matrix} \right. \right\}, \qquad (3.6.11)$$

where I^{**} and J^{**} represented in (3.6.8) and the conditions stated in (3.5.1) are true.

Proof – To prove the above theorem, we use the following results given by Slater [100,p 79,equation (2.5.29)]

$$(1-z)^{-a}\ {}_4F_3\left[a,b,d,c-a;\frac{b}{2}+\frac{d}{2},\frac{b}{2}+\frac{d}{2}+\frac{1}{2},c;\frac{-z^2}{4(1-z)} \right] = \sum_{n'=0}^{\infty} \frac{(c)_{n'}(d)_{n'}}{(d+b)_{n'}} z^{n'}$$

(3.6.12)

We easily arrive at the right hand side of (3.6.11) by proceeding on same lines to those of theorem 1.

3.7 APPLICATIONS AND SPECIAL CASES

(I) If we put c=a in our Theorem 1, the value of $a_{n'}$ in (3.6.4) come out equal to $(b)_{n'}$ and we get the following interesting integral after a little simplification

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\left\{ yx^u(1-x)^v \right\} {}_2F_1\left[a,b;a+\frac{1}{2};x \right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i (a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i (b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta \sum_{n'=0}^{\infty} \frac{(a)_{n'}(b)_{n'}}{\left(a+\frac{1}{2}\right)_{n'}} \frac{1}{n'!}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \left| \begin{matrix} G^{**}, (a_j,\alpha_j)_{1,N}, [\tau_i (a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i (b_{ji},\beta_{ji})]_{M+1,Q_i}, H^{**} \end{matrix} \right. \right\}, \qquad (3.7.1)$$

where G^{**} and H^{**} represented in (3.6.3).

Next on substituting $b = \frac{a}{2}+1$ and $a = -e$ (a non negative integral) in (3.7.1), then we get the following integral

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_\eta^{\mu,\delta,0}\{y x^u (1-x)^v\}$$

$$\times {}_1F_0[-e;-;x]\, \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i (a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i (b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta \sum_{n'=0}^{\infty} \frac{(-e)_{n'}}{n'!}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \left| \begin{matrix} G^{**}, (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i}, H^{**} \end{matrix} \right. \right\}, \tag{3.7.2}$$

where G^{**} and H^{**} represented in (3.6.3).

II. We easily achieve the following result by putting b=c=d in theorem 2

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}\, {}_2F_1\left[a, c-\frac{1}{2}; a+c; 4x(1-x)\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ z\, x^h (1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\, \eta!} \delta^\eta$$

$$\times \sum_{n'=0}^{\infty} \frac{(2c-1)_{n'}}{(a+c)_{n'}} \frac{(2a)_{n'}}{n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\, \Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \left| \begin{matrix} I^{**}, (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i}, J^{**} \end{matrix} \right. \right\}, \tag{3.7.3}$$

where I^{**} and J^{**} represented in (3.6.8).

Again, if we substitute $a = -e$ in (3.7.3), it reduces to the following interesting integral

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}\, {}_2F_1\left[-e, c-\frac{1}{2}; a-e; 4x(1-x)\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ zx^h (1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, [\tau_i(a_{ji},\alpha_{ji})]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, [\tau_i(b_{ji},\beta_{ji})]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^{\eta}$$

$$\times \sum_{n'=0}^{\infty} \frac{(2c-1)_{n'}}{(a-e)_{n'}} \frac{(-2e)_{n'}}{n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \left| \begin{matrix} I^{**},\ (a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i},J^{**} \end{matrix} \right. \right\}, \quad (3.7.4)$$

where I^{**} and J^{**} represented in (3.6.8).

III. We reach at the following integral, if we substitute b=c=d in Theorem 3

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\{y x^u (1-x)^v\}\ {}_2F_1\left[a,c-a;c+\frac{1}{2};\frac{-x^2}{4(1-x)}\right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{ z x^h (1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^{\eta}$$

$$\times \sum_{n'=0}^{\infty} \frac{(c)_{n'}(2a)_{n'}}{(2c)_{n'}\, n'!} \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left\{ z \left| \begin{matrix} I^{**},\ (a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i},J^{**} \end{matrix} \right. \right\}, \quad (3.7.5)$$

where I^{**} and J^{**} represented in (3.6.8).

IV If we put $\tau_i = 1$ in equations (3.5.1) , (3.5.3) and (3.5.5) respectively then Aleph function reduce to I-function and get new special cases

(i) From equation (3.5.1), we get

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x) S_n^{\mu,\delta,0}\left\{y x^u (1-x)^v\right\}$$

$$\times I_{P_i,Q_i;r}^{M,N}\left\{zx^h(1-x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, (a_{ji},\alpha_{ji})_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, (b_{ji},\beta_{ji})_{M+1,Q_i} \end{matrix} \right.\right\} dx = y^{R'+w\eta}\, l^{m+n}$$

$$\times \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\times I_{P_i+2,Q_i+1\,;r}^{M,N+2}\left\{z \left| \begin{matrix} A^{**}, (a_j,\alpha_j)_{1,N}, (a_{ji},\alpha_{ji})_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, (b_{ji},\beta_{ji})_{M+1,Q_i}, B^{**} \end{matrix} \right.\right\}, \qquad (3.7.6)$$

where A^{**} and B^{**} represented in (3.5.2).

(ii) From equation (3.5.3), we get

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_\gamma^{\alpha,\beta}(x)\, S_n^{\mu,\delta,0}\left\{y(1-x)^u(1+x)^v\right\}$$

$$\times I_{P_i,Q_i;r}^{M,N}\left\{z(1-x)^h(1+x)^k \left| \begin{matrix} (a_j,\alpha_j)_{1,N}, (a_{ji},\alpha_{ji})_{N+1,P_i} \\ (b_j,\beta_j)_{1,M}, (b_{ji},\beta_{ji})_{M+1,Q_i} \end{matrix} \right.\right\} dx$$

$$= y^{R'+w\eta}\, l^{m+n}\, 2^{(R'+w\eta)(u+v)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\, t!\,\Gamma(1-\alpha+t)}\, 2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}$$

$$\times I^{M,N+2}_{P_i+2,Q_i+1\,;r}\left\{z\,2^{h+k}\left|\begin{matrix}C^{**},\ (a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i},D^{**}\end{matrix}\right.\right\}, \qquad (3.7.7)$$

where C^{**} and D^{**} represented in (3.5.4).

(iii) From equation (3.5.5), we get

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^2)^{\rho-1}P_\gamma^{\alpha,\beta}(x)\ S_n^{\mu,\delta,0}\left\{y(1-x^2)^k\right\}$$

$$\times I^{M,N}_{P_i,Q_i;r}\left\{z(1-x^2)^h\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right\}dx$$

$$=y^{R'+w\eta}\,l^{m+n}\,\sqrt{(\pi)}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma\,!\ \eta\,!}\delta^\eta$$

$$\times\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{2t}\ t\,!\,\Gamma(1-\alpha+t)}I^{M,N+2}_{P_i+2,Q_i+2\,;r}\left\{z\left|\begin{matrix}E^{**},(a_j,\alpha_j)_{1,N},(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M},(b_{ji},\beta_{ji})_{M+1,Q_i},F^{**}\end{matrix}\right.\right\},$$

(3.7.8)

where E^{**} and F^{**} represented in (3.5.6).

V If we put $\tau_i=1$ and $r=1$ in equations (3.5.1) , (3.5.3) and (3.5.5) respectively then Aleph function reduce to H-function and get new special cases

(i) From equation (3.5.1), we get

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)S_n^{\mu,\delta,0}\{yx^u(1-x)^v\}$$

$$\times H^{M,N}_{P,Q}\left\{zx^h(1-x)^k\left|\begin{matrix}(a_1,\alpha_1),(a_P,\alpha_P)\\(b_1,\beta_1),(b_Q,\beta_Q)\end{matrix}\right.\right\}dx=y^{R'+w\eta}\,l^{m+n}$$

$$\times\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!}\delta^\eta\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^t\;t!\,\Gamma(1-\alpha+t)}$$

$$\times H^{M,N+2}_{P+2,Q+1}\left\{z\left|\begin{matrix}A^{**}\;,\;(a_1,\alpha_1),(a_P,\alpha_P)\\(b_1,\beta_1),(b_Q,\beta_Q),B^{**}\end{matrix}\right.\right\}, \tag{3.7.9}$$

where A^{**} and B^{**} represented in (3.5.2).

(ii) From equation (3.5.3), we get

$$\int_{-1}^{1}(1-x)^\rho(1+x)^\sigma P_\gamma^{\alpha,\beta}(x)\,S_n^{\mu,\delta,0}\left\{y(1-x)^u(1+x)^v\right\}$$

$$\times H^{M,N}_{P,Q}\left\{z(1-x)^h(1+x)^k\left|\begin{matrix}(a_1,\alpha_1),(a_P,\alpha_P)\\(b_1,\beta_1),(b_Q,\beta_Q)\end{matrix}\right.\right\}dx$$

$$=y^{R'+w\eta}\,l^{m+n}\,2^{(R'+w\eta)(u+v)}\sum_{\eta=0}^{m+n}\sum_{\gamma=0}^{\eta}\frac{(-\eta)_\gamma\left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!}\delta^\eta$$

$$\times\sum_{t=0}^{\infty}\frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t\left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{t!\,\Gamma(1-\alpha+t)}2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}$$

$$\times H^{M,N+2}_{P+2,Q+1}\left\{z\,2^{h+k}\left|\begin{matrix}C^{**},\;(a_1,\alpha_1),(a_P,\alpha_P)\\(b_1,\beta_1),(b_Q,\beta_Q),\,D^{**}\end{matrix}\right.\right\}, \tag{3.7.10}$$

where C^{**} and D^{**} represented in (3.5.4).

(iii) From equation (3.5.5), we get

$$\int_0^1(1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^2)^{\rho-1}P_\gamma^{\alpha,\beta}(x)\,S_n^{\mu,\delta,0}\left\{y(1-x^2)^k\right\}$$

$$\times H^{M,N}_{P,Q}\left\{z\,(1-x^2)^h\left|\begin{matrix}(a_1,\alpha_1),(a_P,\alpha_P)\\(b_1,\beta_1),(b_Q,\beta_Q)\end{matrix}\right.\right\}dx$$

$$= y^{R'+w\eta}\, l^{m+n} \sqrt{(\pi)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma \left(\frac{\mu+qn+\xi+\gamma w}{l}\right)_{m+n}}{\gamma!\,\eta!} \delta^\eta$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{2t}\, t!\, \Gamma(1-\alpha+t)} H^{M,N+2}_{P+2,Q+2}\left\{z\middle| \begin{matrix} E^{**},(a_1,\alpha_1),(a_P,\alpha_P) \\ (b_1,\beta_1),(b_Q,\beta_Q),F^{**} \end{matrix}\right\},$$

(3.7.11)

where E^{**} and F^{**} represented in (3.5.6).

VI If we reduce general sequence of Polynomials $S_n^{\mu,\delta,0}$ to unity and Aleph ($\aleph$) function to Fox H function, we get known result given by Anandani [4, 9.343, Eq. (2.2)].

VII If we substitute $q=\xi=m=0$, $l=-1$ and generalized polynomial set $S_n^{\mu,\delta,0}$ reduce to the Gould and Hopper (A.42) in third integral (3.5.3) then we get the following integral

$$\int_0^1 (1-x)^{\alpha/2}(1+x)^{-\beta/2}(1-x^2)^{\rho-1} P_\gamma^{\alpha,\beta}(x) H_n^{(r)}\left\{y(1-x^2)^k,\mu,\delta\right\}$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left\{z(1-x^2)^h \middle| \begin{matrix} (a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i} \end{matrix}\right\} dx$$

$$= y^{-n+w\eta}(-1)^n \sqrt{(\pi)} \sum_{\eta=0}^{m+n} \sum_{\gamma=0}^{\eta} \frac{(-\eta)_\gamma (-\gamma w)_n}{\gamma!\,\eta!} \delta^\eta \sum_{t=0}^{\infty} \frac{\left(\gamma-\frac{\alpha-\beta}{2}+1\right)_t \left(-\gamma-\frac{\alpha-\beta}{2}\right)_t}{2^{2t}\, t!\, \Gamma(1-\alpha+t)}$$

$$\times \aleph^{M,N+2}_{P_i+2,Q_i+2,\tau_i;r}\left\{z\middle| \begin{matrix} (a_j,\alpha_j)_{1,N},\left[\tau_i(a_{ji},\alpha_{ji})\right]_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},\left[\tau_i(b_{ji},\beta_{ji})\right]_{M+1,Q_i} \end{matrix}\right\},$$

(3.7.12)

where

$$\left.\begin{aligned} K^{**} &= (1-\rho+kn-kw\,\eta\,,h\,)\,,(1-\rho+kn-kw\,\eta-t\,,h\,) \\ L^{**} &= \left(1-\rho+kn-kw\,\eta-\frac{t}{2},h\,\right),\left(1-\rho+kn-kw\,\eta-\frac{t}{2}-\frac{1}{2},h\,\right) \end{aligned}\right\}.$$

CHAPTER- 4

FRACTIONAL DIFFERENTIAL CALCULUS OF THE ALEPH-FUNCTION

Publication:

A study of generalized fractional derivative operator involving aleph$(\aleph)$-function, general class of multivariable polynomials and certain multiplication formulae for aleph $(\aleph)$-function of multivariable functions, *International Journal of Modern Mathematical Sciences*, Vol. **15 (3),** 2017, pp. 291-310.

In the present chapter at first we give the definition of the fractional operators and the results which are needed to initiate our three theorems. First two theorems whose integrand involves the product of Aleph-function and multivariable's polynomial $S_V^{U_1,\ldots,U_k}$ using Riemann-Liouville fractional operator and last theorem involving the product of multivariable Aleph-function and multivariable's polynomial $S_V^{U_1,\ldots,U_k}$ using Generalized Saigo derivative operators. Next, we give six corollaries involving useful special functions specially first class of multivariable hypergeometric polynomial, multivariable Jacobi polynomial, multivariable Bessel polynomial. Finally, theorems 1 and 2 are then employed to establish two multiplication formulae for multivariable Aleph-function, from these multiplication formulas we can obtain a known and unknown multiplication formulae as their particular cases. Our Theorems are generalization of the results given by Dube [17], Goyal and Saxena [32], Manocha and Sharma [58, 59], Raina and Koul [82], Srivastava and Goyal [108] and Bhatter [9].

4.1 DEFINITIONS

4.1.1 FRACTIONAL DIFFERENTIAL OPERATOR

In this chapter following two types of fractional operators have been used and further extensively.

(i) Riemann-Liouville Fractional Operator

Oldham and Spanier [75] were defined the Riemann-Liouville Fractional operator for function $f(z)$ of a complex order μ can be represented as follows

$$ {}_{\alpha}D_x^{\mu} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_{\alpha}^{x} (x-t)^{-\mu-1} f(t)\,dt, & Re(\mu) < 0, \\[2ex] \dfrac{d^m}{dx^m}\, {}_{\alpha}D_x^{\mu-m} f(x), & 0 \le Re(\mu) < m, \end{cases} \tag{4.1.1} $$

where m is non - negative integer.

Mishra [70] further generalized the (4.1.1) operator as follow

$$ D_{k,\alpha,x} = x^{k+\alpha} D_x^{\alpha}, \ \alpha \ne \mu . $$

(ii) Generalized Saigo Fractional Derivative Operator

The generalized modified fractional derivative operator due to Samko, Kilbas and Marichev [88] is defined as given below, if we let $0 \le \alpha < 1, \beta, \eta, x \in R, m \in N$

$$D_{0,x,m}^{\alpha,\beta,\eta} f(x) =$$

$$\frac{d}{dx}\left(\frac{x^{m(\beta-\eta)}}{\Gamma(1-\alpha)} \int_0^x (x^m - t^m)^{-\alpha} \; {}_2F_1\left[\begin{matrix}\beta-\alpha;1-\eta \\ 1-\alpha\end{matrix} ;1-\frac{t^m}{x^m}\right] f(t)\, dt^m \right). \tag{4.1.2}$$

If we take m=1in (4.1.2) then the above operator reduces to Saigo derivative operator $D_{0,x}^{\alpha,\beta,\eta}$.

If we substitute m=1 and $\alpha = \beta$, in the equation (4.1.2), it reduces to the well known Riemann-Liouville fractional derivative operator given by Miller and Ross [71], we get

$$D_{0,x}^{\alpha,\beta,\eta} f(x) = D_x^{\alpha} f(x).$$

RESULTS REQUIRED

We will need the following relations in making our results.

$${}_{\alpha}D_x^{\mu}(x^{\mu-1}) = \frac{d^{\alpha}x^{\mu-1}}{dx^{\alpha}} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)} x^{\mu-\alpha-1}, \alpha \neq \mu, \tag{4.1.3}$$

$$D_{k,\alpha,x}^{m}(x^{\mu}) = \prod_{p=0}^{m-1} \frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk+1-\alpha)} x^{\mu+km}, \alpha \neq \mu+1, \tag{4.1.4}$$

where k and α are not necessarily integers.

Binomial expression

$$(t-1)^n = \sum_{h=0}^{n} \frac{(-1)^n (-n)^h}{h!} t^h \tag{4.1.5}$$

To establish our Theorem 3, we also need following result given by Bhatt and Raina [6].

If $0 \le \alpha < 1, m \in N; \beta, \eta, x \in R; k > \max\{0, m(\beta-\eta)\} - m$, then

$$D_{0,x,m}^{\alpha,\beta,\eta}(x^k) = \frac{\Gamma\left(1+\frac{k}{m}\right)\Gamma\left(1+\eta-\beta+\frac{k}{m}\right)}{\Gamma\left(1-\beta+\frac{k}{m}\right)\Gamma\left(1+\eta-\alpha+\frac{k}{m}\right)} x^{k-m\beta} \tag{4.1.6}$$

4.2 MAIN THEOREMS

Theorem 1. If convergence conditions of equation (0.4.3) are hold, then

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} S_V^{U_1,\dots,U_k}\left\{w_1 t^{\rho_1},\dots,w_k t^{\rho_k}\right\} \aleph^{M,N}_{P_i,Q_i;\tau_i;r}\left[zt \left| \begin{matrix} (a_j,A_j)_{1,N}, \left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i} \\ (b_j,B_j)_{1,M}, \left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right] f(xt) \right\}$$

$$= \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V,R_1,\dots,R_k) \frac{w_i^{R_i}}{R_i!} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\Delta+pl)}{\Gamma(\mu+\Delta+pl)}$$

$$\times t^{\lambda+\Delta+ml-1} \left\{ \frac{1}{2\pi i} \int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)\, z^{-s} ds \right\} D_x^n \{f(x)\}$$

$$\times {}_{m+1}F_m \left[\begin{matrix} -n, \lambda+\Delta, \cdots, \lambda+\Delta+(m-1)l \\ \mu+\Delta, \cdots, \mu+\Delta+(m-1)l \end{matrix} ; t \right], \qquad (4.2.1)$$

where

$$\Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{M} \Gamma\left(b_j+B_j s\right) \prod_{j=1}^{N} \Gamma\left(1-a_j-A_j s\right)}{\sum_{i=1}^{r} \tau_i \prod_{j=M+1}^{Q_i} \Gamma\left(1-b_{ji}-B_{ji} s\right) \prod_{j=N+1}^{P_i} \Gamma\left(a_{ji}+A_{ji} s\right)},$$

$$R(\mu+\Delta+p\,l) > 0, R(\lambda+\Delta+p\,l) > 0, |t| < 1,$$

here $\Delta = \sum_{i=1}^{k} \rho_i R_i - s$.

Theorem 2. If convergence conditions of equation (0.4.3) are hold, then

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda} S_V^{U_1,\dots,U_k}\left\{w_1 t^{\rho_1},\dots,w_k t^{\rho_k}\right\} \aleph^{M,N}_{P_i,Q_i;\tau_i;r}\left[zt \left| \begin{matrix} (a_j,A_j)_{1,N}, \left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i} \\ (b_j,B_j)_{1,M}, \left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i} \end{matrix} \right. \right] f(xt) \right\}$$

$$= \sum_{\substack{R_1,\ldots,R_k=0}}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V,R_1,\ldots,R_k) \frac{w_i^{R_i}}{R_i!}$$

$$\times \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\Delta+p\,l)(1-\mu-\Delta-p\,l)_n}{\Gamma(\mu+\Delta+p\,l)(1-\lambda-\Delta-p\,l)_n}$$

$$\times t^{\lambda+\Delta+ml-1} \left\{ \frac{1}{2\pi i} \int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)\, z^{-s} ds \right\} D_x^n \{f(x)\}$$

$$\times {}_{m+1}F_m \left[\begin{matrix} -n, \lambda+\Delta-n, \cdots, \lambda+\Delta+(m-1)l-n \\ \\ \mu+\Delta-n, \cdots, \mu+\Delta+(m-1)l-n \end{matrix} ; t \right], \qquad (4.2.2)$$

where

$R(\mu+\Delta+p\,l-n) > 0, R(\lambda+\Delta+p\,l-n) > 0$, $|t| < 1$,

here $\Delta = \sum_{i=1}^{k} \rho_i R_i - s$ and assuming that the series include in (4.2.1) and (4.2.2) are absolutely convergent.

Proof of Theorem 1

At first we use the well-known Taylor's expansion

$$f(x\,t) = \sum_{n=0}^{\infty} \frac{(t-1)^n}{n!} x^n D_x^n \{f(x)\}, \qquad (4.2.3)$$

where value of $(t-1)^n$ given by equation (4.1.5).

