# New Investigations in Integral Transforms and Fractional Integral Operators Involving Generalized Extended Mittag-Leffler Function and Extended Hurwitz Lerch Zeta Function with Applications to the Solution of Fractional Differential Equations

**Ph.D Thesis**


**NIDHI JOLLY**

**(2015RMA9531)**

**DEPARTMENT OF MATHEMATICS**

**MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY, JAIPUR**


**July 2019**

# New Investigations in Integral Transforms and Fractional Integral Operators Involving Generalized Extended Mittag-Leffler Function and Extended Hurwitz Lerch Zeta Function with Applications to the Solution of Fractional Differential Equations

By


Nidhi Jolly

(2015RMA9531)


under the supervision of

Prof. Rashmi Jain

*Submitted*

*in partial fulfillment of the requirements of the degree of*

*Doctor of Philosophy*

*to the*


Department of Mathematics

MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY, JAIPUR


July 2019

# DECLARATION

I, **Nidhi Jolly**, declare that this thesis titled, **"New Investigations in Integral Transforms and Fractional Integral Operators Involving Generalized Extended Mittag-Leffler Function and Extended Hurwitz Lerch Zeta Function with Applications to the Solution of Fractional Differential Equations"** and the work presented in it, are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this university.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, thesource is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself, jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Jaipur **NIDHI JOLLY**

July 2019 **(2015RMA9531)**

# CERTIFICATE

This is to certify that the thesis entitled **"New Investigations in Integral Transforms and Fractional Integral Operators Involving Generalized Extended Mittag-Leffler Function and Extended Hurwitz Lerch Zeta Function with Applications to the Solution of Fractional Differential Equations"** being submitted by Ms. Nidhi Jolly (2015RMA9531) is a bonafide research work carried out under my supervision and guidance in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the Department of Mathematics, Malaviya National Institute of Technology, Jaipur, India. The matter embodied in this thesis is original and has not been submitted to any other University or Institute for the award of any other degree.

Jaipur
July 2019

**Dr. Rashmi Jain**
**Professor**
**Department of Mathematics**
**Malaviya National Institute of Technology, Jaipur**

ACKNOWLEDGEMENT

I feel great privilege and pleasure in expressing my sincere and deepest sense of gratitude to my supervisor and mentor *Prof. (Mrs.) Rashmi Jain*, Department of Mathematics for her scholarly guidance and affection. It is because of her helpful attitude, incessant directions, invaluable support, uninterrupted and expert supervision that I could accomplish the present manuscript.

I acknowledge my sincere gratitude to *Prof. Uday Kumar R Yaragatti*, Director , MNIT for his kind help, inspiration and encouragement. I am also thankful to the present and former Heads of Department of Mathematics for their timely suggestions, support and facility provided. I express my deep and profound sense of gratitude to my DREC members *Dr. Vatsala Mathur, Dr. Sanjay Bhatter, Dr. Ritu Agarwal*, Department of Mathematics, MNIT, Jaipur, for their persistent encouragement and suggestions.

My heartfelt gratitude is due to *Dr. K.C. Gupta*, former Professor and Head, Department of Mathematics, MNIT, Jaipur for his all time invaluable support, matchless suggestions, uninching and timely help that paved the way for completion of my thesis.

My special regards are due to *Mr. R. K. Jain*, IAS (husband of Prof. Rashmi Jain) and *Mrs. Prakash Gupta* (wife of Prof K.C. Gupta) for showering their affectionate blessings and best wishes for the accomplishment of my work . I am much grateful to all the faculty members, my senior and junior research scholars for their kind cooperation and enthusiastic encouragement.

I am also thankful to all the people who have helped me directly or indirectly during my PhD work.

Words are inadequate to express my sincere gratitude to my beloved parents, *Shri. Rakesh K Jolly* and *Smt. Indu Jolly*, for their keen interest and for being source of tremendous strength and energy to me throughout this work. I would be failing in my duty if I don't express my thanks towards my younger brother *Rahul Jolly* for his kind affection and sharing my responsibilities during this span of life.

Finally, I express my sincere indebtedness to the Almighty for his kind and everlasting blessings.

***(NIDHI JOLLY)***

# ABSTRACT

**Chapter 1** is intended to provide an introduction to various functions, polynomials, integral transforms and fractional integral operators studied by some of the earlier researchers. Further, we present the brief chapter by chapter summary of the thesis. Finally, we give a list of research papers which have either been published or accepted or communicated for publication in reputed journals having a bearing on subject matter of the thesis.

In **Chapter–2**, we first introduce our function of study and call it generalized extended Mittag-Leffler (GEML) function and present it's basic properties. Further, we obtain some integral representations of generalized extended Mittag-Leffler function. Later on, we obtain Laplace transform, Mellin transform and Inverse Mellin transfrom of function of our study. Finally, in this chapter we obtain right-sided Riemann- Liouville fractional integral operator $I^{\gamma}_{a+}$ , right-sided Riemann- Liouville fractional derivative operator $D^{\gamma}_{a+}$ and Hilfer derivative operator $D^{\gamma,\eta}_{a+}$ of our function of study GEML. The results established in this chapter

generalize the findings of and Özarslan and Yilmaz [46]Shukla and Prajapati [59].

In **Chapter–3**, we introduce and study an integral operator whose kernel is generalized extended Mittag-Leffler (GEML) function and point out it's known special cases. Then we derive boundedness property of aforementioned integral operator. Next, we obtain image of some useful functions under the integral operator of our study along with some of it's special cases. Finally, we establish composition relationship of integral operator of our study with right-sided Riemann- Liouville fractional integral operator $I_{a+}^{\gamma}$ and an integral operator $H_{a+;P,Q;\beta}^{w;M,N;\alpha}$ involving the Fox H-function . The results stated in this chapter generalize the findings of Kilbas *et al.* [28] and Srivastava and Tomovski [78].

In **Chapter–4**, we first define a new integral transform whose kernel is extended Hurwitz-Lerch zeta function and name it as the $\mathcal{E}-$transform. This transform yields a number of (new or known) integral transforms as its special cases. Further, we evaluate the Mellin transform of extended Hurwitz-Lerch Zeta function and Inversion formula for $\mathcal{E}-$transform. Next, we present some basic properties of $\mathcal{E}-$transform and prove the Uniqueness theorem for $\mathcal{E}-$transform. Finally, we obtained $\mathcal{E}-$transform of Derivatives and Integrals.

In **Chapter–5**, we study a pair of class of fractional integral operators whose kernel involve the product of $S_V^U[z]$, $E_{\delta,\kappa}^{\vartheta;d}(z;q,\sigma,\zeta)$ and $\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}(z,s,a)$ which stand for $S_V^U$ polynomial, generalized extended Mittag-Leffler function and extended Hurwitz-Lerch Zeta function. Next,we derive three new and interesting composition formulae for the operators of our study. The operators of our study are quite general in nature and may be considered as extensions of a number of simpler fractional integral operators studied from time to time by several authors. By suitably specializing the coefficients and the parameters of functions involved in our fractional integral operators we can get a large number of expressions for the composition of fractional integral operators. Finally, as an application of our main findings we obtain three interesting integrals which are believed to be new.

In **Chapter–6**, we first consider a generalized form of General Family of fractional differential equations and find it's solution. On account of general nature of our findings we can obtain a number of special cases by taking particular values of the parameters involved therein. We mention here two new and three known special cases.

Next, we study some interesting two new and three known special cases of our main findings involving $\overline{H}$-function. Later on, by giving numerical values to the parameters in the functions and the operators involved therein we have plotted some graphs with the help of MATLAB SOFTWARE.

Futher, we obtain solution of another fractional differential equation involving $D_{0+}^{\alpha,\beta_1}$ and $\overline{\mathcal{H}}_{0+;p,q,\beta}^{w;m,n;\gamma}$. Finally, by specializing the parameters occuring therein we can obtain a number of special cases of this result. However, we give here only two known and two new special cases.

# Contents

# LIST OF FIGURES

# 1

# INTRODUCTION TO THE TOPIC OF STUDY AND CHAPTER BY CHAPTER SUMMARY OF THE THESIS

The present chapter deals with an introduction to the topic of the study as well as a brief review of the contributions made by some of the earlier workers on the subject matter presented in this thesis. At the end, a brief chapter by chapter summary of the thesis has been given.

## 1.1 SPECIAL FUNCTIONS

Special functions have vast applications in all branches of engineering, applied sciences, statistics and various other fields. A large number of eminent mathematicians such as Euler, Gauss, Kummer, Ramanujan and several others worked out hard to develop the commonly used special functions like the Gamma function, the elliptic functions, Bessel functions, Whittaker functions and polynomials that go by the name of Jacobi, Legendre, Laguerre, Hermite.

The core of special functions is the Gauss hypergeometric function ${}_2F_1$, introduced by famous mathematicians C. F. Gauss. It is represented by the following series:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a.b}{c}\frac{z}{1!} + \frac{a.(a+1).b.(b+1)}{c.(c+1)}\frac{z^2}{2!} + \cdots \tag{1.1.1}$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \qquad \text{for} \quad n \geq 0; (a)_0 = 1, c \neq 0, -1, -2, \cdots$$

a, b, c and z may be real or complex. Also if either of the numbers a or b is a non-positive integer, the function reduces to a polynomial, but if c is non-positive integer, the function is not defined since all but a finite number of terms of the series become infinite.

This series has a fundamental importance in the theory of special function and is known as Gauss hypergeometric series. It is usually represented by the symbol ${}_2F_1(a,b;c;z)$ the well known Gauss hypergeometric function.

In (1.1.1), if we replace z by $\frac{z}{b}$ and let $b \to \infty$ then

$$\frac{(b)_n}{b^n} z^n \to z^n$$

and we arrive at the following well known Kummer's series

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a.(a+1)}{c.(c+1)}\frac{z^2}{2!} + \cdots \tag{1.1.2}$$

It is represented by the symbol ${}_1F_1(a;c;z)$ and is known as confluent hypergeometric function.

A natural generalization of ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$, which is defined in the following manner:

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; & \\ & z \\ b_1, \cdots, b_q; & \end{matrix}\right] &= {}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.1.3}$$

where p and q are either positive integers or zero and empty product is interpreted as unity, the variable z and all the parameters $a_1, \cdots, a_p; \quad b_1, \cdots, b_q$ are real or complex numbers such that no denominator parameters is zero or a negative integer. The conditions of convergence of the function ${}_pF_q$ are as follow:

(i) when $p \leq q$, the series on the right hand side of (1.1.3) is convergent.

(ii) when $p = q+1$, the series is convergent if $|z| < 1$ and divergent when $|z| > 1$, and on the circle $|z| = 1$, the series is

(a) absolutely convergent if $\Re(w) > 0$

(b) conditionally convergent if $-1 < \Re(w) < 0$ for $z \neq 1$

(c) divergent if $\Re(w) \leq -1$

where $w = \sum\limits_{j=1}^{q} b_j - \sum\limits_{j=1}^{p} a_j$

(iii) when $p > q+1$, the series never converges except when z = 0 and the function is only defined when the series terminates.

A comprehensive account of the functions ${}_2F_1$, ${}_1F_1$ and ${}_pF_q$ can be found in the works of Exton [9], Luke [34], Rainville [53] and Slater [60] their applications can be found in Mathai and Saxena [37].

### 1.1.1 THE FOX H- FUNCTION

The Fox $H-$function is defined by the following Mellin-Barnes type integral [68, p. 10] with the integrand containing products and quotients of the Euler gamma functions. Such a function generalizes most of the known special functions.

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right.\right] = H_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_1,\alpha_1),\cdots,(a_P,\alpha_P) \\ \\ (b_1,\beta_1),\cdots,(b_Q,\beta_Q) \end{matrix} \right.\right]$$

$$:= \frac{1}{2\pi\omega}\int\limits_{\mathfrak{L}} \Theta(\mathfrak{s}) z^{\mathfrak{s}}\, d\mathfrak{s}, \tag{1.1.4}$$

where $\omega = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

and

$$\Theta(\mathfrak{s}) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \mathfrak{s}) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j \mathfrak{s})}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j \mathfrak{s}) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \mathfrak{s})}, \tag{1.1.5}$$

Also M, N, P and Q are non-negative integers satisfying $1 \leqq M \leqq Q$ and $0 \leqq N \leqq P$ ; $\alpha_j (j = 1, \cdots, P)$ and $\beta_j (j = 1, \cdots, Q)$ are assumed to be positive quantities for standardization purposes. The definition of the Fox H-function given by (1.1.4) will, however, have meaning even if some of these quantities are zero. Also, $a_j (j = 1, \cdots, P)$ and $b_j (j = 1, \cdots, Q)$ are complex numbers such that none of the points

$$\mathfrak{s} = \frac{b_h + \nu}{\beta_h} \quad h = 1, \cdots, M; \nu = 0, 1, 2, \cdots \tag{1.1.6}$$

which are the poles of $\Gamma(b_h - \beta_h s), h = 1, \cdots, M$ and the points

$$\mathfrak{s} = \frac{a_i - \eta - 1}{\alpha_i} \quad i = 1, \cdots, N; \eta = 0, 1, 2, \cdots \tag{1.1.7}$$

which are the poles of $\Gamma(1 - a_i + \alpha_i s)$ coincide with one another, i.e

$$\alpha_i (b_h + \nu) \neq b_h (a_i - \eta - 1) \tag{1.1.8}$$

for $\nu, \eta = 0, 1, 2, \cdots$; $h = 1, \cdots, M$; $i = 1, \cdots, N$.

Further, the contour $\mathfrak{L}$ runs from $-\omega\infty$ to $+\omega\infty$ such that the poles $\Gamma(b_h - \beta_h s), h = 1, \cdots, M$, lie to the right left of $\mathfrak{L}$ and the poles of

$\Gamma(1-a_i+\alpha_i s), \quad i=1,\cdots,N$ lie to the left of $\mathfrak{L}$. Such a contour is possible on account of (1.1.8). These assumptions will be adhered to throughout the present work.

## SPECIAL CASES

The following special cases of the Fox $H-$function have been made use in this thesis:

1. **Lorenzo-Hartley G-function** [15, p. 64, Eq. (2.3)]

$$H^{1,1}_{1,2}\left[-az^q \middle| \begin{array}{c} (1-r,1) \\ (0,1), \quad (1+\nu-rq,q) \end{array}\right] = \frac{\Gamma(r)}{z^{rq-\nu-1}} G_{q,\nu,r}[a,z]. \quad (1.1.9)$$

Here, $G_{q,\nu,r}$ is the Lorenzo-Hartley G-function [33].

2. **Generalized Hypergeometric function**[68, p. 18, Eq. (2.6.3)]

$$H^{1,p}_{p,q+1}\left[z \middle| \begin{array}{c} (1-a_j,1)_{1,p} \\ (0,1),(1-b_j,1)_{1,q} \end{array}\right] = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)} {}_pF_q[(a_p);(b_q);-z]. \quad (1.1.10)$$

3. **Generalized Bessel Maitland Function** [38, p. 25, Eq. (1.139)]

$$H^{1,1}_{1,3}\left[\frac{z^2}{4} \middle| \begin{array}{c} (\lambda+\frac{\nu}{2},1) \\ (\lambda+\frac{\nu}{2},1),(\frac{\nu}{2},1),(\mu(\lambda+\frac{\nu}{2})-\lambda-\nu,\mu) \end{array}\right] = J^{\mu}_{\nu,\lambda}(z), \quad (1.1.11)$$

where $J^{\mu}_{\nu,\lambda}$ is the Generalized Bessel Maitland Function [35, p. 128, Eq. (8.2)]

4. **Wright's Generalized Bessel Function**[68, p. 19, Eq. (2.6.10)]

$$H_{0,2}^{1,0}\left[z \,\middle|\, \begin{matrix} -- \\ (0,1),(-\lambda,\nu) \end{matrix}\right] = J_\lambda^\nu(z). \tag{1.1.12}$$

5. **Krätzel Function** [38, p. 25, Eq. (1.141)]

$$H_{0,2}^{2,0}\left[z \,\middle|\, \begin{matrix} -- \\ (0,1),(\frac{\nu}{\rho},\frac{1}{\rho}) \end{matrix}\right] = \rho Z_\rho^\nu(z) \qquad z,\nu \in \mathbb{C}, \rho > 0, \tag{1.1.13}$$

where $Z_\rho^\nu$ is the Krätzel Function [31].

6. **Modified Bessel function of the third kind** [11, p. 155, Eq. (2.6)]

$$H_{1,2}^{2,0}\left[z \,\middle|\, \begin{matrix} \left(1-\dfrac{\sigma+1}{\beta},\dfrac{1}{\beta}\right) & \\ (0,1), & \left(-\gamma-\dfrac{\sigma}{\beta},\dfrac{1}{\beta}\right) \end{matrix}\right] = \lambda_{\gamma,\sigma}^{(\beta)}(z). \tag{1.1.14}$$

### 1.1.2 THE $\overline{H}-$FUNCTION

Though the H-function is sufficiently general in nature, many useful functions notably generalized Riemann Zeta function [8], the polylogarithm of complex order [8], the exact partition of the Gaussian model in statistical mechanics [23], a certain class of Feynman integrals [8] and others do not form its special cases. Inayat Hussain [23] introduced a generalization of the H-function popularly known as $\overline{H}$ -function which includes all the above mentioned functions as its special cases. This function is developing fast and stands on a firm footing through the publications of Buschman and Srivastava [3], Gupta and Soni [16], Gupta, Jain and Agrawal [17], Gupta, Jain and Sharma [18], Jain and Sharma [26], Rathie

[54], Saxena [56, 58] and several others. The $\overline{H}$-function is defined and represented in the following manner:

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[ z \left| \begin{matrix} (e_j, E_j; \in_j)_{1,n}, & (e_j, E_j)_{n+1,p} \\ \\ (f_j, F_j)_{1,m}, & (f_j, F_j; \Im_j)_{m+1,q} \end{matrix} \right. \right]$$

$$:= \frac{1}{2\pi\omega} \int\limits_{\mathfrak{L}} \overline{\Theta}(\xi) z^{\xi} d\xi, \tag{1.1.15}$$

where, $\omega = \sqrt{-1}$,$z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi) = \frac{\prod\limits_{j=1}^{m} \Gamma(f_j - F_j\xi) \prod\limits_{j=1}^{n} \{\Gamma(1 - e_j + E_j\xi)\}^{\in_j}}{\prod\limits_{j=m+1}^{q} \{\Gamma(1 - f_j + F_j\xi)\}^{\Im_j} \prod\limits_{j=n+1}^{p} \Gamma(e_j - E_j\xi)}, \tag{1.1.16}$$

It may be noted that $\overline{\Theta}(\xi)$ contains fractional powers of some of the gamma functions. $m, n, p, q$ are integers such that $1 \leq m \leq q, 0 \leq n \leq p, (E_j)_{1,p}, (F_j)_{1,q}$ and $(\in_j)_{1,n}, (\Im_j)_{m+1,q}$ are positive quantities for standardization purpose. The definition (1.1.15) will however have meaning even if some of these quantities are zero, giving us in turn simple transformation formulae.

$(e_j)_{1,p}$ and $(f_j)_{1,q}$ are complex numbers such that the points

$$\xi = \frac{f_j + k}{F_j} \quad j = 1, \cdots, m; \quad k = 0, 1, 2, \cdots$$

which are the poles of $\Gamma(f_j - F_j\xi)$, and the points

$$\xi = \frac{e_j - 1 - k}{E_j} \quad j = 1, \cdots, n; \quad k = 0, 1, 2, \cdots$$

which are the singularities of $\{\Gamma(1-e_j+E_j\xi)\}^{\in_j}$, do not coincide.

We retain these assumptions throughout the thesis.

The contour $\mathfrak{L}$ is the line from $c-\mathrm{i}\infty$ to $c+\mathrm{i}\infty$, suitably intended to keep the poles of $\Gamma(f_j-F_j\xi) \quad j=1,\cdots,m$ to the right of the path, and the singularities of $\{\Gamma(1-e_j+E_j\xi)\}^{\in_j} \quad j=1,\cdots,n$ to the left of the path.

If $\in_i = \Im_j = 1 \quad (i=1,\cdots,n; \quad j=m+1,\cdots,q)$, the $\overline{H}-$function reduces to the familiar $H-$function.

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}-$function given by (1.1.15) have been given by Gupta, Jain and Agarwal [17]

$$\left.\begin{array}{l}(i)\,|\arg(z)| < \dfrac{1}{2}\Omega\pi \quad \text{and} \quad \Omega > 0\\ (ii)\,|\arg(z)| = \dfrac{1}{2}\Omega\pi \quad \text{and} \quad \Omega \geq 0\\ \quad \text{and (a)} \quad \mu \neq 0 \quad \text{and the contour} \quad \mathfrak{L} \quad \text{is so chosen that} \quad (c\mu+\lambda+1) < 0\\ \qquad \text{(b)} \quad \mu = 0 \quad \text{and} \quad (\lambda+1) < 0,\end{array}\right\}$$

where

$$\Omega = \sum_{j=1}^{m} F_j + \sum_{j=1}^{n} E_j \in_j - \sum_{j=m+1}^{q} F_j \Im_j - \sum_{j=n+1}^{p} E_j$$
$$\mu = \sum_{j=1}^{n} E_j \in_j + \sum_{j=n+1}^{p} E_j - \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{q} F_j \Im_j$$

$$\lambda = \Re\left(\sum_{j=1}^{m} f_j + \sum_{j=m+1}^{q} f_j \Im_j - \sum_{j=1}^{n} e_j \in_j - \sum_{j=n+1}^{p} e_j\right)$$
$$+\frac{1}{2}\left(\sum_{j=1}^{n} \in_j - \sum_{j=m+1}^{q} \Im_j + p - m - n\right).$$

The following series representation for the $\overline{H}-$Function given by Rathie [54] and Saxena [56] has been used in the present work:

$$\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(e_j, E_j; \in_j)_{1,n}, & (e_j, E_j)_{n+1,p} \\ (f_j, F_j)_{1,m}, & (f_j, F_j; \Im_j)_{m+1,q}\end{matrix}\right.\right] = \sum_{t=0}^{\infty}\sum_{h=1}^{m} \overline{\Theta}(\mathfrak{s}_{t,h}) z^{\mathfrak{s}_{t,h}}, \qquad (1.1.17)$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t,h}) = \frac{\prod_{j=1,j\neq h}^{m} \Gamma(f_j - F_j \mathfrak{s}_{t,h}) \prod_{j=1}^{n} \{\Gamma(1 - e_j + E_j \mathfrak{s}_{t,h})\}^{\in_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - f_j + F_j \mathfrak{s}_{t,h})\}^{\Im_j} \prod_{j=n+1}^{p} \Gamma(e_j - E_j \mathfrak{s}_{t,h})} \frac{(-1)^t}{t! F_h}, \quad \mathfrak{s}_{t,h} = \frac{f_h + t}{F_h}$$
$$(1.1.18)$$

In the Sequel, we shall also make use of the following behavior of the $\overline{H}_{p,q}^{m,n}[z]$ function for small and large value of z as recorded by Saxena et al.

[57, p. 112, Eqs.(2.3) and (2.4)].

$$\overline{H}_{p,q}^{m,n}[z] = O[|z|^{\alpha}], \text{for small z, where} \quad \alpha = \min_{1\leq j\leq m} \Re\left(\frac{f_j}{F_j}\right) \qquad (1.1.19)$$

$$\overline{H}_{p,q}^{m,n}[z] = O[|z|^{\beta}], \text{for large z, where} \quad \beta = \max_{1\leq j\leq n} \Re\left(\in_j \left(\frac{e_j - 1}{E_j}\right)\right), \qquad (1.1.20)$$

provided that either of the following conditions are satisfied:

$$\left.\begin{matrix}(i) \quad \mu < 0 \quad \text{and} \quad 0 < |z| < \infty \\ (ii) \quad \mu = 0 \quad \text{and} \quad 0 < |z| < \delta^{-1}\end{matrix}\right\} \qquad (1.1.21)$$

where

$$\mu = \sum_{j=1}^{n} E_j \in_j + \sum_{j=n+1}^{p} E_j - \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{q} F_j \Im_j \tag{1.1.22}$$

$$\delta = \prod_{j=1}^{n} (E_j)^{E_j \in_j} \prod_{j=n+1}^{p} (E_j)^{E_j} \prod_{j=1}^{m} (F_j)^{-F_j} \prod_{m+1}^{q} (F_j)^{-F_j \Im_j}. \tag{1.1.23}$$

## SPECIAL CASES

The following special cases of the $\overline{H}$−function have been made use in this thesis:

(I) **The Polylogarithm of order** $p$ [8, p.30, §1.11, Eq. (14)] and [10, p. 315, Eq. (1.9)]

$$F(z,p) = \sum_{r=1}^{\infty} \frac{z^r}{r^p} = z\overline{H}_{1,2}^{1,1}\left[ -z \left| \begin{matrix} (0,1;p+1) & \\ (0,1), & (-1,1;p) \end{matrix} \right. \right]$$

$$= -\overline{H}_{1,2}^{1,1}\left[ -z \left| \begin{matrix} (1,1;p+1) & \\ (1,1), & (0,1;p) \end{matrix} \right. \right], \tag{1.1.24}$$

Here, $F(z,p)$ is the polylogarithm function of order $p$ .

(II) **The Generalized Wright Hypergeometric Function** [18, p. 271, Eq. (7)]

$${}_p\overline{\Psi}_q \left( \begin{matrix} (e_j, E_j; \in_j)_{1,p}; \\ \\ (f_j, F_j; \Im_j)_{1,q}; \end{matrix} \; z \right) = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \{\Gamma(e_j + E_j r)\}^{\in_j}}{\prod_{j=1}^{q} \{\Gamma(f_j + F_j r)\}^{\Im_j}} \frac{z^r}{r!}$$

$$= \overline{H}^{1,p}_{p,q+1}\left[-z \left| \begin{array}{c} (1-e_j, E_j; \in_j)_{1,p} \\ \\ (0,1), (1-f_j, F_j; \Im_j)_{1,q} \end{array} \right.\right], \qquad (1.1.25)$$

${}_p\overline{\Psi}_q$ reduces to ${}_p\Psi_q$, the familiar Wright's Generalized hypergeometric function [68, p. 19, Eq. (2.6.11)], when all the exponents $(\in_j)_{1,n}$, $(\Im_j)_{m+1,q}$ take the value 1.

$${}_p\Psi_q\left(\begin{array}{c} (e_j, E_j)_{1,p}; \\ \\ (f_j, F_j)_{1,q}; \end{array} z\right) = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(e_j + E_j r)}{\prod_{j=1}^{q}\Gamma(f_j + F_j r)} \frac{z^r}{r!}$$

$$= H^{1,p}_{p,q+1}\left[-z \left| \begin{array}{c} (1-e_j, E_j)_{1,p} \\ \\ (0,1), (1-f_j, F_j)_{1,q} \end{array} \right.\right]. \qquad (1.1.26)$$

(III) **The Generalized Riemann Zeta Function** [8, p. 27, §1.11, Eq. (1)] and [10, pp. 314–315, Eq. (1.6) and (1.7)]

$$\phi(z,p,\eta) = \sum_{r=0}^{\infty} \frac{z^r}{(\eta+r)^p} = \overline{H}^{1,2}_{2,2}\left[-z \left| \begin{array}{cc} (0,1;1), & (1-\eta,1;p) \\ (0,1), & (-\eta,1;p) \end{array} \right.\right]. \qquad (1.1.27)$$

(IV) **Generalized Hurwitz Lerch Zeta Function** [24, pp. 147 & 151, Eqs.(6.2.5) and (6.4.2)]

$$\phi_{\alpha,\beta,\gamma}(z,p,\eta) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!} \frac{z^r}{(\eta+r)^p}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \overline{H}^{1,3}_{3,3}\left[-z \left| \begin{array}{cc} (1-\eta,1;p), (1-\alpha,1;1), & (1-\beta,1;1) \\ \\ (0,1), (1-\gamma,1;1), & (-\eta,1;p) \end{array} \right.\right]. \qquad (1.1.28)$$

(V) **Generalized Wright Bessel Function** [18, p. 271, Eq.(8)]

$$\overline{J}_{\lambda}^{\nu,\mu}(z) = \sum_{r=0}^{\infty} \frac{(-z)^r}{r!(\Gamma(1+\lambda+\nu r))^{\mu}}$$

$$= \overline{H}_{0,2}^{1,0}\left[ z \,\middle|\, \begin{matrix} \\ (0,1),(-\lambda,\nu;\mu) \end{matrix} \right]. \tag{1.1.29}$$

(VI) **A Generalization of the Generalized Hypergeometric Function** [18, p. 271, Eq. (9)]

$${}_p\overline{F}_q \left( \begin{matrix} (e_j, \in_j)_{1,p}; \\ \\ (f_j, \Im_j)_{1,q}; \end{matrix} \; z \right)$$

$$= \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \{(e_j)_r\}^{\in_j}}{\prod_{j=1}^{q} \{(f_j)_r\}^{\Im_j}} \frac{z^r}{r!} = \frac{\prod_{j=1}^{q} \{\Gamma(f_j)\}^{\Im_j}}{\prod_{j=1}^{p} \{\Gamma(e_j)\}^{\in_j}} \overline{H}_{p,q+1}^{1,p}\left[ -z \,\middle|\, \begin{matrix} (1-e_j,1;\in_j)_{1,p} \\ \\ (0,1),(1-f_j,1;\Im_j)_{1,q} \end{matrix} \right]$$

$$= \frac{\prod_{j=1}^{q} \{\Gamma(f_j)\}^{\Im_j}}{\prod_{j=1}^{p} \{\Gamma(e_j)\}^{\in_j}} {}_p\overline{\Psi}_q \left( \begin{matrix} (e_j, 1; \in_j)_{1,p}; \\ \\ (f_j, 1; \Im_j)_{1,q}; \end{matrix} \; z \right).$$

The function ${}_p\overline{F}_q$ reduces to well known ${}_pF_q$ for $\in_j = 1 (j = 1, \cdots, p)$, $\Im_j = 1(j = 1, \cdots, q)$ in it.

Naturally, all functions which are special cases of the $H-$function are also special cases of the $\overline{H}-$function.

### 1.1.3 H-FUNCTION OF TWO VARIABLES

The H-function of two variables is defined and represented in the following manner [74, p. 82, Eqs. (6.1.1-6.1.4)]:

$$H^{0,B:A_1,B_1;A_2,B_2}_{C,D:C_1,D_1;C_2,D_2}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},\alpha_j^{(2)})_{1,C}:(c_j^{(1)},\gamma_j^{(1)})_{1,C_1};(c_j^{(2)},\gamma_j^{(2)})_{1,C_2} \\ & \\ z_2 & (b_j;\beta_j^{(1)},\beta_j^{(2)})_{1,D}:(d_j^{(1)},\delta_j^{(1)})_{1,D_1};(d_j^{(2)},\delta_j^{(2)})_{1,D_2} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\psi(\xi_1,\xi_2)\prod_{i=1}^{2}(\phi_i(\xi_i)z_i^{\xi_i})d\xi_1\,d\xi_2 \qquad (i=1,2), \tag{1.1.30}$$

where $\omega=\sqrt{-1}$,

$$\psi(\xi_1,\xi_2)=\frac{\prod_{j=1}^{B}\Gamma(1-a_j+\sum_{i=1}^{2}\alpha_j^{(i)}\xi_i)}{\prod_{j=1}^{D}\Gamma(1-b_j+\sum_{i=1}^{2}\beta_j^{(i)}\xi_i)\prod_{j=B+1}^{C}\Gamma(a_j-\sum_{i=1}^{2}\alpha_j^{(i)}\xi_i)}, \tag{1.1.31}$$

$$\phi_i(\xi_i)=\frac{\prod_{i=1}^{A_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{B_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=A_i+1}^{D_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod_{j=B_i+1}^{C_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)} \qquad (i=1,2). \tag{1.1.32}$$

All the greek letters occuring on the left-hand side of (1.1.30) are assumed to be positive real numbers for standardization purposes; the definition of the multi-variable $H-$function will, however, be meaningful even if some of these quantities are zero such that

$$\Lambda_i \equiv \sum_{j=1}^{C}\alpha_j^{(i)}+\sum_{j=B_i+1}^{C_i}\gamma_j^{(i)}-\sum_{j=1}^{D}\beta_j^{(i)}-\sum_{j=1}^{D_i}\delta_j^{(i)}>0 \quad (i=1,2), \tag{1.1.33}$$

$$\Omega_i \equiv -\sum_{j=B+1}^{C} \alpha_j^{(i)} + \sum_{j=1}^{B_i} \gamma_j^{(i)} - \sum_{j=B_i+1}^{C_i} \gamma_j^{(i)} - \sum_{j=1}^{D} \beta_j^{(i)} + \sum_{j=1}^{A_i} \delta_j^{(i)} - \sum_{j=A_i+1}^{D_i} \delta_j^{(i)} > 0 \quad (i = 1, 2), \tag{1.1.34}$$

where $B, C, D, A_i, B_i, C_i, D_i$ are non negative integers such that $0 \leq B \leq C$, $D \geq 0, \quad 0 \leq B_i \leq C_i$ and $1 \leq A_i \leq D_i, \quad (i = 1, 2)$.

The sequences of the parameters in (1.1.30) are such that none of the poles of the integrand coincide i.e. the poles of the integrand in (1.1.30) are simple. The contour $\mathfrak{L}_i$ in the complex $\xi_i-$ plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \quad (j = 1, \cdots, A_i)$ are separated from those of $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)}\xi_i)$ $(j = 1, \cdots, B_i)$ and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)}\xi_i) \quad (i = 1, 2; j = 1, \cdots, B)$.

It is known that multiple Mellin-Barnes contour integral representing the multivariable $H-$ function (1.1.36) converges absolutely [73, p. 130, Eq. (1.4)] under the condition (1.1.40) when

$$|\arg(z_i)| < \frac{1}{2}\Omega_i\pi, \quad (i = 1, 2). \tag{1.1.35}$$

The point $z_i = 0 (i = 1, 2)$ and various exceptional parameter values are excluded.

### 1.1.4 THE MULTIVARIABLE $H-$FUNCTION

The multivariable $H-$function occuring in the thesis was introduced and studied by Srivastava and Panda [74, p. 130, Eq. (1.1)]. This function involves r complex variables and will be defined and represented in the following contracted form

[71, pp. 251–252, Eqs. (C.1–C.3)]

$$H^{0,B:A_1,B_1;\cdots;A_r,B_r}_{C,D:C_1,D_1;\cdots;C_r,D_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,C}:(c_j^{(1)},\gamma_j^{(1)})_{1,C_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,D}:(d_j^{(1)},\delta_j^{(1)})_{1,D_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,D_r}\end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int\limits_{\mathfrak{L}_1}\cdots\int\limits_{\mathfrak{L}_r}\psi(\xi_1,\cdots,\xi_r)\prod_{i=1}^{r}(\phi_i(\xi_i)z_i^{\xi_i})d\xi_1\cdots d\xi_r \qquad (i=1,\cdots,r), \tag{1.1.36}$$

where $\omega=\sqrt{-1}$,

$$\psi(\xi_1,\cdots,\xi_r)=\frac{\prod\limits_{j=1}^{B}\Gamma(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod\limits_{j=1}^{D}\Gamma(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod\limits_{j=B+1}^{C}\Gamma(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i)}, \tag{1.1.37}$$

$$\phi_i(\xi_i)=\frac{\prod\limits_{i=1}^{A_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod\limits_{j=1}^{B_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod\limits_{j=A_i+1}^{D_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod\limits_{j=B_i+1}^{C_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)} \qquad (i=1,\cdots,r). \tag{1.1.38}$$

All the greek letters occuring on the left-hand side of (1.1.36) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable $H-$function will, however, be meaningful even if some of these quantities are zero such that

$$\Lambda_i\equiv\sum_{j=1}^{C}\alpha_j^{(i)}+\sum_{j=B_i+1}^{C_i}\gamma_j^{(i)}-\sum_{j=1}^{D}\beta_j^{(i)}-\sum_{j=1}^{D_i}\delta_j^{(i)}>0 \quad (i=1,2,\cdots,r), \tag{1.1.39}$$

$$\Omega_i \equiv -\sum_{j=B+1}^{C} \alpha_j^{(i)} + \sum_{j=1}^{B_i} \gamma_j^{(i)} - \sum_{j=B_i+1}^{C_i} \gamma_j^{(i)} - \sum_{j=1}^{D} \beta_j^{(i)} + \sum_{j=1}^{A_i} \delta_j^{(i)} - \sum_{j=A_i+1}^{D_i} \delta_j^{(i)} > 0 \tag{1.1.40}$$

where $B, C, D, A_i, B_i, C_i, D_i$ are non negative integers such that $0 \leq B \leq C$, $D \geq 0, \quad 0 \leq B_i \leq C_i$ and $1 \leq A_i \leq D_i, \quad (i = 1, \cdots, r)$.

The sequences of the parameters in (1.1.36) are such that none of the poles of the integrand coincide i.e. the poles of the integrand in (1.1.36) are simple. The contour $\mathfrak{L}_i$ in the complex $\xi_i-$ plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \quad (j = 1, \cdots, A_i)$ are separated from those of $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)}\xi_i)$ $(j = 1, \cdots, B_i)$ and $\Gamma(1 - a_j + \sum\limits_{i=1}^{r} \alpha_j^{(i)}\xi_i) \quad (i = 1, \cdots, r; j = 1, \cdots, B)$.

It is known that multiple Mellin-Barnes contour integral representing the multivariable $H-$ function (1.1.36) converges absolutely [73, p. 130, Eq. (1.4)] under the condition (1.1.40) when

$$|\arg(z_i)| < \frac{1}{2}\Omega_i\pi, \quad (i = 1, \cdots, r). \tag{1.1.41}$$

The point $z_i = 0$ $(i = 1, \cdots, r)$ and various exceptional parameter values are excluded.

## SPECIAL CASES

By suitably specializing the various parameters occuring in the multivariable $H-$function defined by (1.1.36), it reduces to the simpler special functions of one and more variables.

Some of them which have been used in this thesis are given below:

(i) If we take $\alpha_j^{(1)} = \alpha_j^{(2)} = ... = \alpha_j^{(r)} \quad (j = 1, ..., D)$ and $\beta_j^{(1)} = \beta_j^{(2)} = ... = \beta_j^{(r)} \quad (j = 1, ..., D)$ in (1.1.36), it reduces to a special multivariable $H-$function studied by Saxena [56].

(ii) If we take $r = 2$, in (1.1.36) , we get $H-$function of two variables defined in [68, p.82, eq.(6.1.1)].

(iii) A relation between $H-$function of two variable and the Appell function [68, p.89,Eq.(6.4.6)] is given as below:

$$H^{0,0:1,2;1,2}_{0,1:2,1;2,1}\left[\begin{array}{c} -x \\ -y \end{array}\middle|\begin{array}{cccc} - : & (1-c,1),(1-c',1); & (1-e,1),(1-e',1) \\ (1-b;1,1) : & (0,1); & (0,1) \end{array}\right]$$

$$= \frac{\Gamma(c)\Gamma(c')\Gamma(e)\Gamma(e')}{\Gamma(b)} F_3(c,e,c',e';b;x,y), \quad |x| < 1, |y| < 1 \qquad (1.1.42)$$

(iv) if we reduce Multivariable H-function into generalized hypergeometric Function [25, p.xi,Eq.(A.18)] as given below:

$$H^{0,C:1,0;\cdots;1,0}_{C,D:0,1;\cdots;0,1}\left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle|\begin{array}{c} (1-a_j;1,\cdots,1)_{1,C} : --;\cdots;-- \\ \\ \\ \\ (1-b_j;1,\cdots,1)_{1,D} : (0,1);\cdots;(0,1) \end{array}\right]$$

$$= \frac{\prod_{j=1}^{C} \Gamma(a_j)}{\prod_{j=1}^{D} \Gamma(b_j)} {}_C F_D \left[ \begin{matrix} (a_C); \\ \\ (b_D); \end{matrix} \; -(z_1 + \cdots + z_r) \right] \tag{1.1.43}$$

### 1.1.5 $S_V^U$ POLYNOMIAL

The $S_V^U$ polynomial was introduced and investigated by Srivastava [61] and is represented in the following manner :

$$S_V^U[x] = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} A_{V,R}}{R!} x^R, \qquad V = 0, 1, 2, .... \tag{1.1.44}$$

where $U$ is an arbitrary positive integer, the coefficients $A_{V,R}$ are constants, real or complex.

### 1.1.6 GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION

A special function of the form $E_\delta(z)$ was introduced by the Swedish mathematician Gosta Mittag-Leffler [43] in 1903 as

$$E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)} \qquad (\delta \in \mathbb{C}, \, \Re(\delta) > 0), \tag{1.1.45}$$

which is direct generalization of the exponential function for $\delta = 1$. In 1905, Wiman [81] gave the generalization of $E_\delta(z)$ known as Wiman's function or generalized Mittag-Leffler function defined as follows:

$$E_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + \kappa)} \qquad (\delta, \kappa \in \mathbb{C}, \; \Re(\delta) > 0, \Re(\kappa) > 0). \tag{1.1.46}$$

Later on, Prabhakar [51] introduced a new generalization of $E_{\delta,\kappa}(z)$ as $E^{\vartheta}_{\delta,\kappa}(z)$,

$$E^{\vartheta}_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{1.1.47}$$

$$(\kappa, \delta, \vartheta \in \mathbb{C},\ \Re(\delta) > 0, \Re(\kappa) > 0, \Re(\vartheta) > 0),$$

where $(\vartheta)_n$ denotes the Pochhammer symbol (see for details, [53], [66] and [67]):

$$(\vartheta)_0 = 1, \qquad (\vartheta)_n = \vartheta(\vartheta+1)(\vartheta+2)\cdots(\vartheta+n-1). \tag{1.1.48}$$

Moreover, generalization of $E^{\vartheta}_{\delta,\kappa}(z)$ was introduced and studied by Shukla and Prajapati [59] defined as

$$E^{\vartheta,r}_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_{rn}}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{1.1.49}$$

$$(\vartheta, \delta, \kappa \in \mathbb{C},\ \Re(\delta) > 0, \Re(\kappa) > 0, \Re(\vartheta) > 0 \,\text{and}\, r \in (0,1) \cup \mathbb{N}),$$

(1.1.49) was further investigated by Srivastava and Tomovski [78]

$$E^{\vartheta,\xi}_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_{\xi n}}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{1.1.50}$$

$$(z, \kappa, \vartheta \in \mathbb{C},\ \Re(\delta) > \max(0, \Re(\xi)-1), \Re(\xi) > 0).$$

Lately, Özarslan and Yilmaz [46] studied extended Mittag-Leffler function and defined it in the following manner:

$$E^{(\vartheta;d)}_{\delta,\kappa}(z;q) = \sum_{n=0}^{\infty} \frac{B_q(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \quad (q \geq 0, \Re(d) > \Re(\vartheta) > 0), \tag{1.1.51}$$

$$\text{where} \qquad B_q(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-q}{t(1-t)}}\, dt \tag{1.1.52}$$

$$(\Re(q) \geq 0, \Re(x) > 0, \Re(y) > 0).$$

Now, we propose to introduce and investigate further *generalization of extended Mittag-Leffler function* in (1.1.51) as $E^{\vartheta;d}_{\delta,\kappa}(z;q,\rho,\zeta)$,

$$E^{\vartheta;d}_{\delta,\kappa}(z;q,\rho,\zeta) = \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{1.1.53}$$

$$(q \geq 0,\ \Re(d) > \Re(\vartheta) > 0,\ \Re(\delta) > 0,\ \Re(\kappa) > 0),$$

where $B_q^{(\rho,\zeta)}(x,y)$ represents generalized Beta type function as follows(see, for details, [65, p. 348, Eq.(1.2)]; see also [48, p. 32, Chapter 4]):

$$B_q^{(\rho,\zeta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1(\rho;\zeta;\frac{-q}{t(1-t)})\, dt \tag{1.1.54}$$

$$(\Re(q) \geq 0, \min(\Re(x), \Re(y), \Re(\zeta), \Re(\rho)) > 0).$$

Further, we also present the contour representation of generalized extended Mittag-Leffler function in the following manner:

$$E^{\vartheta;d}_{\delta,\kappa}(z;q,\rho,\zeta) = \frac{1}{(2\pi i)^2} \frac{\Gamma(\zeta)}{\Gamma(\rho)\Gamma(\vartheta)\Gamma(d-\vartheta)} \int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2} \frac{\Gamma(\rho-\xi_1)}{\Gamma(\zeta-\xi_1)} \frac{\Gamma(\vartheta+\xi_1+\xi_2)\Gamma(d-\vartheta+\xi_1)\Gamma(\xi_1)}{\Gamma(d+\xi_2+2\xi_1)} \cdot \frac{\Gamma(d+\xi_2)\Gamma(-\xi_2)}{\Gamma(\kappa+\delta\xi_2)} (q)^{-\xi_1}(z)^{\xi_2} d\xi_1 d\xi_2 \tag{1.1.55}$$

For, $\rho = \zeta$ generalization of extended Mittag-Leffler function in (1.1.53) reduces to extended Mittag- Leffler function in (1.1.51).

In the sequel, we shall represent Lebesgue measurable real or complex valued functions defined on a finite interval $[a, b]$ of real line $\mathbb{R}$(see, for details, [13] and [27]; see also [55]) in the following manner:

$$L(a,b) = \left\{ f(x) : \|f\|_1 = \int_a^b \mid f(x) \mid dx < \infty \right\}. \tag{1.1.56}$$

### 1.1.7 S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

The S-generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ was introduced and investigated by Srivastava et al. [65, p. 350, Eq. (1.12)]. It is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \qquad (|z| < 1) \tag{1.1.57}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(\alpha), \Re(\beta), \Re(\tau), \Re(\mu)\} > 0; \quad \Re(c) > \Re(b) > 0),$$

in terms of the classical Beta function $B(\lambda, \mu)$ and the S-generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$, which was also defined by Srivastava et al. [65, p. 350, Eq. (1.13)] as follows:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha; \beta; -\frac{p}{t^{\tau}(1-t)^{\mu}}\right) dt \tag{1.1.58}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(x), \Re(y), \Re(\alpha), \Re(\beta), \Re(\tau), \Re(\mu)\} > 0).$$

If we take p = 0 in (1.1.58), it reduces to classical Beta Function and $(\lambda)_n$ denotes the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [67, p. 2 and pp. 4-6]):

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n=0) \\ \lambda(\lambda+1)...(\lambda+n-1), & (n \in \mathbb{N} \quad := \{1,2,3,\cdots\}), \end{cases} \tag{1.1.59}$$

provided that the Gamma quotient exists (see, for details,[71, p. 16 et seq.] and [72, p. 22 et seq.]).