Now multiplying both side of the equation (4.2.3) by $t^{\lambda-1} S_V^{U_1,\ldots,U_k}\left\{w_1 t^{\rho_1},\ldots,w_k t^{\rho_k}\right\} \aleph(zt)$ and use the operator $D^m_{l,\lambda-\mu,t}$ both sides then we get

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} S_V^{U_1,\dots,U_k}\left\{w_1 t^{\rho_1},\dots,w_k t^{\rho_k}\right\} \aleph^{M,N}_{P_i,Q_i;\tau_i;r}\left[zt \left| \begin{matrix} (a_j,A_j)_{1,N}, \left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i} \\ (b_j,B_j)_{1,M}, \left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i} \end{matrix} \right.\right] f(xt)\right\}$$

$$= D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} S_V^{U_1,\dots,U_k}\left\{w_1 t^{\rho_1},\dots, w_k t^{\rho_k}\right\} \frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i;\tau_i;r}(s)(zt)^{-s} ds \sum_{n=0}^{\infty} \frac{(t-1)^n}{n!} x^n D_x^n\{f(x)\}\right\}.$$

(4.2.4)

Then we substitute value of $\aleph(z)$ and $S_V^{U_1,\dots,U_k}$ with the help of (0.4.1) and (0.6.2) respectively. Next, we expand $(t-1)^n$ by using (4.1.5) and changing the order of summation and operator, we get

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} S_V^{U_1,\dots,U_k}\left\{w_1 t^{\rho_1},\dots,w_k t^{\rho_k}\right\} \aleph^{M,N}_{P_i,Q_i;\tau_i;r}\left[zt \left| \begin{matrix} (a_j,A_j)_{1,N}, \left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i} \\ (b_j,B_j)_{1,M}, \left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i} \end{matrix} \right.\right] f(xt)\right\}$$

$$= \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i}\, A(V,R_1,\dots,R_k)\, \frac{w_i^{R_i}}{R_i!} \sum_{n=0}^{\infty}\sum_{h=0}^{n} \frac{(-1)^n(-n)_h\, x^n}{h!\, n!}$$

$$\times D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda+h+\rho_1 R_1+\dots+\rho_k R_k-s-1}\right\} D_x^n\{f(x)\}. \qquad (4.2.5)$$

In right hand side of equation (4.2.5), we use known result (4.1.4), we obtain the following form

$$= \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i}\, A(V,R_1,\dots,R_k)\, \frac{w_i^{R_i}}{R_i!} \sum_{n=0}^{\infty}\sum_{h=0}^{n} \frac{(-1)^n(-n)_h\, x^n}{h!\, n!}$$

$$\times \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\rho_1 R_1+\dots+\rho_k R_k+h+pl-s)}{\Gamma(\mu+\rho_1 R_1+\dots+\rho_k R_k+pl-s)}$$

$$\times\, t^{\lambda+h+\rho_1 R_1+\ldots+\rho_k R_k+ml-s-1}\, D_x^n\{f(x)\}. \tag{4.2.6}$$

With the help of equation (0.1.3), we can also explain above result in terms of generalized hypergeometric function ${}_PF_Q$ and then we get the required result for right hand side of (4.2.1).

To prove Theorem 2 we use following result given by Dube [17] and process same as the theorem 1.

$$t\,f(x\,t) = \sum_{n=0}^{\infty} \frac{\left(1-\frac{1}{t}\right)^n}{n!} D_x^n\left\{x^n f(x)\right\} \tag{4.2.7}$$

Theorem-3 Let $0 \le \alpha < 1, \beta, \eta, x \in \Re, \theta \in N, \mathrm{Re}(\alpha) > 0, w, \delta > 0, \rho_j, \sigma_j > 0\ (j = 1,\ldots,k)$, $\lambda_j, \mu_j > 0, z_j \in C, (j = 1,\ldots,p)$ and if the existence conditions of Aleph ($\aleph$) function are satisfied, then the generalized fractional derivative $D_{0,x,\theta}^{\alpha,\beta,\eta}$ of the product of Multivariable Aleph ($\aleph$) function and $S_V^{U_1,\ldots,U_k}$ exists, then we have

$$D_{0,x,\theta}^{\alpha,\beta,\eta}\Big[t^w\left(t^\nu+\xi^\nu\right)^{-\delta} S_V^{U_1,\ldots,U_k}\left\{Y_1 t^{\rho_1}\left(t^\nu+\xi^\nu\right)^{-\sigma_1},\ldots,Y_k t^{\rho_k}\left(t^\nu+\xi^\nu\right)^{-\sigma_k}\right\}$$

$$\times\aleph\left(z_1 t^{\lambda_1}\left(t^\nu+\xi^\nu\right)^{-\mu_1},\ldots,z_p t^{\lambda_p}\left(t^\nu+\xi^\nu\right)^{-\mu_p}\right)\Big](x)$$

$$=(x)^{w-\theta\beta+\sum_{i=1}^k \rho_i R_i}\,(\xi)^{-\nu\delta-\nu\sum_{i=1}^k \sigma_i R_i} \sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^k U_i R_i \le V}} (-V)_{\sum_{i=1}^k U_i R_i}\, A(V,R_1,\ldots,R_k)\, \frac{Y_i^{R_i}}{R_i!}$$

$$\times \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-x^\nu}{\xi^\nu}\right)^m \aleph_{P_i+3,Q_i+3,\tau_i,R;U_{11}}^{0,N+3;m_1,n_1;\ldots;m_p,n_p}\left[\begin{array}{c|c} z_1\xi^{-\mu_1\nu}x^{\lambda_1} & A^* \\ \vdots & B^* \\ z_p\xi^{-\mu_p\nu}x^{\lambda_p} & \end{array}\right], \tag{4.2.8}$$

where

$$U_{11}=P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)},$$

$$A^*=\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n},\left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i;\mu_1,\ldots,\mu_p\right),\left(-w-m\nu-\frac{\sum_{i=1}^{k}\rho_i R_i}{\theta};\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left(\beta-\eta-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta};\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{n+1,P_i};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1},$$

$$\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r},\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

and

$$B^*=\left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i;\mu_1,\ldots,\mu_p\right),\left(\beta-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta};\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left(\alpha-\eta-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta};\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right)$$

$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right]_{m+1,Q_i};\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,Q_i^{(1)}}$$

$$;\ldots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,Q_i^{(r)}}.$$

Proof: With the help of using equations (0.5.1) and (0.6.2), we have

$$D_{0,x,\theta}^{\alpha,\beta,\eta}\left[t^w\left(t^\nu+\xi^\nu\right)^{-\delta}S_V^{U_1,\ldots,U_k}\left\{Y_1 t^{\rho_1}\left(t^\nu+\xi^\nu\right)^{-\sigma_1},\ldots,Y_k t^{\rho_k}\left(t^\nu+\xi^\nu\right)^{-\sigma_k}\right\}\right.$$

$$\left.\times\aleph\left(z_1 t^{\lambda_1}\left(t^\nu+\xi^\nu\right)^{-\mu_1},\ldots,z_p t^{\lambda_p}\left(t^\nu+\xi^\nu\right)^{-\mu_p}\right)\right](x)$$

$$=D_{0,x,\theta}^{\alpha,\beta,\eta}\left[t^w\left(t^\nu+\xi^\nu\right)^{-\delta}\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k}U_iR_i\le V}}(-V)_{\sum_{i=1}^{k}U_iR_i}A(V,R_1,\ldots,R_k)\frac{Y_i^{R_i}}{R_i!}(t)^{\sum_{i=1}^{k}\rho_iR_i+\sum_{j=1}^{p}\lambda_js_j}\right.$$

$$\times\left(t^{\nu}+\xi^{\nu}\right)^{-\sum_{i=1}^{k}\sigma_i R_i-\sum_{j=1}^{p}\mu_j s_j}\frac{1}{(2\pi i)^p}\int_{L_1}\cdots\int_{L_p}\psi(s_1,\ldots,s_p)\,\varphi_1(s_1)\cdots\varphi_p(s_p)$$

$$\left.\times z_1^{s_1}\ldots z_p^{s_p}\;ds_1\ldots ds_p\right](x)\,. \tag{4.2.9}$$

Interchanging the order of summation and integration (which is satisfied under the conditions, given above) and using the binomial expansion formula is given below

$$(t+\xi)^{-l}=\xi^{-l}\sum_{m=0}^{\infty}\frac{(l)_m}{m\,!}\left(-\frac{t}{\xi}\right)^{m}\;;\;\left|\frac{t}{\xi}\right|<1\,.$$

Then right hand side of (4.2.9) becomes

$$=\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k}U_iR_i\le V}}\frac{(-V)_{\sum_{i=1}^{k}U_iR_i}}{}\;A(V,R_1,\ldots,R_k)\frac{Y_i^{R_i}}{R_i!}$$

$$\times\int_{L_1}\cdots\int_{L_p}\psi(s_1,\ldots,s_p)\,\varphi_1(s_1)\cdots\varphi_p(s_p)\;z_1^{s_1}\ldots z_p^{s_p}$$

$$\times(t)^{\sum_{j=1}^{p}\lambda_j s_j}\;(t^{\nu}+\xi^{\nu})^{-\left(\delta+\sum_{i=1}^{k}\sigma_iR_i+\sum_{j=1}^{p}\mu_js_j\right)}$$

$$\times(\xi)^{-\nu\left(\delta+\sum_{i=1}^{k}\sigma_iR_i+\sum_{j=1}^{p}\mu_js_j\right)}\sum_{m=0}^{\infty}\frac{\left(\delta+\sum_{i=1}^{k}\sigma_iR_i+\sum_{j=1}^{p}\mu_js_j\right)_m}{m!}\left(-\frac{1}{\xi^{\nu}}\right)^{m}$$

$$\times D_{0,x,\theta}^{\alpha,\beta,\eta}\left\{(t)^{w+\sum_{i=1}^{k}\rho_iR_i+\sum_{j=1}^{p}\lambda_js_j+\nu m}\right\}(x)$$

By using the result (4.1.6), we easily get the required result (4.2.8), after a little simplification.

4.3 SPECIAL CASES OF THEOREMS

(i) In equation (4.2.1), if we reduce the general class of polynomial to the first class of multivariable hypergeomertic polynomials $F_D^{(k)}$ [(A.23)], then we obtain the following corollary 1.

Corollary 1

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} F_D^{(k)} \left\{(-V,U_i);(\beta_i,\phi_i);(\gamma,\psi_i);w_1 t^{\rho_1},\ldots,w_k t^{\rho_k}\right\} \aleph(zt) f(xt)\right\}$$

$$= \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} \frac{w_i^{R_i}}{R_i!} \frac{(\beta_1)_{R_1\phi_1}\cdots(\beta_k)_{R_k\phi_k}}{(\gamma)_{R_1\psi_1+\ldots+R_k\psi_k}}$$

$$\times \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\Delta+pl)}{\Gamma(\mu+\Delta+pl)} t^{\lambda+\Delta+ml-1}$$

$$\times \left\{ \frac{1}{2\pi i} \int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s) z^{-s} ds \right\} {}_{m+1}F_m\left\{ \begin{matrix} -n, C^* \\ D^* \end{matrix} ; t \right\} D_x^n \{f(x)\}, \quad (4.3.1)$$

where

$$\left.\begin{aligned} C^* &= \lambda+\Delta, \lambda+\Delta+l, \ldots, \lambda+\Delta+(m-1)l \\ D^* &= \mu+\Delta, \mu+\Delta+l, \ldots, \mu+\Delta+(m-1)l \end{aligned}\right\}, \quad (4.3.2)$$

here

$$\Delta = \sum_{i=1}^{k} \rho_i R_i - s \ .$$

The conditions of validity of (4.3.1) can be easily got from the existence conditions indicate with theorem 1.

(ii) In equation (4.2.1), if we reduce the general class of polynomial to Multivariable Jacobi Polynomial $P_V^{\alpha_1,\beta_1;\ldots;\alpha_k,\beta_k}$ [(A.27)], then we obtain the following corollary 2.

Corollary 2

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1}\, P_{V}^{\alpha_1,\beta_1;\ldots;\alpha_k,\beta_k}\left(1-2w_1 t^{\rho_1},\ldots,1-2w_k t^{\rho_k}\right) \aleph(z\,t)\, f(x\,t)\right\}$$

$$=\frac{\prod_{i=1}^{k}(1+\alpha_i)_V}{(V!)^k}\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k} R_i\le V}}(-V)_{\sum_{i=1}^{k}R_i}\frac{\prod_{i=1}^{k}(1+\alpha_i+\beta_i+V)_{R_i}(w_i)^{R_i}}{\prod_{i=1}^{k}\left\{(1+\alpha_i)_{R_i}R_i!\right\}}$$

$$\times\prod_{p=0}^{m-1}\frac{\Gamma(\lambda+\Delta+pl)}{\Gamma(\mu+\Delta+pl)}\,t^{\lambda+\Delta+ml-1}$$

$$\times\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\left\{\frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)z^{-s}ds\right\}{}_{m+1}F_m\left\{\begin{matrix}-n,C^*\\ D^*\end{matrix};t\right\}D^n_x\{f(x)\},\tag{4.3.3}$$

where C^* and D^* are given by equation (4.3.2).

The conditions of validity of (4.3.2) can be easily got from the existence conditions given with theorem 1.

(iii) If in equation (4.2.1), the generalized class of polynomials is reduced to multivariable Bessel polynomial $y^{\alpha_1,\ldots,\alpha_k}_{V,n_2,\ldots,n_k}$ [(A.29)], then we obtain the following corollary 3.

Corollary 3

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda}\, y^{\alpha_1,\ldots,\alpha_k}_{V,n_2,\ldots,n_k}\left(2w_1 t^{\rho_1},\ldots,2w_k t^{\rho_k}\right) \aleph(zt) f(xt)\right\}$$

$$=\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k} R_i\le V}}(-V)_{\sum_{i=1}^{k}R_i}\frac{w_1^{R_1}(1+\alpha_1+V)_{R_1}}{R_1!}\prod_{i=2}^{k}(1+\alpha_i+n_i)_{R_i}\frac{(w_i)^{R_i}}{R_i!}$$

$$\times \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\Delta+p\,l)\,(1-\mu-\Delta-p\,l)_n}{\Gamma(\mu+\Delta+p\,l)\,(1-\lambda-\Delta-p\,l)_n}\, t^{\lambda+\Delta+m\,l-1} \left\{ \frac{1}{2\pi i} \int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s) z^{-s} ds \right\}$$

$$\times \;{}_{m+1}F_m \left\{ \begin{matrix} -n\,,\, \lambda+\Delta-n\,,\; \lambda+\Delta-n+l,\ldots,\; \lambda+\Delta-n+(m-1)\,l \\ \mu+\Delta-n\,,\; \mu+\Delta-n+l,\ldots,\; \mu+\Delta-n+(m-1)\,l \end{matrix} ;t \right\} D_x^n \{f(x)\}. \qquad (4.3.4)$$

The conditions of validity of (4.3.4) can be easily obtained from the existence conditions of the theorem 2.

(iv) In equation (4.2.8), if we substitute $\beta=\alpha$ and $\eta=0$, then we get the following corollary 4.

Corollary 4

$$D^{\alpha,0}_{0,x,\theta}\left[t^{w}\left(t^{\nu}+\xi^{\nu}\right)^{-\delta} S_V^{U_1,\ldots,U_k}\left\{ Y_1 t^{\rho_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_1},\ldots, Y_k t^{\rho_k}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_k} \right\}\right.$$

$$\left.\times \aleph\left(z_1 t^{\lambda_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_1},\ldots, z_p t^{\lambda_p}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_p} \right)\right](x)$$

$$= x^{w-\theta\alpha+\sum_{i=1}^{k}\rho_i R_i}\, \xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_i R_i} \sum_{\substack{R_1,\ldots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i}\, A(V,R_1,\ldots,R_k) \frac{Y_i^{R_i}}{R_i!}$$

$$\times \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-x}{\xi}\right)^m \aleph^{0,N+2;m_1,n_1;\ldots;m_p,n_p}_{P_i+2,Q_i+2,\tau_i,R;U_{11}} \left[\begin{matrix} z_1\, \xi^{-\mu_1 \nu}\, x^{\lambda_1} \\ \vdots \\ z_p\, \xi^{-\mu_p \nu}\, x^{\lambda_p} \end{matrix} \middle| \begin{matrix} E^* \\ F^* \end{matrix} \right], \qquad (4.3.5)$$

where

$$U_{11} = P_{i^{(1)}}, Q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)};\ldots; P_{i^{(r)}}, Q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)},$$

$$E^* = \left(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)}\right)_{1,n}, \left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,...,\mu_p\right), \left(\frac{-w-m\nu-\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},...,\frac{\lambda_p}{\theta}\right),$$

$$,\left[\tau_i\left(a_{ji},\alpha_{ji}^{(1)},...,\alpha_{ji}^{(r)}\right)\right]_{n+1,P_i};\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i(1)}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}}$$

$$;...;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i(r)}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

and

$$F^* = \left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,...,\mu_p\right), \left(\alpha-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},...,\frac{\lambda_p}{\theta}\right), \left[\tau_i\left(b_{ji},\beta_{ji}^{(1)},...,\beta_{ji}^{(r)}\right)\right]_{m+1,Q_i}$$

$$;\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i(1)}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,Q_i^{(1)}}$$

$$;...;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i(r)}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,Q_i^{(r)}}.$$

The conditions of validity of (4.3.5) can be easily obtained from the existence conditions given with theorem 3.

(v) If we substitute $\theta = 1$ in equation (4.2.8), we obtain the following corollary 5.

Corollary 5

$$D_{0,x,1}^{\alpha,\beta,\eta}\Bigg[t^{w}\left(t^{\nu}+\xi^{\nu}\right)^{-\delta} S_V^{U_1,...,U_k}\left\{Y_1 t^{\rho_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_1},...,Y_k t^{\rho_k}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_k}\right\}$$

$$\times\aleph\left(z_1 t^{\lambda_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_1},...,z_p t^{\lambda_p}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_p}\right)\Bigg](x)$$

$$= x^{w-\beta+\sum_{i=1}^{k}\rho_i R_i}\ \xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_i R_i}\sum_{\substack{R_1,...,R_k=0\\ \sum_{i=1}^{k}U_i R_i\le V}}(-V)_{\sum_{i=1}^{k}U_i R_i}\ A(V,R_1,...,R_k)\frac{Y_i^{R_i}}{R_i!}$$

$$\times \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-x}{\xi}\right)^{m} \aleph^{0,N+3;m_1,n_1;\ldots;m_p,n_p}_{P_i+3,Q_i+3,\tau_i,R;U_{11}} \left[\begin{matrix} z_1 \xi^{-\mu_1 \nu} x^{\lambda_1} \\ \vdots \\ z_p \xi^{-\mu_p \nu} x^{\lambda_p} \end{matrix} \middle| \begin{matrix} G^* \\ H^* \end{matrix}\right], \tag{4.3.6}$$

where

$$U_{11} = P_{i^{(1)}}, Q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \ldots; P_{i^{(r)}}, Q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)},$$

$$G^* = \left(a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,n}, \left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i, \mu_1, \ldots, \mu_p\right), \left(-w-m\nu-\sum_{i=1}^{k}\rho_i R_i, \lambda_1, \ldots, \lambda_p\right),$$

$$\left(\beta-\eta-\left(w+m\nu+\sum_{i=1}^{k}\rho_i R_i\right), \lambda_1, \ldots, \lambda_p\right), \left[\tau_i\left(a_{ji}, \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(1)}\right)\right]_{n+1,P_i}; \left[\left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,n_1}\right],$$

$$\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}}; \ldots; \left[\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r}\right], \left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

and

$$H^* = \left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i, \mu_1, \ldots, \mu_p\right), \left(\beta-\left(w+m\nu+\sum_{i=1}^{k}\rho_i R_i\right), \lambda_1, \ldots, \lambda_p\right), \left(\alpha-\eta-\left(w+m\nu+\sum_{i=1}^{k}\rho_i R_i\right), \lambda_1, \ldots, \lambda_p\right),$$

$$\left[\tau_i\left(b_{ji}, \beta_{ji}^{(1)}, \ldots, \beta_{ji}^{(r)}\right)\right]_{m+1,Q_i}; \left[\left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,m_1}\right], \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,Q_i^{(1)}}$$

$$; \ldots; \left[\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}\right], \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,Q_i^{(r)}}.$$

The conditions of validity of (4.3.6) can be easily obtained from the existence conditions given with theorem 3.

Corollary 6

On putting $\beta = \alpha$ and $\eta = 0$ in equation (4.3.6), the equation (4.3.6), can be also reduced to following result for the Riemann-Liouville Fractional derivative operator:

$$D_{0,x,1}^{\alpha,}\left[t^{w}\left(t^{\nu}+\xi^{\nu}\right)^{-\delta} S_V^{U_1,\ldots,U_k}\left\{Y_1 t^{\rho_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_1}, \ldots, Y_k t^{\rho_k}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_k}\right\}\right.$$

$$\left.\times \aleph\left(z_1 t^{\lambda_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_1}, \ldots, z_p t^{\lambda_p}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_p}\right)\right](x)$$

$$= x^{w-\alpha+\nu m+\sum_{i=1}^{k}\rho_i R_i}\,\xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_i R_i}\sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k}U_iR_i\le V}(-V)_{\sum_{i=1}^{k}U_iR_i}\,A(V,R_1,\ldots,R_k)\,\frac{Y_i^{R_i}}{R_i!}$$

$$\times\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-x}{\xi}\right)^m \aleph^{0,N+2;m_1,n_1;\ldots;m_p,n_p}_{P_i+2,Q_i+2,\tau_i,R;U_{11}}\left[\begin{array}{c|c} z_1\xi^{-\mu_1\nu}x^{\lambda_1} & I^* \\ \vdots & \\ z_p\xi^{-\mu_p\nu}x^{\lambda_p} & J^* \end{array}\right], \qquad (4.3.7)$$

where

$$U_{11}=P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;P_{i^{(r)}},Q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)},$$

$$I^*=\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n},\left(1-\delta-m-\sum_{i=1}^{k}\sigma_iR_i,\mu_1,\ldots,\mu_p\right),\left(-w-m\nu-\sum_{i=1}^{k}\rho_iR_i,\lambda_1,\ldots,\lambda_p\right),$$

$$\left[\tau_i\left(a_{ji},\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{n+1,P_i};\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}}$$

$$;\ldots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

and

$$J^*=\left(1-\delta-\sum_{i=1}^{k}\sigma_iR_i,\mu_1,\ldots,\mu_p\right),\left(\alpha-\left(w+m\nu+\sum_{i=1}^{k}\rho_iR_i\right),\lambda_1,\ldots,\lambda_p\right),$$

$$\left[\tau_i\left(b_{ji},\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(1)}\right)\right]_{m+1,Q_i};\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i(1)}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,Q_i^{(1)}}$$

$$;\ldots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i(1)}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,Q_i^{(r)}}.$$

(vi) In equation (4.2.8), if we put $R=1,\nu=1$ and $\tau_i=1$ the Aleph ($\aleph$) function reduce to H function then we find the known result due to Bhatter [8].

$$D^{\alpha,\beta,\eta}_{0,x,\theta}\left[t^w(t+\xi)^{-\delta}S_V^{U_1,\ldots,U_k}\left\{Y_1t^{\rho_1}(t+\xi)^{-\sigma_1},\ldots,Y_kt^{\rho_k}(t+\xi)^{-\sigma_k}\right\}\right.$$

$$\times H\left(z_1 t^{\lambda_1}(t+\xi)^{-\mu_1},\ldots,z_p t^{\lambda_p}(t+\xi)^{-\mu_p}\right)\Bigg](x)$$

$$= x^{w-\theta\beta+\sum_{i=1}^{k}\rho_i R_i}\ \xi^{-\delta-\sum_{i=1}^{k}\sigma_i R_i} \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i}\ A(V,R_1,\ldots,R_k)\frac{Y_i^{R_i}}{R_i!}$$

$$\times \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-x}{\xi}\right)^m H^{0,N+3;m_1,n_1;\ldots;m_p,n_p}_{P+3,Q+3,P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{matrix} z_1\xi^{-\mu_1}x^{\lambda_1}\\ \vdots \\ z_p\xi^{-\mu_p}x^{\lambda_p}\end{matrix}\left|\begin{matrix}K^*\\ L^*\end{matrix}\right.\right], \qquad (4.3.8)$$

where

$$K^* = \left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\ldots,\mu_p\right),\left(\frac{-w-m-\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left(\beta-\eta-\frac{w+m+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,P};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r},$$

and

$$L^* = \left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\ldots,\mu_p\right),\left(\beta-\frac{w+m+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left(\alpha-\eta-\frac{w+m+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r}.$$

(vii) If we put k=1 and $\aleph(z)=1$ in equation (4.2.1) and (4.2.2), then we find the known results due to Goyal and Saxena [108].

(viii) If we reduce the generalized class of polynomial to unity and $\aleph(z)=1$ in equation (4.2.1) and (4.2.2), then we arrive at known results due to Dube [17].

4.4.1 APPLICATIONS OF THEOREM 1 AND 2

Now we create two multiplication formulae as application of Theorem 1 and 2 respectively.

Let $f(x) = x^{\sigma-1} \aleph\left[xy_1, ..., xy_n\right]$ (4.4.1)

Now using (4.1.3) on $f(x)$, then we get

$$D_x^n \left\{ x^{\sigma-1} \aleph \begin{bmatrix} x\,y_1 \\ \vdots \\ x\,y_n \end{bmatrix} \right\} = x^{\sigma-n-1} \aleph^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P_i+1,Q_i+1\,;\tau_i,R;U_{11}} \left[\begin{matrix} x\,y_1 \\ \vdots \\ x\,y_n \end{matrix} \middle| \begin{matrix} (1+\sigma;1,1),** \\ \\ (1-\sigma+n;1,1),** \end{matrix} \right],$$

(4.4.2)

with the help of equations (0.6.2), (0.5.1) and (4.1.4), we have

$$D^m_{l,\lambda-\mu,t} \left\{ t^{\lambda+\sigma-2}\, x^{\sigma-1}\, S_V^{U_1,\ldots,U_k} \left\{ w_1 t^{\rho_1}, \ldots, w_k t^{\rho_k} \right\} \aleph(xt, xy_1 t, \ldots, xy_{n-1} t) \right\}$$

$$= t^{\lambda+\sigma+\rho_1 R_1+\ldots+\rho_k R_k+ml-2} \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i}\, A(V, R_1, \ldots, R_k)$$

$$\times \frac{w_i^{R_i}}{R_i!}\, x^{\sigma-1}\, \aleph^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P_i+m,Q_i+m;\tau_i,R,U_{11}} \left[\begin{matrix} x\,y_1 t \\ \vdots \\ x\,y_n t \end{matrix} \middle| \begin{matrix} M^* \\ N^* \end{matrix} \right], \qquad (4.4.3)$$

where

$$\left.\begin{aligned} M^* &= (2-\lambda-\sigma-\rho_1 R_1-\ldots-\rho_k R_k, 1,\ldots,1), (2-\lambda-\sigma-\rho_1 R_1-\ldots-\rho_k R_k - l, 1,\ldots,1), \ldots, \\ &\quad (2-\lambda-\sigma-\rho_1 R_1-\ldots-\rho_k R_k-(m-1)\,l, 1,\ldots,1), ** \\ N^* &= (2-\mu-\sigma-\rho_1 R_1-\ldots-\rho_k R_k, 1,\ldots,1), (2-\mu-\sigma-\rho_1 R_1-\ldots-\rho_k R_k - l, 1,\ldots,1), \ldots, \\ &\quad (2-\mu-\sigma-\rho_1 R_1-\ldots-\rho_k R_k-(m-1)\,l, 1,\ldots,1), ** \end{aligned}\right\},$$

(4.4.4)

the asterisk $**$ in (4.4.2) and (4.4.4) specifies that the parameters at these places are the same as the parameters of the multivariable's Aleph-function (0.5.1).

Putting the values of differential operators (4.4.2) and (4.4.3) in theorem 1 and 2, Assuming $y_1 \to \frac{y_1}{x}, \ldots, y_n \to \frac{y_n}{x}$ and comparing the coefficients of $w_i^{R_i}$ both sides, next we replace $\lambda \to \lambda - \rho_1 R_1 - \ldots - \rho_k R_k$, $\mu \to \mu - \rho_1 R_1 - \ldots - \rho_k R_k$ and $\sigma \to 1-\sigma$, we get the following Multiplication formula for multivariable's Aleph-function.

Multiplication Formula 1

$$\aleph^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P_i+m,Q_i+m;\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 t & O^* \\ \vdots & \\ y_n t & P^* \end{array}\right]$$

$$= t^{\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda + p\,l)}{\Gamma(\mu + p\,l)} \; {}_{m+1}F_m \left\{ \begin{array}{l} -n, \lambda, \lambda+l, \ldots, \lambda+(m-1)l \\ \mu, \mu+l, \ldots, \mu+(m-1)l \end{array} ; t \right\}$$

$$\times \aleph^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P_i+1,Q_i+1,\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 & (\sigma,1,\ldots,1),** \\ \vdots & \\ y_n & (\sigma+n,1,\ldots,1),** \end{array}\right], \tag{4.4.5}$$

where

$$\left.\begin{array}{l} O^* = (1-\lambda+\sigma,1,\ldots,1),(1-\lambda+\sigma-l,1,\ldots,1),\ldots,(1-\lambda+\sigma-(m-1)l,1,\ldots,1),** \\ P^* = (1-\mu+\sigma,1,\ldots,1),(1-\mu+\sigma-l,1,\ldots,1),\ldots,(1-\mu+\sigma-(m-1)l,1,\ldots,1),** \end{array}\right\} \tag{4.4.6}$$

and the asterisk $**$ in (4.4.5) and (4.4.6) specifies that the parameters at these places are the same as the parameters of the multivariable's Aleph-function (0.5.1).

Multiplication Formula 2

$$\aleph^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P_i+m,Q_i+m;\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 t & Q^* \\ \vdots & \\ y_n t & R^* \end{array}\right] = t^{\sigma-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{p=0}^{m-1} \left[\frac{\Gamma(\lambda+p\,l)\,(1-\mu-p\,l)_n}{\Gamma(\mu+p\,l)\,(1-\lambda-p\,l)_n} \right]$$

$$\times {}_{m+1}F_m\left\{\begin{matrix}-n,\lambda-n,\lambda-n+l,...,\lambda-n+(m-1)\,l\\ \mu-n,\mu-n+l,...,\mu-n+(m-1)\,l\end{matrix};t\right\}\aleph^{0,N+1,m_1,n_1;...;m_r,n_r}_{P_i+1,Q_i+1,\tau_i;R,U_{11}}\left[\begin{matrix}y_1\\ \vdots\\ y_n\end{matrix}\middle|\begin{matrix}(\sigma+n,1,...,1\ ),**\\ \\ (\sigma,1,...,1\ ),**\end{matrix}\right]$$

(4.4.7)

where

$$\left.\begin{matrix}Q^*=(-\lambda+\sigma,1,...,1\,),(-\lambda+\sigma-l,1,...,1),...,(-\lambda+\sigma-(m-1)l,1,...,1),**\\ R^*=(-\mu+\sigma,1,...,1),(-\mu+\sigma-l,1,...,1),...,(1-\mu+\sigma-(m-1)l,1,...,1),**\end{matrix}\right\} \qquad (4.4.8)$$

and the asterisk $**$ in (4.4.7) and (4.4.8) specifies that the parameters at these places are the same as the parameters of the multivariable's Aleph-function (0.5.1).

4.5 SPECIAL CASES

If we put $\tau_i=1$ and $R=1$ in equation (4.4.5) and (4.4.7) then Aleph ($\aleph$) function reduces to H function and we find a known result due to Bhatter [9]

(i) **Multiplication Formula 1**

$$H^{0,N+m,m_1,n_1;...;m_r,n_r}_{P+m,Q+m;P_1,Q_1;...;P_p,Q_p}\left[\begin{matrix}y_1t\\ \vdots\\ y_nt\end{matrix}\middle|\begin{matrix}S^*\\ T^*\end{matrix}\right]$$

$$=t^{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\prod_{p=0}^{m-1}\frac{\Gamma(\lambda+p\,l)}{\Gamma(\mu+p\,l)}\ {}_{m+1}F_m\left\{\begin{matrix}-n,\lambda,\lambda+l,...,\lambda+(m-1)\,l\\ \mu,\mu+l,...,\mu+(m-1)\,l\end{matrix};t\right\}$$

$$\times H^{0,N+1,m_1,n_1;...;m_r,n_r}_{P+1,Q+1,P_1,Q_1;...;P_p,Q_p}\left[\begin{matrix}y_1\\ \vdots\\ y_n\end{matrix}\middle|\begin{matrix}(\sigma,1,...,1\ ),\left(a_j,\alpha_j^{(1)},...,\alpha_j^{(r)}\right)_{1,P};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1},\\ ;...;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r}\\ (\sigma+n,1,...,1\ ),\left(b_j,\beta_j^{(1)},...,\beta_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1}\\ ;...;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r}\end{matrix}\right],$$

(4.5.1)

where

$$\left.\begin{aligned} S^{*} &= (1-\lambda+\sigma,1,\ldots,1),(1-\lambda+\sigma-l,1,\ldots,1),\ldots,(1-\lambda+\sigma-(m-1)l,1,\ldots,1),\\ &\quad \left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,P};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r}\\ T^{*} &= (1-\mu+\sigma,1,\ldots,1),(1-\mu+\sigma-l,1,\ldots,1),\ldots,(1-\mu+\sigma-(m-1)l,1,\ldots,1),\\ &\quad \left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r} \end{aligned}\right\}. \tag{4.5.2}$$

(ii) **Multiplication Formula 2**

$$H^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P+m,Q+m;P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{matrix} y_1 t\\ \vdots\\ y_n t\end{matrix}\,\middle|\,\begin{matrix}U^{*}\\ V^{*}\end{matrix}\right]$$

$$= t^{\sigma-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+p\,l)\,(1-\mu-p\,l)_n}{\Gamma(\mu+p\,l)\,(1-\lambda-p\,l)_n}\right]{}_{m+1}F_m\left\{\begin{matrix}-n,\lambda-n,\lambda-n+l,\ldots,\lambda-n+(m-1)l\\ \mu-n,\mu-n+l,\ldots,\mu-n+(m-1)l\end{matrix};t\right\}$$

$$\times H^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P+1,Q+1,P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{matrix} y_1\\ \vdots\\ y_n\end{matrix}\,\middle|\,\begin{matrix}(\sigma+n,1,\ldots,1),\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,P};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1},\\ ;\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r}\\ (\sigma,1,\ldots,1),\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1}\\ ;\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r}\end{matrix}\right], \tag{4.5.3}$$

where

$$\left.\begin{aligned}U^{*} &= (-\lambda+\sigma,1,\ldots,1),(-\lambda+\sigma-l,1,\ldots,1),\ldots,(-\lambda+\sigma-(m-1)l,1,\ldots,1),\\ &\qquad \left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,P};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1},;\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r}\\ V^{*} &= (-\mu+\sigma,1,\ldots,1),(-\mu+\sigma-l,1,\ldots,1),\ldots,(1-\mu+\sigma-(m-1)l,1,\ldots,1),\\ &\qquad \left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r}\end{aligned}\right\}. \tag{4.5.4}$$

.

.

.

CHAPTER-5

INTEGRAL PROPERTIES OF THE ALEPH-FUNCTION, MULTIVARIABLE'S GENERAL CLASS OF POLYNOMIALS ASSOCIATED WITH THE FEYNMAN INTEGRALS

Publication:

On some integral properties of aleph function, Multivariable's general class of polynomials associated with Feynman integrals, International Journal of Physics and Mathematical Sciences, 2014 Vol. 4 (2) April-June, pp. 109-117.

The aim of this chapter is to discuss certain integral properties of Aleph-function and multivariable's general class of polynomials, proposed by Inayat – Hussain which contains a certain class of Feynman integrals. We establish two certain new double integrals. In first integral whose integrand involves the product of multivariable's polynomials $S_L^{h_1,\ldots,h_\nu}$ and Aleph-function and in second integral whose integrand involves the product of multivariable's general class of polynomials $S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}$ and Aleph –function. Also we obtain new and known integrals as their special cases.

5.1 RESULTS REQUIRED

Edwards [18, pp.145, 177 and 243] has defined the following relations which are useful to find our results

(i) $$\int_0^1\int_0^1 y^m (1-x)^{m-1}(1-y)^{n-1}(1-xy)^{1-m-n}\,dx\,dy = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \qquad (5.1.1)$$

(ii) $$\int_0^\infty\int_0^\infty \phi(x+y)(x)^m(y)^n\,dx\,dy = \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+1)}\int_0^\infty \psi(z)\,z^{m+n+1}dz, \qquad (5.1.2)$$

(iii) $$\int_0^1\int_0^1 f(xy)(1-x)^{m-1}(1-y)^{n-1}y^m = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\int_0^1 f(z)\,(1-z)^{m+n-1}dz. \qquad (5.1.3)$$

5.2 FIRST MAIN RESULT

(A) In part section we derive four integrals whose kernels is product of $S_L^{h_1,\ldots,h_\nu}$ and Aleph - function

$$(A_1)\quad \int_0^1\int_0^1 \left\{\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right\} S_L^{h_1,\ldots,h_\nu}\left\{\frac{1-p}{1-pq}wq\right\} \aleph_{p_i,q_i,\tau_i,r}^{m,n}\left\{\frac{1-q}{1-pq}w\right\}dpdq$$

$$= \sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\ldots,k_\nu) \prod_{i=1}^{\nu} \frac{w^{k_i}}{k_i!} \Gamma(\sigma+k_i)$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[w \left| \begin{matrix} (1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},\left(1-k_i-\sigma-\rho,1\right) \end{matrix} \right. \right]. \quad (5.2.1)$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right]>0,\left(1\le j\le m\right)$, the above integral is valid, where ρ,σ are positive.

Proof:- With the help of equations (0.6.2) and (0.4.1), we can write

$$S_L^{h_1,\ldots,h_\nu}\left\{\frac{1-p}{1-pq}wq\right\} \aleph^{m,n}_{p_i,q_i,\tau_i,r}\left\{\frac{1-q}{1-pq}w\right\}$$

$$= \sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\ldots,k_\nu) \prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\left(\frac{1-p}{1-pq}w\right)^{k_i}\right\}$$

$$\times \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_j s\right)}{\sum_{i=1}^{r}\tau_i \prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)}\left(\frac{1-q}{1-pq}w\right)^{-s} ds. \quad (5.2.2)$$

Now we multiply both sides of equation (5.2.2) by $\left[\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right]$ and integrating with respect to p and q with limit 0 to 1 for both the variables

$$\int_0^1\int_0^1 \left\{\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right\} S_L^{h_1,\ldots,h_\nu}\left\{\frac{1-p}{1-pq}wq\right\} \aleph^{m,n}_{p_i,q_i,\tau_i,r}\left\{\frac{1-q}{1-pq}w\right\}dpdq$$

$$= \sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\ldots,k_\nu) \prod_{i=1}^{\nu}\left\{\frac{w^{k_i}}{k_i!}\right\}$$

$$\times\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(w)^{-s}ds\left\{\int_0^1\int_0^1\left(\frac{(1-p)q}{1-pq}\right)^{\sigma+k_i}\left(\frac{1-q}{1-pq}\right)^{\rho-s}\left(\frac{1-pq}{(1-p)(1-q)}\right)dp\,dq\right\}.$$

Now we use the result (5.1.1) on right hand side in above inner most integral, we get

$$=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{w^{k_i}}{k_i!}\right\}$$

$$\times\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(w)^{-s}ds\,\frac{\Gamma(\sigma+k_i)\,\Gamma(\rho-s)}{\Gamma(\sigma+k_i+\rho-s)},$$

after a little simplification , we obtain the required result (5.2.1).

$$(A_2)\quad\int_0^\infty\int_0^\infty f(\eta+w)\,\eta^{\sigma-1}w^{\rho-1}S_L^{h_1,\ldots,h_\nu}(\eta)\,\aleph^{m,n}_{p_i,q_i,\tau_i,r}(w)\,d\eta\,dw$$

$$=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\frac{1}{k_i!}\Gamma(\sigma+k_i)\int_0^\infty f(\xi)\,\xi^{\sigma+\rho+k_i-1}d\xi$$

$$\times\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[\xi\left|\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-k_i-\sigma-\rho,1)\end{matrix}\right.\right]. \qquad (5.2.3)$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right]>0,(1\le j\le m)$, the above integral is valid, where ρ,σ are positive.

Proof:- With the help of equations (0.6.2) and (0.4.1), we can write

$$S_L^{h_1,\ldots,h_r}[\eta]\,\aleph^{m,n}_{p_i,q_i,\tau_i,r}[w]=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}(\eta)^{k_i}\right\}$$

$$\times\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_j s\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)}(w)^{-s}ds. \tag{5.2.4}$$

Now we multiply both sides of the equation (5.2.4) by $\left[f(\eta+w)\,\eta^{\sigma-1}\,w^{\rho-1}\right]$ and integrating with respect to η and w between 0 to ∞ for both the variables

$$\int_0^\infty\int_0^\infty f(\eta+w)\,\eta^{\sigma-1}\,w^{\rho-1}\,S_L^{h_1,\dots,h_\nu}(\eta)\,\aleph^{m,n}_{p_i,q_i,\tau_i,r}(w)\,d\eta\,dw$$

$$=\sum_{k_1,\dots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}\,A(L,k_1,\dots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times\frac{1}{2\pi i}\int_L\Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(w)^{-s}ds\left\{\int_0^\infty\int_0^\infty f(\eta+w)(\eta)^{\sigma+k_i-1}(w)^{\rho-s-1}\,d\eta\,dw\right\}.$$

Interchanging the order of integration and summation in above equation and use the known result (5.1.2), we have

$$=\sum_{k_1,\dots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}\,A(L,k_1,\dots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times\frac{\Gamma(\sigma+k_i)\,\Gamma(\rho-s)}{\Gamma(\sigma+k_i+\rho-s)}\int_0^\infty f(\xi)\,(1-\xi)^{\sigma+k_i+\rho-s-1}d\xi,$$

after a little simplification, we get the required result (5.2.3).

$$(A_3)\qquad \int_0^1\int_0^1 \psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}\,S_L^{h_1,\dots,h_\nu}[w(1-\eta)]\,\aleph^{m,n}_{p_i,q_i,\tau_i,r}[(1-w)]d\eta\,dw$$

$$= \sum_{k_1,\dots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\dots,k_\nu) \prod_{i=1}^{\nu} \frac{1}{k_i!} \Gamma(\sigma+k_i) \int_0^1 \psi(\xi)(1-\xi)^{\sigma+\rho+k_i-1} d\xi$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[(1-\xi)\middle| \begin{matrix} (1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},\left(1-k_i-\sigma-\rho,1\right) \end{matrix}\right]. \quad (5.2.5)$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right]>0, (1\le j\le m)$, the above integral is valid, where ρ,σ are positive.

Proof: - With the help of the equations (0.6.2) and (0.4.1), we can write

$$S_L^{h_1,\dots,h_\nu}[w(1-\eta)]\,\aleph^{m,n}_{p_i,q_i,\tau_i,r}[1-w]$$

$$= \sum_{k_1,\dots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\dots,k_\nu) \prod_{i=1}^{\nu} \left\{\frac{1}{k_i!}\{w(1-\eta)\}^{k_i}\right\}$$

$$\times \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i \prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)}(1-w)^{-s}ds. \quad (5.2.6)$$

Now we multiply both sides of equation (5.2.6) by $\psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}$ and integrating with respect to η and w with limit 0 to 1 for both the variables

$$\int_0^1\int_0^1 \psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma} S_L^{h_1,\dots,h_\nu}[w(1-\eta)]\,\aleph^{m,n}_{p_i,q_i,\tau_i,r}[(1-w)]d\eta\, dw$$

$$= \sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\ldots,k_\nu) \prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)\, ds \left\{\int_0^1\int_0^1 \psi(\eta w)(1-\eta)^{\sigma+k_i-1}(1-w)^{\rho-s-1} w^{\sigma+k_i}\, d\eta\, dw\right\}.$$

Interchanging the order of integration and summation in above equation and use the known result (5.1.3), we get

$$= \sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\ldots,k_\nu) \prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times \frac{\Gamma(\sigma+k_i)\,\Gamma(\rho-s)}{\Gamma(\sigma+k_i+\rho-s)}\int_0^1 \psi(\xi)\,(1-\xi)^{\sigma+k_i+\rho-s-1}\, d\xi\,,$$

after a little simplification, we obtain the required result (5.2.5).

$$(A_4)\quad \int_0^1\int_0^1 \left\{\left(\frac{q(1-p)}{1-pq}\right)^{a+\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right\} S_L^{h_1,\ldots,h_\nu}\left\{\frac{q(1-p)}{1-pq}\right\} \aleph^{m,n}_{p_i,q_i,\tau_i,r}\left\{\frac{q(1-p)}{1-pq}w\right\} dp\, dq$$

$$= \sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu} h_i k_i \le L} (-L)_{\sum_{i=1}^{\nu} h_i k_i} A(L,k_1,\ldots,k_\nu)\ \Gamma(\rho+1)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[w \left| \begin{matrix} (1-a-\sigma-k_i;1),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(-a-k_i-\sigma-\rho,1) \end{matrix} \right.\right]. \qquad (5.2.7)$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right] > 0, (1 \le j \le m)$, the above integral is valid, where a,ρ,σ are positive.

Proof:- With the help of equations (0.6.2) and (0.4.1), we can write

$$S_L^{h_1,\ldots,h_\nu}\left\{\frac{q(1-p)}{1-pq}\right\}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left\{\frac{q(1-p)}{1-pq}w\right\}$$

$$=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\left(\frac{q(1-p)}{1-pq}\right)^{k_i}\right\}$$

$$\times\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)}\left(\frac{q(1-p)}{1-pq}w\right)^{-s}ds. \qquad (5.2.8)$$

Now we multiply both sides of equation (5.2.8) by $\left\{\left(\frac{q(1-p)}{1-pq}\right)^{a+\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right\}$ and integrating with respect to p and q with limit 0 to 1 for both the variables

$$\int_0^1\int_0^1\left\{\left(\frac{q(1-p)}{1-pq}\right)^{a+\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right\}S_L^{h_1,\ldots,h_\nu}\left\{\frac{q(1-p)}{1-pq}\right\}\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left\{\frac{q(1-p)}{1-pq}w\right\}dpdq$$

$$=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times\frac{1}{2\pi i}\int_L\Omega_{p_i,q_i;\tau_i;r}^{m,n}(s)(w)^{-s}ds\int_0^1\int_0^1\left(\frac{(1-p)q}{1-pq}\right)^{a+\sigma+k_i-s}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)dp\,dq$$

$$=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(w)^{-s}ds\int_0^1\int_0^1 \left(\frac{q-pq}{1-pq}\right)^{\sigma+k_i-s}\left(1-\frac{q-pq}{1-pq}\right)^{(\rho+1)-1}\left(\frac{1}{(1-p)}\right)dp\,dq\,.$$

Interchanging the order of integration and summation in above equation and use the known result (5.1.1), we get

$$=\sum_{k_1,\ldots,k_\nu=0}^{\sum_{i=1}^{\nu}h_ik_i\le L}(-L)_{\sum_{i=1}^{\nu}h_ik_i}\,A(L,k_1,\ldots,k_\nu)\prod_{i=1}^{\nu}\left\{\frac{1}{k_i!}\right\}$$

$$\times\frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i;\tau_i;r}(s)(w)^{-s}ds\,\frac{\Gamma(a+\sigma+k_i-s)\,\Gamma(\rho+1)}{\Gamma(1+a+\sigma+k_i+\rho-s)}$$

after a little simplification , we obtain the required result (5.2.7).