For $\tau = \mu$, the S-generalized Gauss hypergeometric function defined by (1.1.57) reduces to the following generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau)}(a,b;c;z)$ studied earlier by Parmar [49, p. 44]:

$$F_p^{(\alpha,\beta;\tau)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\tau)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{1.1.60}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(\alpha),\Re(\beta),\Re(\tau)\} > 0; \quad \Re(c) > \Re(b) > 0).$$

which, in the *further* special case when $\tau = 1$, reduces to the following extension of the generalized Gauss hypergeometric function (see, e.g., [48, p. 4606, Section 3] ; see also [47, p. 39]):

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \qquad (|z|<1) \tag{1.1.61}$$

$$(\Re(p) \geqslant 0; \quad \min\{\Re(\alpha),\Re(\beta)\} > 0; \quad \Re(c) > \Re(b) > 0).$$

Upon setting $\alpha = \beta$ in (1.1.61), we arrive at the following Extended Gauss hypergeometric function (see [5, p. 591, Eqs. (2.1) and (2.2)]:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \qquad (|z| < 1) \tag{1.1.62}$$

$$(\Re(p) \geqslant 0; \quad \Re(c) > \Re(b) > 0).$$

### 1.1.8 EXTENDED HURWITZ-LERCH ZETA FUNCTION

The extended Hurwitz-Lerch Zeta function introduced by Srivastava et al. [77, p. 503, Eq.(6.2)](see also [62] and [70]) is recalled here in slightly modified form:

$$\Phi^{(\varrho_1,\ldots,\varrho_P,\upsilon_1,\ldots,\upsilon_Q)}_{k_1,\ldots,k_P;l_1,\ldots,l_Q}(z,s,a) = \Phi^{k_j;\varrho_j P}_{l_j;\upsilon_j Q}(z,s,a) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{P} (k_j)_{m\varrho_j}}{m! \prod_{j=1}^{Q} (l_j)_{m\upsilon_j}} \frac{z^m}{(m+a)^s} \tag{1.1.63}$$

$$(P, Q \in \mathbb{N}_0; k_j \in \mathbb{C}(j = 1, \cdots, P); a, l_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j = 1, \cdots, Q);$$

$$\varrho_j, \upsilon_n \in \mathbb{R}^+(j = 1 \cdots P; n = 1 \cdots, Q); \Im > -1 \text{ when } s, z \in \mathbb{C};$$

$$\Im = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \mho; \Im = -1 \text{ and } \Re(\varpi) > \tfrac{1}{2} \text{ when } |z| = \mho$$

$$\text{where } \mho := (\prod_{j=1}^{P} \varrho_j^{-\varrho_j})(\prod_{j=1}^{Q} \upsilon_j^{\upsilon_j}) \text{ and } \Im := \sum_{j=1}^{Q} \upsilon_j - \sum_{j=1}^{P} \varrho_j,$$

$$\varpi := s + \sum_{j=1}^{Q} l_j - \sum_{j=1}^{P} k_j + \tfrac{P-Q}{2}),$$

the integral representation of extended Hurwitz-Lerch zeta function was also given by Srivastava et al. ([77],Theorem 8)see also([63], Theorem 6).

$$\Phi^{(\lambda_j,\rho_j;P)}_{(\mu_j,\sigma_j;Q)}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_P\Psi^*_Q\left[\begin{array}{ll}(\lambda_1,\rho_1),\cdots,(\lambda_P,\rho_P); & \\ & ze^{-t}\\ (\mu_1,\sigma_1),\cdots,(\mu_Q,\sigma_Q); & \end{array}\right]dt \tag{1.1.64}$$

$$(\min\{\Re(a),\Re(s)\}>0),$$

where ${}_P\Psi^*_Q$ $(P,Q\in\mathbb{N}_0)$ denotes the Fox-Wright function,which is generalization of the familiar generalized hypergeometric function ${}_PF_Q$ $(P,Q\in\mathbb{N}_0)$defined by [8, p.183]

$${}_P\Psi^*_Q\left[\begin{array}{ll}(\lambda_j,\rho_j)_{1,P}; & \\ & z\\ (\mu_j,\sigma_j)_{1,Q}; & \end{array}\right]:=\sum_{n=0}^\infty\frac{(\lambda_1)_{\rho_1 n}\cdots(\lambda_P)_{\rho_P n}}{(\mu_1)_{\sigma_1 n}\cdots(\mu_Q)_{\sigma_Q n}}\frac{z^n}{n!}=\frac{\Gamma(\mu_1)\cdots\Gamma(\mu_Q)}{\Gamma(\lambda_1)\cdots\Gamma(\lambda_P)}{}_P\Psi_Q\left[\begin{array}{ll}(\lambda_j,\rho_j)_{1,P}; & \\ & z\\ (\mu_j,\sigma_j)_{1,Q}; & \end{array}\right]. \tag{1.1.65}$$

In a subsequent work by Srivastava [64], this last integral representation formula (1.1.64) was suitably modified in order to introduce and investigate the various properties of a significantly more general class of the $\lambda-$extended Hurwitz-Lerch zeta functions defined by

$$\begin{aligned}\Phi^{(\lambda_j,\rho_j;P)}_{(\mu_j,\sigma_j;Q)}(z,s,a;b,\lambda)&=\Phi^{(\rho_1,\ldots,\rho_P,\sigma_1,\ldots,\sigma_Q)}_{\lambda_1,\ldots,\lambda_P;\mu_1,\ldots,\mu_Q}(z,s,a;b,\lambda)\\&:=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}exp\left(-at-\frac{b}{t^\lambda}\right){}_P\Psi^*_Q\left[\begin{array}{ll}(\lambda_1,\rho_1),\cdots,(\lambda_P,\rho_P); & \\ & ze^{-t}\\ (\mu_1,\sigma_1),\cdots,(\mu_Q,\sigma_Q); & \end{array}\right]dt\end{aligned} \tag{1.1.66}$$

$$(\min\{\Re(a),\Re(s)\}>0;\Re(b)\geq 0;\lambda\geq 0),$$

so that, obviously, we have the following relationship:

$$\begin{aligned}\Phi^{(\rho_1,\ldots,\rho_P,\sigma_1,\ldots,\sigma_Q)}_{\lambda_1,\ldots,\lambda_P;\mu_1,\ldots,\mu_Q}(z,s,a;0,\lambda)&=\Phi^{(\rho_1,\ldots,\rho_P,\sigma_1,\ldots,\sigma_Q)}_{\lambda_1,\ldots,\lambda_P;\mu_1,\ldots,\mu_Q}(z,s,a)\\&=e^b\Phi^{(\rho_1,\ldots,\rho_P,\sigma_1,\ldots,\sigma_Q)}_{\lambda_1,\ldots,\lambda_P;\mu_1,\ldots,\mu_Q}(z,s,a;b,0).\end{aligned} \tag{1.1.67}$$

By using the following corrected version of an integral formula [38, p. 10, Eq. (1.53)]:

$$\int_0^{\infty} t^{a-1} exp\left(-bt - \frac{c}{t^\rho}\right) dt = \frac{1}{\rho b^a} H_{0,2}^{2,0}\left[ bc^{\frac{1}{\rho}} \middle| \begin{matrix} - \\ (a,1),(0,\frac{1}{\rho}) \end{matrix} \right] \tag{1.1.68}$$

$$(\min\{\Re(a), \Re(b), \Re(c)\} > 0; \rho \geq 0),$$

an explicit representation was proven by Srivastava [64, p. 1489, Eq. (2.1)]:

$$\Phi_{\lambda_1,\dots,\lambda_P;\mu_1,\dots,\mu_Q}^{(\rho_1,\dots,\rho_P,\sigma_1,\dots,\sigma_Q)}(z,s,a;b,\lambda) = \frac{1}{\lambda\Gamma(s)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{P} (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{Q} (\mu_j)_{n\sigma_j}} \cdot H_{0,2}^{2,0}\left[ (a+n)b^{\frac{1}{\lambda}} \middle| \begin{matrix} - \\ (s,1),(0,\frac{1}{\lambda}) \end{matrix} \right] \frac{z^n}{n!} \tag{1.1.69}$$

$$(\lambda > 0),$$

in terms of Fox's H-function defined by (1.1.4), it being assumed that each member of the assertion (1.1.69) exists. The following Mellin-Barnes type contour integral representation was also presented by Srivastava [64, p. 1489, Eq. (2.2)]:

$$\Phi_{\lambda_1,\dots,\lambda_P;\mu_1,\dots,\mu_Q}^{(\rho_1,\dots,\rho_P,\sigma_1,\dots,\sigma_Q)}(z,s,a;b,\lambda) = \frac{\prod_{j=1}^{Q} \Gamma(\mu_j)}{2\pi i \lambda \Gamma(s) \prod_{j=1}^{P} \Gamma(\lambda_j)} \cdot \int_{-i\infty}^{i\infty} \frac{\Gamma(\mathtt{s}) \prod_{j=1}^{P} \Gamma(\lambda_j - \mathtt{s}\rho_j)}{(a-\mathtt{s})^s \prod_{j=1}^{Q} \Gamma(\mu_j - \mathtt{s}\sigma_j)} \cdot H_{0,2}^{2,0}\left[ (a-\mathtt{s})b^{\frac{1}{\lambda}} \middle| \begin{matrix} - \\ (s,1),(0,\frac{1}{\lambda}) \end{matrix} \right] (-z)^{-\mathtt{s}} \, d\mathtt{s} \tag{1.1.70}$$

$$(\lambda > 0),$$

provided that each member of the assertion (1.1.70) exists.

## SPECIAL CASES OF EXTENDED HURWITZ-LERCH ZETA FUNCTION

(i) **General Hurwitz-Lerch Zeta function :** If we take $Q = 0$ and $P = \rho_j = k_j = 1$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(z, s, a) = \Phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^s}, \tag{1.1.71}$$

where $\Phi(z, s, a)$ is defined in [8, p.27, Eq.(1.11)].

(ii) **Riemann Zeta function :** Again, if we take $Q = 0$ and $P = \rho_j = k_j = z = a = 1$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(1, s, 1) = \zeta(s) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^s}, \tag{1.1.72}$$

where $\zeta(s)$ is defined in [8, Chapter 1](see, for details, [66, Chapter 2]).

(iii) **Hurwitz (or generalized) Zeta function :** Further, if we take $Q = 0$ and $P = \rho_j = k_j = z = 1$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(1, s, a) = \zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}, \tag{1.1.73}$$

where $\zeta(s, a)$ is defined in [8, Chapter 1]( see also, [66, Chapter 2]).

(iv) **Lerch Zeta function :** If we take $Q = 0$, $P = \rho_j = k_j = a = 1$ and $z = e^{2\pi i\xi}$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(e^{2\pi i\xi}, s, 1) = \ell_s(\xi) = \sum_{m=0}^{\infty} \frac{e^{2m\pi i\xi}}{(m+1)^s}, \tag{1.1.74}$$

where $\ell_s(\xi)$ is defined in [8, Chapter 1](see, for details, [66, Chapter 2]).

(v) **Lipschitz-Lerch Zeta function :** If we take $Q = 0$, $P = \rho_j = k_j = 1$ and $z = e^{2\pi i\xi}$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(e^{2\pi i\xi}, s, a) = \phi(\xi, s, a) = \sum_{m=0}^{\infty} \frac{e^{2m\pi i\xi}}{(m+a)^s}, \tag{1.1.75}$$

where $\phi(\xi, s, a)$ is defined in [66, p. 122, Eq. (2.5)](see, for details, [80, p. 280, Example 8]).

## 1.2 INTEGRAL TRANSFORMS

If $f(x)$ denotes of a prescribed class of functions defined on a given interval $[a, b]$ and $K(x, s)$ denotes a definite function of $x$ in that interval for each value of $s$, a parameter whose domain is prescribed, then the linear integral transform $T[f(x); s]$ of the function $f(x)$ is defined in the following manner:

$$T[f(x); s] = \int_a^b K(x, s) f(x) dx \tag{1.2.1}$$

wherein the class of functions and the domain of parameter $s$ are so prescribed that the above integral exists. In (1.2.1), $K(x, s)$ is known as the kernel of the transform, $T[f(x); s]$ is the image of $f(x)$ in the said transform; and $f(x)$ is the original of $T[f(x); s]$.

**Inversion formula for the transform**

If an integral equation can be determined that

$$f(x) = \int_{\alpha}^{\beta} \phi(s,x) T[f(x);s] ds \tag{1.2.2}$$

then (1.2.2) is termed as the inversion formula of (1.2.1).

### 1.2.1 LAPLACE TRANSFORM

One of the simplest and most important integral transform is the well known Laplace Transform. It has been a subject of wide and extensive study on account of its applications in applied mathematics and physics.

The Laplace Transform of a function is defined as follows:

$$L\{f(x);s\} = \int_{0}^{\infty} e^{-sx} f(x) dx \qquad (\Re(s) > 0), \tag{1.2.3}$$

provided that the integral (1.2.3) exists and and the inversion formula is given by:

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{sx} L\{f(x);s\} ds \tag{1.2.4}$$

provided that the above integral exists.

The standard works of Doestch [7] in three volumes give the detailed and complete account of Laplace Transform.

### 1.2.2 MELLIN TRANSFORM

The well known Mellin Transform is defined by:

$$M\left\{f(x);s\right\} = \int_{0}^{\infty} x^{s-1} f(x)dx \tag{1.2.5}$$

and the inversion formula is given by:

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} M\left\{f(x);s\right\} ds \tag{1.2.6}$$

provided that the above integral exists.

### 1.2.3 THE EXTENDED HURWITZ-LERCH ZETA TRANSFORM OR THE $\mathcal{E}-$TRANSFORM

We define the $\mathcal{E}-$transform, that is, the extended Hurwitz-Lerch zeta transform as follows:

$$\mathcal{E}_{(\mu_j,\sigma_j;Q)}^{(\lambda_j,\rho_j;P)}(z,s,a)[f(t)](\mathtt{s}) := \int_{0}^{\infty} \Phi_{(\mu_j,\sigma_j;Q)}^{(\lambda_j,\rho_j;P)}(\mathtt{s}t,s,a)f(t)dt =: \varphi(\mathtt{s}) \qquad (f(t)\in\Lambda), \tag{1.2.7}$$

in the neighbourhood of $t = z$, where $\Lambda$ denotes the class of admissible functions $f(t)$, which are integrable in every finite interval in $(0,\infty)$ with the following order estimates:

$$f(t) = \begin{cases} O(t^{\omega}) & (t \to 0) \\ O(t^{\kappa}e^{-\mu t}) & (|t| \to \infty) \end{cases} \tag{1.2.8}$$

provided that the existence conditions given with (1.1.64) for the extended Hurwitz-Lerch zeta function are satisfied, $\Re(\omega) > -1$ and

$$\Re(\mu) > 0 \quad \text{or} \quad \Re(\mu) = 0 \quad \text{and} \quad \max_{1\leq j\leq p}\left\{\Re\left(\kappa + \tfrac{\lambda_j - 1}{\rho_j} + 1\right)\right\} < 0.$$

If we reduce the extended Hurwitz-Lerch zeta function to general Fox-Wright function of the type ${}_P\Psi^*_Q$ defined by (1.1.65)[18, p.271, Eqs. (7) and (9)], we are led to the known integral transfroms studied earlier in [1, p. 44, Eqs. (1.3.5) and (1.3.6)].

## 1.3 INTEGRAL OPERATORS

In this section, $\Lambda$ will denote the class of function f(t) for which

$$f(t) = \begin{cases} O\{|t|^{\zeta}\}; & \max\{|t|\} \to 0 \\ \\ O\{|t|^{w_1} e^{-w_2|t|}\}; & \min\{|t|\} \to \infty \end{cases} \tag{1.3.1}$$

In our present investigation we make use of an integral operator with $\overline{H}$-function in its kernel defined as follows:

$$\left(\overline{\mathcal{H}}^{w;m,n;\gamma}_{a+;p,q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\,\overline{H}^{m,n}_{p,q}[w(x-t)^{\gamma}]\varphi(t)dt \tag{1.3.2}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C}\backslash\{0\};\ 1 \leqq m \leqq q;\ 0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\tfrac{\gamma f_j}{F_j}\right)\right\} > 0\right).$$

If we take $w = 1, m = 1$ and $a = 0$ in (1.3.2), we obtain an integral operator introduced by Harjule(see for details [19, p.80, Eq.(5.1.10)]).

Next, if we reduce $\overline{H}$-function to the polylogarithm function of order $\eta$ [8, p.30] in (1.3.2), we obtain the following

$$\left(\mathcal{F}^{w;1,1;\gamma}_{a+;1,2;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\, F[w(x-t)^\gamma, \eta]\varphi(t)dt \tag{1.3.3}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\}\right),$$

provided that the integral exists.

Further, if we reduce $\overline{H}$-function to the generalized Wright hypergeometric function [18, p.271, Eq.(7)] in (1.3.2), we get

$$\left(\overline{\psi}^{w;p;\gamma}_{a+;q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\ {}_p\overline{\psi}_q \left[\begin{array}{c} (e_j, E_j; \in_j)_{1,p} \\ \\ (f_j, F_j; \Im_j)_{1,q} \end{array} ; w(x-t)^\gamma\right]\varphi(t)dt \tag{1.3.4}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\right),$$

provided that the integral exists.

Next, if we reduce $\overline{H}$-function to the generalized Riemann zeta function [8, p.27, section 1.11, Eq.(1)] in (1.3.2), we obtain

$$\left(\phi^{w;1,2;\gamma}_{0+;2,2;\beta}\,\varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\, \phi(w(x-t)^\gamma, \varrho, \eta)\varphi(t)dt \tag{1.3.5}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\}\right),$$

provided that the integral exist.

Again, if we reduce $\overline{H}$-function to the generalized Wright Bessel function [18, p.271, Eq.(8)] in (1.3.2), we obtain

$$\left(\overline{J}^{w;1,0;\gamma}_{0+;0,2;\beta}\,\varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\,\overline{J}^{\zeta,\epsilon}_{\vartheta}(w(x-t)^{\gamma})\varphi(t)dt \tag{1.3.6}$$

$$\left(\Re(\beta) > 0;\; w \in \mathbb{C} \setminus \{0\}\right),$$

provided that the integral exist.

### 1.3.1 AN INTEGRAL OPERATOR INVOLVING THE GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AS IT'S KERNEL

An integral operator with GEML function given by (1.1.53) as it's kernel and $x > a$ $(a \in \mathbb{R}^+ = [0, \infty))$ is defined as follows:

$$(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta} f)(x) = \int_a^x (x-z)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(x-z)^{\delta}; q, \rho, \zeta) f(z)\; dz, \tag{1.3.7}$$

$$(\vartheta, \omega \in \mathbb{C}, \Re(\delta) > 0, \Re(\kappa) > 0)$$

provided that conditions of GEML function in (1.1.53) are satisfied.

### SPECIAL CASES

(i) Considering $\rho = \zeta$ in (1.3.7), we obtain an integral operator introduced by Rahman *et al.* [52] as

$$(\varepsilon^{\omega;\vartheta;d}_{a+;\delta,\kappa} f)(x) = \int_a^x (x-z)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(x-z)^{\delta}; q) f(z)\; dz. \tag{1.3.8}$$

(ii) Next, on taking $q = 0$ in (1.3.8) we get an integral operator given by Srivastava and Tomovski [78]:

$$(\varepsilon^{\omega;\vartheta}_{a+;\delta,\kappa} f)(x) = \int_a^x (x-z)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(x-z)^{\delta}) f(z)\, dz. \tag{1.3.9}$$

(iii) Further, if we take $\omega = 0$, the integral operator in (1.3.9) reduces to Riemann-Liouville fractional integral operator as defined in (1.4.1).

## 1.4 FRACTIONAL CALCULUS

The term Fractional calculus has its origin to the letter written by L'hospital in 1695 to Leibniz, wherein he enquired whether a meaning could be ascribed to $\frac{d^n f(x)}{dx^n}$ if n were a fraction. Because the answer to the questions was affirmative, various authors started working on the subject. In the initial stage of development, the order n was taken to be fraction. Although now n is taken as an arbitrary number, the subject is still known as fractional calculus. The works of Caputo [4], Gorenflo and Vessela [12], Kiryakova [30], McBridge [39], Miller and Ross [42], Nishimoto [44], Oldham and Spanier [45], Podlubny [50] and Samko, Kilbas and Marichev [55], provide a comprehensive account of the development and applications in the field of fractional calculus.

The following well known Fractional integral operator has been widely studied.

The Riemann-Liouville fractional integral and derivative operator $I^p_{a+}$ and $D^p_{a+}$, which are defined by (see, for details, [29], [42] and [55])

$$(I^{\mu}_{a+}f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt \qquad \big(\Re(\mu) > 0\big) \tag{1.4.1}$$

and

$$(D^{\mu}_{a+}f)(x) = \left(\frac{d}{dx}\right)^n (I^{n-\mu}_{a+}f)(x) \qquad \big(\Re(\mu) > 0;\ n = [\Re(\mu)] + 1\big), \tag{1.4.2}$$

($[x]$ denotes the greatest integer in the real number $x$)

will be required during the course of our study.

Hilfer [20] generalized the operator in (1.4.2) and defined a general fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 \leqq \nu \leqq 1$ with respect to $x$ as follows:

$$(D^{\mu,\nu}_{a+}f)(x) = \left(I^{\nu(1-\mu)}_{a+}\,\frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+}f\right)\right)(x). \tag{1.4.3}$$

Eq.(1.4.3) yields the classical Riemann-Liouville fractional derivative operator $D^{\mu}_{a+}$ when $\nu = 0$ and for $\nu = 1$ it reduces to the fractional derivative operator introduced by Joseph Liouville (1809-1882) in 1832, which is called the Liouville-Caputo fractional derivative operator (see [13], [29] and [79]).

In the present work we have introduced and developed a pair of unified Fractional Integral Operators whose kernels involve the product $S^U_V$ polynomial, generalized extended Mittag-Leffler function and extended Hurwitz-Lerch Zeta function and defined by (1.1.44), (1.1.53) and (1.1.63) respectively.

$$I_x^{\eta,\rho}\{f(t)\} = x^{-\eta-\rho-1}\int_0^x t^{\eta}(x-t)^{\rho} S_V^U\left[y_1\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right] E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{t}{x}\right)^{\rho_0};q,\sigma,\zeta\right)$$
$$*\,\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}\left(y_2\left(\frac{t}{x}\right)^{\eta_2}\left(1-\frac{t}{x}\right)^{\rho_2},s,a\right) f(t)dt \quad (1.4.4)$$

where $f(t)\in\Lambda$,

$$\min\{\Re\left(\eta+\varsigma+1,\rho+1\right)\}>0 \quad \text{and} \quad \min(\rho_0,\eta_1,\rho_1)>0 \quad \text{and}$$

$$J_t^{\eta',\rho'}\{f(z)\} = t^{\eta'}\int_t^{\infty} z^{-\eta'-\rho'-1}(z-t)^{\rho'} S_{V'}^{U'}\left[y_1'\left(\frac{t}{z}\right)^{\eta_1'}\left(1-\frac{t}{z}\right)^{\rho_1'}\right] E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{z}\right)^{\rho_0'};q',\sigma',\zeta'\right)$$
$$*\,\Phi_{\mu_j';\sigma_j' Q'}^{\lambda_j';\rho_j' P'}\left(y_2'\left(\frac{t}{z}\right)^{\eta_2'}\left(1-\frac{t}{z}\right)^{\rho_2'},s',a'\right) f(z)dz \quad (1.4.5)$$

provided that

$$\Re(w_2)>0 \quad \text{or} \quad \Re(w_2)=0 \quad \text{and} \quad \min\{\Re\left(\eta'-w_1\right)\}>0;$$
$$\Re(\rho'+1)>0,\ \min(\rho_0',\eta_1',\rho_1')\geq 0$$

On suitably specializing the parameters involved in the functions $S_V^U[z]$, $E_{\delta,\kappa}^{\vartheta;d}(z;q,\sigma,\zeta)$ and $\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}(z,s,a)$ our fractional integral operators can be easily reduced to left- and right-sided generalized fractional integral operators involving the Gauss hypergeometric function and classical Riemann-Liouville left- and right-sided fractional integral operators.

## 1.5 FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional differential equations have gained considerable importance due to their application in various disciplines, such as physics, mechanics, chemistry, engineering, etc. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives (see the monographs of Kilbas et al. [29], Miller and Ross [42], Oldham and Spanier [45], Podlubny [50] and Samko et al.[55]). Numerous problems in these areas are modeled mathematically by systems of fractional differential equations.

A growing number of works in science and engineering deal with dynamical systems described by fractional order equations that involve derivatives and integrals of non-integer order [Benson et al. [2], Metzler & Klafter [41], Zaslavasky [82]]. These new models are more adequate than the previously used integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances [50]. This is the most significant advantage of the fractional order models in comparison with integer order models, in which such effects are neglected. In the context of flow in porous media, fractional space derivatives exhibit large motions through highly conductive layers or fractures, while fractional time derivatives describe particles that remain motionless for extended period of time [40].

Recent applications of fractional differential equations to a number of systems have given opportunity for physicists to study even more complicated systems.

For example, the fractional diffusion equation allow describing complex systems with anomalous behavior in much the same way as simpler systems.

### 1.5.1 A GENERAL FAMILY OF FRACTIONAL DIFFERENTIAL EQUATIONS

The following family of fractional differential equations [79, p.803, Eq.(3.7)] was introduced and studied by several authors [18, 22] on account of their importance in dielectric relaxation in glasses.

$$a\left(D_{0+}^{\alpha_1,\beta_1}\,y\right)(x)+b\left(D_{0+}^{\alpha_2,\beta_2}\,y\right)(x)+cy(x)=g(x) \tag{1.5.1}$$

where

$$\left(0<\alpha_1 \leqq \alpha_2<1;\; 0 \leqq \beta_1,\beta_2 \leqq 1 \,\text{and}\, a,b,c \in \mathbb{R}\right)$$

in the space of Lebesgue integrable functions ( see [13, 78]) $y \in L(0,\infty)$ with the initial conditions:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}\,y\right)(0+)=C_i \qquad (i=1,2), \tag{1.5.2}$$

where, without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leqq (1-\beta_2)(1-\alpha_2).$$

if $C_1<\infty$, then $C_2=0$ unless $(1-\beta_1)(1-\alpha_1)=(1-\beta_2)(1-\alpha_2)$.

In the thesis, we shall study a generalized form of fractional differential equation (1.5.1) and aim at finding it's solution by using Laplace Transfrom method.

## 1.6 BRIEF CHAPTER BY CHAPTER SUMMARY OF THE THESIS

Now we present a brief summary of the work carried out in Chapter 2 to 6.

In **Chapter–2**, we first give a brief survey of Mittag-Leffler function and it's various generalizations introduced from time to time. Next, we introduce our function of study and call it generalized extended Mittag-Leffler (GEML) function and present some basic properties of the function of our study. Further, we obtain some integral representations of generalized extended Mittag-Leffler function. Later on, we obtain Laplace transform, Mellin transform and Inverse Mellin transfrom of function of our study. Finally, in this chapter we obtain right-sided Riemann- Liouville fractional integral operator $I^{\gamma}_{a+}$ , right-sided Riemann- Liouville fractional derivative operator $D^{\gamma}_{a+}$ and Hilfer derivative operator $D^{\gamma,\eta}_{a+}$ of our function of study GEML. The results established in this chapter generalize the findings of Shukla and Prajapati [59] and Özarslan and Yilmaz [46].

In **Chapter–3**, Firstly, we give an introduction to all the functions and integral operator which will be required in the sequel. Next, we introduce and study an integral operator whose kernel is generalized extended Mittag-Leffler (GEML) function and point out it's known special cases. Then we derive boundedness property of aforementioned integral operator. Further, we obtain image of some useful functions under the integral operator of our study along with some of it's special cases. Finally, we establish composition relationship of integral operator of our study with right-sided Riemann- Liouville fractional integral operator $I^{\gamma}_{a+}$

and an integral operator $H^{w;M,N;\alpha}_{a+;P,Q;\beta}$ involving the Fox H-function . The results stated in this chapter generalize the findings of Kilbas *et al.* [28] and Srivastava and Tomovski [78].

In **Chapter–4**, we first define extended Hurwitz-Lerch zeta function and give it's integral representation. Next, we define a new integral transform whose kernel is extended Hurwitz-Lerch zeta function and name it as the $\mathcal{E}-$transform. The transform of our study yields known integral transforms [1, p. 44, Eqs. (1.3.5) and (1.3.6)]. Further, we evaluate the Mellin transform of extended Hurwitz-Lerch Zeta function and Inversion formula for $\mathcal{E}-$transform. Next, we present some basic properties and prove the Uniqueness theorem for $\mathcal{E}-$transform. Finally, we obtain $\mathcal{E}-$transform of Derivatives and Integrals.

In **Chapter–5**, we study a pair of class of fractional integral operators whose kernel involve the product of $S_V^U[z]$, $E^{\vartheta;d}_{\delta,\kappa}(z;q,\sigma,\zeta)$ and $\Phi^{\lambda_j;\rho_j P}_{\mu_j;\sigma_j Q}(z,s,a)$ which stand for $S_V^U$ polynomial, generalized extended Mittag-Leffler function and extended Hurwitz-Lerch Zeta function. Next, we derive three new and interesting composition formulae for the operators of our study. The operators of our study are quite general in nature and may be considered as extensions of a number of simpler fractional integral operators studied from time to time by several authors. Later on, by suitably specializing the coefficients and the parameters of functions involved in our fractional integral operators we can get a large number of expressions for the composition of fractional integral operators. Finally, as an application of our main findings we obtain three interesting integrals which are believed to be new.

In **Chapter–6**, we define the $\overline{H}$-function, the Mittag-Leffler function, it's various generalizations and certain fractional integral operators that we will be using in our study. Further, we define Laplace transform and provide some necessary results required in finding solutions of fractional differential equations involving Hilfer derivative operator and an integral operator involving $\overline{H}$-function.

Furthermore, we consider a generalized form of General Family of fractional differential equations and find it's solution. On account of general nature of our findings we can obtain a number of special cases by taking particular values of the parameters involved therein. We mention here two new and three known special cases.

Next, we study some interesting two new and three known special cases of our main findings involving $\overline{H}$-function. Later on, by giving numerical values to the parameters in the functions and the operators involved therein we have plotted some graphs with the help of MATLAB SOFTWARE.

Futher, we obtain solution of another fractional differential equation involving $D_{0+}^{\alpha,\beta_1}$ and $\overline{\mathcal{H}}_{0+;p,q,\beta}^{w;m,n;\gamma}$. Finally, by specializing the parameters occuring therein we can obtain a number of special cases of this result. However, we give here only two known and two new special cases.

# 2

# GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND ASSOCIATED FRACTIONAL CALCULUS

The main findings of this chapter have bearing on the following paper as detailed below:

1. **N. JOLLY and R. JAIN** (2017). STUDY OF GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND IT'S PROPERTIES, *South East Asian J. of Math. & Math. Sci.*, **14(1)**, 47-58 .

In this chapter, we first give a brief survey of Mittag-Leffler function and it's various generalizations introduced from time to time. Next, we introduce our function of study and call it generalized extended Mittag-Leffler (GEML) function and present some basic properties of the function of our study. Further, we obtain some integral representations of generalized extended Mittag-Leffler function. Later on, we obtain Laplace transform, Mellin transform and Inverse Mellin transfrom of function of our study. Finally, in this chapter we obtain right-sided Riemann- Liouville fractional integral operator $I_{a+}^{\gamma}$ , right-sided Riemann-Liouville fractional derivative operator $D_{a+}^{\gamma}$ and Hilfer derivative operator $D_{a+}^{\gamma;\eta}$ of our function of study GEML. The results established in this chapter generalize the findings of Shukla and Prajapati [59] and Özarslan and Yilmaz [46].

## 2.1 INTRODUCTION

### GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION

A special function of the form $E_\delta(z)$ was introduced by the Swedish mathematician Gosta Mittag-Leffler [43] in 1903 as

$$E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)} \qquad (\delta \in \mathbb{C},\, \Re(\delta) > 0), \tag{2.1.1}$$

which is direct generalization of the exponential function for $\delta = 1$. In 1905, Wiman [81] gave the generalization of $E_\delta(z)$ known as Wiman's function or generalized Mittag-Leffler function defined as follows

$$E_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + \kappa)} \qquad (\delta, \kappa \in \mathbb{C},\; \Re(\delta) > 0, \Re(\kappa) > 0). \tag{2.1.2}$$

Later on, Prabhakar [51] introduced a new generalization of $E_{\delta,\kappa}(z)$ as $E^{\vartheta}_{\delta,\kappa}(z)$,

$$E^{\vartheta}_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!} \tag{2.1.3}$$

$$(\kappa, \delta, \vartheta \in \mathbb{C},\; \Re(\delta) > 0, \Re(\kappa) > 0, \Re(\vartheta) > 0),$$

where $(\vartheta)_n$ denotes the Pochhammer symbol (see for details, [53], [66] and [67]):

$$(\vartheta)_0 = 1, \qquad (\vartheta)_n = \vartheta(\vartheta + 1)(\vartheta + 2)\cdots(\vartheta + n - 1). \tag{2.1.4}$$

Moreover, generalization of $E^{\vartheta}_{\delta,\kappa}(z)$ was introduced and studied by Shukla and Prajapati [59] defined as

$$E_{\delta,\kappa}^{\vartheta,r}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_{rn}}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{2.1.5}$$

$$(\vartheta, \delta, \kappa \in \mathbb{C},\ \Re(\delta) > 0, \Re(\kappa) > 0, \Re(\vartheta) > 0 \,\text{and}\, r \in (0,1) \cup \mathbb{N}),$$

(2.1.5) was further investigated by Srivastava and Tomovski [78]

$$E_{\delta,\kappa}^{\vartheta,\xi}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_{\xi n}}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{2.1.6}$$

$$(z, \kappa, \vartheta \in \mathbb{C},\ \Re(\delta) > \max(0, \Re(\xi) - 1), \Re(\xi) > 0).$$

Lately, Özarslan and Yilmaz [46] studied extended Mittag-Leffler function and defined it in the following manner:

$$E_{\delta,\kappa}^{(\vartheta;d)}(z;q) = \sum_{n=0}^{\infty} \frac{B_q(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \quad (q \geq 0, \Re(d) > \Re(\vartheta) > 0), \tag{2.1.7}$$

$$\text{where} \qquad B_q(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-q}{t(1-t)}}\, dt \tag{2.1.8}$$

$$(\Re(q) \geq 0, \Re(x) > 0, \Re(y) > 0).$$

Now, in this chapter, we propose to introduce and investigate further *generalization of extended Mittag-Leffler function* in (2.1.7) as $E_{\delta,\kappa}^{\vartheta;d}(z; q, \rho, \zeta)$,

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta) = \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \tag{2.1.9}$$

$$(q \geq 0,\ \Re(d) > \Re(\vartheta) > 0,\ \Re(\delta) > 0,\ \Re(\kappa) > 0),$$

where $B_q^{(\rho,\zeta)}(x,y)$ represents generalized Beta type function as follows(see, for details, [65, p. 348, Eq.(1.2)]; see also [48, p. 32, Chapter 4]):

$$B_q^{(\rho,\zeta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\rho;\zeta;\frac{-q}{t(1-t)})\ dt \tag{2.1.10}$$

$$(\Re(q) \geq 0, \min(\Re(x), \Re(y), \Re(\zeta), \Re(\rho)) > 0).$$

Further, we also present the contour representation of generalized extended Mittag-Leffler function in the following manner:

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta) = \frac{1}{(2\pi i)^2}\frac{\Gamma(\zeta)}{\Gamma(\rho)\Gamma(\vartheta)\Gamma(d-\vartheta)}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\frac{\Gamma(\rho-\xi_1)}{\Gamma(\zeta-\xi_1)}\frac{\Gamma(\vartheta+\xi_1+\xi_2)\Gamma(d-\vartheta+\xi_1)\Gamma(\xi_1)}{\Gamma(d+\xi_2+2\xi_1)} \cdot \frac{\Gamma(d+\xi_2)\Gamma(-\xi_2)}{\Gamma(\kappa+\delta\xi_2)}(q)^{-\xi_1}(z)^{\xi_2}d\xi_1 d\xi_2 \tag{2.1.11}$$

For, $\rho = \zeta$ generalization of extended Mittag-Leffler function in (2.1.9) reduces to extended Mittag- Leffler function in (2.1.7).

In the sequel, we shall represent Lebesgue measurable real or complex valued functions defined on a finite interval $[a,b]$ of real line $\mathbb{R}$(see, for details, [13] and [27]; see also [55]) in the following manner:

$$L(a,b) = \left\{ f(x) : \|f\|_1 = \int_a^b \mid f(x) \mid dx < \infty \right\}. \tag{2.1.12}$$

## 2.2 BASIC PROPERTIES OF GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION

**Theorem 2.2.1.** *For $d, \vartheta, \kappa, \delta \in \mathbb{C}$, $\Re(d) > \Re(\vartheta) > 0$, $\Re(\delta) > 0$, $\Re(\kappa) > 0$ and $q \geq 0$, we have*

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta) = \kappa E_{\delta,\kappa+1}^{\vartheta;d}(z;q,\rho,\zeta) + \delta z \frac{d}{dz} E_{\delta,\kappa+1}^{\vartheta;d}(z;q,\rho,\zeta), \tag{2.2.1}$$

*provided that the conditions of generalized extended Mittag-Leffler function in (2.1.9) are satisfied.*

*Proof.* Using (2.1.9) in the right hand side of (2.2.1), we obtain

$$\begin{aligned}
&\kappa E_{\delta,\kappa+1}^{\vartheta;d}(z;q,\rho,\zeta) + \delta z \frac{d}{dz} E_{\delta,\kappa+1}^{\vartheta;d}(z;q,\rho,\zeta) \\
&\quad = \kappa E_{\delta,\kappa+1}^{\vartheta;d}(z;q,\rho,\zeta) + \delta z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa+1)} \frac{z^n}{n!} \\
&\quad = \kappa E_{\delta,\kappa+1}^{\vartheta;d}(z;q,\rho,\zeta) + \sum_{n=0}^{\infty} \frac{\delta n B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa+1)} \frac{z^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{(\delta n+\kappa) B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa+1)} \frac{z^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!}.
\end{aligned} \tag{2.2.2}$$

Again, applying (2.1.9) in (2.2.2) we get the desired result. □

**Theorem 2.2.2.** *If $d, \vartheta, \kappa, \delta \in \mathbb{C}$, $\Re(d) > \Re(\vartheta) > 0$, $\Re(\delta) > 0$, $\Re(\kappa) > 0$ and $q \geq 0$ then for $m \in \mathbb{N}$,*

$$\left(\frac{d}{dz}\right)^m E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta) = (d)_m E_{\delta m+\kappa,\kappa}^{\vartheta+m;d+m}(z;q,\rho,\zeta), \tag{2.2.3}$$

$$\left(\frac{d}{dz}\right)^m [z^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega z^{\delta};q,\rho,\zeta)] = z^{\kappa-m-1}E^{\vartheta;d}_{\delta,\kappa-m}(\omega z^{\delta};q,\rho,\zeta), \quad Re(\kappa-m)>0, \tag{2.2.4}$$

*provided that the conditions of generalized extended Mittag-Leffler function in (2.1.9) are satisfied.*

*Proof.* Applying (2.1.9) in the left hand side of (2.2.3) and differentiating $m$ times, we get

$$\begin{aligned}\left(\frac{d}{dz}\right)^m E^{\vartheta;d}_{\delta,\kappa}(z;q,\rho,\zeta) &= \sum_{n=m}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^{n-m}}{(n-m)!} \\ &= \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n+m,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d)_{n+m}}{\Gamma(\delta(n+m)+\kappa)} \frac{z^n}{n!} \\ &= (d)_m \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n+m,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d+m)_n}{\Gamma(\delta(n+m)+\kappa)} \frac{z^n}{n!}.\end{aligned} \tag{2.2.5}$$

Now, reinterpreting (2.2.5) in the form of generalized extended Mittag-Leffler function, we obtain right hand side of (2.2.3).
Next, Considering left hand side of (2.2.4) and applying (2.1.9), we obtain

$$\left(\frac{d}{dz}\right)^m [z^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega z^{\delta};q,\rho,\zeta)]$$

$$= \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{\omega^n z^{\delta n+\kappa-1}}{n!}, \tag{2.2.6}$$

$$= z^{\kappa-m-1} \sum_{n=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d)_n}{\Gamma(\delta n+\kappa-m)} \frac{\omega^n z^{\delta n}}{n!}. \tag{2.2.7}$$

Reinterpreting (2.2.7) in the form of generalized extended Mittag-Leffler function, we obtain right hand side of (2.2.4). □

**Remark 2.2.1.** *On taking $\rho=\zeta, q=0$ and $\vartheta=1$ in (2.2.4), we obtain the result given by Shukla and Prajapati [59, p. 801,theorem 2.2, Eq.(2.2.3)].*

**Theorem 2.2.3.** *If $\vartheta,\kappa,\delta \in \mathbb{C}, \Re(d)>\Re(\vartheta)>0$, $\Re(\delta)>0, \Re(\alpha)>0$,*

$\Re(\kappa) > 0, min(\Re(\vartheta + n), \Re(d - \vartheta), \Re(\zeta), \Re(\rho)) > 0$ *and* $\Re(q) > 0$ *then*

$$\frac{1}{\Gamma(\alpha)} \int_0^1 u^{\kappa-1}(1-u)^{\alpha-1} E_{\delta,\kappa}^{\vartheta;d}(zu^{\delta}; q, \rho, \zeta) du = E_{\delta,\kappa+\alpha}^{\vartheta;d}(z; q, \rho, \zeta). \tag{2.2.8}$$

*If* $\vartheta, \kappa, \delta, \alpha \in \mathbb{C}, \Re(d) > \Re(\vartheta) > 0$, $\Re(\delta) > 0, \Re(\alpha) > 0, \Re(\kappa) > 0$, $min(\Re(\vartheta + n), \Re(d - \vartheta), \Re(\zeta), \Re(\rho)) > 0$ *and* $\Re(q) > 0$ *then*

$$\int_0^z t^{\kappa-1} E_{\delta,\kappa}^{\vartheta;d}(\omega t^{\delta}; q, \rho, \zeta) dt = z^{\kappa} E_{\delta,\kappa+1}^{\vartheta;d}(\omega z^{\delta}; q, \rho, \zeta). \tag{2.2.9}$$

*In particular,*

$$\int_0^z t^{\kappa-1} E_{\delta,\kappa}^{\vartheta;d}(\omega t^{\delta}; q, \rho, \zeta) dt = z^{\kappa} E_{\delta,\kappa+1}^{(\vartheta;d)}(\omega z^{\delta}; q). \tag{2.2.10}$$

*Proof.* Applying (2.1.9) in left hand side of (2.2.8) and using definition of Beta function, we obtain the desired result after a little simplification.
Next, for the proof of assertion (2.2.9), we proceed by using the definition of generalized extended Mittag-Leffler function given by (2.1.9), evaluating the intergral obtained and reinterpreting it in the form of (2.1.9), gives us the desired result.
Substituting $\vartheta = d$ in equation (2.2.9), we obtain the result (2.2.10) after some simplification. □

**Remark 2.2.2.** *If we reduce generalized extended Mittag-Leffler function to two parametric Mittag-Leffler function in (2.2.9), we get the result obtained by Shukla and Prajapati [59, p. 803,theorem 2.4, Eq.(2.4.5)].*

## 2.3 INTEGRAL REPRESENTATION OF THE GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION

**Theorem 2.3.1.** *For the GEML function, we have*

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta) = \frac{1}{B(\vartheta, d-\vartheta)} \int_0^1 t^{\vartheta-1}(1-t)^{d-\vartheta-1} {}_1F_1\left(\rho;\zeta;\frac{-q}{t(1-t)}\right) E_{\delta,\kappa}^{d}(tz)dt, \tag{2.3.1}$$

*where* $\Re(q) \geq 0$, $\Re(d) > \Re(\vartheta) > 0$, $\Re(\delta) > 0$, $\Re(\kappa) > 0$, *provided that conditions of GEML function in (2.1.9) are satisfied.*

*Proof.* To prove the above result, we proceed by using (2.1.10) in the left hand side of (2.3.1)

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta)$$
$$= \sum_{n=0}^{\infty} \frac{1}{B(\vartheta, d-\vartheta)} \left\{ \int_0^1 t^{\vartheta+n-1}(1-t)^{d-\vartheta-1} {}_1F_1\left(\rho;\zeta;\frac{-q}{t(1-t)}\right) dt \right\} \frac{(d)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!}. \tag{2.3.2}$$

Interchaging the order of summation and integration in (2.3.2)(which is permissible under the assumptions stated in the above theorem), we get

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta)$$
$$= \int_0^1 t^{\vartheta-1}(1-t)^{d-\vartheta-1} {}_1F_1\left(\rho;\zeta;\frac{-q}{t(1-t)}\right) \sum_{n=0}^{\infty} \frac{(d)_n}{B(\vartheta, d-\vartheta)} \frac{(tz)^n}{\Gamma(\delta n+\kappa) n!} dt. \tag{2.3.3}$$

Finally, applying (2.1.3) in (2.3.3), we get the desired result. □

**Corollary 2.3.1.** *If we take $t = \frac{u}{1+u}$ in (2.3.1), we get*

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta) = \frac{1}{B(\vartheta,d-\vartheta)}\int_0^\infty \frac{u^{\vartheta-1}}{(1-u)^d}\,{}_1F_1\left(\delta;\sigma;\frac{-q(1+u)^2}{u}\right)E_{\delta,\kappa}^{d}\left(\frac{uz}{1+u}\right)du. \tag{2.3.4}$$

**Corollary 2.3.2.** *Taking $t = sin^2\theta$ in (2.3.1), we obtain the following integral representation:*

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta)$$
$$= \frac{2}{B(\vartheta,d-\vartheta)}\int_0^{\frac{\pi}{2}}(sin\theta)^{2\vartheta-1}(cos\theta)^{2d-2\vartheta-1}\,{}_1F_1\left(\rho;\zeta;\frac{-q}{sin^2\theta cos^2\theta}\right)E_{\delta,\kappa}^{d}\left(zsin^2\theta\right)d\theta. \tag{2.3.5}$$

## 2.4 INTEGRAL TRANSFORMS OF THE FUNCTION OF OUR STUDY

## LAPLACE TRANSFORM

**Theorem 2.4.1.** *The Laplace transform of generalized extended Mittag-Leffler function is defined as*

$$\int_0^\infty z^{a-1}e^{-sz}E_{\delta,\kappa}^{\vartheta;d}(xz^\sigma;q,\rho,\zeta)dz = \frac{s^{-a}\Gamma(\zeta)}{\Gamma(\rho)\Gamma(\vartheta)\Gamma(d-\vartheta)}\,H_{1,1:1,3;2,2}^{0,1:2,1;1,2}\left[\begin{array}{c|c} q & A^* \\ \\ \frac{-x}{s^\sigma} & B^* \end{array}\right], \tag{2.4.1}$$

*where*

$$A^* = (1-\vartheta;1,1) : (1-\rho,1);(1-a,\sigma),(1-d,1)$$
$$B^* = (1-d;1,2) : (0,1),(d-\vartheta,1),(1-\zeta,1);(0,1)(1-\kappa,\delta),$$

*and* $\vartheta, \kappa, \delta, a, \sigma \in \mathbb{C}, \Re(d) > \Re(\vartheta) > 0, \Re(\delta) > 0, \Re(\alpha) > 0, \Re(\kappa) > 0,$ $\Re(a) > 0, \Re(\sigma) > 0$ *and* $\Re(q) > 0$*, provided that the conditions of generalized extended Mittag-Leffler function mentioned in (2.1.9) are satisfied.*
*where the* $H$*-function of two variables occurring in the right hand side of (2.4.1) is defined in chapter zero by (1.1.30).*