5.3 SECOND MAIN RESULT

(B) If we use the multivariable's general class of polynomials by (0.6.3) and Aleph-function (0.4.1), then we also get new results

$$(B_1)\quad \int_0^1\int_0^1 \left\{\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right\}S^{m_1,\ldots,m_\nu}_{n_1,\ldots,n_\nu}\left\{\frac{1-p}{1-pq}wq\right\}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left\{\frac{1-q}{1-pq}w\right\}dpdq$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}\,w^{k_i}\,\Gamma(\sigma+k_i)$$

$$\times\aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[w\left|\begin{matrix}(1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},\left(1-k_i-\sigma-\rho,1\right)\end{matrix}\right.\right]. \tag{5.3.1}$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right]>0,(1\le j\le m)$, the above integral is valid, where ρ,σ are positive.

Proof: - With the help of equations (0.6.3) and (0.4.1), we can write

$$S_{n_1,\dots,n_\nu}^{m_1,\dots,m_\nu}\left\{\frac{1-p}{1-pq}wq\right\}\aleph_{p_i,q_i,\tau_i,r}^{m,n}\left\{\frac{1-q}{1-pq}w\right\}$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\left\{\frac{(-n_i)_{m_i k_i}}{k_i!}A_{n_i,k_i}\left(\frac{1-p}{1-pq}wq\right)^{k_i}\right\}$$

$$\times\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)}\left(\frac{1-q}{1-pq}w\right)^{-s}ds. \tag{5.3.2}$$

Now we multiply both sides of equation (5.3.2) by $\left[\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right]$ and integrating with respect to p and q with limit 0 to 1 for both the variables. Interchanging the order of integral and summation and use the known result (5.1.1) in obtain equation, we get the required result (5.3.1), after little simplification.

$$(B_2)\quad\int_0^\infty\int_0^\infty f(\eta+w)\,\eta^{\sigma-1}w^{\rho-1}S_{n_1,\dots,n_\nu}^{m_1,\dots,m_\nu}(\eta)\,\aleph_{p_i,q_i,\tau_i,r}^{m,n}(w)\,d\eta\,dw$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_i k_i}}{k_i!}A_{n_i,k_i}\Gamma(\sigma+k_i)\int_0^\infty f(\xi)\,\xi^{\sigma+\rho+k_i-1}d\xi$$

$$\times\aleph_{p_i+1,q_i+1,\tau_i,r}^{m,n+1}\left[\xi\left|\begin{matrix}(1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},\left(1-k_i-\sigma-\rho;1\right)\end{matrix}\right.\right]. \tag{5.3.3}$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right]>0,\left(1\le j\le m\right)$, the above integral is valid, where ρ,σ are positive.

Proof:- With the help of equations (0.6.3) and (0.4.1), we can write

$$S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}[\eta]\,\aleph_{p_i,q_i,\tau_i,r}^{m,n}[w] = \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\left\{\frac{(-n_i)_{m_i k_i}}{k_i!}A_{n_i,k_i}\eta^{k_i}\right\}$$

$$\times\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_j s\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)}(w)^{-s}ds. \tag{5.3.4}$$

Now we multiply both sides of equation (5.3.4) by $\left[f(\eta+w)\eta^{\sigma-1}w^{\rho-1}\right]$ and integrating with respect to η and w between 0 to ∞ for both the variables. Interchanging the order of integral and summation and use the known result (5.1.2) on obtain equation, we get the required result (5.3.3), after little simplification.

$$(B_3)\quad \int_0^1\int_0^1 \psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}[w(1-\eta)]\,\aleph_{p_i,q_i,\tau_i,r}^{m,n}[(1-w)]\,d\eta\,dw$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_i k_i}}{k_i!}A_{n_i,k_i}\Gamma(\sigma+k_i)\int_0^\infty f(\xi)(1-\xi)^{\sigma+\rho+k_i-1}d\xi$$

$$\times\aleph_{p_i+1,q_i+1,\tau_i,r}^{m,n+1}\left[(1-\xi)\left|\begin{matrix}(1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},\left(1-k_i-\sigma-\rho;1\right)\end{matrix}\right.\right], \tag{5.3.5}$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right]>0,\left(1\le j\le m\right)$, the above integral is valid, where ρ,σ are positive.

Proof:- With the help of equations (0.6.3) and (0.4.1), we can write

$$S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}[w(1-\eta)]\,\aleph_{p_i,q_i,\tau_i,r}^{m,n}[1-w]$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \left\{ \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} \{w(1-\eta)\}^{k_i} \right\}$$

$$\times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - A_j s\right)}{\sum_{i=1}^{r} \tau_i \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} + A_{ji} s\right) \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji} - B_{ji} s\right)} (1-w)^{-s} ds, \tag{5.3.6}$$

Now we multiply both sides of equation (5.3.6) by $\psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}$ and integrating with respect to η and w between 0 to 1 for both the variables, now interchanging the order of integral and summation and use the known result (5.1.3) in obtain equation, we get the required result (5.3.5), after little simplification.

$$(B_4) \quad \int_0^1 \int_0^1 \left\{ \left(\frac{q(1-p)}{1-pq} \right)^{a+\sigma} \left(\frac{1-q}{1-pq} \right)^{\rho} \left(\frac{1}{(1-p)} \right) \right\} S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu} \left\{ \frac{q(1-p)}{1-pq} \right\} \aleph_{p_i,q_i,\tau_i,r}^{m,n} \left\{ \frac{q(1-p)}{1-pq} w \right\} dp\,dq$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} \Gamma(\rho+1)$$

$$\times \aleph_{p_i+1,q_i+1,\tau_i,r}^{m,n+1} \left[w \left| \begin{matrix} \left(1-a-\sigma-k_i;1\right), \left(a_j, A_j\right)_{1,n}, \left[\tau_i\left(a_{ji}, A_{ji}\right)\right]_{n+1,,p_i;r} \\ \left(b_j, B_j\right)_{1,m}, \left[\tau_i\left(b_{ji}, B_{ji}\right)\right]_{m+1,q_i;r}, \left(-a-k_i-\sigma-\rho,1\right) \end{matrix} \right. \right], \tag{5.3.7}$$

Provided that for the conditions (0.4.3) and $\mathrm{Re}\left[\sigma+\rho+\frac{b_j}{B_j}\right] > 0, \left(1 \le j \le m\right)$, the above integral is valid, where a, ρ, σ are positive.

Proof: - With the help of (0.6.3) and (0.4.1), we can write

$$S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu} \left\{ \frac{q(1-p)}{1-pq} \right\} \aleph_{p_i,q_i,\tau_i,r}^{m,n} \left\{ \frac{q(1-p)}{1-pq} w \right\}$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \left\{ \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} \left(\frac{q(1-p)}{1-pq} \right)^{k_i} \right\}$$

$$\times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - A_j s\right)}{\sum_{i=1}^{r} \tau_i \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} + A_{ji} s\right) \prod_{j=m+1}^{q_i} \Gamma\left(1 - b_{ji} - B_{ji} s\right)} \left(\frac{q(1-p)}{1-pq} w \right)^{-s} ds, \qquad (5.3.8)$$

Now we multiply both sides of equation (5.3.8) by $\left\{ \left(\frac{q(1-p)}{1-pq} \right)^{a+\sigma} \left(\frac{1-q}{1-pq} \right)^{\rho} \left(\frac{1}{(1-p)} \right) \right\}$ and integrating with respect to p and q with limit 0 to 1 for both the variables, now interchanging the order of integral and summation and use the known result (5.1.1) on obtain equation, we get the required result (5.3.7), after little simplification.

5.4 SPECIAL CASES

1. If we choose $\tau_i = 1$ in equation (5.3.1), (5.3.3), (5.3.5) and (5.3.7) then Aleph – function reduces to I-function we find the known result due to Agrawal [3]

$$(C_1) \quad \int_0^1 \int_0^1 \left\{ \left(\frac{(1-p)q}{1-pq} \right)^{\sigma} \left(\frac{1-q}{1-pq} \right)^{\rho} \left(\frac{1-pq}{(1-p)(1-q)} \right) \right\} S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu} \left\{ \frac{1-p}{1-pq} wq \right\} I_{p_i,q_i,r}^{m,n} \left\{ \frac{1-q}{1-pq} w \right\} dpdq$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} w^{k_i} \Gamma(\sigma + k_i)$$

$$\times I_{p_i+1,q_i+1,r}^{m,n+1} \left[w \,\middle|\, \begin{matrix} (1-\rho;1), \left(a_j, A_j\right)_{1,n}, \left(a_{ji}, A_{ji}\right)_{n+1,p_i;r} \\ \left(b_j, B_j\right)_{1,m}, \left(b_{ji}, B_{ji}\right)_{m+1,q_i;r}, \left(1-k_i-\sigma-\rho, 1\right) \end{matrix} \right], \qquad (5.4.1)$$

$$(C_2) \quad \int_0^\infty \int_0^\infty f(\eta + w) \eta^{\sigma-1} w^{\rho-1} S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}(\eta)\, I_{p_i,q_i,r}^{m,n}(w) d\eta\, dw$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} \Gamma(\sigma+k_i) \int_0^\infty f(\xi)\, \xi^{\sigma+\rho+k_i-1} d\xi$$

$$\times I^{m,n+1}_{p_i+1,q_i+1,r}\left[\xi \left| \begin{matrix} (1-\rho;1),(a_j,A_j)_{1,n},(a_{ji},A_{ji})_{n+1,p_i;r} \\ (b_j,B_j)_{1,m},(b_{ji},B_{ji})_{m+1,q_i;r},(1-k_i-\sigma-\rho;1) \end{matrix} \right.\right], \qquad (5.4.2)$$

(C_3) $$\int_0^1 \int_0^1 \psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1} w^{\sigma} S^{m_1,\dots,m_\nu}_{n_1,\dots,n_\nu}[w(1-\eta)]\, I^{m,n}_{p_i,q_i,r}[(1-w)]\, d\eta\, dw$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} \Gamma(\sigma+k_i) \int_0^\infty f(\xi)(1-\xi)^{\sigma+\rho+k_i-1} d\xi$$

$$\times I^{m,n+1}_{p_i+1,q_i+1,r}\left[(1-\xi) \left| \begin{matrix} (1-\rho;1),(a_j,A_j)_{1,n},(a_{ji},A_{ji})_{n+1,p_i;r} \\ (b_j,B_j)_{1,m},(b_{ji},B_{ji})_{m+1,q_i;r},(1-k_i-\sigma-\rho;1) \end{matrix} \right.\right], \qquad (5.4.3)$$

(C_4) $$\int_0^1 \int_0^1 \left\{ \left(\frac{q(1-p)}{1-pq}\right)^{a+\sigma} \left(\frac{1-q}{1-pq}\right)^{\rho} \left(\frac{1}{(1-p)}\right) \right\} S^{m_1,\dots,m_\nu}_{n_1,\dots,n_\nu}\left\{\frac{q(1-p)}{1-pq}\right\} I^{m,n}_{p_i,q_i,r}\left\{\frac{q(1-p)}{1-pq} w\right\} dp\, dq$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_\nu=0}^{[n_\nu/m_\nu]} \prod_{i=1}^{\nu} \frac{(-n_i)_{m_i k_i}}{k_i!} A_{n_i,k_i} \Gamma(\rho+1)$$

$$\times I^{m,n+1}_{p_i+1,q_i+1,r}\left[w \left| \begin{matrix} (1-a-\sigma-k_i;1),(a_j,A_j)_{1,n},(a_{ji},A_{ji})_{n+1,,p_i;r} \\ (b_j,B_j)_{1,m},(b_{ji},B_{ji})_{m+1,q_i;r},(-a-k_i-\sigma-\rho,1) \end{matrix} \right.\right]. \qquad (5.4.4)$$

2. If we put $\tau_i = 1$ and $r = 1$ in equation (5.3.1), (5.3.3), (5.3.5) and (5.3.7) then Aleph – function reduce to H-function then we find the known result due to Agrawal [3]

$$(D_1)\quad \int_0^1\int_0^1 \left\{\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right\} S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}\left\{\frac{1-p}{1-pq}wq\right\} H_{p,q}^{m,n}\left\{\frac{1-q}{1-pq}w\right\}dpdq$$

$$= \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}\, w^{k_i}\Gamma(\sigma+k_i)$$

$$\times H_{p+1,q+1}^{m,n+1}\left[w\,\middle|\,\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j)_{m+1,q},(1-k_i-\sigma-\rho,1)\end{matrix}\right], \tag{5.4.5}$$

$$(D_2)\quad \int_0^\infty\int_0^\infty f(\eta+w)\eta^{\sigma-1}w^{\rho-1}S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}(\eta)\,H_{p,q}^{m,n}(w)d\eta\,dw$$

$$= \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}\,\Gamma(\sigma+k_i)\int_0^\infty f(\xi)\xi^{\sigma+\rho+k_i-1}d\xi$$

$$\times H_{p+1,q+1}^{m,n+1}\left[\xi\,\middle|\,\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j)_{m+1,q},(1-k_i-\sigma-\rho;1)\end{matrix}\right], \tag{5.4.6}$$

$$(D_3)\quad \int_0^1\int_0^1 \psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}S_{n_1,\ldots,n_\nu}^{m_1,\ldots,m_\nu}[w(1-\eta)]\,H_{p,q}^{m,n}[(1-w)]d\eta\,dw$$

$$= \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_ik_i}}{k_i!}A_{n_i,k_i}\,\Gamma(\sigma+k_i)\int_0^\infty f(\xi)(1-\xi)^{\sigma+\rho+k_i-1}d\xi$$

$$\times H_{p+1,q+1}^{m,n+1}\left[(1-\xi)\,\middle|\,\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n},(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m},(b_j,B_j)_{m+1,q},(1-k_i-\sigma-\rho;1)\end{matrix}\right], \tag{5.4.7}$$

$$(D_4)\quad \int_0^1\int_0^1 \left\{\left(\frac{q(1-p)}{1-pq}\right)^{a+\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right\} S_{n_1,\dots,n_\nu}^{m_1,\dots,m_\nu}\left\{\frac{q(1-p)}{1-pq}\right\} H_{p,q}^{m,n}\left\{\frac{q(1-p)}{1-pq}w\right\} dp\,dq$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_\nu=0}^{[n_\nu/m_\nu]}\prod_{i=1}^{\nu}\frac{(-n_i)_{m_i k_i}}{k_i!}A_{n_i,k_i}\,\Gamma(\rho+1)$$

$$\times H_{p+1,q+1}^{m,n+1}\left[w\left|\begin{matrix}(1-a-\sigma-k_i;1),(a_j,A_j)_{1,n},(a_j,A_j)_{n+1,,p}\\ (b_j,B_j)_{1,m},(b_j,B_j)_{m+1,q},(-a-k_i-\sigma-\rho,1)\end{matrix}\right.\right]. \qquad (5.4.8)$$

3. If we put $\nu=1$ in equation (5.2.1), (5.2.3), (5.2.5) and (5.2.7), then we find the known result due to Khan and Sharma [49, equation (2.1), (2.3), (2.5), (2.7)]

$$(E_1)\quad \int_0^1\int_0^1 \left\{\left(\frac{(1-p)q}{1-pq}\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right\} S_L^h\left\{\frac{1-p}{1-pq}wq\right\} \aleph_{p_i,q_i,\tau_i,r}^{m,n}\left\{\frac{1-q}{1-pq}w\right\} dp\,dq$$

$$=\sum_{k=0}^{[L/h]}\frac{(-L)_{hk}}{k!}A_{L,k}\;w^k\,\Gamma(\sigma+k)$$

$$\times \aleph_{p_i+1,q_i+1,\tau_i,r}^{m,n+1}\left[w\left|\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r}\\ (b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-k-\sigma-\rho,1)\end{matrix}\right.\right], \quad (5.4.9)$$

$$(E_2)\quad \int_0^\infty\int_0^\infty f(\eta+w)\eta^{\sigma-1}w^{\rho-1}\,S_L^h(\eta)\,\aleph_{p_i,q_i,\tau_i,r}^{m,n}(w)\,d\eta\,dw$$

$$=\sum_{k=0}^{[L/h]}\frac{(-L)_{hk}}{k!}A_{L,k}\quad\Gamma(\sigma+k)\int_0^\infty f(\xi)\;\xi^{\sigma+\rho+k-1}d\xi$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[\xi \middle| \begin{matrix} (1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},(1-k-\sigma-\rho,1) \end{matrix}\right], \quad (5.4.10)$$

$$(E_3) \quad \int_0^1\int_0^1 \psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1} w^{\sigma} S_L^h[w(1-\eta)]\aleph^{m,n}_{p_i,q_i,\tau_i,r}[(1-w)]d\eta\, dw$$

$$=\sum_{k=0}^{[L/h]}\frac{(-L)_{hk}}{k!}A_{L,k}\quad \Gamma(\sigma+k)\int_0^1 f(\xi)\ (1-\xi)^{\sigma+\rho+k-1}d\xi$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[(1-\xi) \middle| \begin{matrix} (1-\rho;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},(1-k-\sigma-\rho,1) \end{matrix}\right], \quad (5.4.11)$$

$$(E_4) \quad \int_0^1\int_0^1 \left\{\left(\frac{q(1-p)}{1-pq}\right)^{a+\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right\} S_L^h\left\{\frac{q(1-p)}{1-pq}\right\}\aleph^{m,n}_{p_i,q_i,\tau_i,r}\left\{\frac{q(1-p)}{1-pq}w\right\}dp\,dq$$

$$=\sum_{k=0}^{[L/h]}\frac{(-L)_{hk}}{k!}A_{L,k}\quad \Gamma(\rho+1)$$

$$\times \aleph^{m,n+1}_{p_i+1,q_i+1,\tau_i,r}\left[w \middle| \begin{matrix} (1-a-\sigma-k;1),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},(-a-k-\sigma-\rho,1) \end{matrix}\right], \quad (5.4.12)$$

4. If we put $\tau_i = 1$ and $S_n^2(x) \to x^{\frac{n}{2}} H_n\left[\frac{1}{2\sqrt{x}}\right]$ in which $m_1,...,m_\nu = 2$, $n_1,...,n_\nu = n, \nu = 1$ and $A_{n_i,k_i} = (-1)^k$ in our results (5.3.1), (5.3.3), (5.3.5) and (5.3.7) then reduce to given results due to Agrawal [3]

5. If we put $\tau_i = 1$ and $S_n^2(x) \to x^{\frac{n}{2}} H_n\left[\frac{1}{2\sqrt{x}}\right]$ in which $m_1,...,m_\nu = 1,\ n_1,...,n_\nu = n, \nu = 1$

and $A_{n_i,k_i} = \binom{n+\alpha}{n}\frac{1}{(\alpha+1)_k}$ in our results (5.3.1), (5.3.3), (5.3.5) and (5.3.7) then reduce to given results due to Agrawal [3]

6. If we choose $\tau_i = 1, n = p_i = 0, \nu = 1, m = 1, q_i = 2, b_1 = 0, B_1 = 1, b_{m+1,1} = -\lambda, B_{m+1,1} = \mu$ in our results (5.3.1), (5.3.3), (5.3.5) and (5.3.7) then reduce to given results due to Agrawal [3].

CHAPTER-6

FRACTIONAL INTEGRAL TRANSFORMATIONS OF THE ALEPH-FUNCTION

Publication:

Fractional integral transformations of Mittag- Leffler type E-function with Multivariable Polynomial and Aleph function, International journal of mathematical Archive, 8(5), (2017), 1-11.

In the present chapter, we study about different types of fractional integral transformations. At first we derive Riemann-Liouville fractional integral transformation of the E-function, Multivariable polynomial and Aleph function then we obtain various new and known special cases. Finally establish Erdelyi- Kober fractional integral transformation and generalized fractional integral transformation of the E-function, Multivariable polynomial and Aleph function respectively then we get many new and known special cases.

6.1 DEFINITIONS

6.1.1 RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL TRANSFORM

Samko, Kilbas and Marichev [88] define the Riemann-Liouville fractional integral operator $(I_{c+}^{\theta}\Psi)(x)$ as follows

$$(I_{c+}^{\theta}\Psi)(x) = \frac{1}{\Gamma\theta}\int_{c}^{x}(x-t)^{\theta-1}\Psi(t)\,dt\,, \tag{6.1.1}$$

where $\theta \in C$ and $R(\theta) > 0$.

6.1.2 ERDELYI-KOBER FRACTIONAL INTEGRAL TRANSFORM

Samko, Kilbas and Marichev [88] define the Erdelyi-Kober fractional integral operator $(\Xi_{0+}^{\eta,\theta})(x)$ as follows

$$\left(\Xi_{0+}^{\eta,\theta} f\right)(x) = \frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_{0}^{x}(x-t)^{\eta-1}t^{\theta}\,f(t)\,dt\,, \tag{6.1.2}$$

where $\eta,\theta \in C; R(\eta) > 0$ and $R(\theta) > 0$.

6.1.3 THE GENERAL MULTIVARIABLE POLYNOMIAL

Srivastava and Garg [107, p.686, eq. (1.4), (1987)], introduced the Multivariable polynomial $S_{V}^{U_1,\ldots,U_l}(x_1,\ldots,x_l)$ in the following manner

$$S_{V}^{U_1,\ldots,U_l}(x_1,\ldots,x_l) = \sum_{\substack{R_1,\ldots,R_l=0 \\ \sum_{i=1}^{l} U_iR_i \le V}} (-V)_{\sum_{i=1}^{l} U_iR_i}\, A(V,R_1,\ldots,R_l)\,\frac{x_i^{R_i}}{R_i!}\,, \tag{6.1.3}$$

where $U_1,...,U_l$ an arbitrary positive integers , $V = 0,1,2\ldots$ and the coefficients $A(V, R_1,...,R_l)$ are arbitrary constants (real or complex).