*Proof.* To prove (2.4.1), we first apply (2.1.9) in left hand side of (2.4.1), and obtain

$$\int_0^\infty z^{a-1} e^{-sz} E_{\delta,\kappa}^{\vartheta;d}(xz^\sigma; q, \rho, \zeta) dz$$

$$= \sum_{n=0}^\infty \frac{B_q^{(\rho,\zeta)}(\vartheta + n, d - \vartheta)}{B(\vartheta, d - \vartheta)} \frac{(d)_n}{\Gamma(\delta n + \kappa)} \frac{x^n}{n!} \int_0^\infty z^{n\sigma+a-1} e^{-sz} dz$$

$$= s^{-a} \sum_{n=0}^\infty \frac{B_q^{(\rho,\zeta)}(\vartheta + n, d - \vartheta)}{B(\vartheta, d - \vartheta)} \frac{\Gamma(a + n\sigma)(d)_n}{\Gamma(\delta n + \kappa) n!} \left(\frac{x}{s^\sigma}\right)^n. \tag{2.4.2}$$

Further, evaluating the function $B_q^{(\rho,\zeta)}(\vartheta + n, d - \vartheta)$ in the right hand side of (2.4.2) with the help of (2.1.10) in series form, we get

$$\int_0^\infty z^{a-1} e^{-sz} E_{\delta,\kappa}^{\vartheta;d}(xz^\sigma; q, \rho, \zeta) dz$$

$$= \frac{s^{-a}\Gamma(\zeta)}{\Gamma(\rho)\Gamma(\vartheta)\Gamma(d - \vartheta)} \sum_{n=0}^\infty \sum_{r=0}^\infty \frac{\Gamma(\rho + r)\Gamma(\vartheta + n - r)\Gamma(d - \vartheta - r)}{\Gamma(\zeta + r)\Gamma(d + n - 2r)}$$
$$\cdot \frac{\Gamma(a + n\sigma)\Gamma(d + n)}{\Gamma(\delta n + \kappa)} \frac{(-q)^r}{r!} \left(\frac{x}{s^\sigma}\right)^n. \tag{2.4.3}$$

Finally, applying definition of $H$-function of two variables (1.1.30) for $r = 2$ in (2.4.3), we get the desired result. □

**Remark 2.4.1.** *On taking,* $\sigma = \delta$ *and* $\kappa = a$ *in (2.4.1), we obtain the result given by (2.2.8).*

# MELLIN TRANSFORM

**Theorem 2.4.2.** *The Mellin transform of the generalized extended Mittag-Leffler function is as follows:*

$$\mathcal{M}(E_{\delta,\kappa}^{\vartheta;d}(\omega z^{\delta};q,\rho,\zeta);s)=\frac{\Gamma(s)(\rho)_{-s}(d-\vartheta)_{s}}{\Gamma(\vartheta)(\zeta)_{-s}}\,{}_{2}\Psi_{2}\left[\begin{array}{cc}(\vartheta+s,1),(d,1); & \\ & z\\ (\kappa,s),(d+2s,1); & \end{array}\right], \tag{2.4.4}$$

$$(\Re(q)\geq 0,\ \Re(d)>\Re(\vartheta)>0,\ \Re(\delta)>0,\Re(\kappa)>0,$$
$$min(\Re(\vartheta+n),\Re(d-\vartheta),\Re(\zeta),\Re(\rho))>0),$$

*where ${}_{2}\Psi_{2}$ denotes a special case of the Generalized Hypergeometric function defined by (1.1.26).*

*Proof.* In order to prove (2.4.4), we proceed by obtaining the Mellin transform of the generalized extended Mittag-Leffler function

$$\mathcal{M}(E_{\delta,\kappa}^{\vartheta;d}(\omega z^{\delta};q,\rho,\zeta);s)=\int_{0}^{\infty}q^{s-1}E_{\delta,\kappa+1}^{\vartheta;d}(\omega z^{\delta};q,\rho,\zeta)dq. \tag{2.4.5}$$

Now,using equation (2.1.9) in (2.4.5) and interchanging the order of integration (which is permissible under the conditions stated in the theorem), we get

$$\begin{aligned}&\mathcal{M}(E_{\delta,\kappa}^{\vartheta;d}(\omega z^{\delta};q,\rho,\zeta);s)\\&=\frac{1}{B(\vartheta,d-\vartheta)}\int_{0}^{1}t^{\vartheta-1}(1-t)^{d-\vartheta-1}E_{\delta,\kappa}^{d}(tz)\left[\int_{0}^{\infty}q^{s-1}{}_{1}F_{1}\left(\rho;\zeta;\frac{-q}{t(1-t)}\right)dq\right]dt.\end{aligned} \tag{2.4.6}$$

Substituting $u = \frac{q}{t(1-t)}$ in equation (2.4.6), we get

$$\begin{aligned}
&\mathcal{M}(E^{\vartheta;d}_{\delta,\kappa}(\omega z^{\delta}; q, \rho, \zeta); s) \\
&= \frac{1}{B(\vartheta, d-\vartheta)} \int_0^1 t^{\vartheta+s-1}(1-t)^{d+s-\vartheta-1} E^{d}_{\delta,\kappa}(tz) \left[ \int_0^\infty u^{s-1}{}_1F_1(\rho; \zeta; -u)\, du \right] dt \\
&= \frac{\Gamma(s)\Gamma(\rho-s)\Gamma(\zeta)}{B(\vartheta, d-\vartheta)\Gamma(\rho)\Gamma(\zeta-s)} \int_0^1 t^{\vartheta+s-1}(1-t)^{d+s-\vartheta-1} E^{d}_{\delta,\kappa}(tz)\, dt. \qquad (2.4.7)
\end{aligned}$$

Further, using (2.1.3) in (2.4.7) and evaluating the integral obtained by using definition of Beta function, we obtain

$$\mathcal{M}(E^{\vartheta;d}_{\delta,\kappa}(\omega z^{\delta}; q, \rho, \zeta); s) = \frac{\Gamma(s)\Gamma(\rho-s)\Gamma(\zeta)\Gamma(d+s-\vartheta)}{B(\vartheta, d-\vartheta)\Gamma(\rho)\Gamma(\zeta-s)\Gamma(d)} \sum_{r=0}^{\infty} \frac{\Gamma(\vartheta+r+s)\Gamma(d+r)}{\Gamma(\delta r+\kappa)\Gamma(d+r+2s)} \frac{z^r}{r!}. \qquad (2.4.8)$$

Reinterpreting the summation $r$ over gamma's in (2.4.8) as Wright Generalized Hypergeometric function (1.1.27), we get the desired result. □

## INVERSE MELLIN TRANSFORM

Taking inverse Mellin transform on both sides of (2.4.4), we have

$$E^{\vartheta;d}_{\delta,\kappa}(z; q, \rho, \zeta) = \frac{1}{2\pi i\, \Gamma(\vartheta)} \int_{\nu-i\infty}^{\nu+i\infty} \frac{\Gamma(s)(\rho)_{-s}(d-\vartheta)_s}{(\zeta)_{-s}} {}_2\Psi_2 \left[ \begin{matrix} (\vartheta+s, 1), (d, 1); \\ \\ (\kappa, s), (d+2s, 1); \end{matrix} \; z \right] q^{-s} ds, \qquad (2.4.9)$$

where $\nu > 0$.

**Remark 2.4.2.** *If we consider, $\rho = \zeta$ in (2.4.4), we obtain the result given by Özarslan and Yilmaz [46, p.4, Theorem 5].*

## FRACTIONAL CALCULUS

Right-sided Riemann- Liouville fractional integral operator $I_{a+}^{\gamma}$ and right-sided Riemann- Liouville fractional derivative operator $D_{a+}^{\gamma}$ studied earlier by Samko *et al.*[55] will be defined and represented in the present chapter in the following manner:

$$(I_{a+}^{\gamma} f)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(t)}{(x-t)^{1-\gamma}}\, dt \qquad (\gamma \in \mathbb{C}, \Re(\gamma) > 0), \tag{2.4.10}$$

and

$$(D_{a+}^{\gamma} f)(x) = \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\gamma} f)(x) \qquad (\gamma \in \mathbb{C}, \Re(\gamma) > 0, n = [\Re(\gamma)] + 1), \tag{2.4.11}$$

respectively. Hilfer (see [20, 21])generalized the Riemann-Liouville fraction derivative operator in (2.4.11) as $D_{a+}^{\gamma,\eta}$ with order $0 < \gamma < 1$ and $0 \leq \eta \leq 1$ by

$$(D_{a+}^{\gamma,\eta} f)(x) = (I_{a+}^{\eta(1-\gamma)} \frac{d}{dx} (I_{a+}^{(1-\eta)(1-\gamma)} f))(x). \tag{2.4.12}$$

The following result on right-sided Riemann- Liouville fractional integral operator $I_{a+}^{\gamma}$ was given by Mathai and Haubold [36] as

$$I_{a+}^{\gamma} (\tau - a)^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\gamma+\mu)} (x-a)^{\gamma+\mu-1}, \tag{2.4.13}$$

where $\gamma, \mu \in \mathbb{C}, \Re(\gamma) > 0$ and $\Re(\mu) > 0$. Another result of our interest was established by Srivastava and Tomovski [78] as

$$D_{a+}^{\gamma,\eta} [(\tau - a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} (x-a)^{\beta-\gamma-1}, \tag{2.4.14}$$

where $x > a$, $0 < \gamma < 1$, $0 \leq \eta \leq 1$, $\Re(\gamma) > 0$ and $\Re(\beta) > 0$.

## 2.5 RIGHT-SIDED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL OPERATOR $I^{\gamma}_{a+}$ AND DERIVATIVE OPERATOR $D^{\gamma}_{a+}$ AND HILFER DERIVATIVE OPERATOR $D^{\gamma,\eta}_{a+}$ OF THE FUNCTION OF OUR STUDY

**Theorem 2.5.1.** *If* $x > a$ $(a \in \mathbb{R}^+ = [0, \infty))$, $\vartheta$, $\gamma$, $\kappa$, $\omega \in \mathbb{C}$, $\Re(\kappa) > 0$, $\Re(\delta) > 0$ *and* $\Re(\gamma) > 0$ *then*

$$I^{\gamma}_{a+}[(\tau - a)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau - a)^{\delta}; q, \rho, \zeta)](x) = (x - a)^{\gamma+\kappa-1} E^{\vartheta;d}_{\delta,\kappa+\gamma}(\omega(x - a)^{\delta}; q, \rho, \zeta), \tag{2.5.1}$$

$$D^{\gamma}_{a+}[(\tau - a)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau - a)^{\delta}; q, \rho, \zeta)](x) = (x - a)^{\kappa-\gamma-1} E^{\vartheta;d}_{\delta,\kappa-\gamma}(\omega(x - a)^{\delta}; q, \rho, \zeta) \tag{2.5.2}$$

*and*

$$D^{\gamma,\eta}_{a+}[(\tau - a)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau - a)^{\delta}; q, \rho, \zeta)](x) = (x - a)^{\kappa-\gamma-1} E^{\vartheta;d}_{\delta,\kappa-\gamma}(\omega(x - a)^{\delta}; q, \rho, \zeta), \tag{2.5.3}$$

*provided that conditions of GEML function in (2.1.9) are satisfied.*

*Proof.* In order to prove result (2.5.1), we proceed by using (2.4.10) and (2.1.9) in the left hand side of (2.5.1) and upon interchanging the order of summation and integration (which is permissible under the assumptions stated in the above theorem), we obtain

$$I^{\gamma}_{a+}[(\tau-a)^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau-a)^{\delta};q,\rho,\zeta)](x)$$
$$=\frac{1}{\Gamma(\gamma)B(\vartheta,d-\vartheta)}\sum_{n=0}^{\infty}\frac{B^{(\rho,\zeta)}_{q}(\vartheta+n,d-\vartheta)}{\Gamma(\delta n+\kappa)}\frac{(d)_n\omega^n}{n!}\cdot\int_a^x(\tau-a)^{\kappa+\delta n-1}(x-\tau)^{\gamma-1}d\tau. \tag{2.5.4}$$

Again, applying (2.4.10) to (2.5.4) and using the result (2.4.13), we get

$$I^{\gamma}_{a+}[(\tau-a)^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau-a)^{\delta};q,\rho,\zeta)](x)$$
$$=(x-a)^{\gamma+\kappa-1}\sum_{n=0}^{\infty}\frac{B^{(\rho,\zeta)}_{q}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)\Gamma(\delta n+\kappa+\gamma)}\frac{(d)_n(w(x-a)^{\delta})^n}{n!}. \tag{2.5.5}$$

reinterpreting the above expression in terms of GEML function given by (2.1.9) we arrive at assertion (2.5.1).
Further, in order to the prove (2.5.2), we use (2.4.11) and previously proved result (2.5.1) in the left hand side of (2.5.2)

$$D^{\gamma}_{a+}[(\tau-a)^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau-a)^{\delta};q,\rho,\zeta)](x)$$
$$=\left(\frac{d}{dx}\right)^n\left[(x-a)^{n-\gamma+\kappa-1}E^{\vartheta;d}_{\delta,\kappa-\gamma+n}(\omega(x-a)^{\delta};q,\rho,\zeta)\right]$$
$$=(x-a)^{\kappa-\gamma-1}E^{\vartheta;d}_{\delta,\kappa-\gamma}(\omega(x-a)^{\delta};q,\rho,\zeta), \tag{2.5.6}$$

from (2.5.6) we get the desired result.
Finally, for the proof of assertion (2.5.3), we use (2.1.9) in the left hand side of (2.5.3)

$$D^{\gamma;\eta}_{a+}[(\tau-a)^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega(\tau-a)^{\delta};q,\rho,\zeta)](x)$$
$$=D^{\gamma;\eta}_{a+}\left[\sum_{n=0}^{\infty}\frac{B^{(\rho,\zeta)}_{q}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)\Gamma(\delta n+\kappa)}\frac{(d)_n w^n}{n!}(\tau-a)^{\delta n+\kappa-1}\right], \tag{2.5.7}$$

and obtain (2.5.7) to which we apply (2.4.14) to get the required result. □

# 3

# AN INTEGRAL OPERATOR INVOLVING GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION

The main findings of this chapter have bearing on the following paper as detailed below:

1. **M. K. BANSAL, N. JOLLY, R. JAIN and D. KUMAR** (2018). AN INTEGRAL OPERATOR INVOLVING GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND ASSOCIATED FRACTIONAL CALCULUS, *The Journal of Analysis*, https://doi.org/10.1007/s41478-018-0119-0 .

In this chapter, we first give a brief introduction to all the functions and integral operator which will be required in the sequel. Next, we introduce and study an integral operator whose kernel is generalized extended Mittag-Leffler (GEML) function and point out it's known special cases. Then we derive boundedness property of aforementioned integral operator. Next, we obtain image of some useful functions namely extended Hurwitz-Lerch Zeta function, S-generalized Gauss hypergeometric function and Fox H-function under the integral operator of our study along with some of it's special cases. Finally, we establish composition relationship of integral operator of our study with right-sided Riemann- Liouville fractional integral operator $I^{\gamma}_{a+}$ and an integral operator $H^{w;M,N;\alpha}_{a+;P,Q;\beta}$ involving the Fox H-function. The results stated in this chapter generalize the findings of Kilbas *et al.* [28] and Srivastava and Tomovski [78].

## 3.1 INTRODUCTION

### S-GENERALIZED GAUSS HYPERGEOMETRIC FUNCTION

The S-generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z)$ was introduced and investigated by Srivastava et al. [65, p. 350, Eq. (1.12)]. It is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \tag{3.1.1}$$

$$(\Re(p) \geq 0; \quad \min \Re(\alpha), \Re(\beta), \Re(\tau), \Re(\mu)\} > 0; \quad \Re(c) > \Re(b) > 0),$$

in terms of the classical Beta function $B(\lambda,\mu)$ and the S-generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$, which was also defined by Srivastava et al. [65, p. 350, Eq.(1.13)] as follows:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^{\tau}(1-t)^{\mu}}\right)dt \tag{3.1.2}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(x), \Re(y), \Re(\alpha), \Re(\beta), \Re(\tau), \Re(\mu)\} > 0)$$

For $\tau = \mu$, the S-generalized Gauss hypergeometric function defined by (3.1.1) reduces to the following generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau)}(a,b;c;z)$ studied earlier by Parmar [49, p.44]

$$F_p^{(\alpha,\beta;\tau)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\tau)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \qquad (|z|<1) \tag{3.1.3}$$

$$(\Re(p) \geq 0; \quad \min\{\Re(\alpha), \Re(\beta), \Re(\tau)\} > 0; \quad \Re(c) > \Re(b) > 0).$$

## EXTENDED HURWITZ-LERCH ZETA FUNCTION

The extended Hurwitz-Lerch Zeta function introduced by Srivastava et al. [77, p. 503, Eq.(6.2)](see also [70] and [62]) is recalled here in slightly modified form:

$$\Phi^{(\varrho_1,\ldots,\varrho_P,\upsilon_1,\ldots,\upsilon_Q)}_{k_1,\ldots,k_P;l_1,\ldots,l_Q}(z,s,a) = \Phi^{k_j;\varrho_j P}_{l_j;\upsilon_j Q}(z,s,a) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{P}(k_j)_{m\varrho_j}}{m! \prod_{j=1}^{Q}(l_j)_{m\upsilon_j}} \frac{z^m}{(m+a)^s} \quad (3.1.4)$$

$$(P, Q \in \mathbb{N}_0; k_j \in \mathbb{C}(j = 1, \cdots, P); a, l_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j = 1, \cdots, Q);$$

$$\varrho_j, \upsilon_n \in \mathbb{R}^+(j = 1 \cdots P; n = 1 \cdots, Q); \Im > -1 \text{ when } s, z \in \mathbb{C};$$

$$\Im = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \mho; \ \Im = -1 \text{ and } \Re(\varpi) > \tfrac{1}{2} \text{ when } |z| = \mho$$

$$\text{where } \mho := (\prod_{j=1}^{P} \varrho_j^{-\varrho_j})(\prod_{j=1}^{Q} \upsilon_j^{\upsilon_j}) \text{ and } \Im := \sum_{j=1}^{Q} \upsilon_j - \sum_{j=1}^{P} \varrho_j,$$

$$\varpi := s + \sum_{j=1}^{Q} l_j - \sum_{j=1}^{P} k_j + \tfrac{P-Q}{2}).$$

## SPECIAL CASES OF EXTENDED HURWITZ-LERCH ZETA FUNCTION

(i) **General Hurwitz-Lerch Zeta function :** If we take $Q = 0$ and $P = \rho_j = k_j = 1$ in (1.1.63), we get

$$\Phi^{1;P}_{-}(z,s,a) = \Phi(z,s,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^s}, \quad (3.1.5)$$

where $\Phi(z,s,a)$ is defined in [8, p.27, Eq.(1.11)].

(ii) **Riemann Zeta function :** Again, if we take $Q = 0$ and $P = \rho_j = k_j = z =$

$a = 1$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(1, s, 1) = \zeta(s) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^s}, \tag{3.1.6}$$

where $\zeta(s)$ is defined in [8, Chapter 1](see, for details, [66, Chapter 2]).

(iii) **Hurwitz (or generalized) Zeta function :** Further, if we take $Q = 0$ and $P = \rho_j = k_j = z = 1$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(1, s, a) = \zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}, \tag{3.1.7}$$

where $\zeta(s, a)$ is defined in [8, Chapter 1]( see also, [66, Chapter 2]).

(iv) **Lerch Zeta function :** If we take $Q = 0$, $P = \rho_j = k_j = a = 1$ and $z = e^{2\pi i\xi}$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(e^{2\pi i\xi}, s, 1) = \ell_s(\xi) = \sum_{m=0}^{\infty} \frac{e^{2m\pi i\xi}}{(m+1)^s}, \tag{3.1.8}$$

where $\ell_s(\xi)$ is defined in [8, Chapter 1](see, for details, [66, Chapter 2]).

(v) **Lipschitz-Lerch Zeta function :** If we take $Q = 0$, $P = \rho_j = k_j = 1$ and $z = e^{2\pi i\xi}$ in (1.1.63), we get

$$\Phi_{-}^{1;P}(e^{2\pi i\xi}, s, a) = \phi(\xi, s, a) = \sum_{m=0}^{\infty} \frac{e^{2m\pi i\xi}}{(m+a)^s}, \tag{3.1.9}$$

where $\phi(\xi, s, a)$ is defined in [66, p. 122, Eq. (2.5)](see, for details, [80, p. 280, Example 8]).

## 3.2 AN INTEGRAL OPERATOR INVOLVING THE GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AS IT'S KERNEL AND IT'S PROPERTIES

An integral operator with GEML function given by (2.1.9) as it's kernel and $x > a$ $(a \in \mathbb{R}^+ = [0, \infty))$ is defined as follows

$$(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta} f)(x) = \int_a^x (x-z)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(x-z)^{\delta}; q, \rho, \zeta) f(z)\, dz \tag{3.2.1}$$

$$(\vartheta, \omega \in \mathbb{C}, \Re(\delta) > 0, \Re(\kappa) > 0),$$

provided that conditions of GEML function in (2.1.9) are satisfied.

### SPECIAL CASES

(i) Considering $\rho = \zeta$ in (3.2.1), we obtain an integral operator introduced by Rahman *et al.* [52] as

$$(\varepsilon^{\omega;\vartheta;d}_{a+;\delta,\kappa} f)(x) = \int_a^x (x-z)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(x-z)^{\delta}; q) f(z)\, dz. \tag{3.2.2}$$

(ii) Next, on taking $q = 0$ in (3.2.2) we get an integral operator given by Srivastava and Tomovski [78]:

$$(\varepsilon^{\omega;\vartheta}_{a+;\delta,\kappa}f)(x) = \int_a^x (x-z)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega(x-z)^{\delta}) f(z)\ dz. \tag{3.2.3}$$

(iii) Further, if we take $\omega = 0$, the integral operator in (3.2.3) reduces to Riemann-Liouville fractional integral operator as defined in (2.4.10).