6.1.4 ALEPH $(\aleph)$ - FUNCTION IN SERIES

Chaurasia [15] has given the series representation of the Aleph function

$$\aleph^{M,N}_{P_i,Q_i,c_i,r}[z] = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}z^{-s}, \tag{6.1.4}$$

with $s = \eta_{G,g} = \dfrac{b_G + g}{B_G}$, $P_i < Q_i$, $|z| < 1$

$$\text{and } \Omega^{M,N}_{P_i,Q_i,c_i,r}(s) = \frac{\prod\limits_{j=1, j\neq G}^{M}\Gamma\left(b_j + B_j s\right)\prod\limits_{j=1}^{N}\Gamma\left(1 - a_j - A_j s\right)}{\sum\limits_{i=1}^{r}c_i\prod\limits_{j=M+1}^{Q_i}\Gamma\left(1 - b_{ji} - B_{ji}s\right)\prod\limits_{j=N+1}^{P_i}\Gamma\left(a_{ji} + A_{ji}s\right)}. \tag{6.1.5}$$

6.1.5 MITTAG- LEFFLER TYPE E-FUNCTION

Bhatter and Faisal [7], have defined a M-L type E-function in the following way

$${}_{\tau}E^{h}_{k}(z) = {}_{\tau}E^{h}_{k}\left[z\,\middle|\,\begin{matrix}(\rho, a);\left(\gamma_i, q_i, s_i\right)_{1,h}\\ (\alpha, \beta);\left(\delta_j, p_j, r_j\right)_{1,k}\end{matrix}\right] = {}_{\tau}E^{h}_{k}\left[z\,\middle|\,\begin{matrix}(\rho,a);\left(\gamma_1,q_1,s_1\right),\ldots,\left(\gamma_h,q_h,s_h\right)\\ (\alpha,\beta);\left(\delta_1,p_1,r_1\right),\ldots,\left(\delta_k,p_k,r_k\right)\end{matrix}\right]$$

$$= \frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\cdots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\cdots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n + \beta)}, \tag{6.1.6}$$

where $z, \alpha, \beta, \gamma_i, \delta_j \in C, R(\alpha) \geq 0, R(\beta) > 0, R(\gamma_i) > 0, R(\delta_j) > 0, q_i \geq 0$

$$p_j \geq 0; s_i \geq 0, r_j \geq 0; a, \tau \in R; \rho \in \{0,1\}, \left(\sum_{i=0}^{h}q_i s_i < \sum_{j=1}^{k}p_j r_j + R(\alpha)\right) \text{ or}$$

$$\left(\sum_{i=0}^{h}q_i s_i = \sum_{j=1}^{k}p_j r_j + R(\alpha) \text{ when } \prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}\left(p_j\right)^{p_j r_j}\right]^{-1}\left|z^{\alpha}\right| < 1\right), \tag{6.1.7}$$

here $i = 1,2,...,h; j = 1,2,...,k.$

6.1.6 RESULTS REQUIRED

(i) $$\int_a^b (z-a)^{m-1}(b-z)^{n-1}\,dz = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}(b-a)^{m+n-1}, \tag{6.1.8}$$

(ii) $$\int_0^b (z)^{m-1}(b-z)^{n-1}\,dz = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}(b)^{m+n-1}. \tag{6.1.9}$$

6.2 THE IMAGE OF ALEPH -FUNCTION UNDER THE RIEMANN – LIOUVILLE (R-L) OPERATOR I_{c+}^{θ}

Theorem 1. If convergence condition of (6.1.3), (6.1.6) and (6.1.4) are satisfied also $\theta \in C$ and $R(\theta) > 0$, then the R-L transform I_{c+}^{θ} of the Multivariable polynomial , E-function and Aleph function is

$$\left[I_{C+}^{\theta}\left\{{}_{\tau}E_{k}^{h}(t-c)\; S_{V}^{U_1,\ldots,U_l}\left((t-c)^{R_1},\ldots,(t-c)^{R_l}\right)\aleph_{P_i,Q_i,c_i,r}^{M,N}(t-c)\right\}\right](x)$$

$$=\frac{1}{\left(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\theta}}\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\;A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}(x-c)^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_{\tau+\theta}E_{k+1}^{h+1}\left[(x-c)\,\middle|\,\begin{matrix}(\rho,a)\,;(\gamma_i,q_i,s_i)_{1,h}\,,(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1)\end{matrix}\right]. \tag{6.2.1}$$

Proof:-With the help of equations (6.1.3), (6.1.6) and (6.1.4), we can give the R-L transform I_{c+}^{θ} of the Multivariable polynomial, E-function and Aleph function as follows

$$\left[I_{C+}^{\theta}\left\{{}_{\tau}E_{k}^{h}(t-c)\, S_{V}^{U_1,\dots,U_l}\left((t-c)^{R_1},\dots,(t-c)^{R_l}\right) \aleph_{P_i,Q_i,c_i,r}^{M,N}(t-c)\right\}\right](x)$$

$$=\left[\frac{1}{\Gamma\theta}\int_{c}^{x}(x-t)^{\theta-1}\sum_{n=0}^{\infty}\Phi(n)\,(t-c)^{an+\tau}\sum_{R_1,\dots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\dots,R_l)\right.$$

$$\left.\times\prod_{i=1}^{l}\frac{(t-c)^{R_i}}{R_i!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}(t-c)^{-\eta_{G,g}}\right]dt,$$

where

$$\Phi(n)=\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}\cdots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}\cdots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)}. \tag{6.2.2}$$

Then

$$\left[I_{C+}^{\theta}\left\{{}_{\tau}E_{k}^{h}(t-c)\, S_{V}^{U_1,\dots,U_l}\left((t-c)^{R_1},\dots,(t-c)^{R_l}\right) \aleph_{P_i,Q_i,c_i,r}^{M,N}(t-c)\right\}\right](x)$$

$$=\frac{1}{\left(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\theta}}\sum_{R_1,\dots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\dots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}\sum_{n=0}^{\infty}\Phi(n)\frac{\left(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{an}}{\left(\tau+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{an}}(x-c)^{\tau+an+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$=\frac{1}{\left(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_\theta}\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}(x-c)^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_{\tau+\theta}E^{h+1}_{k+1}\left[(x-c)\left|\begin{matrix}(\rho,a)\,;\,(\gamma_i,q_i,s_i)_{1,h},(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1)\end{matrix}\right.\right].$$

(6.2.3)

6.2.1 SPECIAL CASES OF THEOREM 1

I. In the equation (6.2.1), if we substitute $h=0,\tau=0,k=m-1,\rho=0,a=1,s_i=0,r_j=1$ then we get R-L transform I^{θ}_{c+} of the "multiindex M-L type functions" (0.8.4), as follows

$$\left[I^{\theta}_{C+}\left\{E_{(1/\rho_i),(\mu_i)}(t-c)\,S_V^{U_1,\ldots,U_l}\left((t-c)^{R_1},\ldots,(t-c)^{R_l}\right)\aleph^{M,N}_{P_i,Q_i,c_i,r}(t-c)\right\}\right](x)$$

$$=\frac{1}{\left(\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_\theta}\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}(x-c)^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times {}_{\theta}E_{m}^{1}\left[(x-c)\,\middle|\,\begin{matrix}(0,1)\,;\,(\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),...,(\mu_{m-1},1/\rho_{m-1},1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\end{matrix}\right].$$

(6.2.4)

II. In the equation (6.2.1), if we substitute $h=1,\tau=0,k=m,\rho=0,a=1,s_i=0,r_j=1$ then we get R-L transform I_{c+}^{θ} of the M-L type function (0.8.5), as follows

$$\left[I_{C+}^{\theta}\left\{E_{\gamma,k}\left[(\alpha_1,\beta_1),...,(\alpha_m,\beta_m);t\right]S_V^{U_1,...,U_l}(t^{R_1},...,t^{R_l})\aleph_{P_i,Q_i,c_i,r}^{M,N}(t)\right\}\right](x)$$

$$=\frac{1}{\left(\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\theta}}\sum_{R_1,...,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,...,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{m}\Gamma(\beta_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}(x)^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times {}_{\theta}E_{m+1}^{2}\left[x\,\middle|\,\begin{matrix}(0,1)\,;(\gamma,k,1),\,(\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\\(1,1);(\beta_1,\alpha_1,1),...,(\beta_m,\alpha_m,1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\end{matrix}\right]. \quad (6.2.5)$$

III. In the equation (6.2.1), if we substitute $h=0,\tau=M,k=\nu-1,\rho=0,a=\Lambda,s_i=0,r_j=1$ then we get R-L transform I_{c+}^{θ} of the M-L type function (0.8.6), as follows

$$\left[I_{C+}^{\theta}\left\{\left(HE_{\mu_1,...,\mu_\nu}^{\lambda_1,...,\lambda_\nu};t\right)S_V^{U_1,...,U_l}(t^{R_1},...,t^{R_l})\aleph_{P_i,Q_i,c_i,r}^{M,N}(t)\right\}\right](x)$$

$$= \frac{1}{\left(M + \sum_{i=1}^{l} R_i - \eta_{G,g} + 1\right)_\theta} \sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l} U_i R_i \leq V} (-V)_{\sum_{i=1}^{l} U_i R_i} A(V, R_1, \ldots, R_l) \frac{1}{R_1!} \cdots \frac{1}{R_l!}$$

$$\times \frac{1}{\prod_{j=1}^{\nu-1} \Gamma\left(1+\mu_j\right)} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \frac{(-1)^g \, \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g! \, B_G} (x)^{\theta + \sum_{i=1}^{l} R_i - \eta_{G,g}}$$

$$\times {}_M E^1_\nu \left[\frac{x}{\Lambda} \middle| \begin{matrix} (1,\Lambda) \; ; \; (M + \sum_{i=1}^{l} R_i - \eta_{G,g} + 1, \Lambda, 1) \\ (\lambda_\nu, 1+\mu_\nu) \; ; (1+\mu_i, \lambda_i, 1)_{1,\nu-1}, (M + \sum_{i=1}^{l} R_i - \eta_{G,g} + \theta + 1, \Lambda, 1) \end{matrix} \right].$$

(6.2.6)

IV. If we put multivariable polynomial and Aleph function is unity in (6.2.1), (6.2.4), (6.2.5) and (6.2.6) then we get the known results due to Bhatter [7, equations (4.2.1) , (4.2.5) , (4.2.6) and (4.2.7) respectively].

(i) $$\left[I^\theta_{C+} \left({}_\tau E^h_k (t-c) \right) \right](x) = \frac{1}{(\tau+1)_\theta}$$

$$\times {}_{\tau+\theta} E^{h+1}_{k+1} \left[(x-c) \middle| \begin{matrix} (\rho, a) \; ; (\gamma_i, q_i, s_i)_{1,h}, (\tau+1, a, 1) \\ (\alpha, \beta) ; (\delta_j, p_j, r_j)_{1,k}, (\tau+\theta+1, a, 1) \end{matrix} \right], \qquad (6.2.7)$$

(ii) $$\left[I^\theta_{C+} \left\{ E_{(1/\rho_i),(\mu_i)} (t-c) \right\} \right](x) = \frac{1}{\theta! \prod_{j=1}^{m-1} \Gamma(\mu_j)}$$

$$\times {}_\theta E^1_m \left[(x-c) \middle| \begin{matrix} (0,1) \; ; \; (1,1,1) \\ (1/\rho_m, \mu_m) ; (\mu_1, 1/\rho_1, 1), \ldots, (\mu_{m-1}, 1/\rho_{m-1}, 1), (\theta+1, 1, 1) \end{matrix} \right], \qquad (6.2.8)$$

(iii) $$\left[I_{C+}^{\theta} \left\{ E_{\gamma,k}\left[(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_m);t\right] \right\} \right](x) = \frac{1}{\theta\,!\prod_{j=1}^{m}\Gamma(\beta_j)}$$

$$\times_{\theta} E_{m+1}^{2}\left[x \middle| \begin{matrix} (0,1)\,;(\gamma,k,1),\,(1,1,1) \\ (1,1);(\beta_1,\alpha_1,1),\dots,(\beta_m,\alpha_m,1),(\theta+1,1,1) \end{matrix} \right], \tag{6.2.9}$$

(iv) $$\left[I_{C+}^{\theta} \left\{ \left(HE_{\mu_1,\dots,\mu_\nu}^{\lambda_1,\dots,\lambda_\nu};t \right) \right\} \right](x) = \frac{x^{\theta}}{(M+1)_{\theta}\prod_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)}$$

$$\times_{M} E_{\nu}^{1}\left[\frac{x}{\Lambda} \middle| \begin{matrix} (1,\Lambda)\ ;\ (M+1,\Lambda,1) \\ (\lambda_\nu,1+\mu_\nu);(1+\mu_i,\lambda_i,1)_{1,\nu-1},(M+\theta+1,\Lambda,1) \end{matrix} \right]. \tag{6.2.10}$$

6.3. THE IMAGE OF ALEPH- FUNCTION UNDER ERDELYI- KOBER (E-K) OPERATOR $\Xi_{0+}^{\eta,\theta}$

Theorem 2. If convergence conditions of (6.1.3), (6.1.6) and (6.1.4) are satisfied also $\eta,\theta \in C$, $R(\eta) > 0$ and $R(\theta) > 0$ then the E-K transform $\Xi_{0+}^{\eta,\theta}$ of the Multivariable polynomial, E-function and Aleph function is

$$\left[\Xi_{0+}^{\eta,\theta} \left\{ {}_{\tau}E_{k}^{h}(t)\, S_{V}^{U_1,\dots,U_l}(t^{R_1},\dots,t^{R_l})\, \aleph_{P_i,Q_i,c_i,r}^{M,N}(t) \right\} \right](x)$$

$$= \frac{1}{\left(\tau+\theta+\sum_{i=1}^{l} R_i - \eta_{G,g} + 1 \right)_{\eta}} \sum_{R_1,\dots,R_l=0}^{\sum_{i=1}^{l} U_i R_i \le V} (-V)_{\sum_{i=1}^{l} U_i R_i}\, A(V,R_1,\dots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times \sum_{G=1}^{M}\sum_{g=0}^{\infty} \frac{(-1)^{g}\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G} (x)^{\sum_{i=1}^{l} R_i - \eta_{G,g}}$$

$$\times {}_{\tau}E_{k+1}^{h+1}\left[x\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\theta+\eta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1)\end{matrix}\right.\right]. \tag{6.3.1}$$

Proof: With the help of equations (6.1.3), (6.1.6) and (6.1.4), we establish the E-K transform $\Xi_{0+}^{\eta,\theta}$ of the Multivariable polynomial, E-function and Aleph function as follows

$$\left[\Xi_{0+}^{\eta,\theta}\left\{{}_{\tau}E_k^h(t)\, S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l})\,\aleph_{P_i,Q_i,c_i,r}^{M,N}(t)\right\}\right](x)$$

$$=\left[\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}\, t^{\theta}\sum_{n=0}^{\infty}\Phi(n)\, t^{an+\tau}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\, A(V,R_1,\ldots,R_l)\frac{t^{R_i}}{R_i!}\right.$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}\, t^{-\eta_{G,g}}\,dt\right]$$

$$=\left[\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\sum_{n=0}^{\infty}\Phi(n)\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\, A(V,R_1,\ldots,R_l)\frac{t^{R_1}}{R_1!}\cdots\frac{t^{R_l}}{R_l!}\right.$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}\int_0^x (x-t)^{\eta-1}\, t^{(\theta+an+\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1)-1}\,dt\right]. \tag{6.3.2}$$

With the help of known result (6.1.9), right hand side of equation (6.3.2) change in the following form

$$=\left[\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\sum_{n=0}^{\infty}\Phi(n)\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\, A(V,R_1,\ldots,R_l)\frac{t^{R_1}}{R_1!}\cdots\frac{t^{R_l}}{R_l!}\right.$$

$$\times \sum_{G=1}^{M}\sum_{g=0}^{\infty} \frac{(-1)^g\, \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G} \frac{\Gamma(\theta+an+\tau+\sum_{i=1}^{l} R_i-\eta_{G,g}+1)\Gamma(\eta)}{\Gamma(\theta+\eta+an+\tau+\sum_{i=1}^{l} R_i-\eta_{G,g}+1)}\, x^{\eta+\theta+an+\tau+\sum_{i=1}^{l} R_i-\eta_{G,g}}\Bigg].$$

(6.3.3)

After little simplification, we get the required result (6.3.1).

6.3.1 SPECIAL CASES OF THEOREM 2

I. In the equation (6.3.1), if we substitute $h=0, \tau=0, k=m-1, \rho=0, a=1, s_i=0, r_j=1$

then we get E-K transform $\Xi^{\eta,\theta}_{0+}$ of the M-L type function (0.8.4), as follows

$$\left[\Xi^{\eta,\theta}_{0+}\left\{E_{(1/\rho_i),(\mu_i)}(t)\, S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l})\, \aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}+1\right)_\eta} \sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l} U_iR_i\le V} (-V)_{\sum_{i=1}^{l} U_iR_i}\, A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times \frac{1}{\prod_{j=1}^{m-1}\Gamma(\mu_j)} \sum_{G=1}^{M}\sum_{g=0}^{\infty} \frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}\,(x)^{\sum_{i=1}^{l} R_i-\eta_{G,g}}$$

$$\times {}_0E^1_m\left[x \,\middle|\, \begin{matrix} (0,1)\,;\,(\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}+1,1,1) \\ (1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}+1,1,1) \end{matrix}\right].$$

(6.3.4)

II. In the equation (6.3.1), if we substitute $h=1, \tau=0, k=m, \rho=0, a=1, s_i=0, r_j=1$

then we get E-K transform $\Xi^{\eta,\theta}_{0+}$ of the M-L type function (0.8.5), as follows

$$\left[\Xi^{\eta,\theta}_{0+}\left\{E_{\gamma,k}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);t\right] S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l})\, \aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(\theta+\sum\limits_{i=1}^{l} R_i-\eta_{G,g}+1\right)_\eta}\sum_{R_1,\ldots,R_l=0}^{\sum\limits_{i=1}^{l}U_iR_i\le V}(-V)_{\sum\limits_{i=1}^{l}U_iR_i}\;A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod\limits_{j=1}^{m}\Gamma(\beta_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}(x)^{\sum\limits_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_0E^2_{m+1}\left[x\,\middle|\,\begin{matrix}(0,1);(\gamma,k,1),\,(\theta+\sum\limits_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\\(1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\theta+\eta+\sum\limits_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\end{matrix}\right].$$

(6.3.5)

III. In the equation (6.3.1),if we substitute $h=0,\tau=M,k=\nu-1,\rho=0,a=\Lambda,s_i=0,r_j=1$ then we get E-K transform $\Xi^{\eta,\theta}_{0+}$ of the M-L type function (0.8.6), as follows

$$\left[\Xi^{\eta,\theta}_{0+}\left\{\left(HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu};t\right)S^{U_1,\ldots,U_l}_V(t^{R_1},\ldots,t^{R_l})\aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(M+\theta+\sum\limits_{i=1}^{l} R_i-\eta_{G,g}+1\right)_\eta}\sum_{R_1,\ldots,R_l=0}^{\sum\limits_{i=1}^{l}U_iR_i\le V}(-V)_{\sum\limits_{i=1}^{l}U_iR_i}\;A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod\limits_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}(x)^{\sum\limits_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_ME^1_\nu\left[\frac{x}{\Lambda}\,\middle|\,\begin{matrix}(1,\Lambda);(M+\theta+\sum\limits_{i=1}^{l}R_i-\eta_{G,g}+1,\,\Lambda,1)\\(\lambda_\nu,1+\mu_\nu);(1+\mu_i,\lambda_i,1)_{1,\nu-1},(M+\eta+\sum\limits_{i=1}^{l}R_i-\eta_{G,g}+\theta+1,\Lambda,1)\end{matrix}\right].$$

(6.3.6)

IV. If we substitute multivariable polynomial and Aleph function is unity in equations (6.3.1), (6.3.4), (6.3.5) and (6.3.6) then we get the results due to Bhatter [7, equations (4.3.1) , (4.3.5), (4.3.6) and (4.3.7) respectively]

(i) $$\left(\Xi_{0+}^{\eta,\theta}\left[{}_{\tau}E_{k}^{h}(t)\right]\right)(x)=\frac{1}{(\tau+\theta+1)_{\eta}}$$

$$\times{}_{\tau}E_{k+1}^{h+1}\left[x\left|\begin{matrix}(\rho,a)\;;(\gamma_i,q_i,s_i)_{1,h}\,,(\tau+\theta+1,a,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\theta+\eta+1,a,1)\end{matrix}\right.\right], \quad (6.3.7)$$

(ii) $$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{(1/\rho_i),(\mu_i)}(t)\right\}\right](x)=\frac{1}{(\theta+1)_{\eta}\prod_{j=1}^{m-1}\Gamma(\mu_j)}$$

$$\times{}_{0}E_{m}^{1}\left[x\left|\begin{matrix}(0,1)\;;\;(\theta+1,1,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),...,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+1,1,1)\end{matrix}\right.\right], \quad (6.3.8)$$

(iii) $$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{\gamma,k}\left[(\alpha_1,\beta_1),...,(\alpha_m,\beta_m);t\right]\right\}\right](x)=\frac{1}{(\theta+1)_{\eta}\prod_{j=1}^{m}\Gamma(\beta_j)}$$

$$\times{}_{0}E_{m+1}^{2}\left[x\left|\begin{matrix}(0,1)\;;(\gamma,k,1),\;(\theta+1,1,1)\\(1,1);(\beta_1,\alpha_1,1),...,(\beta_m,\alpha_m,1),(\theta+\eta+1,1,1)\end{matrix}\right.\right], \quad (6.3.9)$$

(iv) $$\left[\Xi_{0+}^{\eta,\theta}\left\{\left(HE_{\mu_1,...,\mu_\nu}^{\lambda_1,...,\lambda_\nu};t\right)\right\}\right](x)=\frac{1}{(M+\theta+1)_{\eta}\prod_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)}$$

$$\times{}_{M}E_{\nu}^{1}\left[\frac{x}{\Lambda}\left|\begin{matrix}(1,\Lambda)\,;(M+\theta+1,\Lambda,1)\\(\lambda_\nu,1+\mu_\nu)\,;(1+\mu_i,\lambda_i,1)_{1,\nu-1},(M+\eta+\theta+1,\Lambda,1)\end{matrix}\right.\right]. \quad (6.3.10)$$

6.4 THE IMAGE OF THE ALEPH- FUNCTION UNDER THE GENERALIZED INTEGRAL OPERATOR

Theorem 3 If convergence condition (6.1.3), (6.1.6) and (6.1.4) are fulfilled also $\eta,\theta,\sigma \in C,\ R(\eta)>0,\ R(\sigma)>0, R(\theta)>0$ and $t, x, \nu \in R$ then

$$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\,{}_\tau E_k^h\left\{\nu(z-t)^\sigma\right\} S_V^{U_1,\ldots,U_l}\left(\prod_{i=1}^{l}(z-t)^{R_i}\right) \aleph_{P_i,Q_i,c_i,r}^{M,N}(z-t)\,dz$$

$$=\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l} U_iR_i\le V}} (-V)_{\sum_{i=1}^{l} U_iR_i} A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}$$

$$\times B(\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}+\sigma\tau,\eta)\ (x-t)^{\eta+\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}-1}$$

$$\times\,{}_\tau E_{k+1}^{h+1}\left[\nu(x-t)^\sigma \,\middle|\, \begin{matrix}(\rho,a)\,;(\gamma_i,q_i,s_i)_{1,h}\,,(\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\end{matrix}\right].$$

(6.4.1)

Proof: we put value of multivariable polynomial, Aleph function and E- function by using the equations (6.1.3), (6.1.4) and (6.1.6) respectively, the left hand side of equation (6.4.1), we get

$$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\,{}_\tau E_k^h\left\{\nu(z-t)^\sigma\right\} S_V^{U_1,\ldots,U_l}\left(\prod_{i=1}^{l}(z-t)^{R_i}\right) \aleph_{P_i,Q_i,c_i,r}^{M,N}(z-t)\,dz$$

$$=\left[\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\sum_{n=0}^{\infty}\Phi(n)\left\{\nu(z-t)^\sigma\right\}^{an+\tau}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l} U_iR_i\le V}}(-V)_{\sum_{i=1}^{l} U_iR_i}\ A(V,R_1,\ldots,R_l)\right.$$

$$\times \prod_{i=1}^{l} \frac{(z-t)^{R_i}}{R_i!} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \frac{(-1)^g \, \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G} (z-t)^{-\eta_{G,g}} dz \Bigg],$$

$$= \Bigg[\sum_{n=0}^{\infty} \Phi(n)\, \nu^{an+\tau} \sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l} U_i R_i \le V} (-V)_{\sum_{i=1}^{l} U_i R_i} \, A(V,R_1,\ldots,R_l) \frac{1}{R_1!} \cdots \frac{1}{R_l!}$$

$$\times \sum_{G=1}^{M} \sum_{g=0}^{\infty} \frac{(-1)^g \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G} \int_t^x (x-z)^{\eta-1} (z-t)^{\theta+\sum_{i=1}^{l} R_i + \sigma a n + \sigma\tau - \eta_{G,g} - 1} dz \Bigg]. \tag{6.4.2}$$

By using known result (6.1.8) and after little simplification we get the required result (6.4.1).

Corollary 1 If we put $t = 0$ in theorem 3, then we get

$$\int_0^x (x-z)^{\eta-1} z^{\theta-1} \, {}_\tau E_k^h\{\nu z^\sigma\} \, S_V^{U_1,\ldots,U_l}(z^{R_1},\ldots,z^{R_l}) \, \aleph^{M,N}_{P_i,Q_i,c_i,r}(z)\, dz$$

$$= \sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l} U_i R_i \le V} (-v)_{\sum_{i=1}^{l} U_i R_i} \, A(V,R_1,\ldots,R_l) \frac{1}{R_1!} \cdots \frac{1}{R_l!} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \frac{(-1)^g \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}$$

$$\times B\left(\theta + \sum_{i=1}^{l} R_i - \eta_{G,g} + \sigma\tau, \eta\right) (x)^{\eta+\theta+\sum_{i=1}^{l} R_i - \eta_{G,g} - 1}$$

$$\times {}_\tau E_{k+1}^{h+1}\left[\nu x^\sigma \,\middle|\, \begin{matrix} (\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau+\sum_{i=1}^{l} R_i-\eta_{G,g},a\sigma,1) \\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau+\sum_{i=1}^{l} R_i-\eta_{G,g},a\sigma,1) \end{matrix} \right]. \tag{6.4.3}$$

Corollary 2 If we put $t = 0, \eta = 1$ in theorem 3, then we get

$$\int_0^x z^{\theta-1}\, {}_{\tau}E_k^h\{\nu z^{\sigma}\}\, S_V^{U_1,\dots,U_l}(z^{R_1},\dots,z^{R_l})\, \aleph_{P_i,Q_i,c_i,r}^{M,N}(z)\,dz$$

$$= \sum_{R_1,\dots,R_l=0}^{\sum_{i=1}^{l} U_i R_i \le V} (-V)_{\sum_{i=1}^{l} U_i R_i} A(V,R_1,\dots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}$$

$$\times\left(\frac{(x)^{\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}}}{\sigma\tau+\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}}\right){}_{\tau}E_{k+1}^{h+1}\left[\nu x^{\sigma}\middle|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau+\sum_{i=1}^{l} R_i-\eta_{G,g},a\sigma,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(1+\theta+\sigma\tau+\sum_{i=1}^{l} R_i-\eta_{G,g},a\sigma,1)\end{matrix}\right].$$

(6.4.4)

6.4.1 SPECIAL CASES OF THEOREM 3

I. If we substitute $h=0,\tau=0,k=m-1,\rho=0,a=1,s_i=0,r_j=1$ in the equation (6.4.1), we get general integral transform of the M-L type function (0.8.4), as follows

$$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\, E_{(1/\rho_i),(\mu_i)}\{\nu(z-t)^{\sigma}\}$$

$$\times S_V^{U_1,\dots,U_l}\left((z-t)^{R_1},\dots,(z-t)^{R_l}\right)\aleph_{P_i,Q_i,c_i,r}^{M,N}(z-t)\,dz$$

$$= \sum_{R_1,\dots,R_l=0}^{\sum_{i=1}^{l} U_i R_i \le V} (-V)_{\sum_{i=1}^{l} U_i R_i} A(V,R_1,\dots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}$$

$$\times\frac{B(\theta+\sum_{i=1}^{l} R_i-\eta_{G,g},\eta)}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}(x-t)^{\eta+\theta+\sum_{i=1}^{l} R_i-\eta_{G,g}-1}$$

$$\times {}_0E_m^1\left[\nu(x-t)^{\sigma}\left|\begin{matrix}(0,1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\sigma,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),...,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\sigma,1)\end{matrix}\right.\right].$$

(6.4.5)

II. In the equation (6.4.1), if we substitute $h=1,\tau=0,k=m,\rho=0,a=1,s_i=0,r_j=1$ then we get general integral transform of the M-L type function (0.8.5), as follows

$$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\, E_{\gamma,k}\{(\alpha_1,\beta_1),...,(\alpha_m,\beta_m);(z-t)\}$$

$$\times S_V^{U_1,...,U_l}\left\{(z-t)^{R_1},...,(z-t)^{R_l}\right\}\aleph_{P_i,Q_i,c_i,r}^{M,N}(z-t)\,dz$$

$$=\sum_{R_1,...,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,...,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}$$

$$\times\frac{B(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\eta)}{\prod_{j=1}^{m}\Gamma(\beta_j)}(x-t)^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times {}_0E_{m+1}^2\left[(x-t)\left|\begin{matrix}(0,1),(\gamma,k,1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},1,1)\\(1,1);(\beta_1,\alpha_1,1),...,(\beta_m,\alpha_m,1),(\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},1,1)\end{matrix}\right.\right].$$

(6.4.6)

III. In the equation (6.4.1), if we substitute $h=0,\tau=M,k=\nu-1,\rho=0,a=\Lambda,s_i=0,r_j=1$ then we obtain general integral transform of the M-L type function (0.8.6), as follows

$$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\, HE_{\mu_1,...,\mu_\nu}^{\lambda_1,...,\lambda_\nu}\left\{\nu(z-t)^{\sigma}\right\}$$

$$\times S_V^{U_1,\ldots,U_l}\left\{(z-t)^{R_1},\ldots,(z-t)^{R_l}\right\} \aleph_{P_i,Q_i,c_i,r}^{M,N}(z-t)\,dz$$

$$=\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l} U_iR_i\le V}} (-V)_{\sum_{i=1}^{l} U_iR_i} A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}$$

$$\times\frac{B(\theta+\sigma M+\sum_{i=1}^{l}R_i-\eta_{G,g},\eta)}{\prod_{j=1}^{\nu-1}\Gamma(1+\mu_j)}(x-t)^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times\,{}_M E_\nu^1\left[\frac{\nu(x-t)^\sigma}{\Lambda}\,\middle|\,\begin{matrix}(1,\Lambda);(\sigma M+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\sigma\Lambda,1)\\(\lambda_\nu,1+\mu_\nu);(1+\mu_i,\lambda_i,1)_{1,\nu-1},(\sigma M+\eta+\sum_{i=1}^{l}R_i-\eta_{G,g}+\theta,\sigma\Lambda,1)\end{matrix}\right]$$

(6.4.7)

IV. If we put multivariable polynomial and Aleph function is unity in (6.4.1), (6.4.4), (6.4.5) and (6.4.6) then we get the results due to Bhatter [7, equation (4.4.1) , (4.4.7) , (4.4.8) and (4.4.9) respectively]

(i) $$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\,{}_\tau E_k^h\left\{\nu(z-t)^\sigma\right\}dz=B(\theta+\sigma\tau,\eta)\,(x-t)^{\eta+\theta-1}$$

$$\times\,{}_\tau E_{k+1}^{h+1}\left[\nu(x-t)^\sigma\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau,a\sigma,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau,a\sigma,1)\end{matrix}\right].$$

(6.4.8)

(ii) $$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}E_{(1/\rho_i),(\mu_i)}\left\{\nu(z-t)^\sigma\right\}dz=\frac{(x-t)^{\eta+\theta-1}B(\theta,\eta)}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}$$

$$\times {}_0E_m^1\left[\ \nu(x-t)^{\sigma}\Big|\begin{matrix}(0,1),(\theta,\sigma,1)\\ (1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta,\sigma,1)\end{matrix}\right].$$

(6.4.9)

(iii) $$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\,E_{\gamma,k}\left\{(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);(z-t)\right\}dz=\frac{(x-t)^{\eta+\theta-1}B(\theta,\eta)}{\prod_{j=1}^{m}\Gamma(\beta_j)}$$

$$\times {}_0E_{m+1}^2\left[\ (x-t)\Big|\begin{matrix}(0,1),(\gamma,k,1),(\theta,1,1)\\ (1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\eta+\theta,1,1)\end{matrix}\right].$$

(6.4.10)

(iv) $$\int_t^x (x-z)^{\eta-1}(z-t)^{\theta-1}\,HE_{\mu_1,\ldots,\mu_\nu}^{\lambda_1,\ldots,\lambda_\nu}\left\{\nu(z-t)^{\sigma}\right\}dz=\frac{(x-t)^{\eta+\theta-1}B(\theta+\sigma M,\eta)}{\prod_{j=1}^{\nu-1}\Gamma(1+\mu_j)}$$

$$\times {}_ME_\nu^1\left[\frac{\nu(x-t)^{\sigma}}{\Lambda}\Big|\begin{matrix}(1,\Lambda);(\gamma,k,1),(\sigma M+\theta,\sigma\Lambda,1)\\ (\lambda_\nu,1+\mu_\nu);(1+\mu_i,\lambda_i,1)_{1,\nu-1},(\sigma M+\eta+\theta,\sigma\Lambda,1)\end{matrix}\right].$$

(6.4.11)

.

APPENDIX-A

THE GENERAL CLASS OF POLYNOMIALS

S_V^U,

MULTIVARIABLE POLYNOMIAL

AND

THE GENERALIZED CLASS OF SEQUENCE

$S_n^{\mu,\delta,\tau}$

PART-A

A GENERAL CLASS OF POLYNOMIALS

The general class of polynomials introduced by Srivastava [104] (see also [105] and [106]) is defined as the following way:

$$S_V^U[x] = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\, A_{V,R}}{R!} x^R, V = 0,1,2,...; \tag{A.1}$$

where the coefficients $A_{V,R}$ are arbitrary constants, real or complex and U is an arbitrary positive integer.

SPECIAL CASES OF THE POLYNOMIALS $S_V^U[x]$

The general class of polynomials can be reduced to the generalized hypergeometric polynomials and the classical orthogonal polynomials, on considerably specializing of the coefficients $A_{V,R}$ present in (A.1) which is cited in the chapter referred to above.

(i) Hermite Polynomial

If we put $U = 2$, $A_{V,R} = (-1)^R$ in (A.1), we have

$$S_V^2[x] \to x^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{x}}\right), \tag{A.2}$$

where $H_V[x]$ is the Hermite polynomial Szego [127, p.106, Eq. (5.5.4)] , define as follows

$$H_V[x] = \sum_{R=0}^{[V/2]} \frac{(-1)^R\, V!\,(2x)^{V-2R}}{R!\,(V-2R)!}$$

$$= (2x)^V {}_2F_0\left[\frac{-V}{2}, \frac{-V+1}{2}; -; \frac{-1}{x^2}\right].$$

(ii) The Jacobi Polynomial

On assuming $U = 1$, $A_{V,R} = \binom{V+\alpha}{V} \frac{(\alpha+\beta+V+1)_R}{(\alpha+1)_R}$ in (A.1), we have

$$S_V^1[x] \to P_V^{(\alpha,\beta)}(1-2x). \tag{A.3}$$

Szego [127, p.68, Eq. (4.3.2)] defined the Jacobi polynomials $P_V^{(\alpha,\beta)}$, which is define as

$$P_V^{(\alpha,\beta)}(x) = \sum_{R=0}^{V} \binom{V+\alpha}{V-R}\binom{V+\beta}{R}\left(\frac{x-2}{2}\right)^R\left(\frac{x+1}{2}\right)^{V-R}$$

$$= \frac{(1+\alpha)_V}{V!}\sum_{R=0}^{V}\frac{(-V)_R\,(1+\alpha+\beta+V)_R}{(1+\alpha)_R\;R!}\left(\frac{1-x}{2}\right)^R,$$

also the polynomials $S_V^U[x]$ defined by (A.1) can also be reduced to several special cases of the Jacobi polynomials $P_V^{(\alpha,\beta)}$ for example, the Legendre polynomials $P_V(x)$, the Tchebychef polynomials $T_V(x)$, the Gegenbauer polynomial $C_V^{v}(x)$ and $U_V(x)$ of the first and second kinds

$$C_V^{\alpha+1/2}(x) = \binom{V+\alpha}{V}^{-1}\binom{V+2\alpha}{V}P_V^{(\alpha,\alpha)}(x), \tag{A.4}$$

$$P_V(x) = P_V^{(0,0)}(x), \tag{A.5}$$

$$T_V(x) = \binom{V-1/2}{V}^{-1} P_V^{(-1/2,-1/2)}(x), \tag{A.6}$$

$$U_V(x) = \frac{1}{2}\binom{V+1/2}{V+1}^{-1} P_V^{(1/2,1/2)}(x). \tag{A.7}$$

(iii) The Laguerre Polynomial

On assuming $U=1$, $A_{V,R} = \binom{V+\alpha}{V}\frac{1}{(\alpha+1)_R}$ in (A.1), we have

$$S_V^1[x] \to L_V^{(\alpha)}(x). \tag{A.8}$$

Szego [127, p.101, Eq. (5.1.6)] defined the Laguerre polynomial $L_V^{(\alpha)}(x)$, as follows

$$L_V^{(\alpha)}(x)=\frac{(1+\alpha)_V}{V!}\ {}_1F_1[-V;1+\alpha;x].$$

(iv) The Bessel Polynomial

Considering $U=1$, $A_{V,R}=(\alpha+V-1)_R$ in (A.1) ,

$$S_V^1[x]\rightarrow y_V(-\beta x,\alpha,\beta). \tag{A.9}$$

Krall and Frink [53, p.108, Eq. (34)] defined the Bessel polynomial $y_V(x,\alpha,\beta)$ as follows

$$y_V(x,\alpha,\beta)=\sum_{R=0}^{V}\frac{(-V)_R\ (\alpha+V-1)_R}{R!}\left(\frac{-x}{\beta}\right)^R$$
$$={}_2F_0\left[-V;\alpha+V\text{-}1;-;\frac{-x}{\beta}\right].$$

(v) The Gould and Hopper Polynomial (Generalized Hermite Polynomial)

Considering $A_{V,R}=1$ in (A.1), we have

$$S_V^U[x]\rightarrow\left(-\frac{x}{h}\right)^{V/U}g_V^U\left[\left(-\frac{h}{x}\right)^{1/U},h\right]. \tag{A.10}$$

Gould and Hopper [29, p.58, Eq. (6.2)] defined Gould and Hopper polynomials $g_V^U[x,h]$, as follows

$$g_V^U[x,h]=\sum_{R=0}^{[V/U]}\frac{V!}{R!(V\text{-}UR)!}\ h^R\ x^{V-UR}$$
$$=x^V\ {}_UF_0\left[\Delta(U;-V);-;h\left(-\frac{U}{x}\right)^U\right].$$

(vi)The Brafman Polynomial

Considering $A_{V,R} = \frac{(\alpha_1)_R \cdots (\alpha_p)_R}{(\beta_1)_R \cdots (\beta_q)_R}$ in (A.1),

$$S_V^U[x] \to B_V^U\left[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x\,U^U\right]. \tag{A.11}$$

Brafman [11, p.186] defined the Brafman polynomial, as follows

$$B_V^U\left[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x\right] = {}_{U+p}F_p\left[\Delta(U;-V),\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x\right],$$

here $\Delta(U;V)$ constrict the array of U parameters $\frac{V}{U},\frac{V+1}{U},\dots,\frac{V+U-1}{U}$, $U \geq 1$ the set $\Delta(0;V)$ being unoccupied.

(vii) The Konhauser Biorthogonal Polynomial

If we substitute $U = 1$, $A_{V,R} = \frac{1}{V!}\frac{\Gamma(1+\alpha+kV)}{\Gamma(1+\alpha+kR)}$ in (A.1),

$$S_V^1[x] \to Z_V^\alpha(x^{1/k};k). \tag{A.12}$$

Konhauser [51, p.304, Eq. (5)] defined the biorthogonal polynomial, as follows

$$Z_V^\alpha(x;k) = \frac{\Gamma(1+\alpha+kV)}{V!}\sum_{R=0}^{V}(-1)^R\binom{V}{R}\frac{x^{kR}}{\Gamma(1+\alpha+kR)}$$

$$= \frac{(1+\alpha)_{kV}}{V!}\,{}_1F_k\left(\begin{matrix}-V;\\ \Delta(k;\alpha+1);\end{matrix}\left(\frac{x}{k}\right)^k\right).$$

(viii) Shively Polynomial

Putting $U = 1$, $A_{V,R} = \frac{(\lambda+V)_V}{V!}\frac{(\alpha_1)_R \cdots (\alpha_p)_R}{(\lambda+V)_R(\beta_1)_R \cdots (\beta_q)_R}$ in (A.1),

$$S_V^U[x] \to S_V^{(\lambda)}[x]. \tag{A.13}$$

Srivastava and Manocha [110, p.187, Eq. (49); 13, p. 54] defined the Shively polynomial $S_V^{(\lambda)}[x]$, as follows

$$S_V^{(\lambda)}[x] = \frac{(\lambda+V)_V}{V!}\,{}_{p+1}F_{q+1}\left(\begin{matrix}-V,\alpha_1,\dots,\alpha_p;\\ \lambda+V,\beta_1,\dots,\beta_q;\end{matrix}x\right).$$

(ix) Bateman Polynomials

(a) Substituting $U=1$, $A_{V,R}=\frac{(1+V)_R}{R!\,R!}$,

$$S_V^1[x]\rightarrow Z_V[x]. \qquad (A.14)$$

Srivastava and Manocha [110, p.183, Eq. (42); 1, pp.574 & 575] defined the Bateman polynomial $Z_V[x]$, as follows

$$Z_V[x]={}_2F_2\left(\begin{matrix}-V,V+1;\\1,1\ ;\end{matrix}\ x\right),$$

(b) Assuming $U=1$, $A_{V,R}=\frac{1}{V!}\frac{\Gamma(\frac{\lambda}{2}+1+\sigma+V)}{\Gamma(\lambda+R+1)\ \Gamma(\frac{\lambda}{2}+1+\sigma+R)}$ in (A.1),

$$S_V^U[x]\rightarrow x^{-\lambda/2}\,J_V^{(\lambda,\sigma)}(\sqrt{x}), \qquad (A.15)$$

where

$$J_V^{(\lambda,\sigma)}(x)=\begin{pmatrix}\frac{\lambda}{2}+V+\sigma\\V\end{pmatrix}\frac{x^{\lambda}}{\Gamma(\lambda+1)}\ {}_1F_2\left(\begin{matrix}-V;\\\lambda+1,\frac{\lambda}{2}+\sigma+1;\end{matrix}\ x^2\right).$$

(x) Cesaro Polynomial

Considering $U=1$, $A_{V,R}=\frac{(s+1)_V\ R!}{V!\,(-s-V)_R}$,

$$S_V^1[x]\rightarrow g_V^{(s)}(x). \qquad (A.16)$$

Srivastava and Manocha [110, p.449, Eq. (20)] defined the Cesaro polynomial $g_V^{(s)}(x)$, as follows

$$g_V^{(s)}(x)=\begin{pmatrix}V+s\\V\end{pmatrix}{}_2F_1\left(\begin{matrix}-V,1;\\-s-V;\end{matrix}\ x\right).$$

(xi) Generalized Hypergeometric Polynomial by Fasenmyer

Putting $U=1$ and $A_{V,R}=\dfrac{(V+1)_V\,(\alpha_1)_R\cdots(\alpha_p)_R}{(1/2)_R\,R!\,(\beta_1)_R\cdots(\beta_q)_R}$,

$$S_V^U[x]\rightarrow f_V\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x\right]. \tag{A.17}$$

Srivastava and Manocha [110, p.182, Eq. (41); 6, p.806, Eq. (1)] defined the Generalized Hypergeometric polynomial $f_V\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x\right]$, as follows

$$f_V\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x\right]={}_{p+2}F_{q+2}\left(\begin{matrix}-V,V+1,\alpha_1,\ldots,\alpha_p; \\ 1/2,1,\beta_1,\ldots,\beta_q;\end{matrix}\;x\right).$$

(xii) Krawtchouk Polynomial

Taking $U=1$, $A_{V,R}=\dfrac{(-y)_R}{(-N)_R}$, in (A.1)

$$S_V^1[x]\rightarrow K_V\left[y,x^{-1};N\right]. \tag{A.18}$$

Srivastava and Manocha [110, p.75, Eq. (2)] defined Krawtchouk polynomial $K_V[y,x;N]$, as follows

Where $K_V[y,x;N]={}_2F_1\left(\begin{matrix}-V,-y; \\ -N;\end{matrix}\;x^{-1}\right)$,

$0<x<1, y=0,1,\ldots,N.$

(xiii) Meixner Polynomial

Considering $U=1$, $A_{V,R}=\dfrac{(-y)_R}{(\beta)_R}$, in(A.1)

$$S_V^1[x]\rightarrow M_V\left[y;\beta,(1-x)^{-1}\right]. \tag{A.19}$$

Srivastava and Manocha [110, p. 75, Eq. (3)] defined Meixner polynomial $M_V(y,\beta,x)$, as follows

$$M_V[y,\beta;x] = {}_2F_1\left(\begin{matrix}-V,-y; \\ -\beta;\end{matrix}\ 1-x^{-1}\right),$$

$$0 < x < 1, y = 0,1,\ldots, N, \beta > 0 .$$

(xiv) Gottlieb Polynomial

If we put $U = 1$, $A_{V,R} = \frac{(-y)_R}{R!}$, in (A.1)

$$S_V^1[x] \rightarrow (1-x)^V I_V[y; \log(1-x)\ ]. \tag{A.20}$$

Srivastava and Manocha [110, p.185, Eq. (47); 7, p. 454.Eq. (2.3)] defined by Gottlieb Polynomial $I_V(y,t)$, as follows

$$I_V[y;\ t\ ] = e^{-Vt}\ {}_2F_1(-V,-y;1;1-e^{-t}) .$$

The Polynomials $S_V^U[x]$ can also be reduced to other hypergeometric polynomials such as extended Jacobi polynomials [117, part I, p. 24 ; 117, Part II, p. 106, Eq. (1.3)] and their generalizations [116,p.471,Eqs. (4.2) and (4.3)] and [113, Part II, p. 107, Eq. (1.11); 22, Part II, p.108, Eq. (1.17)] etc.