**Theorem 3.2.1.** *Under the various parametric constraints stated already with the definition (3.2.1), let the function $\varphi$ be in the space $L(\mathfrak{a},\mathfrak{b})$ of Lebesgue measurable functions on a finite interval $[\mathfrak{a},\mathfrak{b}](\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ given by*

$$L(\mathfrak{a},\mathfrak{b}) = \left\{ \mathfrak{f} : ||\mathfrak{f}||_1 = \int_a^b |\mathfrak{f}(x)| dx < \infty \right\}. \tag{3.2.4}$$

*Then the integral operator $\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}$ is bounded on $L(\mathfrak{a},\mathfrak{b})$ and*

$$||\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}\varphi||_1 \leqq \mathfrak{m}.||\varphi||_1, \tag{3.2.5}$$

*where the constant $\mathfrak{m}(0 < \mathfrak{m} < \infty)$ is given by*

$$\mathfrak{m} = \left( (b-a)^{Re(\kappa)} \sum_{n=0}^{\infty} \frac{|B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)|}{B(\vartheta, d-\vartheta)} \frac{|(d)_n|}{\Gamma(\delta n+\kappa)} \frac{|\omega(b-a)^{\Re(\delta)}|^n}{n!} \right). \tag{3.2.6}$$

*Proof.* It is sufficient to prove that

$$||\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}\varphi||_1 = \int_a^b \left| \int_a^x (x-t)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}[\omega(x-t)^{\delta}; q,\rho,\zeta]\varphi(t)dt \right| dx < \infty$$
$$(q \geq 0,\ \Re(d) > \Re(\vartheta) > 0,\ \Re(\delta) > 0,\ \Re(\kappa) > 0),$$

We apply the definitions (3.2.1) and (3.2.4) in conjuction with the definition (2.1.9) of the GEML function. Upon interchanging the order of integration by means of the Dirichlet formula[42, p.56], we thus find that

$$
\begin{aligned}
||\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}\varphi||_1 &\leqq \int_a^b |\varphi(t)| \left( \int_t^b (x-t)^{\Re(\kappa)-1} \left| E^{\vartheta;d}_{\delta,\kappa}[\omega(x-t)^{\delta}; q,\rho,\zeta] \right| dx \right) dt \\
&= \int_a^b |\varphi(t)| \left( \int_0^{b-t} \tau^{\Re(\kappa)-1} |E^{\vartheta;d}_{\delta,\kappa}[\omega(\tau)^{\delta}; q,\rho,\zeta]| d\tau \right) dt \\
&\leqq \int_a^b |\varphi(t)| \left( \int_0^{b-a} \tau^{\Re(\kappa)-1} |E^{\vartheta;d}_{\delta,\kappa}[\omega(\tau)^{\delta}; q,\rho,\zeta]| d\tau \right) dt \\
&\leqq \left( \sum_{n=0}^{\infty} \frac{|B^{(\rho,\zeta)}_q(\vartheta+n, d-\vartheta)|}{B(\vartheta, d-\vartheta)} \frac{|(d)_n|}{\Gamma(\delta n+\kappa)} \frac{|\omega|^n}{n!} . \int_0^{b-a} \tau^{n\Re(\delta)+\Re(\kappa)-1} d\tau \right) .||\varphi||_1 \\
&= \mathfrak{m}.||\varphi||_1 \qquad (\Re(\kappa)>0).
\end{aligned}
\tag{3.2.7}
$$

This completes the proof of the boundedness property of the integral operator $\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}$ as asserted by Theorem 3.2.1. □

**Remark 3.2.1.** *Throughout the present investigation, it is tactily assumed that, in such situations as those occurring in the definitions (2.4.10), (2.4.11) and (3.2.1), the number* $\mathfrak{a}$ *in the function space* $L(\mathfrak{a}, \mathfrak{b})$ *coincides precisely with the lower terminal* $a$ *in the integrals involved in the definitions (2.4.10), (2.4.11) and (3.2.1).*

**Remark 3.2.2.** *The results obtained by Kilbas et al. [28] and Srivastava and Tomovski [78] can be deduced as special cases of Theorem 3.2.1.*

## 3.3 IMAGES

In this section, we will find images of some useful functions under our operator defined by (3.2.1)

**(I)** $$\left(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[(z-a)^{\alpha-1}\Phi_{l_j;\upsilon_j Q}^{k_j;\varrho_j P}(\beta z,s,a)]\right)(x)$$

$$= \frac{\Gamma(\zeta)\Gamma(\alpha)}{\Gamma(\rho)\Gamma(d)}\frac{(x-a)^{\kappa+\alpha-1}}{B(\vartheta,d-\vartheta)}\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{P}(k_j)_{m\varrho_j}}{m!\prod_{j=1}^{Q}(l_j)_{m\upsilon_j}}\frac{(\beta x)^m}{(m+a)^s m!\Gamma(-m)}$$

$$\cdot H_{2,2:1,2;1,3;3,1}^{0,2:1,1;1,1;1,2}\left[\begin{array}{c|cc} \omega(x-a)^{\delta} & A^*: & C^* \\ q^{-1} & & \\ \frac{a}{x}-1 & B^*: & D^* \end{array}\right], \qquad (3.3.1)$$

where

$A^* = (1-\vartheta;1,1,0),(1-\kappa;\delta,0,1)$

$C^* = (1-d,1);(1-d+\vartheta,1),(\zeta,1),(1,1);(1+m,1)$

$B^* = (1-d;1,2,0),(1-\alpha-\kappa;\delta,0,1)$

$D^* = (1-\kappa,\delta),(0,1);(\rho,1);(0,1),$

provided that conditions given by (3.2.1) are satisfied.

where the multivariable $H$-function for $r=3$ occurring in the right hand side of (3.3.1) is defined in chapter zero by (1.1.36).

*Proof.* By defination (3.2.1), we have

$$\left(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[(z-a)^{\alpha-1}\Phi_{l_j;\upsilon_j Q}^{k_j;\varrho_j P}(\beta z,s,a)]\right)(x)$$
$$= \int_a^x (x-z)^{\kappa-1}(z-a)^{\alpha-1}E_{\delta,\kappa}^{\vartheta;d}(\omega(x-z)^{\delta};q,\rho,\zeta)\,\Phi_{l_j;\upsilon_j Q}^{k_j;\varrho_j P}(\beta z,s,a))\,dz$$
(3.3.2)

Next, using (2.1.9) and (3.1.4) and interchanging the order of summation and integration (which is permissible under the assumptions stated in the above theorem), we get the following expression after a little simplification

$$
\begin{aligned}
&(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[(z-a)^{\alpha-1}\Phi^{k_j;\varrho_j P}_{l_j;\upsilon_j Q}(\beta z,s,a)])(x) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d)_n\omega^n}{\Gamma(\delta n+\kappa)n!} \frac{\prod\limits_{j=1}^{P}(k_j)_{m\varrho_j}}{m!\prod\limits_{j=1}^{Q}(l_j)_{m\upsilon_j}} \frac{\beta^m}{(m+a)^s} \\
&\qquad \cdot \int_a^x (x-z)^{\delta n+\kappa-1}(z-a)^{\alpha-1}z^m dz
\end{aligned}
\tag{3.3.3}
$$

Further, substituting $u = \frac{x-z}{x-a}$ in the right hand side of (3.3.3) and evaluating the u-integral with the help of [53, p. 47, Theorem 16], we obtain the following expression:

$$
\begin{aligned}
&(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[(z-a)^{\alpha-1}\Phi^{k_j;\varrho_j P}_{l_j;\upsilon_j Q}(\beta z,s,a)])(x) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)} \frac{(d)_n\omega^n}{n!} \frac{\prod\limits_{j=1}^{P}(k_j)_{m\varrho_j}}{m!\prod\limits_{j=1}^{Q}(l_j)_{m\upsilon_j}} \frac{\beta^m}{(m+a)^s}(x-a)^{\delta n+\kappa+\alpha-1}x^m \\
&\qquad \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+\delta n+\kappa)}\, {}_2F_1\left(\begin{matrix} -m,\delta n+\kappa; \\ \alpha+\delta n+\kappa; \end{matrix}\ \frac{x-a}{x}\right).
\end{aligned}
\tag{3.3.4}
$$

Interpreting the functions present in (3.3.4) in form of multivariable H-function (1.1.36) for $r=3$, we obtain the desired result. □

**(II)** $$(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[(z-a)^{\sigma-1}F_P^{(\alpha,\beta;\tau,\mu)}(a,b;c;\lambda z)])(x)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{B_P^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{(d)_n(a)_m\omega^n\lambda^m}{\Gamma(\delta n+\kappa)n!m!}(x-a)^{\delta n+\kappa+\lambda-1}x^m$$
$$\cdot\frac{\Gamma(\sigma)}{\Gamma(\sigma+\delta n+\kappa)}{}_2F_1\left(\begin{matrix}-m,\delta n+\kappa; \\ \sigma+\delta n+\kappa;\end{matrix}\ \frac{x-a}{x}\right), \tag{3.3.5}$$

provided that conditions given by (3.2.1) are satisfied.

**The proof of (3.3.5) will follow on similar lines as those given in the proof of (3.3.1).**

**(III)** $$\left(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}\left[\tau^{\beta}H^{M,N}_{P,Q}\left[A\tau^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\ (b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]\right]\right)(x)$$
$$= x^{\beta}(x-a)^{\kappa}\frac{\Gamma(\zeta)}{\Gamma(\rho)\Gamma(d-\vartheta)\Gamma(\vartheta)}H^{0,3:N,M;1,0;1,2;1,1}_{3,2:Q,P+1;0,1;3,1;2,1}\left[\begin{matrix}\frac{1}{Ax^{\sigma}}\\ \frac{-(x-a)}{x}\\ \frac{1}{q}\\ \omega(x-a)^{\delta}\end{matrix}\left|\begin{matrix}A^*: & C^*\\ \\ \\ B^*: & D^*\end{matrix}\right.\right], \tag{3.3.6}$$

where

$A^* = (1+\beta;\sigma,\eta,0,0),(1-\vartheta;0,0,1,1),(1-\kappa;0,1,0,\delta)$

$C^* = (1-b_j,\beta_j)_{1,Q};-;(1,1),(1-d+\vartheta,1),(\zeta,1);(1-d,1)$

$B^* = (-\kappa;0,1,0,\delta),(1-d;0,0,2,1)$

$D^* = (1-a_j,\alpha_j)_{1,P},(1+\beta,\sigma);(0,1);(\rho,1);(0,1)(1-\kappa,\delta),$

provided that conditions given by (3.2.1) are satisfied. where the multivariable $H$-function for $r = 4$ occurring in the right hand side of (3.3.1) is defined in chapter zero by (1.1.36).

**The proof of (3.3.5) will follow on similar lines as those given in the proof of (3.3.1).**

## SPECIAL CASES OF THE MAIN FINDINGS

(i) In image (I), if we reduce extended Hurwitz-Lerch Zeta function in (3.3.1) to general Hurwitz-Lerch Zeta function (3.1.5),by taking $Q = 0$ and $P = \rho_j = \kappa_j = 1$, we can easily obtain the following interesting result after a little simplification:

$$\begin{aligned}
&(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[(z-a)^{\alpha-1}\Phi(\beta z, s, a)](x) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{B_q^{(\rho,\zeta)}(\vartheta+n, d-\vartheta)}{B(\vartheta, d-\vartheta)} \frac{(d)_n\omega^n}{n!} \frac{\beta^m}{(m+a)^s}(x-a)^{\delta n+\kappa+\alpha-1}x^m \\
&\qquad \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+\delta n+\kappa)} {}_2F_1\left(\begin{matrix} -m, \delta n+\kappa; \\ \alpha+\delta n+\kappa; \end{matrix} \ \frac{x-a}{x}\right).
\end{aligned} \tag{3.3.7}$$

(ii) Next, in image (I) if we reduce extended Hurwitz-Lerch Zeta function in (3.3.1) to general Riemann Zeta function (3.1.6) by taking $Q = 0, P = \rho_j = \kappa_j = 1$ and $\beta z = a = 1$, we can easily obtain the following interesting result after a little simplification:

$$(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[(z-a)^{\alpha-1}\zeta(s)])(x)$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n\omega^n}{\Gamma(\delta n+\kappa)n!}\frac{1}{(m+1)^s}(x-1)^{\delta n+\kappa+\alpha-1}\beta(\delta n+\kappa,\alpha). \tag{3.3.8}$$

(iii) Again, in image (I) if we reduce extended Hurwitz-Lerch Zeta function in (3.3.1) to Hurwitz (or generalized) Zeta function (3.1.7) by taking $Q=0, P=\rho_j=\kappa_j=1$ and $\beta z=1$, we can easily obtain the following interesting result after a little simplification:

$$(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[(z-a)^{\alpha-1}\zeta(s,a)])(x)$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n\omega^n}{n!}\frac{1}{(m+a)^s}(x-a)^{\delta n+\kappa+\alpha-1}\beta(\delta n+\kappa,\alpha). \tag{3.3.9}$$

(iv) Further, in image (I) if we reduce extended Hurwitz-Lerch Zeta function in (3.3.1) to Lerch Zeta function (3.1.8) by taking $Q=0, P=\rho_j=\kappa_j=a=1$ and $\beta z=e^{2\pi i\xi}$, we can easily obtain the following interesting result after a little simplification:

$$(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[(z-a)^{\alpha-1}\ell_s(\xi)])(x)$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n\omega^n}{n!}\frac{e^{2\pi im\xi}}{(m+1)^s}(x-1)^{\delta n+\kappa+\alpha-1}\beta(\delta n+\kappa,\alpha). \tag{3.3.10}$$

(v) In image (I), if we reduce extended Hurwitz-Lerch Zeta function in (3.3.1) to Lerch Zeta function (3.1.9) by taking $Q = 0, P = \rho_j = \kappa_j = 1$ and $\beta z = e^{2\pi i\xi}$, we can easily obtain the following interesting result after a little simplification:

$$\begin{aligned}&(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[(z-a)^{\alpha-1}\ell_s(\xi)])(x)\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B^{(\rho,\zeta)}_q(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n\omega^n}{n!}\frac{e^{2\pi im\xi}}{(m+a)^s}(x-a)^{\delta n+\kappa+\alpha-1}\beta(\delta n+\kappa,\alpha).\end{aligned} \tag{3.3.11}$$

# COMPOSITION RELATIONSHIP

**Theorem 3.3.1.** *If $\vartheta$, $\gamma$, $\delta$, $\kappa$, $\omega \in \mathbb{C}$, $\Re(\kappa) > 0, \Re(\gamma) > 0$, $\Re(\delta) > 0$, then*

$$(I^{\gamma}_{a+}[\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f])(x) = (\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa+\gamma;\zeta}f)(x) = (\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[I^{\gamma}_{a+}f])(x), \tag{3.3.12}$$

*where $f \in L(a, b)$.*

*Proof.* In order to prove the left hand side of above mentioned assertion, we proceed by using (3.2.1) in (2.4.10)

$$(I^{\gamma}_{a+}[\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f])(x) = \frac{1}{\Gamma(\gamma)}\int_a^x \frac{(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f)(z)}{(x-z)^{1-\gamma}}dz \tag{3.3.13}$$

$$= \frac{1}{\Gamma(\gamma)}\int_a^x (x-z)^{\gamma-1}\left[\int_a^z (z-u)^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega(z-u)^{\delta};q,\rho,\zeta)f(u)du\right]dz. \tag{3.3.14}$$

Now, by interchanging the order of integration (which is permissible under the assumptions stated in the above theorem), we obtain

$$(I^{\gamma}_{a+}[\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f])(x) = \int_a^x \left[\frac{1}{\Gamma(\gamma)}\int_u^x (x-z)^{\gamma-1}(z-u)^{\kappa-1}E^{\vartheta;d}_{\delta,\kappa}(\omega(z-u)^{\delta};q,\rho,\zeta)dz\right]f(u)du. \tag{3.3.15}$$

Substituting $(z-u)=\sigma$ in the above equation, we have

$$(I_{a+}^{\gamma}[\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}f])(x)=\int_{a}^{x}\left[\frac{1}{\Gamma(\gamma)}\int_{0}^{x-u}(x-u-\sigma)^{\gamma-1}(\sigma)^{\kappa-1}E_{\delta,\kappa}^{\vartheta;d}(\omega\sigma^{\delta};q,\rho,\zeta)d\sigma\right]f(u)du.$$

$$=\int_{a}^{x}(I_{0+}^{x-u}\sigma^{\kappa-1}E_{\delta,\kappa}^{\vartheta;d}(\omega\sigma^{\delta};q,\rho,\zeta))(x-u)f(u)du \tag{3.3.16}$$

Further, applying (2.5.1) in (3.3.16), we have

$$\begin{aligned}(I_{a+}^{\gamma}[\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}f])(x)&=\int_{a}^{x}(x-u)^{\gamma+\kappa-1}E_{\delta,\kappa+\gamma}^{\vartheta;d}(\omega(x-u)^{\delta};q,\rho,\zeta)f(u)du\\&=(\varepsilon_{a+;\delta,\kappa+\gamma;\zeta}^{\omega;\vartheta;d;\rho}f)(x).\end{aligned} \tag{3.3.17}$$

this proves the first part of (3.3.12).
In order to prove the second part, we proceed by using (3.2.1) in the right hand side of (3.3.12)

$$\begin{aligned}(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[I_{a+}^{\gamma}f])(x)&=\int_{a}^{x}(x-z)^{\kappa-1}E_{\delta,\kappa}^{\vartheta;d}(\omega(x-z)^{\delta};q,\rho,\zeta)[I_{a+}^{\gamma}f](z)dz\\&=\int_{a}^{x}(x-z)^{\kappa-1}E_{\delta,\kappa}^{\vartheta;d}(\omega(x-z)^{\delta};q,\rho,\zeta)\left(\frac{1}{\Gamma(\gamma)}\int_{a}^{z}\frac{f(u)}{(z-u)^{1-\gamma}}du\right)dz.\end{aligned} \tag{3.3.18}$$

Next, interchanging the order of integration (which is permissible under the assumptions stated in the above theorem), we have

$$(\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega;\vartheta;d;\rho}[I_{a+}^{\gamma}f])(x)=\int_{a}^{x}\left[\frac{1}{\Gamma(\gamma)}\int_{u}^{x}(x-z)^{\kappa-1}(z-u)^{\gamma-1}E_{\delta,\kappa}^{\vartheta;d}(\omega(x-z)^{\delta};q,\rho,\zeta)dz\right]f(u)du. \tag{3.3.19}$$

Substituting $(x-\tau)=\varepsilon$, we obtain

$$(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[I^{\gamma}_{a+}f])(x)=\int\limits_a^x\left[\frac{1}{\Gamma(\gamma)}\int\limits_0^{x-u}(\varepsilon)^{\kappa-1}(x-\varepsilon-u)^{\gamma-1}E^{\vartheta;d}_{\delta,\kappa}(\omega\varepsilon^{\delta};q,\rho,\zeta)d\varepsilon\right]f(u)du. \tag{3.3.20}$$

Again, by using (2.4.10) and (2.5.1), we get

$$(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[I^{\gamma}_{a+}f])(x)=(\varepsilon^{\omega;\vartheta;d;\rho}_{a+;\delta,\kappa+\gamma;\zeta}f)(x), \tag{3.3.21}$$

from (3.3.17) and (3.3.21) we get the desired proof. □

**Remark 3.3.1.** *The above theorem generalizes the results obtained by Kilbas et al. [28] and Srivastava and Tomovski [78].*

## AN INTEGRAL OPERATOR $H^{w;M,N;\alpha}_{a+;P,Q;\beta}$ INVOLVING THE FOX H-FUNCTION

In our present investigation, we make use of a special case of general families of fractional integral operators having Fox-H function in their kernels (see [75, p. 15, Eq. (6.3)]), defined and represented in the following manner:

$$(H^{w;M,N;\alpha}_{a+;P,Q;\beta}\varphi)(x)=\int\limits_a^x(x-t)^{\beta-1}H^{M,N}_{P,Q}[\omega(x-t)^{\alpha}]\varphi(t)dt \tag{3.3.22}$$

$$\left(\Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\};\ 1\leqq M\leqq Q;\ 0\leqq N\leqq P;\Re(\beta)+\min_{1\leqq j\leqq M}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\}>0\right).$$

**Theorem 3.3.2.** *If* $\vartheta,\kappa,\delta,\beta\in\mathbb{C},w\in\mathbb{C}\setminus\{0\},\Re(\beta)>0,\Re(\delta)>0,\Re(\kappa)>$

0 *and* $\Re(\beta) + \min_{1 \leqq j \leqq M} \left\{ \Re\left(\frac{\alpha b_j}{\beta_j}\right) \right\} > 0$ *then*

$$(H^{\omega_2;M,N;\alpha}_{a+;P.Q;\beta}[\varepsilon^{\omega_1;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f])(x)$$

$$= \frac{\Gamma(\zeta)}{\Gamma(\rho)\Gamma(d-\vartheta)\Gamma(\vartheta)} \int_a^x (x-\tau)^{\beta+\kappa-1} H^{0,1:1,2;1,1;M,N+1}_{1,2:3,1;1,1;P,Q} \left[ \begin{array}{c|cc} q^{-1} & A^* : & C^* \\ \omega_1(x-\tau)^{\delta} & & \\ \omega_2(x-\tau)^{\alpha} & B^* : & D^* \end{array} \right] f(\tau)d\tau$$

$$= (\varepsilon^{\omega_1;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}[H^{\omega_2;M,N;\alpha}_{a+;P,Q;\beta}f])(x), \tag{3.3.23}$$

*where* $f \in L(a,b)$ *and*

$A^* = (1-\vartheta; 1, 1, 0)$

$C^* = (1,1), (1-d+\vartheta, 1), (\zeta, 1); (1-d, 1); (a_j, \alpha_j)_{1,P}, (1-\beta, \alpha)$

$B^* = (1-\kappa-\beta; 0, \delta, \alpha), (1-d; 2, 1, 0)$

$D^* = (\rho, 1); (0, 1); (b_j, \beta_j)_{1,Q}$,

*where the multivariable $H$-function for $r = 3$ occurring in the right hand side of (3.3.23) is defined in chapter zero by (1.1.36).*

*Proof.* In order to prove the left hand side of above mentioned assertion, we proceed by using (3.3.22) in (3.3.23)

$$(H^{\omega_2;M,N;\alpha}_{a+;P.Q;\beta}[\varepsilon^{\omega_1;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f])(x)$$

$$= \int_a^x (x-t)^{\beta-1} H^{M,N}_{P,Q}[\omega_2(x-t)^{\alpha}](\varepsilon^{\omega_1;\vartheta;d;\rho}_{a+;\delta,\kappa;\zeta}f)(t)dt$$

$$= \int_a^x (x-t)^{\beta-1} H^{M,N}_{P,Q}[\omega_2(x-t)^{\alpha}] \left[ \int_a^t (t-\tau)^{\kappa-1} E^{\vartheta;d}_{\delta,\kappa}(\omega_1(t-\tau)^{\delta}; q, \rho, \zeta) f(\tau)d\tau \right] dt. \tag{3.3.24}$$

Now, by interchanging the order of integration (which is permissible under the assumptions stated in the above theorem), we obtain

$$(H_{a+;P.Q;\beta}^{\omega_2;M,N;\alpha}[\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega_1;\vartheta;d;\rho}f])(x) = \int_a^x \left[ \int_\tau^x (x-t)^{\beta-1}(t-\tau)^{\kappa-1} H_{P,Q}^{M,N}[\omega_2(x-t)^\alpha] E_{\delta,\kappa}^{\vartheta;d}(\omega_1(t-\tau)^\delta; q,\rho,\zeta)dt \right] f(\tau)d\tau. \tag{3.3.25}$$

Using (2.1.9) and (1.1.4) in the right hand side of the above equation and changing the order of summation and integration (which is permissible under the assumptions stated in the above theorem) and then substituting $(t-\tau)=\sigma$, we have

$$(H_{a+;P.Q;\beta}^{\omega_2;M,N;\alpha}[\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega_1;\vartheta;d;\rho}f])(x) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_a^x \frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n}{\Gamma(\delta n+\kappa)}\frac{\omega_1^n}{n!}\int_{\mathfrak{L}}\Theta(\mathfrak{s})\omega_2^{\mathfrak{s}} \cdot \int_0^{x-\tau}(x-\sigma-\tau)^{\beta+\alpha\mathfrak{s}-1}\sigma^{\kappa+\delta n-1}d\sigma d\mathfrak{s} d\tau. \tag{3.3.26}$$

Further, substituting $\frac{\sigma}{x-\tau}=z$, using beta function and contour representation of generalized extended Mittag-Leffler function (2.1.11), we have

$$\begin{aligned}&(H_{a+;P.Q;\beta}^{\omega_2;M,N;\alpha}[\varepsilon_{a+;\delta,\kappa;\zeta}^{\omega_1;\vartheta;d;\rho}f])(x)\\ &= \frac{1}{(2\pi i)^3}\frac{\Gamma(\zeta)}{\Gamma(\rho)\Gamma(d-\vartheta)\Gamma(\vartheta)}\int_a^x (x-\tau)^{\beta+\kappa-1}f(\tau)\int_{\mathfrak{L}}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}\Theta(\mathfrak{s})(\omega_2(x-\tau)^\alpha)^{\mathfrak{s}}(\omega_1(x-\tau)^\delta))^{\xi_2}(q_1)^{-\xi_1}\\ &\cdot\frac{\Gamma(\beta+\alpha\mathfrak{s})}{\Gamma(\kappa+\beta+\delta\xi_2+\alpha\mathfrak{s})}\frac{\Gamma(-\xi_2)\Gamma(\xi_1)\Gamma(\rho-\xi_1)}{\Gamma(\zeta-\xi_1)}\frac{\Gamma(\vartheta+\xi_2+\xi_1)\Gamma(d-\vartheta+\xi_1)\Gamma(d+\xi_2)}{\Gamma(d+\xi_2+2\xi_1)}d\mathfrak{s}\, d\xi_1 d\xi_2 d\tau,\end{aligned} \tag{3.3.27}$$

reinterpreting the right hand side of (3.3.27) in the form of multivariable H-function (1.1.36) for $r=3$, we obtain the desired result. □

# 3. AN INTEGRAL OPERATOR INVOLVING GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION

# 4

# A NEW INTEGRAL TRANSFORM ASSOCIATED WITH THE EXTENDED HURWITZ-LERCH ZETA FUNCTION

The main findings of this chapter have bearing on the following paper detailed below:

1. **H.M. SRIVASTAVA, N. JOLLY, M.K. BANSAL and R. JAIN** (2018). A STUDY OF NEW INTEGRAL TRANSFORM ASSOCIATED WITH $\lambda-$EXTENDED HURWITZ-LERCH ZETA FUNCTION, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales*, https://doi.org/10.1007/s13398-018-0570-4.