For more details, one can also refer to papers by Srivastava and Singh [122, pp. 158-162] and Srivastava and Garg [107, p.686].

MULTIVARIABLE ANALOGUE OF $S_V^U[x]$

Srivastava and Garg [107, p.686, eq. (1.4)] have defined the multivariable polynomial as follows

$$S_V^{U_1,\ldots,U_k}(x_1,\ldots,x_k)=\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k}U_iR_i\le V}}(-V)_{\sum_{i=1}^{k}U_iR_i}\,A(V,R_1,\ldots,R_k)\,\frac{x_1^{R_1}}{R_1!}\cdots\frac{x_k^{R_k}}{R_k!},$$

(A.21)

where the coefficients are $A(V,R_1,\ldots,R_k)$ are arbitrary constants (real or complex) and $V=0,1,2\ldots$and $U_1,\ldots,U_k$ arbitrary positive integers. The class of multivariable polynomials can be reduced to several multivariable polynomials by suitably specializing the coefficients $A(V,R_1,\ldots,R_k)$, occurring in (A.21), defined by different authors.

(a) Multivariable Hypergeometric Polynomials $F_D^{(k)}$

In (A.21), if we consider

$$A(V,R_1,\ldots,R_k)=\frac{(\beta_1)_{R_1\phi_1}\cdots(\beta_k)_{R_k\phi_k}}{(\gamma)_{R_1\psi_1+\ldots+R_k\psi_k}},$$

then

$$S_V^{U_1,\ldots,U_k}(x_1,\ldots,x_k)\to F_D^{(k)}\left[(-V:U_i):(\beta_i,\phi_i);(\gamma,\psi_i);x_1,\ldots,x_k\right]. \tag{A.22}$$

Srivastava and Manocha [110, pp. 462-463, Eq. 9.4 (4)] defined the first class of multivariable hypergeometric polynomials $F_D^{(k)}$, as follows

$$F_D^{(k)}\left[(-V:U_i):(\beta_i,\phi_i);(\gamma,\psi_i);x_1,\ldots,x_k\right]$$

$$= \sum_{\substack{R_1,\ldots,R_k = 0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i} \frac{(\beta_1)_{R_1\phi_1} \cdots (\beta_k)_{R_k\phi_k}}{(\gamma)_{R_1\psi_1 + \ldots + R_k\psi_k}} \frac{x_1^{R_1}}{R_1!} \cdots \frac{x_k^{R_k}}{R_k!}. \qquad \text{(A.23)}$$

(b) Generalized Lauricella Polynomial

In (A.21), if we consider

$$A(V,R_1,\ldots,R_k) = \frac{\prod_{s=1}^{M} (\beta_s)_{\phi_s^{(1)} R_1 + \ldots + \phi_s^{(k)} R_k} \prod_{s=1}^{M_1} \left(\beta_s^{(1)}\right)_{\delta_s^{(1)} R_1} \cdots \prod_{s=1}^{M_k} \left(\beta_s^{(k)}\right)_{\delta_s^{(k)} R_k}}{\prod_{s=1}^{N} (\gamma_s)_{\psi_s^{(1)} R_1 + \ldots + \psi_s^{(k)} R_k} \prod_{s=1}^{N_1} \left(\gamma_s^{(1)}\right)_{\lambda_s^{(1)} R_1} \cdots \prod_{s=1}^{N_k} \left(\gamma_s^{(k)}\right)_{\lambda_s^{(k)} R_k}},$$

then

$$S_V^{U_1,\ldots,U_k}\left[x_1,\ldots,x_k\right] \to F_{N;N_1,\ldots,N_k}^{M+1;M_1,\ldots,M_k}\left[\begin{matrix} x_1 \\ \vdots \\ x_k \end{matrix} \middle| \begin{matrix} (-V:U_1,\ldots,U_k), \\ \left(\beta_s;\phi_s^{(1)},\ldots,\phi_s^{(k)}\right)_{1,M};\left(\beta_s^{(1)};\delta_s^{(1)}\right)_{1,M_1},\ldots,\left(\beta_s^{(k)};\delta_s^{(k)}\right)_{1,M_k} \\ \left(\gamma_s;\psi_s^{(1)},\ldots,\psi_s^{(k)}\right)_{1,N};\left(\gamma_s^{(1)};\lambda_s^{(1)}\right)_{1,N_1},\ldots,\left(\gamma_s^{(k)};\lambda_s^{(k)}\right)_{1,N_k} \end{matrix}\right].$$

(A.24)

Srivastava and Daoust [106, p.454] defined the $F_{N;N_1,\ldots,N_k}^{M+1;M_1,\ldots,M_k}$ polynomial form of generalized Lauricella function, as follows

$$F_{N;N_1,\ldots,N_k}^{M+1;M_1,\ldots,M_k}\left[\begin{matrix} x_1 \\ \vdots \\ x_k \end{matrix} \middle| \begin{matrix} (-V:U_1,\ldots,U_k),\left(\beta_s;\phi_s^{(1)},\ldots,\phi_s^{(k)}\right)_{1,M};\left(\beta_s^{(1)};\delta_s^{(1)}\right)_{1,M_1},\ldots,\left(\beta_s^{(k)};\delta_s^{(k)}\right)_{1,M_k} \\ ,\left(\gamma_s;\psi_s^{(1)},\ldots,\psi_s^{(k)}\right)_{1,N};\left(\gamma_s^{(1)};\lambda_s^{(1)}\right)_{1,N_1},\ldots,\left(\gamma_s^{(k)};\lambda_s^{(k)}\right)_{1,N_k} \end{matrix}\right]$$

$$= \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i} \frac{\prod_{s=1}^{M} (\beta_s)_{\varphi_s^{(1)} R_1 + \dots + \varphi_s^{(k)} R_k}}{\prod_{s=1}^{N} (\gamma_s)_{\psi_s^{(1)} R_1 + \dots + \psi_s^{(k)} R_k}}$$

$$\times \frac{\prod_{s=1}^{M_1} \left(\beta_s^{(1)}\right)_{\delta_s^{(1)} R_1} \cdots \prod_{s=1}^{M_k} \left(\beta_s^{(k)}\right)_{\delta_s^{(k)} R_k}}{\prod_{s=1}^{N_1} \left(\gamma_s^{(1)}\right)_{\lambda_s^{(1)} R_1} \cdots \prod_{s=1}^{N_k} \left(\gamma_s^{(k)}\right)_{\lambda_s^{(k)} R_k}} \prod_{i=1}^{k} \frac{x_i^{R_i}}{R_i!} . \tag{A.25}$$

(c) Multivariable Jacobi Polynomial

In (A.21) if we take $U_1 = \cdots = U_k = 1$ and

$$A(V, R_1, \dots, R_k) = \frac{\prod_{i=1}^{k} (1+\alpha_i)_V}{(V!)^k} \frac{\prod_{i=1}^{k} (1+\alpha_i+\beta_i+V)_{R_i}}{\prod_{i=1}^{k} (1+\alpha_i)_{R_i}} ,$$

then

$$S_V^{1,\cdots,1}(x_1, \cdots, x_k) \to P_V^{\alpha_1,\beta_1;\dots;\alpha_k,\beta_k} \left[1-2x_1, \dots, 1-2x_k \right] . \tag{A.26}$$

Srivastava [121, p.65, Eq. (1.4)] defined the Jacobi Polynomial $P_V^{\alpha,\beta;\dots;\alpha_k,\beta_k}$ of k variables, as follows

$$P_V^{\alpha_1,\beta_1;\dots;\alpha_k,\beta_k}(x_1, \dots, x_k) = \frac{\prod_{i=1}^{k} (1+\alpha_i)_V}{(V!)^k} \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i}$$

$$\times \frac{\prod_{i=1}^{k}(1+\alpha_i+\beta_i+V)_{R_i}}{\prod_{i=1}^{k}(1+\alpha_i)_{R_i}\, R_i!}\prod_{i=1}^{k}\left(\frac{1-x_i}{2}\right)^{R_i}. \tag{A.27}$$

(d) Multivariable Bessel Polynomial

In (A.21) if we put $U_1 = \cdots = U_k = 1$ and

$$A(V,R_1,\ldots,R_k) = (1+\alpha_1+V)_{R_1}\prod_{i=2}^{k}(1+\alpha_i+n_i)_{R_i},$$

then

$$S_V^{1,\cdots,1}(x_1,\cdots,x_k) \to y_{V,n_2,\ldots,n_k}^{\alpha,\ldots,\alpha_k}\left[-2x_1,\ldots,-2x_k\right]. \tag{A.28}$$

Srivastava [119, p.164, Eq. (2.3)] defined the Bessel Polynomial $y_{V,n_2,\ldots,n_k}^{\alpha,\ldots,\alpha_k}$ of k variables ,as follows

$$y_{V,n_2,\ldots,n_k}^{\alpha,\ldots,\alpha_k}(x_1,\ldots,x_k) = \sum_{\substack{R_1,\ldots,R_k=0 \\ \sum_{i=1}^{k} R_i \le V}} (-V)_{\sum_{i=1}^{k} R_i}\frac{(1+\alpha_1+V)_{R_1}}{R_1!\ldots R_k!}$$

$$\times \prod_{i=2}^{k}\left(1+\alpha_i+n_i\right)_{R_i}\prod_{i=1}^{k}\left(\frac{-x_i}{2}\right)^{R_i}. \tag{A.29}$$

(e) Multivariable Hermite Polynomial

In (A.21) if we substitute $U_1 = \cdots = U_k = 2$ and

$$A(V,R_1,\ldots,R_k) = (-1)^{R_1+\ldots+R_k},$$

then

$$S_V^{2,\cdots,2}(x_1,\cdots,x_k) \to (x_1)^{V/2} H_V\left[X_1,\ldots,X_k\right]. \tag{A.30}$$

where

$$X_1 = \frac{1}{2\sqrt{x_1}}, X_j = \frac{x_j}{x_1}, \quad j = 1, \ldots, k .$$

Srivastava [120, p.97, Eq. (2.4)] defined the Multivariable Hermite polynomial $H_V(X_1, \ldots, X_k)$,as follows:

$$H_V(X_1, \ldots, X_k) = x_1^V \sum_{\substack{R_1, \ldots, R_k = 0 \\ \sum_{i=1}^{k} 2R_i \le V}} (-V)_{\sum_{i=1}^{k} 2R_i} \frac{(2)^{V - 2(R_1 + \ldots + R_k)}}{R_1! \ldots R_k!}$$

$$\times \left(\frac{-1}{x_1^2}\right)^{R_1} \prod_{i=2}^{k} \left(\frac{-x_i}{x_1^2}\right)^{R_i} . \qquad \text{(A.31)}$$

PART-B

THE GENERALIZED SEQUENCE OF FUNCTIONS

Raijada [80, p.64, Eq. (2.18)] defined the Rodrigues type formula in terms of generalized polynomial set is represented as follows

$$S_n^{\alpha,\beta,\tau}[x;r,s,q,A,B,m,k,l]$$
$$= (Ax+B)^{-\alpha}\left(1-\tau x^r\right)^{-\beta/\tau} T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}\left(1-\tau x^r\right)^{\frac{\beta}{\tau}+sn}\right], \tag{A.32}$$

with the differential operator $T_{k,l}$ defined as

$$T_{k,l} = x^l\left[k + x\frac{d}{dx}\right]. \tag{A.33}$$

Raijada [80, p.71, Eq. (2.3.4)] expressed the generalized sequence of functions in terms of series is as follows

$$S_n^{\alpha,\beta,\tau}[x;r,s,q,A,B,m,k,l]$$
$$= \sum_{\nu,u,e,p} \theta(\nu,u,e,p)\, x^R \left(1-\tau x^r\right)^{sn-\nu}, \tag{A.34}$$

where

$$\theta(\nu,u,e,p) = B^{qn} \sum_{\nu=0}^{m+n} \sum_{u=0}^{\nu} \sum_{e=0}^{m+n} \sum_{p=0}^{e} \frac{(-1)^e(-e)_p(\alpha)_e(-\alpha-qn)_p(-\beta/\tau-sn)_\nu}{\nu!\,u!\,e!\,p!\,(1-\alpha-e)_p}$$
$$\times l^{(m+n)}\left(\frac{A}{B}\right)^e \left(\frac{p+k+ru}{l}\right)_{m+n} (-\tau)^\nu, \tag{A.35}$$

here

$$R = l(m+n) + r\nu + e.$$

Many research workers, such as Krall and Frink [53], Singh [98], Chatterja [13], Dhillon [16] , Gould and Hopper [29], and Singh and Srivastava [99], etc. extend the Generalized sequence of functions is given in (A.32). We mention below the following important specials cases of (A.32):

(i) If we put $A = 1, B = 0$ in (A.32) and (A.33), then we obtain the following form

$$S_n^{\alpha,\beta,\tau}[x;r,s,q,1,0,m,k,l]$$

$$= x^{-\alpha}\left(1-\tau x^r\right)^{-\beta/\tau} T_{k,l}^{m+n}\left[x^{\alpha+qn}\left(1-\tau x^r\right)^{\frac{\beta}{\tau}+sn}\right]$$

$$= \sum_{\nu=0}^{m+n}\sum_{u=0}^{\nu} \theta(\nu,u)\, x^{R'}\left(1-\tau x^r\right)^{sn-\nu}, \tag{A.36}$$

where

$$\theta(\nu,u) = \frac{l^{(m+n)}(-\nu)_u}{\nu!\,u!}\left(\frac{\alpha+qn+k+ru}{l}\right)_{m+n}\left(\frac{-\beta}{\tau}-sn\right)_\nu (-\tau)^\nu, \tag{A.37}$$

here $R' = l(m+n)+qn+r\nu$.

(ii) Further putting $\tau \to 0$ in (A.32) and (A.33) and using the well known results

$$\mathrm{Lt}_{\tau\to 0}\left(1-\tau x^r\right)^{\frac{\beta}{\tau}} = \exp\left(-\beta x^r\right),$$

$$\mathrm{Lt}_{|b|\to\infty}(b)_n\left(\frac{z}{b}\right)^n = z^n.$$

Then, important polynomial set is given as follows

$$S_n^{\alpha,\beta,0}[x;r,q,A,B,m,k,l]$$

$$= (Ax+B)^{-\alpha}\exp\left(\beta x^r\right) T_{k,l}^{m+n}\left[(Ax+B)^{\alpha+qn}\exp\left(-\beta x^r\right)\right] \tag{A.38}$$

$$= \sum_{\nu,u,e,p} \theta_1(\nu,u,e,p)\, x^R,$$

where

$$\sum_{\nu,u,e,p}\theta_1(\nu,u,e,p) = B^{qn}\, l^{m+n}\sum_{\nu=0}^{m+n}\sum_{u=0}^{\nu}\sum_{e=0}^{m+n}\sum_{p=0}^{e}\frac{(-1)^e(-\nu)_u(-e)_p(-\alpha-qn)_p}{\nu!\,u!\,e!\,p!}$$

$$\times\beta^\nu\left(\frac{i+k+ru}{l}\right)_{m+n}\left(\frac{A}{B}\right)^e. \tag{A.39}$$

(iii) Further considering $A = 1, B = 0$ in (A.38), we obtain the interesting special case

$$S_n^{\alpha,\beta,0}[x;r,q,1,0,m,k,l]$$

$$= x^{-\alpha}\exp\left(\beta x^r\right) T_{k,l}^{m+n}\left[x^{\alpha+qn}\exp\left(-\beta x^r\right)\right] \tag{A.40}$$

$$= \sum_{\nu=0}^{m+n}\sum_{u=0}^{\nu}\theta_1(\nu,u)\,x^{R'},$$

where

$$\theta_1(\nu,u) = l^{m+n}\frac{(-\nu)_u}{\nu!\,u!}\left(\frac{\alpha+qn++k+ru}{l}\right)_{m+n}\beta^{\nu}, \tag{A.41}$$

here

$R' = l(m+n)+qn+r\nu$.

(iv) if we put $q=k=m=0$ and $l=-1$ in (A.40), Gould and Hopper [29] defined the class of the polynomials ,we get the following interesting result

$$S_n^{\alpha,\beta,0}[x;r,0,1,0,0,0,-1]$$

$$= x^{-\alpha}\exp\left(\beta x^r\right) D_x^n\left[x^{\alpha}\exp\left(-\beta x^r\right)\right]$$

$$=(-1)^n H_n^{(r)}(x,\alpha,\beta)$$

$$=(-x)^{-n}\sum_{\nu=0}^{m+n}\sum_{u=0}^{\nu}\frac{(-\nu)_u}{\nu!\,u!}(-\alpha-ru)_n\left(\beta x^r\right)^{\nu}. \tag{A.42}$$

(v) Further, putting $q=1, k=m=0$ and $l=-1$ in (A.40), we get the interesting polynomials set

$$S_n^{\alpha,\beta,0}[x;r,1,1,0,0,0,-1]$$

$$= x^{-\alpha}\exp\left(\beta x^r\right) D_x^n\left[x^{\alpha+qn}\exp\left(-\beta x^r\right)\right]$$

$$= n!\, L_n^{(r)}(x, r, \beta)$$

$$= (-1)^{-n} \sum_{\nu=0}^{u} \sum_{u=0}^{e} \frac{(-v)_u}{\nu!\, u!} (-\alpha - n - ru)_n \left(\beta x^r\right)^{\nu}. \tag{A.43}$$

Singh and Srivastava [122] defined the class of polynomials $L_n^{(r)}(x, r, \beta)$, is a generalization of well known Laguerre polynomials.

(vi) We obtain the class of polynomials defined dur to Chatterja [13], if we put $k = m = 0$ and $l = -1$ in (A.40).

$$S_n^{\alpha, \beta, 0}[x; r, q, 1, 0, 0, 0, -1]$$

$$= x^{-\alpha} \exp\left(\beta x^r\right) D_x^n \left[x^{\alpha + qn} \exp\left(-\beta x^r\right)\right]$$

$$= F_n^{(r)}(x, \alpha, q, \beta)$$

$$= \left(-x^{q-1}\right)^{-n} \sum_{\nu=0}^{n} \sum_{u=0}^{\nu} \frac{(-v)_u}{\nu!\, u!} (-\alpha - qn - ru)_n \left(\beta x^r\right)^{\nu}. \tag{A.44}$$

.

REFERENCES

REFERENCES

1. **Agrawal, B.D. and Chaubey, J.P.;** Operational derivation of generating relations for generalized polynomials, Indian J. Pure Appl. Math. **11**, (1980) 1155-1157.
2. **Agrawal, B.D. and Chaubey, J.P.;** Bilateral generating relations for a function defined by generalized Rodrigues formula,Indian J. Pure Appl. Math. **12**, (1981) 377-379.
3. **Agrawal, P.;** Certain integral properties of generalized class of polynomials and contour integral associated with Feynman integrals, International Journal of Physics and Mathematical science **1**, (2011) 1-8.
4. **Anandani, P.;** On some integrals involving generalized Legendre's associated functions and H - function, Proc. Nat. Acad. Sci. India, Sect. A, **39** (III) , (1969) 341-348.
5. **Ayant, F.Y.;** An integral associated with the Aleph-functions of several variables.International Journal of Mathematics Trends and Technology (IJMTT), **31**(3), (2016) 142-154.
6. **Bhatt, S. and Raina, R.K.;** A New class of analytic functions involving certain fractional derivative operations , Acta Math Univ. Comenianae, **LXVIII,1**, (1999) 179-193.
7. **Bhatter, S. and Faisal, S.M.;** Fractional integral transformations of Mittag-Leffler type E-function, South East Asian Journal of Mathematical Sciences **11**(1), (2015) 31-38.
8. **Bhatter, S. and Shekhawat, S.;** A Study of generalized fractional derivative operator involving Multivariable H-function and general class of Multivariable Polynomials, Journal of the applied Mathematics, Statistics and Informatics (JAMSI) , **7**, (2011) 1-15.
9. **Bhatter, S. and Shekhawat, S.;** Certain multiplication formulae for multivariable H function, J.Rajasthan Acad. Phy. Sci. **8**(2), (2009) 211-218.
10. **Bhatter, S. and Shekhawat, S.;** The general Eulerian integral, Int. Journal of Math. Analysis, **4**, (2010) 393-402.
11. **Brafman, F.;** Some generating functions for Laguerre and Hemilte Polynomials, Canad. J. Math. **9**, (1957) 80-187.
12. **Caputo, M.;** Linear Model of Dissipation whose Q is almost Frequency Independent-II, Geophys. J.R. Astr. Soc. **13**, (1967) 529-539.

13. **Chatterja, S.K.;** Ouelques functions generatrices des Polynomials d' Hermite, du point de vue de lalgebre de Lie., C.R. Acad. Sci. Paris Ser. A-B **268**, (1969), A600-A602.

14. **Chaurasia, V.B.L. and Gill, V.;** New pathway fractional operators involving H-function, Journal of fractional calculus and applications; **4** (1), (2013) 160-168.

15. **Chaurasia, V.B.L. and Singh, Y.;** New generalization of integral equations of fredholm type using Aleph-function Int. J. of Modern Math. Sci. **9**(3), (2014) 208-220.

16. **Dhillon, S.S.;** A Study of generalization of special functions of mathematical physics and their Applications, PhD Thesis, Bundelkhand Univ. 1989.

17. **Dube, A. P.;** Certain multiplication formulae for Kampe De Feriet functions, Indian J. Pure Appl. Math. **5** (7), (1974) 682-689.

18. **Edwards J.;** A Treatise on the Integrals Calculus, Chelsea Pub. Co. 2 New York (1922).

19. **Erdelyi, A.;** On fractional integration and its application to the theory of Hankel Transforms, Quart J. Math. (Oxford) Ser. **2** (2), (1940) 293-303.

20. **Erdelyi, A.;** On some functional transformations, Univ. Politec. Torino, Rend. Sem. Mat. **10**, (1950-51) 217-234.

21. **Erdelyi, A.;** Fractional integrals of generalized functions in Fractional Calculus and its applications (Lecture Notes in Math. Vol. 457), New York, Springer-Verlag (1975).

22. **Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G.;** Higher Transcendental Function, McGraw Hill Book Co., New York-Toronto-London, **1**, (1953).

23. **Exton, H.;** Multiple hypergeometric functions and applications, Halsted Press (Ellis Horwood, Chichester), John Wiley and Sons, New York, London Sydney and Toronto, 1976.

24. **Fox, C.;** The G and H-functions as symmetrical fourier kernels, Trans. Amer. Math. Soc. **98**, (1961) 395-429.

25. **Garg, M. and Purohit, M.;** A Study of multidimensional fractional integral operators and generalized stieltjes transform, Kyungpook Math. J.**40**, (2000) 115-124.

26. **Garg, S.;** Multidimensional fractional integral operators and a general class of multivariable polynomials, Simon Steven Quart. J. Pure Appl. Math. **66**, (1992) 327-339.

27. **Glasser , M.L.;** A Novel class of Bessel function integrals, J. Math. Phys. **25**, (1985) 2933-2934. (Erratum) 2082.