In this chapter, we first define extended Hurwitz-Lerch zeta function and give it's integral representation. Next, we define a new integral transform whose kernel is extended Hurwitz-Lerch zeta function and name it as the $\mathcal{E}-$transform. The transform of our study yield known integral transforms [1, p. 44, Eqs. (1.3.5) and (1.3.6)]. Further, we evaluate the Mellin transform of extended Hurwitz-Lerch Zeta function and Inversion formula for $\mathcal{E}-$transform. Next, we present some basic properties and prove the Uniqueness theorem for $\mathcal{E}-$transform. Finally, we obtain $\mathcal{E}-$transform of Derivatives and Integrals.

# 4.1 INTRODUCTION

The extended Hurwitz-Lerch zeta function introduced by Srivastava et al. [77, p. 503, Eq.(6.2)](see also [62] and [70]) is recalled here in slightly modified form:

$$\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a) = \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a) = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{4.1.1}$$

$$(p,q \in \mathbb{N}_0; \lambda_j \in \mathbb{C}(j=1,\cdots,p); a,\mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-(j=1,\cdots,q);$$

$$\rho_j,\sigma_k \in \mathbb{R}^+(j=1\cdots p; k=1\cdots,q); \Delta > -1 \text{ when } s,z \in \mathbb{C};$$

$$\Delta = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \bigtriangledown^*; \ \Delta = -1 \text{ and } \Re(\Xi) > \tfrac{1}{2} \text{ when } |z| = \bigtriangledown^*$$

$$\text{where } \bigtriangledown^* := (\prod_{j=1}^{p}\rho_j^{-\rho_j})(\prod_{j=1}^{q}\sigma_j^{\sigma_j}) \text{ and } \Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j,$$
$$\Xi := s + \sum_{j=1}^{q}\mu_j - \sum_{j=1}^{p}\lambda_j + \tfrac{p-q}{2}),$$

the integral representation of extended Hurwitz-Lerch zeta function was also given by Srivastava et al. ([77],Theorem 8)see also([63], Theorem 6).

$$\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}{}_p\Psi^*{}_q\left[\begin{matrix}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p);\\ \\(\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q);\end{matrix}\ ze^{-t}\right]dt \tag{4.1.2}$$

$$(\min\{\Re(a),\Re(s)\} > 0),$$

where ${}_p\Psi^*{}_q$ $(p,q \in \mathbb{N}_0)$ denotes the Fox-Wright function,which is generalization of the familiar generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$defined by [8, p.183]

$$ {}_p\Psi^*{}_q \left[ \begin{array}{c} (\lambda_j, \rho_j)_{1,p}; \\ \\ (\mu_j, \sigma_j)_{1,q}; \end{array} z \right] := \sum_{n=0}^{\infty} \frac{(\lambda_1)_{\rho_1 n} \cdots (\lambda_p)_{\rho_p n}}{(\mu_1)_{\sigma_1 n} \cdots (\mu_q)_{\sigma_q n}} \frac{z^n}{n!} = \frac{\Gamma(\mu_1) \cdots \Gamma(\mu_q)}{\Gamma(\lambda_1) \cdots \Gamma(\lambda_p)} {}_p\Psi_q \left[ \begin{array}{c} (\lambda_j, \rho_j)_{1,p}; \\ \\ (\mu_j, \sigma_j)_{1,q}; \end{array} z \right]. \tag{4.1.3} $$

## 4.2 A SET OF MAIN RESULTS

In this section, we first introduce the new integral transform of our study whose kernel involves the extended Hurwitz-Lerch zeta function defined by (4.1.2).

### 4.2.1 THE EXTENDED HURWITZ-LERCH ZETA TRANSFORM OR THE $\mathcal{E}-$TRANSFORM

The extended Hurwitz-Lerch zeta transform studied in this chapter will be defined and represented in the following manner:

$$ \mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z, s, a)[f(t)](\mathtt{s}) := \int_0^\infty \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(\mathtt{s}t, s, a) f(t) dt =: \varphi(\mathtt{s}) \qquad (f(t) \in \Lambda), \tag{4.2.1} $$

in the neighbourhood of $t = z$, where $\Lambda$ denotes the class of admissible functions $f(t)$, which are integrable in every finite interval in $(0,\infty)$ with the following order estimates:

$$ f(t) = \begin{cases} O(t^\omega) & (t \to 0) \\ O(t^\kappa e^{-\mu t}) & (|t| \to \infty), \end{cases} \tag{4.2.2} $$

provided that the existence conditions given with (4.1.2) for the extended Hurwitz-Lerch zeta function are satisfied, $\mathfrak{R}(\omega) > -1$ and

$$\Re(\mu) > 0 \quad \text{or} \quad \Re(\mu) = 0 \quad \text{and} \quad \max_{1 \leq j \leq p} \left\{ \Re \left( \kappa + \tfrac{\lambda_j - 1}{\rho_j} + 1 \right) \right\} < 0.$$

We shall refer this transform as $\mathcal{E}-$transform.

If we reduce the extended Hurwitz-Lerch zeta function to general Fox-Wright function of the type ${}_p\Psi_q^*$ defined by (4.1.3)[18, p.271, Eqs. (7) and (9)], we are led to the known integral transfroms studied earlier in [1, p. 44, Eqs. (1.3.5) and (1.3.6)].

## 4.2.2 THE MELLIN TRANSFORM OF THE EXTENDED HURWITZ-LERCH ZETA FUNCTION

The Mellin transform of a suitably constrained function $f(t)$ is defined by (see, for example, [6, p.340, Eq.(8.2.5)]):

$$\mathfrak{M}[f(t)](\mathsf{s}) := \int_0^\infty t^{\mathsf{s}-1} f(t) dt =: F_{\mathfrak{M}}(\mathsf{s}) \qquad (\Re(\mathsf{s}) > 0), \tag{4.2.3}$$

provided that the improper integral in (4.2.3) exists.

If

$$\int_0^\infty t^{\ell-1} |f(t)| dt < \infty,$$

for some $\ell > 0$, then the Mellin Inversion Formula reads as follows [14, p.1194]:

$$f(z) = \mathfrak{M}^{-1}[f(\mathsf{s})](z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-\mathsf{s}} F_{\mathfrak{M}}(\mathsf{s}) d\mathsf{s} \qquad (c > \ell), \tag{4.2.4}$$

where $F_{\mathfrak{M}}(\mathsf{s})$ is given by Eq. (4.2.3), provided that the above integral exists.

**Theorem 4.2.1.** *If* $\Re(\mathrm{s}) < \min\limits_{1\leq j\leq p}\left\{\Re\left(\frac{\lambda_j}{\rho_j}+a\right)\right\}$ *and* $\Re(a-\mathrm{s}) > 0$, *then*

$$\mathfrak{M}[\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(zt,s,a)](\mathrm{s}) = \frac{(-z)^{-\mathrm{s}}\Gamma(\mathrm{s})\prod\limits_{j=1}^{q}\Gamma(\mu_j)\prod\limits_{j=1}^{p}\Gamma(\lambda_j-\mathrm{s}\rho_j)}{\prod\limits_{j=1}^{P}\Gamma(\lambda_j)\prod\limits_{j=1}^{q}\Gamma(\mu_j-\mathrm{s}\sigma_j)}\left(\frac{\Gamma(a-\mathrm{s})}{\Gamma(a-\mathrm{s}+1)}\right)^{s}, \tag{4.2.5}$$

*provided that each member of the assertions (4.2.5) exists.*

*Proof.* Our demonstration of the assertion (4.2.5) of Theorem 4.2.1 is based upon the Mellin Inversion Formula (4.2.3) in conjunction with the integral representation (4.1.2) for the extended Hurwitz-Lerch zeta function.
For direct derivation of the assertion (4.2.5) of Theorem 4.2.1, we can apply the representations (4.1.2) together with the following familiar Eulerian integral:

$$\int_0^\infty t^{\rho-1}e^{-\sigma t} = \frac{\Gamma(\rho)}{\sigma^\rho} \qquad (\min\{\Re(\rho),\Re(\sigma)\} > 0), \tag{4.2.6}$$

The details involved are fairly straightforward and are, therefore, being omitted here. □

## 4.3 INVERSION FORMULA FOR $\mathcal{E}-$TRANSFORM

**Theorem 4.3.1.** *If* $t^{\ell-1}f(t) \in L(0,\infty)$, *the function* $f(t)$ *is of bounded variation in the neighbourhood of the point* $t = z$ *and*

$$\varphi(\mathrm{s}) = \mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a)[f(t)](\mathrm{s}) := \int_0^\infty \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(\mathrm{s}t,s,a)f(t)dt \qquad (f(t)\in\Lambda), \tag{4.3.1}$$

*then*

$$\frac{1}{2}[f(z+0)+f(z-0)] = \frac{1}{2\pi i}\int\limits_{\mathfrak{L}} (-1)^{\ell-1} \frac{\prod\limits_{j=1}^{p}\Gamma(\lambda_j)\prod\limits_{j=1}^{q}\Gamma(\mu_j-\sigma_j+\ell\sigma_j)}{\Gamma(1-\ell)\prod\limits_{j=1}^{q}\Gamma(\mu_j)\prod\limits_{j=1}^{p}\Gamma(\lambda_j-\rho_j+\ell\rho_j)} \cdot \left(\frac{\Gamma(a+\ell)}{\Gamma(a+\ell-1)}\right)^{s} z^{-\ell}\Omega(\ell)d\ell, \quad (4.3.2)$$

*where*

$$\Omega(\ell) = \int\limits_{0}^{\infty} \mathtt{s}^{-\ell}\varphi(\mathtt{s})d\mathtt{s}, \qquad (4.3.3)$$

*provided that existence conditions for the extended Hurwitz-Lerch zeta function given by (4.1.2) are satisfied, the extended Hurwitz-Lerch zeta function transform of* $|f(z)|$ *exists, and* $\min\{1-\ell, \Re(\ell+a-1), \Re(s)\} > 0.$

*Proof.* In order to prove the inversion formula (4.3.2), we substitute the value of $\varphi(\mathtt{s})$ from (4.3.1) into the right-hand side of (4.3.3), we get

$$\Omega(\ell) := \int\limits_{0}^{\infty} \mathtt{s}^{-\ell}\varphi(\mathtt{s})d\mathtt{s} = \int\limits_{0}^{\infty} \mathtt{s}^{-\ell}\int\limits_{0}^{\infty} \Phi_{b_j;\beta_j q}^{a_j;\alpha_j p}(\mathtt{s}t,s,a)f(t)dtd\mathtt{s}. \qquad (4.3.4)$$

Upon interchanging the order of the t-integral and the $\mathtt{s}$-integrals in (4.3.4), which is permissible under the conditions stated above, if we evaluate the $\mathtt{s}$-integral by using (4.2.5), we obtain

$$\Omega(\ell) := \int\limits_{0}^{\infty} \frac{\Gamma(1-\ell)\prod\limits_{j=1}^{q}\Gamma(\mu_j)\prod\limits_{j=1}^{p}\Gamma(\lambda_j-\rho_j+\ell\rho_j)}{\prod\limits_{j=1}^{p}\Gamma(\lambda_j)\prod\limits_{j=1}^{q}\Gamma(\mu_j-\sigma_j+\ell\sigma_j)} \left(\frac{\Gamma(a+\ell-1)}{\Gamma(a+\ell)}\right)^{s}(-t)^{\ell-1}f(t)dt. \qquad (4.3.5)$$

Finally, by applying the Mellin Inversion formula (4.2.4) to this last $t-$integral (4.3.5), we get the desired result (4.3.2) after a little simplification. □

## 4.4 BASIC PROPERTIES OF THE $\mathcal{E}$-TRANSFORM

In this section, we present linearity property and scaling property of the $\mathcal{E}$-transform.

### 4.4.1 LINEARITY PROPERTY

We state the following linearity property.

**Theorem 4.4.1.** *Let* $\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a)[f_k(t)](\mathtt{s}) \quad (k=1,\cdots,n)$ *denote the* $\mathcal{E}-$*transform of the functions* $f_k(t) \quad (k=1,\cdots,n)$*. Then*

$$\begin{aligned}&\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a)[c_1 f_1(t)+\cdots+c_k f_k(t)](\mathtt{s})\\&=c_1\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a)[f_1(t)](\mathtt{s})+\cdots+c_n\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a)[f_n(t)](\mathtt{s}),\end{aligned} \tag{4.4.1}$$

*for any constants* $c_1,\cdots,c_n$.

*Proof.* The proof of the linearity property (4.4.1) asserted by Theorem 4.4.1 follows readily from the definition (4.2.1) of the $\mathcal{E}-$transform. □

### 4.4.2 SCALING PROPERTY

Now, we state the following scaling property of the $\mathcal{E}-$transform which is derivable in a straight forward manner from it's definition (4.2.1).

**Theorem 4.4.2.** *Let* $\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a;b,\lambda)[f(t)](\mathtt{s})$ *denote the* $\mathcal{E}-$*transform (4.2.1) of the function* $f(t)$*. Then*

$$\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a;b,\lambda)[\xi f(t)](\mathtt{s})=\frac{1}{\xi}\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a;b,\lambda)[f(t)]\left(\frac{\mathtt{s}}{\xi}\right) \quad (\xi>0). \tag{4.4.2}$$

Now we shall define an important function that will be required in the proof of uniqueness theorem.

## 4.5 THE $\overline{H}-$FUNCTION

We recall here that, in the course of evaluation (in two different ways) of some Feynman integrals which are known to arise naturally in perturbation calculations of the equilibrium properties of a magnetic field of phase transitions, Inayat-Hussain [23] was led eventually to a further generalization of Fox's $H-$function defined by (1.1.4). Inayat-Hussain's general $\overline{H}-$function is known to contain such special cases as (for example) the polylogarithm function of complex order and the exact partition function of the Gaussian model in statistical mechanics. In terms of a suitable Mellin-Barnes contour integral of the type $\mathcal{L}$, which we have already encountered in our present investigation, it is defined as follows (see, for details, the subsequent systematic investigation of this general $\overline{H}-$function by Buschman [3]; see also [76]):

$$\overline{H}^{m,n}_{p,q}\left[z\left|\begin{array}{cc}(e_j,E_j;\in_j)_{1,n}, & (e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m}, & (f_j,F_j;\Im_j)_{m+1,q}\end{array}\right.\right]=\frac{1}{2\pi\omega}\int_{\mathfrak{L}}\overline{\Theta}(\xi)z^{\xi}d\xi \tag{4.5.1}$$

where, $\omega=\sqrt{-1}$, $z\in\mathbb{C}\setminus\{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi)=\frac{\prod\limits_{j=1}^{m}\Gamma(f_j-F_j\xi)\prod\limits_{j=1}^{n}\{\Gamma(1-e_j+E_j\xi)\}^{\in_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-f_j+F_j\xi)\}^{\Im_j}\prod\limits_{j=n+1}^{p}\Gamma(e_j-E_j\xi)}, \tag{4.5.2}$$

and which obviously contains fractional powers of some of the $\Gamma-$functions. Here, and in what follows, $e_j\quad(j=1,\cdots,p)$ and $f_j\quad(j=1,\cdots,q)$,

$$\min\{E_j,F_j\}>0\quad(j=1,\cdots,p;k=1,\cdots,q)$$

and the exponents (or powers)

$$\epsilon_j \quad (j = 1, \cdots, n) \quad and \quad \Im_j \quad (j = m+1, \cdots, q)$$

can take on non-integers values. Clearly, when all of the exponents (or powers)

$$\epsilon_j \quad (j = 1, \cdots, n) \quad and \quad \Im_j \quad (j = m+1, \cdots, q)$$

take on integer values, the $\overline{H}-$function defined by (4.5.1) would reduce to the relatively more familiar $H-$function defined by (1.1.4), which has indeed been investigated and applied in the mathematical, physical, engineering and statistical sciences rather extensively.

Now, we state and prove an important lemma that will be required to establish our Uniqueness theorem.

## 4.6 THE UNIQUENESS THEOREM

**Lemma 4.6.1.** *If*

$$\int_0^\infty \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(-\eta t, s, a) f(t) dt = 0, \tag{4.6.1}$$

*then*

$$f(t) \equiv 0 \quad (t \in (0, \infty)),$$

*provided that* $f(t)$ *is continuous in* $(0, \infty)$, $f(t) \in \Lambda$ *and the existence conditions for the extended Hurwitz-Lerch zeta function:*

$$\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z, s, a) \tag{4.6.2}$$

*are satisfied.*

*Proof.* We first multiply the left-hand side of (4.6.1) by

$$\eta^{\varrho-1}\overline{H}^{1,q+1}_{q+2,p+2}\left[\frac{y^\sigma}{\eta}\left|\begin{array}{c}(-a+\varrho,1;s),(1-\mu_j+\sigma_j\varrho,\sigma_j)_{1,q},(\varrho,1)\\ \\ (0,\sigma),(1-a+\varrho,1;s),(1-\lambda_j+\rho_j\varrho,\rho_j)_{1,p}\end{array}\right.\right]$$

and then integrate with respect to $\eta$ from $\eta=0$ to $\eta=\infty$ under the following parametric constraints:

$$\min_{1\leq j\leq q+1}\left\{\Re\left(\frac{\mu_j}{\sigma_j},a+1\right)\right\}>0 \quad \text{and} \quad \max_{1\leq j\leq p}\left\{\Re\left(\varrho+\frac{\lambda_j}{\rho_j}\right)\right\}<0,$$

the $\overline{H}-$function being defined above by (4.5.1). We thus have

$$\int_0^\infty \eta^{\varrho-1}\overline{H}^{1,q+1}_{q+2,p+2}\left[\frac{y^\sigma}{\eta}\left|\begin{array}{c}C^*\\ \\ D^*\end{array}\right.\right]\int_0^\infty \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(-\eta t,s,a)f(t)dtd\eta=0, \tag{4.6.3}$$

where, for convenience,

$$C^*=(-a+\varrho,1;s),(1-\mu_j+\sigma_j\varrho,\sigma_j)_{1,q},(\varrho,1)$$

and

$$D^*=(0,\sigma),(1-a+\varrho,1;s),(1-\lambda_j+\rho_j\varrho,\rho_j)_{1,p}.$$

Now, if we interchange the order of integration in the left-hand side of (4.6.3), we get

$$\int_0^\infty f(t)dt\int_0^\infty \eta^{\varrho-1}\overline{H}^{1,q+1}_{q+2,p+2}\left[\frac{y^\sigma}{\eta}\left|\begin{array}{c}C^*\\ \\ D^*\end{array}\right.\right]\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(-\eta t,s,a)d\eta=0, \tag{4.6.4}$$

Expressing the $\overline{H}-$function in the form of its Mellin-Barnes contour integral representation given by (4.5.1) and changing the order of the resulting $\mathtt{s}-$integral, we obtain

$$\int_0^\infty f(t)\ \frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\Gamma(-\sigma\mathtt{s})\{\Gamma(1+a-\varrho+\mathtt{s})\}^s \prod_{j=1}^{q}\Gamma(\mu_j-\sigma_j\varrho+\sigma_j\mathtt{s})}{\{\Gamma(a-\varrho+\mathtt{s})\}^s \prod_{j=1}^{p}\Gamma(\lambda_j-\rho_j\varrho+\rho_j\mathtt{s})\Gamma(\varrho-\mathtt{s})} y^{\sigma\mathtt{s}}$$

$$\cdot \int_0^\infty \eta^{\varrho-\mathtt{s}+1}\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(-\eta t,s,a)d\eta\ d\mathtt{s}\ dt = 0.$$

We now evaluate the $\eta-$integral with the help of (4.2.5). We thus obtain the following result after a little simplification:

$$\int_0^\infty t^{-\varrho\sigma+\sigma-1}e^{-yt}f(t^\sigma)dt = 0, \tag{4.6.5}$$

which, by applying Lerch's Theorem [32, p. 339], immediately yields the following result:

$$f(t^\sigma) \equiv 0. \tag{4.6.6}$$

This evidently leads us to the assertion of the above Lemma. □

**Theorem 4.6.1.** *(Uniqueness theorem) If the functions $f_1(t)$ and $f_2(t)$ are continuous in the semi-closed interval* $[0,\infty), \eta > 0$ *and*

$$\int_0^\infty \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(-\eta t,s,a)f_1(t)dt = \int_0^\infty \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(-\eta t,s,a)f_2(t)dt, \tag{4.6.7}$$

*in which both integrals are convergent, then*

$$f_1(t) = f_2(t) \qquad (0 \leqq t < \infty). \tag{4.6.8}$$

*Proof.* Theorem 4.6.1 follows as a direct consequence of the Lemma. We choose to omit the details involved. □

## 4.7 $\mathcal{E}-$TRANSFORM OF DERIVATIVES AND INTEGRALS

By using integration by parts and the principle of mathematical induction on $m$, we can easily derive the following result.

**Theorem 4.7.1.** *If $f^{(m)}(z)$ is the derivative of the function $f(z)$ of order $m$ with respect to $z$, then*

$$\mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a)[f^{(m)}(t)](\mathtt{s}) = (-\mathtt{s})^m \frac{\prod_{j=1}^{q}\Gamma(\mu_j)\prod_{j=1}^{p}\Gamma(\lambda_j+m\rho_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)\prod_{j=1}^{q}\Gamma(\mu_j+m\sigma_j)}\mathcal{E}^{(\lambda_j+m\rho_j,\rho_j;p)}_{(\mu_j+m\sigma_j,\sigma_j;q)}(z,s,a+m)[f(t)](\mathtt{s}), \tag{4.7.1}$$

*provided that there exists $\varepsilon_1$ and $\varepsilon_2$ such that*

$$\lim_{t\to 0}\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(\mathtt{s}t,s,a)f^m(t)dt = 0, \qquad \lim_{t\to\infty}\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(\mathtt{s}t,s,a+1)f^m(t)dt = 0$$

$$\text{and} \qquad \mathcal{E}^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a+1)[f^{(m)}(t)](\mathtt{s}),$$

*exists when $\varepsilon_1 < \Re(\mathtt{s}) < \varepsilon_2$.*

*Proof.* We will be omitting the details of the proof as it is fairly simple. □

Finally, we state the following easily derivable result.