28. **Gorenflo, R. and Vessalla, S.;** Abel integral equations: analysis and applications, Lecture notes in mathematics, **1461**, Springer Verlag Berlin, 1991.

29. **Gould, H.W. and Hopper, A.T.;** Operational formulas connected with two generalizations Hermite polynomials, Duke Math. J, **29**, (1962) 51-63.

30. **Goyal, S.P. and Jain, R.M.;** Fractional integral operators and the generalized hypergeometric functions, Indian J. pure Applied Math. **18**(3), (1987) 251-259.

31. **Goyal, S.P., Jain, R.M. and Gaur, N.;** Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials, Indian J. pure applied Math. **22**(5), (1991) 403-411.

32. **Goyal, S.P. and Saxena, S.;** Fractional derivative operators and multiplication formulas, Proc. Nat. Acad. Sci. India **65**(A), (1995) 95-103.

33. **Gradshteyn, I.S. and Ryzhik, I.M.;** Tables of integrals, series and products, academic Press Inc., New York, 1980.

34. **Gupta, K.C.;** On the H-function, Ann. Soc. Sci., Bruxelles **79**, (1965) 97-106.

35. **Gupta, K.C.;** New relationship of H-Function with functions of practical utility in fractional calculus, Ganita Sandesh **15**, (2001) 63-66.

36. **Gupta, K.C and Jain, R.;** A study of multidimensional fractional integral operators involving general class of polynomials and a generalized Hyper geometric function , Ganita Sandesh **5** (2) , (1991) 55-64.

37. **Gupta, K.C. and Jain, R. and Agrawal, P.;** A unified study of fractional integral operators involving general polynomials and A Mulivariable H-Function, Soochow J. of Math**. 21** (1), (1995) 29-40.

38. **Gupta, K.C. and Soni, R.C.;** On composition of some general fractional integral operators, Proc. Indian Acad. Sci. (Math. Sci.) **104** (2), (1994) 339-349.

39. **Inayat-Hussain, A.A.;** New properties of Hypergeometic series derivable from Feynman Integrals; II A Generalization of the H-function, J. Phys. A; Math. Gen. **20**, (1987) 4119-4128.

40. **Jain, R. and Arekar, k.;** Pathway integral operator associated with Aleph-function general polynomial, Global Journal of Science Frontier Research Mathematics and Decision Sciences, **13**, (2013).

41. **Joshi, C.M. and Prajapat, M.L.;** On a unified presentation of certain classical polynomials, The Mathematics Student **45**, (1977).

42. **Kalla, S.L.;** Integral operators involving Fox's H-Function, Acta Mexicana Cien. Tec. **3**, **(**1969) 117- 122.

43. **Kalla, S.L. Haidey, V., and Virchenko, N.O.**; A generalized Multi parameter function of Mittag-Leffler type. Integral Transforms Special Functions **23**(12), (2012) 901-911.

44. **Kalla, S.L. and Saxena, R.K.;** Integral operators involving hypergeometric functions, Math. Zeitschr **108**, (1969) 231-234.

45. **Kilbas, A.A. and Saigo, M.;** H-Transforms, Theory and Applications, Chapman & Hall/ CRC, London New York, 2004.

46. **Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J.;** Theory and Applications of fractional Differential Equations, Elsevier, Amsterdam, 2006.

47. **Kiryakova, V.S.;** Generalized fractional calculus and applications, Research notes in Math. Series **301**, Pitman Longman, Harlow Wiley, New York, 1994.

48. **Kiryakova, V.S.;** Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. J. Comp. Appl. Math., **118**(1-2), (2000) 241-259.

49. **Khan, Y. and Sharma, P.;** Some Integral Properties of Aleph Function, Multivariable's General Class of Polynomials Associated with Feynman Integrals. IJMSI, **2**(1), (2014) 21-27

50. **Kober, H.;** On fractional integrals and derivatives, Quart. J. Math. Oxford Ser. **11**, (1940) 193-211.

51. **Konhauser, J.D.E.;** Biorthogonal polynomials suggested by the Laguerre polynomials. Pacific J. Math. **21**, (1967) 303-314.

52. **Koul, C.L.;** On fractional integrals operator of functions of two variables, Proc. Nat. Acad. Sci. Indian **41** A6, (1971) 233-240.

53. **Krall, H.L. and Frink, O**.**;** A new class of orthogonal polynomials: the Bessel polynomials, Trans. Amer. Math. Soc. **65**, (1949) 100-115.

54. **Lovoie, J.L., Osler, T.J. and Trembley, R.;** Fractional calculus and special functions, SIAM Rev. **18**, (1976) 240-268.

55. **Luke, Y.L.;** The Special functions and Their Approximations, Vols. I and II, Academic Press, New York and London, 1969.

56. **MacRobert, T.M.;** Functions of complex variables, Macmilln, London and St. Martion's Press, New York, 1962.

57. **Manocha, H.L.;** Some expansions by fractional derivatives, Mathematica (cluj) **9** (32), (1967) 303-309.

58. **Manocha, H.L. and Sharma, B.L.;** Summation of infinite series, J. Astral. Math. Soc. **6**, (1966) 470-467.

59. **Manocha, H.L. and Sharma, B.L.;** Fractional derivatives and summation. J. Indian Math. Soc. (N.S.), **38**, (1974) 371-382.

60. **Marichev, O.I.;** Handbook of Integral Transform of Higher Transcendental Functions, Theory and Algorithmic Tables. Halsted Wiley, New York 1983.

61. **Mathai, A. M.;** A pathway to matrix-variate gamma and normal densities. Linear Algebra Appl., **396**, (2005) 317-328.

62. **Mathai, A. M. and Haubold, H. J.;** On generalized distributions and pathways. Phys. Lett. A **372**, (2008) 2109-2113.

63. **Mathai A. M. and Haubold H. J.;** Pathway model, super statistics, Tsallis statis-tics and a Generalize measure of entropy. Phys., **A 375**, (2007) 110-122.

64. **Mathai, A.M. and Saxena, R.K.;** Generalized Hyper geometric functions with applications in Statistical and Physical Sciences, Lecture Notes in Math. **348**. Springer, Berlin, 1973.

65. **Mathai, A.M. and Saxena, R.K.;** The H- Functions with its applications in Statistics and Other Disciplines, Wiley Eastern, New Delhi, 1978.

66. **Mc Bride, A.C. and Roach, G.F.;** (editors), Fractional Calculus, University of Stratcheltde , Glasgow, Pitman Advanced Publishing Program, 1985.

67. **Meijer, C.S.;** On the G-function, I-VIII, Nederl. Akad. Wetensch. Proc. **49,** 227-237; 344-356; 457-469; 632-641; 765-772; 936-943; 1063-1072; 1165-1175=Indag. Math. **8,** (1946) 124-134; 213-225; 312-324; 391-400; 468-475; 595-602; 661-670; 713-723.

68. **Meulenbeld, B.;** Generalized Legendre's associated function for real values of all arguments numerically less than Unity, Nedere. Akad. Van. Welensch. Proc. Ser. A. **61**(5), (1958) 557-563.

69. **Meulenbeld, B. and Robin, L.;** Nouveaux Results Aux Function de Legendre Generalizes, Nedere. Akad Van. Welensch. Amesterdum. Proc. Sec. **A 64**, (1961) 333-347.

70. **Mishra, A.P.;** On a Fractional Differential Operator, Ganita Sandesh **26**(2), (1975) 1-18.

71. **Miller, K.S. and Ross, B.;** An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons, New York 1993.

72. **Mittag-Leffler, G.M.;** Sur lanouvelle function $E_{\alpha}(x)$, C.R. Acad. Sci. Paris, **137**, (1903) 554-558.

73. **Nair, S. S.;** Pathway fractional integration operator, Fract. Calc. Appl. Anal.,**12**(3), (2009) 237-252.

74. **Nishimoto, K.;** Fractional Calculus. Vol. I (1984), Vol. II (1987), Vol.III (1989), Vol. IV (1991), Vol. V (1996), Descartes Press, Koriyama, Japan.

75. **Oldham, K.B. and Spanier, J.;** The fractional calculus: Theory and Applications of differentiation and integration to arbitrary order, in: Matematics in science and engineering, academic press, New York and London, 1994.

76. **Patil, K.R. and Thakre, N.K.;** Bilateral generating function for a function defined by generalized Rodriques Formula, Indian J. Pure Appl. Math. **8,** (1977) 425-429.

77. **Podlubny, I.;** Fractional Differential Equations, Academic Press, New York (1999).

78. **Prathima J., Vasudevan, N.T. and Shantha K.K.;** A Study of I-Function of Several Complex Variables, Hindawi Publishing Corporation International journal of Engineering Mathematics, **14**,(2014).

79. **Prudnikov, A.P., Brycekov, A. Yu. and Mariceov, O.**I.; Integrals and Series of Elementary Functions (Moscow; Nauka) (in Russian) , 1983.

80. **Raijada, S.K.;** A study of unified representation of special functions of mathematical physics and their use in statistical and boundary value Problems, PhD, thesis, Bundelkhand University, 1991.

81. **Raina, R.K.;** On composition of certain fractional integral operators, Indian J. Pure Applied Math. **15**(5), (1984), 509-516.

82. **Raina, R.K. and Koul, C.L.;** Fractional derivatives of the H-functions, Jnanabha (Professor Charles Fox Memorial) **7**, (1977) 97-105.

83. **Raina, R.K. and Kriyakova, V.S.;** On the Wely fractional calculus, Proc. Amer. Math. Soc. **73**, (1979) 188-192.

84. **Rainville, E.D.;** Special Functions. Macmillan Company, New York 1960; Reprinted by Chelsea Publ. Co. Bronx, New York 1971.

85. **Ross, R. (Ed.);** Fractional Calculus and its Applications, Proceedings of the International Conference held at the University of New Haven in June 1974. Lecture Notes in Math. **457**, Springer-Verlag, Berline, Heidelberg and New York, 1975.

86. **Saigo, M.;** A remark on integral operators involving the Gauss Hypergeometric function, Math. Rep. Kyushu Univ. **11**, (1978) 135-143.

87. **Saigo, M.;** A certain boundary value problem for the Euler Darboux equations III, Mathematica Japonica. **26** (1), **(**1981) 103-119.

88. **Samko, S.G. Kilbas, A.A. and Marichev, O.I.;** Fractional integrals and derivatives and some of their applications.Nauka ; Tekhnica, Minsk, 1987 (in Russian); English translation, Gordon and Breach, Reading 1993.

89. **Saxena, R.K.;** Fractional integration of the Aleph function V\A Pathway operator, International Journal of Physics and Mathematical Science, **2** (1), (2012) 163-172.

90. **Saxena, R.K., Ram, J., and Kumar, D.;** Generalized fractional calculus of the product of the two Aleph functions associated with the Appell function, Tamkang Journal of Mathematics, **45** (2) , (2013) 137-148.

91. **Saxena, R.K. and Ram, J.;** Generalized fractional integral of the product of two Aleph-Functions, An International Journal of Applications and Applied Mathematics, **8**(2), (2013) 631-646.

92. **Saxena, R.K. and Kumbhat, R.K.;** Integral operators involving H-functions, Indian J. Pure Appl. Math. **5**, (1974) 1-6.

93. **Saxena, V.P.;** A formal solution of certain new pair of dual integral equations involving H-Functions, Nat. Acad. Sci. India Sect, A **52**, (1982) 366-375.

94. **Saxena, V.P.;** The I-Function, Anamaya Publishers, New Delhi, 2008.

95. **Saxena, R.K. and Nishimoto, K.;** N-fractional calculus of generalized Mittag-Leffler functions, J.Fract. Calc. **37**, (2010) 43-52.

96. **Sharma, C.K. and Ahmad, S.S.;** On the multivariable I-function. Acta ciencia Indica Math, **20**(2) ,(1994) 113-116.

97. **Shukla, A.K. and Prajapati, J.C.;** On a generalization of Mittag- Leffler function and its properties, J., Math Anal. Appl. **336**, (2007) 797-811.

98. **Singh, A.;** A Study of Special Functions of Mathematical Physics and their Applications in Combinational Analysis, PhD Thesis, Bundelkhand Univ. India, 1981.

99. **Singh, R.P. and Srivastava, K.N.;** A note in generalization of Laguerre and Humbert Polynomials, Ricerca (Napoli) **14** (2) (1963), 11-21; Errata, ibid. **15** (1964) 63.

100. **Slater, L.J.;**Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge, London and New York (1966).

101. **Sneddon, I.N.;** Special Functions of Mathematical Physics and Chemistry, Oliver and Boyd. New York, 1961.

102 **Sneddon I.N.;** Mixed boundary value problems in potential theory, North Holland Pudl. Co. Amsterdam, 1966.

103 **Srivastava, H.M.;** A Multilinear generating function for the Konhausar sets of Biortogonal Polynomial suggested by the Laguerre Polynomials, pacific J. Math., **177,** (1985) 183-191.

104. Srivastava, H.M.; A Contour integral involving Fox H-function, Indian J. Math **14,** (1972) 1-6.

105. Srivastava, H.M.; The Weyl fractional integral of general class of polynomials boll. Un. Math. Ital. **(6)** 2B, (1983) 219-228.

106. Srivastava, H.M. and Daoust, M.C.; Certain Generalized Neumann expansions associated with the kampe de Feriet Function, Nederl. Akad. Wetensch. Proc. Ser. A 72=Indag. Math. **31**, (1969) 449-4457.

107. Srivastava, H.M. and Garg, M.; Some integral involving a general class of Polynomials and the Multivariable H-function, Rev. Romaine Phys., **32**,(1987) 685-692.

108. Srivastava, H.M. and Goyal, S.P.; Fractional derivatives of the H-function of several variables, J.Math. Anal. and Appl. **112**, (1985) 641-651.

109. Srivastava, H.M., Gupta, K.C. and Goyal, S.P.; The H-functions of one and two variables with applications, South Asian Publishers, New Delhi 1982.

110. Srivastava, H.M. and Manocha, H.L.; A treatise on generating functions, Halsted Press (Ellis Hortwood Ltd. Chichester), John Wiley and sons, New York. Chichester, Bribane and Toronto 1984.

111. Srivastava, H.M. and Panda, R.; Certain multidimensional integral transformations, I and II, Nederl. Akad. Wetensch. Proc. Ser. A 81=Indag. Math. **40**, (1978) 118-131 and 132-144.

112. Srivastava, H.M. and Panda, R.; On the unified presentation of certain classical polynomials, boll. Uni. Mat. Ital, **4** (12), (1975) 306-314.

113. Srivastava, H.M. and Panda, R.; Some bilateral generalizing functions for a class of generalized hyergeometric polynomials, J. Reine Angew. Math. **283/284**, (1976) 265-274.

114. Srivastava, H.M. and Panda, R.; Expansion theorems for the H-function of several complex variables, J. Reine Angew. Math. **288**, (1976) 129-145.

115. Srivastava, H.M. and Panda, R.; Some multiple integral transformations involving the H- Function of several variables, Nederl. Akad. Wetensch. Proc. Ser. A 82=Indag. Math. **41**, (1979) 353-362.

116. Srivastava, H.M. and Panda, R.; A Note on certain results involving a general class of polynomials, Boll. Un. Mat. Ital. **(5)** 16A, (1979) 467-474.

117. Srivastava, H.M. and Pathan , M.A., Some Bilateral generating functions for the extended Jacobi polynomials, I and II, comment, Math. Univ. St. Paul, **28**, fasc, **1** (1979), 23-30; ibid 29 fasc, **2** (1980), 105-114.

118. Srivastava, H.M. and Raina, R.K.; Evolution of a certain class of Eulerian integrals, J. Phys. A: Math. Gen. **26**, (1993) 691-696.

119. Srivastava, H.S.P.; Multivariable Bessel polynomial, Proc. of 3^{rd} Annual Conf (SSFA), (2002) 163-173.

120. Srivastava, H.S.P.; On Hermite polynomials of several variables, J. Indian Acad.Math. **(24)** (1), (2002) 95-105.

121. Srivastava, H.S.P.; On Jacobi polynomials of several variables, Int. Trans. Sp. Funct. **10** (1), (2000) 61-70.

122. Srivastava, H.M. and Singh, N.P.; The Integration of certain products of the multivariable H-function with general class of polynomials, Rend. Circ. Mat. Palermo **2** (32), (1983) 157-187.

123. Srivastava, H.M. Sago, M. and Owa, S.; A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl., **131**,(1988) 412-420.

124. Srivastava, H.M. and Saxena, R. K.; Operators of fractional integration and their Applications, Elsevier. Appl. Math. Comput. **118,** (2001) 1-5.

125. Sudland, B., Baumann, B. and Nonnenmacher ,T.F.; Fractional driftless, Fokker -Planck equation with power law diffusion coefficient. In: Computer Algebra in Scientific Computing (CASC Konstanz, 2001), edited by VG Ganga, EW Mayr, WG Varozhtsov (Springer, and Berlin) (2001) 513-525.

126. Sudland, N., Bauman, B. and Nonnenmacher, T.F.; Open problem, who know about the Aleph Functions? Fract. Calc. Appl. Anal.**1** (4), (1998) 401-402.

127. Szego, G.; Orthogonal polynomials, Amer Math Soc. Colloq. Publ. Vol. **23,** (1975) Fourth edition, Amer. Math. Soc. Providence, Rhode Island.

128. **Wille, L.T.;** A novel class of G-function, J. Math. Phys. **29**, (1988) 599-603.

129. **Wiman, A.;** Uber den fundamental Satz in der theorie der Funktionen ¨ $E_{\alpha}(x)$, Acta Math., **29,** (1905) 191-201.

130. Wright, E.M.; The asymptotic expansion of the generalized Bessel Function, proceeding of the London Mathematical Society (Ser.2), **38**, 257-260.

.

.

BIOGRAPHICAL PROFILE OF RESEARCHER

Name	:	RAKESH KUMAR BOHRA
Permanent Address	:	Village Post-Bhander, Vaya-Nana, Tehshil- Bali, District- Pali, Rajasthan, 306504 India
Mob. No.	:	+91- 9309281941
e-mail	:	rakeshbohra11@gmail.com
Academic Degrees	:	B. Sc., M.Sc., (Mathematics) and RSET (Rajasthan State Eligibility Test)
Research Area	:	Special functions, Fractional Calculus, Integral Transformations

<u>Teaching Experience</u>

July 29, 2013 to till date	:	Assistant Professor, Department of Mathematics, Global College of Engineering, Jaipur.
July 26, 2010 to July 28, 2013	:	Assistant Professor, Department of Mathematics, Poornima College of Engineering, Jaipur.
June 2009 to June 2010	:	Lecturer, Department of Mathematics Geetanjali Engg College, Udaipur.

Publications

1. Multidimensional fractional integral operators involving general class of Polynomial and Aleph $(\aleph)$ function, *International Journal of Mathematics and its Applications,* Vol. **5, 2(B)**, 2017, pp. 293-300.
2. Pathway fractional integral operator of the product of two Aleph-functions, *IOSR Journal of Mathematics* (IOSR-JM), Vol. **13(2)**, 2017, pp.27-30.
3. On some composition formulae for multidimensional fractional integral operators associated aleph $(\aleph)$- function and general class of polynomial, *International Journal of Computational and Applied Mathematics*, Vol. **12(2),** 2017, pp. 255-272.
4. A Study of unified integrals involving the generalized polynomial set, generalized Legendre's associated function and Aleph $(\aleph)$- function With Applications, *South East Asian J. of Math.& Math. Sci,* Vol.**13 (1),** 2017, pp. 27-46.
5. Fractional integral transformations of Mittag- Leffler type E-function with Multivariable Polynomial and Aleph function, *International Journal of Mathematical Archive" (IJMA)* Vol. **8(5)** , 2017, pp. 1-11.
6. A study of generalized fractional derivative operator involving aleph $(\aleph)$-function, general class of multivariable polynomials and certain multiplication formulae for aleph $(\aleph)$-function of multivariable functions, *International Journal of Modern Mathematical Sciences,* Vol. **15 (3),** 2017, pp. 291-310.
7. The general Eulerian integral with Aleph $(\aleph)$-Function, *South East Asian J. of Math.& Math. Sci,* Vol. **12(1),** 2016, pp. 57-66.
8. Pathway fractional integral operator associated with Aleph-function, Multivariable's general class of polynomial, Mittag- Leffler function with H-function, *International J. of Math. Sci. & Engg. Appls.* (IJMSEA), Vol.**10 (1),** 2016, pp. 15-21.
9. On some integral properties of aleph function, Multivariable's general class of polynomials associated with feynman integrals, *International Journal of Physics and Mathematical Sciences,* Vol. **4 (2)** April-June, 2014, pp. 109-117
10. New Pathway fractional integral operator associated with Aleph-function, Multivariable's general class of polynomial with H-function*, International Journal of Mathematics Trends and Technology* (IJMTT), Vol. **14,** 2014, pp.84-87

LIST OF RESEARCH PAPERS

1. Multidimensional fractional integral operators involving general class of Polynomial and Aleph $(\aleph)$function, *International Journal of Mathematics and its Applications,* Vol. **5, 2(B)**, 2017, pp. 293-300.
2. Pathway fractional integral operator of the product of two Aleph-functions, *IOSR Journal of Mathematics* (IOSR-JM), Vol. **13(2)**, 2017, pp.27-30.
3. On some composition formulae for multidimensional fractional integral operators associated aleph $(\aleph)$- function and general class of polynomial, *International Journal of Computational and Applied Mathematics*, Vol. **12(2),** 2017, pp. 255-272.
4. A Study of unified integrals involving the generalized polynomial set, generalized Legendre's associated function and Aleph $(\aleph)$- function With Applications, *South East Asian J. of Math.& Math. Sci,* Vol.**13 (1),** 2017, pp. 27-46.
5. Fractional integral transformations of Mittag- Leffler type E-function with Multivariable Polynomial and Aleph function, *International Journal of Mathematical Archive" (IJMA)* Vol. **8(5)** , 2017, pp. 1-11.
6. A study of generalized fractional derivative operator involving aleph $(\aleph)$-function, general class of multivariable polynomials and certain multiplication formulae for aleph $(\aleph)$-function of multivariable functions, *International Journal of Modern Mathematical Sciences,* Vol. **15 (3),** 2017, pp. 291-310.
7. The general Eulerian integral with Aleph $(\aleph)$-Function, *South East Asian J. of Math.& Math. Sci,* Vol. **12(1),** 2016, pp. 57-66.
8. Pathway fractional integral operator associated with Aleph-function, Multivariable's general class of polynomial, Mittag- Leffler function with H-function, *International J. of Math. Sci. & Engg. Appls.* (IJMSEA), Vol.**10 (1),** 2016, pp. 15-21.
9. On some integral properties of aleph function, Multivariable's general class of polynomials associated with feynman integrals, *International Journal of Physics and Mathematical Sciences,* Vol. **4 (2)** April-June, 2014, pp. 109-117

10. New Pathway fractional integral operator associated with Aleph-function, Multivariable's general class of polynomial with H-function, *International Journal of Mathematics Trends and Technology* (IJMTT), Vol. **14,** 2014, pp.84-87.