**Theorem 4.7.2.** *If*

$$F(z) = \int_0^z f(t)dt, \tag{4.7.2}$$

*so that*

$$F(0) = 0 \qquad \text{and} \qquad F^{(1)}(z) = f(z), \tag{4.7.3}$$

*then*

$$\mathcal{E}_{(\mu_j,\sigma_j;q)}^{(\lambda_j,\rho_j;p)}(z,s,a+1)\left[\int\limits_0^t f(u)du\right](\mathtt{s}) = -\frac{1}{\mathtt{s}}\frac{\prod\limits_{j=1}^{p}\Gamma(\lambda_j)\prod\limits_{j=1}^{q}\Gamma(\mu_j+\sigma_j)}{\prod\limits_{j=1}^{q}\Gamma(\mu_j)\prod\limits_{j=1}^{p}\Gamma(\lambda_j+\rho_j)}\mathcal{E}_{(\mu_j,\sigma_j;q)}^{(\lambda_j,\rho_j;p)}(z,s,a)[f(t)](\mathtt{s}), \tag{4.7.4}$$

*Proof.* Our demonstration of the assertion (4.7.4) of Theorem 4.7.2 is fairly straightforward. We, therefore, skip the details involved. □

## 4. A NEW INTEGRAL TRANSFORM ASSOCIATED WITH THE EXTENDED HURWITZ-LERCH ZETA FUNCTION

# 5

# A STUDY OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS INVOLVING THE PRODUCT OF FUNCTIONS $E_{\delta,\kappa}^{\vartheta;d}(z;q,\sigma,\zeta)$, $\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}(z,s,a)$ and $S_V^U[z]$

---

The main findings of this chapter have bearing on the following paper detailed below:

1. **N. Jolly and R. JAIN** (2016). AN INVESTIGATION OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS, *Palestine Journal of Mathematics*, (Accepted).

In this chapter, we study a pair of class of fractional integral operators whose kernel involve the product of $S_V^U[z]$, $E_{\delta,\kappa}^{\vartheta;d}(z;q,\sigma,\zeta)$ and $\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}(z,s,a)$ which stand for $S_V^U$ polynomial, generalized extended Mittag-Leffler function and extended Hurwitz-Lerch Zeta function. Next, we derive three new and interesting composition formulae for the operators of our study. The operators of our study are quite general in nature and may be considered as extensions of a number of simpler fractional integral operators studied from time to time by several authors. By suitably specializing the coefficients and the parameters of functions involved in our fractional integral operators we can get a large number of expressions for the composition of fractional integral operators. Finally, as an application of our main findings we obtain three interesting integrals which are believed to be new.

## 5.1 INTRODUCTION

In this chapter, $\Lambda$ will denote the class of function f(t) for which

$$f(t) = \begin{cases} O\{|t|^{\varsigma}\}; & \max\{|t|\} \to 0 \\ \\ O\{|t|^{w_1} e^{-w_2|t|}\}; & \min\{|t|\} \to \infty. \end{cases} \tag{5.1.1}$$

The fractional integral operators studied herein will be defined and represented in the following manner:

$$I_x^{\eta,\rho}\{f(t)\} = x^{-\eta-\rho-1} \int_0^x t^{\eta}(x-t)^{\rho} S_V^U\left[y_1\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right] E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{t}{x}\right)^{\rho_0}; q,\sigma,\zeta\right)$$
$$* \Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}\left(y_2\left(\frac{t}{x}\right)^{\eta_2}\left(1-\frac{t}{x}\right)^{\rho_2}, s, a\right) f(t)dt, \tag{5.1.2}$$

where $f(t) \in \Lambda$,

$$\min\{\Re(\eta+\varsigma+1, \rho+1)\} > 0 \quad \text{and} \quad \min(\rho_0, \eta_1, \rho_1) > 0 \quad \text{and}$$

$S_V^U[z]$, $E_{\delta,\kappa}^{\vartheta;d}(z;q,\sigma,\zeta)$ and $\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}(z,s,a)$ occurring in (5.1.2) stand for $S_V^U$ polynomial, generalized extended Mittag-Leffler function and extended Hurwitz-Lerch Zeta function and defined by (1.1.44) in chapter 1, (2.1.9) chapter 2 and (3.1.4) chapter 3 respectively.

$$J_t^{\eta',\rho'}\{f(z)\} = t^{\eta'} \int_t^{\infty} z^{-\eta'-\rho'-1}(z-t)^{\rho'} S_{V'}^{U'}\left[y_1'\left(\frac{t}{z}\right)^{\eta_1'}\left(1-\frac{t}{z}\right)^{\rho_1'}\right] E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{z}\right)^{\rho_0'}; q',\sigma',\zeta'\right)$$
$$* \Phi_{\mu_j';\sigma_j' Q'}^{\lambda_j';\rho_j' P'}\left(y_2'\left(\frac{t}{z}\right)^{\eta_2'}\left(1-\frac{t}{z}\right)^{\rho_2'}, s', a'\right) f(z)dz, \tag{5.1.3}$$

provided that

$$\Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \min\{\Re\,(\eta' - w_1)\} > 0;$$

$$\Re(\rho' + 1) > 0,\ \min(\rho_0', \eta_1', \rho_1') \geq 0.$$

On suitably specializing the parameters involved in the functions $S_V^U[z]$, $E_{\delta,\kappa}^{\vartheta;d}\,(z;q,\sigma,\zeta)$ and $\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}\,(z,s,a)$ our fractional integral operators can be easily reduced to left- and right-sided generalized fractional integral operators involving the Gauss hypergeometric function and classical Riemann-Liouville left- and right-sided fractional integral operators.

# 2. MAIN RESULTS

**Result 1**

$$I_x^{\eta',\rho'}\,[J_t^{\eta,\rho}\{f(z)\}] = \frac{1}{x}\int_0^x F\left(\frac{z}{x}\right) f(z)dz + \int_x^\infty \frac{1}{z}F^*\left(\frac{x}{z}\right) f(z)dz, \tag{5.1.4}$$

where

$$F(t) = \frac{\Gamma(\zeta)\Gamma(\zeta')\Gamma(\eta+\eta'+\eta_1 R+\eta_1' R'+\eta_2 n+\eta_2' m+1)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{R=0}^{[V/U]}\sum_{R'=0}^{[V'/U']}$$

$$\frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!}\frac{\prod_{j=1}^{P}(\lambda_j)_{n\rho_j}\prod_{j=1}^{P'}(\lambda_j')_{m\rho_j'}}{n!\,m!\prod_{j=1}^{Q}(\mu_j)_{n\sigma_j}\prod_{j=1}^{Q'}(\mu_j')_{m\sigma_j'}}\frac{1}{(n+a)^s}\frac{1}{(m+a')^{s'}}y_1^R(y_1')^{R'}$$

$$(y_2)^n (y_2')^m t^{\eta+\eta_1 R+\eta_2 n}(1-t)^{\rho+\rho'+\rho_1 R+\rho_1' R'+n\rho_2+m\rho_2'+1}$$

$$\cdot H^{0,4:1,0;1,1;1,1;1,2;1,2}_{4,4:0,1;1,2;1,2;3,1;3,1}\left[\begin{array}{c|c} \begin{array}{c} -t \\ -w(1-t)^{\rho_0} \\ -w'(1-t)^{\rho_0'} \\ \frac{1}{q} \\ \frac{1}{q'} \end{array} & \begin{array}{c} A^*:\quad C^* \\ \\ \\ \\ B^*:\quad D^* \end{array} \end{array}\right], \tag{5.1.5}$$

where

$$A^* = (1-\eta-\eta'-\rho-\rho'-(\rho_1'+\eta_1')R'-(\rho_1+\eta_1)R+(\rho_2+\eta_2)n-(\rho_2'+\eta_2')m;1,\rho_0,\rho_0',0,0),$$
$$(-\rho'-\rho_1'R'-\rho_0'm;1,0,\rho_0',0,0),(1-\vartheta;0,1,0,1,0),(1-\vartheta';0,0,1,0,1)$$

$$B^* = (1-\eta-\eta'-\rho-\rho'-(\rho_1'+\eta_1')R'-(\rho_1+\eta_1)R+(\rho_2+\eta_2)n-(\eta_2'+\rho_2')m;0,\rho_0,\rho_0',1,0),$$
$$(-1-\eta-\eta'-\rho'-\eta_1 R-(\rho_1'+\eta_1')R'-\eta_2 n-\eta_2' m;1,0,\rho_0',0,0),(1-d;0,1,0,2,0),$$
$$(1-d';0,0,1,0,2)$$

$$C^* = -;(1-d,1);(1-d',1);(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1)$$

$$D^* = (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(\rho,1);(\rho',1),$$

and $F^*(t)$ can be obtained from F(t) by interchanging the parameters with dashes with those without dashes and following conditions are satisfied

$$\left.\begin{array}{r} \text{where}\quad f(t)\in\Lambda \\ \Re(\eta'+\eta+\varsigma)>-2,\Re(\rho+\rho'+\rho_0'+\rho_0)>-2 \\ \Re(w_2)>0\quad\text{or}\quad\Re(w_2)=0\quad\text{and}\quad\Re(\eta'-w_1)>0 \end{array}\right\}$$

**Result 2**

$$I_x^{\eta',\rho'}[I_t^{\eta,\rho}\{f(z)\}] = \frac{1}{x}\int_0^x G\left(\frac{z}{x}\right) f(z)dz, \tag{5.1.6}$$

where

$$\begin{aligned}G(X) =&\frac{\Gamma(\zeta)\Gamma(\zeta')}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{R=0}^{[V/U]}\sum_{R'=0}^{[V'/U']}\frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!}\\
&\cdot\frac{\prod_{j=1}^{P}(\lambda_j)_{n\rho_j}\prod_{j=1}^{P'}(\lambda'_j)_{m\rho'_j}}{n!\,m!\prod_{j=1}^{Q}(\mu_j)_{n\sigma_j}\prod_{j=1}^{Q'}(\mu'_j)_{m\sigma'_j}}\frac{1}{(n+a)^s}\frac{1}{(m+a')^{s'}}y_1^R(y'_1)^{R'}(y_2)^n(y'_2)^m X^{\eta'+\eta'_1R'+\eta'_2m}\\
&\cdot(1-X)^{\rho+\rho'+\rho_1R+\rho'_1R'+n\rho_2+m\rho'_2+1}H^{0,4:1,0;1,1;1,2;1,2;1,2}_{4,3:0,1;1,2;2,3;3,1;3,1}\left[\begin{array}{c|c} & A^{**}:\quad C^{**}\\ X & \\ w(1-X)^{\rho_0} & \\ w'(1-X)^{\rho'_0} & \\ \frac{1}{q} & \\ \frac{1}{q'} & \\ & B^{**}:\quad D^{**}\end{array}\right],\end{aligned} \tag{5.1.7}$$

where

$$\left.\begin{aligned}A^{**} &= (\eta-\eta'-\rho'-\rho'_1R'-\rho'_2m+\eta_2n+\eta_1R-\eta'_1R'-\eta'_2m;1,0,\rho'_0,0,0),\\
&(-\rho-\rho_1R-\rho_2n;1,\rho_0,0,0,0),(1-\vartheta;0,1,0,1,0),(1-\vartheta';0,0,1,0,1)\\
B^{**} &= (-2-\rho-\rho'-\rho_1R-\rho_2n-\rho'_1R'-\rho'_2m;1,\rho_0,\rho'_0,0,0),\\
&(1-d;0,1,0,2,0),(1-d';0,0,1,0,2)\\
C^{**} &= -;(1-d,1);(1-d',1),(-\rho-\rho'_1R-\rho'_2m,\rho'_0);\\
&(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1)\end{aligned}\right\}$$

$$\left.\begin{aligned} D^{**}=(0,1);(0,1),(1-\kappa,\delta);\\ (0,1),(1-\kappa',\delta'),(\eta-\eta'-\rho'-\rho_1'R'-\rho_2'm+\eta_2 n+\eta_1 R-\eta_1'R'-\eta_2'm;\rho_0');\\ (\sigma,1);(\sigma',1)\end{aligned}\right\}$$

and following conditions are satisfied

$$\left.\begin{aligned} \text{where}\quad f(t)\in\Lambda\\ \Re(\eta'+\eta+\varsigma)>-2,\Re(\rho+\rho'+\rho_0'+\rho_0)>-2\\ \min\{\eta_1,\eta_1',\rho_1,\rho_1'\}\geq 0\end{aligned}\right\}$$

**Result 3**

$$J_x^{\eta,\rho}[J_t^{\eta',\rho'}\{f(z)\}]=\int_x^{\infty}\frac{1}{z}G^*\left(\frac{x}{z}\right)f(z)dz, \tag{5.1.8}$$

where $G(x)$ is given by (5.2.2), $f(t)\in\Lambda$ exists and following conditions are satisfied

$$\left.\begin{aligned} \Re(w_2)>0\quad\text{or}\quad\Re(w_2)=0\quad\text{and}\quad\Re(\eta+\eta'-w_1)>0;\\ \Re(\rho+\rho'+\rho_0'+\rho_0)>-2,\ \min\{\eta_1,\eta_1',\rho_1,\rho_1'\}\geq 0\end{aligned}\right\}$$

**Proof of (5.1.4),(5.1.8) & (5.1.6):** In order to prove **Result 1**, we proceed by expressing I- and J-operators in the left hand side of (5.1.4) in their integral forms with the help of (5.1.2) and (5.1.3), we obtain

$$
\begin{aligned}
I_x^{\eta',\rho'}[J_t^{\eta,\rho}\{f(z)\}] &= x^{\eta-\rho-1}\int_0^x t^{\eta}(x-t)^{\rho} S_V^U\left[y\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right] E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{t}{x}\right)^{\rho_0};q,\sigma,\zeta\right)\\
&\cdot \Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}\left(y_2\left(\frac{t}{x}\right)^{\eta_2}\left(1-\frac{t}{x}\right)^{\rho_2},s,a\right) t^{\eta'}\int_t^{\infty} z^{-\eta'-\rho'-1}(z-t)^{\rho'} S_{V'}^{U'}\left[y'\left(\frac{t}{z}\right)^{\eta_1'}\left(1-\frac{t}{z}\right)^{\rho_1'}\right]\\
&\cdot E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{z}\right)^{\rho_0'};q',\sigma',\zeta'\right)\Phi_{\mu_j';\sigma_j' Q'}^{\lambda_j';\rho_j' P'}\left(y_2'\left(\frac{t}{z}\right)^{\eta_2'}\left(1-\frac{t}{z}\right)^{\rho_2'},s',a'\right) f(z)dzdt
\end{aligned}
\tag{5.1.9}
$$

Next, we interchange the order of t- and z- integrals (which is permissible under the conditions stated) and obtain the following after a little simplification:

$$
\begin{aligned}
I_x^{\eta',\rho'}[J_t^{\eta,\rho}\{f(z)\}] &= \int_0^x f(z)\int_0^z g(x,z,t)dtdz + \int_x^{\infty} f(z)\int_0^x g(x,z,t)dtdz\\
&= \int_0^x f(z) I_1 dz + \int_x^{\infty} f(z) I_2 dz \qquad \text{(say)}
\end{aligned}
\tag{5.1.10}
$$

where

$$
\begin{aligned}
g(x,z,t) &= x^{-\eta-\rho-1}t^{\eta+\eta'}(x-t)^{\rho}(z-t)^{\rho'}z^{-\eta'-\rho'-1} S_V^U\left[y\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right]\\
&\cdot E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{t}{x}\right)^{\rho_0};q,\sigma,\zeta\right)\Phi_{\mu_j;\sigma_j Q}^{\lambda_j;\rho_j P}\left(y_2\left(\frac{t}{x}\right)^{\eta_2}\left(1-\frac{t}{x}\right)^{\rho_2},s,a\right)\\
&\cdot S_{V'}^{U'}\left[y'\left(\frac{t}{z}\right)^{\eta_1'}\left(1-\frac{t}{z}\right)^{\rho_1'}\right] E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{z}\right)^{\rho_0'};q',\sigma',\zeta'\right)\\
&\cdot \Phi_{\mu_j';\sigma_j' Q'}^{\lambda_j';\rho_j' P'}\left(y_2'\left(\frac{t}{z}\right)^{\eta_2'}\left(1-\frac{t}{z}\right)^{\rho_2'},s',a'\right)
\end{aligned}
\tag{5.1.11}
$$

and

$$I_1 = \int_0^z g(x,z,t)dt \qquad \text{and} \qquad I_2 = \int_0^x g(x,z,t)dt.$$

To evaluate $I_1$, we first express both the generalised extended Mittag-Leffler functions in terms of their respective contour integral forms using (2.1.9). Next, express both the $S^V_U$ polynomials and extended Hurwitz-Lerch Zeta functions in terms of their respective series with the help of (1.1.44) and (1.1.63). Further, on interchanging the order of summations and contour integral, we get

$$I_1 = \left(\frac{1}{2\pi\omega}\right)^4 \frac{\Gamma(\zeta)\Gamma(\zeta')}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{R=0}^{[V/U]}\sum_{R'=0}^{[V'/U']}$$
$$\frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!} \frac{\prod_{j=1}^{P}(\lambda_j)_{n\rho_j}\prod_{j=1}^{P'}(\lambda'_j)_{m\rho'_j}}{n!\,m!\prod_{j=1}^{Q}(\mu_j)_{n\sigma_j}\prod_{j=1}^{Q'}(\mu'_j)_{n\sigma'_j}} \frac{1}{(n+a)^s}\frac{1}{(m+a')^{s'}} y_1^R (y'_1)^{R'}(y_2)^n(y'_2)^m$$
$$\int_{\mathfrak{L}_1}\cdots\int_{\mathfrak{L}_4}\int_0^z t^{\eta+\eta'+\eta_1 R+\eta'_1 R'+\eta_2 n+\eta'_2 m} x^{-\eta-\rho-(\eta_1+\rho_1)R-\rho_0\xi_2-(\eta_2+\rho_2)n-1}(x-t)^{\rho+\rho_1 R+\rho_0\xi_2+\rho_2 n}$$
$$(z-t)^{\rho'+\rho'_1 R'+\rho'_0\xi_4+\rho'_2 m} z^{-\eta'-\rho'-(\eta'_2+\rho'_2)m-(\eta'_1+\rho'_1)R'-\rho'_0\xi_4-1} \frac{\Gamma(-\xi_1)\Gamma(\xi_2)\Gamma(\sigma-\xi_2)}{\Gamma(\zeta-\xi_2)}$$
$$\frac{\Gamma(\vartheta+\xi_1+\xi_2)\Gamma(d-\vartheta+\xi_2)\Gamma(d+\xi_1)}{\Gamma(d+\xi_1+2\xi_2)\Gamma(\kappa+\delta\xi_1)} q^{-\xi_2}(-\omega)^{\xi_1} \frac{\Gamma(-\xi_3)\Gamma(\xi_4)\Gamma(\sigma'-\xi_4)}{\Gamma(\zeta'-\xi_4)}$$
$$\frac{\Gamma(\vartheta'+\xi_4+\xi_3)\Gamma(d'-\vartheta'+\xi_4)\Gamma(d'+\xi_3)}{\Gamma(d'+\xi_3+2\xi_4)\Gamma(\kappa'+\delta'\xi_3)}(q')^{-\xi_4}(-\omega')^{\xi_3} dt d\xi_1\cdots d\xi_4 \tag{5.1.12}$$

Now, we substitute $t = uz$ in (5.1.12) and evaluate the u-integral using the known result[53, p. 47, Eq.(16)]. Finally, re-interpreting the result in terms of the Multivariable H-function we obtain $I_1$.

In order to evaluate $I_2$, we proceed on similar lines as mentioned above substituting $t = ux$. On substituting the values of $I_1$ and $I_2$ in (5.1.10), we get the required result.

**Similarly, we can prove the results (5.1.6) and (5.1.8) with the help of the results given by [53, p.60, Eq.(5)], so we omit the details.**

## 5.2 APPLICATIONS

In this section we first give the application of **result 1**. Thus, on taking both $S_U^V$ polynomials and extended Hurwitz-Lerch Zeta functions equal to unity and taking $f(z) = (1-\gamma z)^l$ therein, we arrive at the following after a little simplification:

$$x^{-\eta-1}\int_0^x t^{\eta+l}\left(1-\frac{t}{x}\right)^{\rho}\left(1-\frac{1}{\gamma t}\right)^{\rho'+l+1} H^{0,4:1,1;1,1;1,0;1,2;1,2}_{4,3:1,2;3,1;0,1;3,1;3,1}\left[\begin{array}{c|c} -w\left(1-\frac{t}{x}\right)^{\rho_0} & I^*:\quad K^* \\ -w'\left(1-\frac{1}{\gamma t}\right)^{\rho_0'} & \\ \frac{-1}{\gamma t} & \\ \frac{1}{q} & \\ \frac{1}{q'} & J^*:\quad L^* \end{array}\right] dt$$

$$= (1-\gamma x)^{\rho+\rho'+l+2}\left\{ H^{0,6:1,1;1,1;1,1;1,0;1,2;1,2}_{6,6:1,1;1,2;1,2;0,1;3,1;3,1}\left[\begin{array}{c|c} -1 & P^*:\quad R^* \\ -w(1-\gamma x)^{\rho_0} & \\ -w'(1-\gamma x)^{\rho_0'} & \\ -\gamma x & \\ \frac{1}{q} & \\ \frac{1}{q'} & Q^*:\quad S^* \end{array}\right]\right.$$

$$+\left(\frac{-1}{\gamma x}\right)^{\rho'+\rho+2} H^{0,6:1,1;1,1;1,1;1,0;1,2;1,2}_{6,6:1,1;1,2;1,2;0,1;3,1;3,1}\left[\begin{array}{c|c} -1 & E^*:\quad G^* \\ -w\left(1-\frac{1}{\gamma x}\right)^{\rho_0} & \\ -w'\left(1-\frac{1}{\gamma x}\right)^{\rho_0'} & \\ -\frac{1}{\gamma x} & \\ \frac{1}{q} & \\ \frac{1}{q'} & F^*:\quad H^* \end{array}\right]\Bigg\}, \quad (5.2.1)$$

where

$$\left.\begin{array}{r} I^*=(1-\vartheta;1,0,0,1,0),(1-\vartheta';0,1,0,0,1), \\ (-\rho';0,\rho_0',1,0,0),(-\rho'-\eta';0,\rho_0',1,0,0) \\ J^*=(1-d;1,0,0,2,0),(1-d';0,1,2,0,0),(l-\eta'-\rho';0,\rho_0',1,0,0) \\ K^*=(1-d,1);(1-d',1);-; \\ (1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1) \\ L^*=(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta'),(1-\rho-\eta'-l,\rho_0');(0,1);(\sigma,1);(\sigma',1) \end{array}\right\}$$

$$\left.\begin{array}{r} P^*=(-1-\eta-\eta'-\rho-\rho';1,\rho_0,\rho_0',0,0,0),(-\rho';1,0,\rho_0',0,0,0), \\ (1-\vartheta;0,1,0,0,1,0),(1-\vartheta';0,0,1,0,0,1), \\ (1-\rho-\rho'-\eta-l;1,\rho_0,\rho_0',0,0,0),(-1-\rho-\rho';0,\rho_0,\rho_0',1,0,0) \\ Q^*=(-1-\eta-\eta'-\rho-\rho';0,\rho_0,\rho_0',0,0,0),(-1-\eta-\eta'-\rho';1,0,\rho_0',0,0,0), \\ (1-d;0,1,0,0,2,0),(1-d';0,0,1,0,0,2), \\ (-1-\rho-\rho'-\eta-l;1,\rho_0,\rho_0',0,0,0),(-1-\eta-\rho-\rho';1,\rho_0,\rho_0',0,0,0) \end{array}\right\}$$

$$\left.\begin{aligned} R^* &= (1-\eta,1);(1-d,1);(1-d',1);-;\\ &(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1)\\ S^* &= (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(0,1);(\sigma,1);(\sigma',1) \end{aligned}\right\}$$

$$\left.\begin{aligned} E^* &= (-1-\eta-\eta'-\rho-\rho';1,0,\rho_0,\rho_0',0,0),(-\rho;1,0,\rho_0,0,0,0),\\ &(1-\vartheta;0,0,1,0,0,1),(1-\vartheta';0,1,0,0,1,0),(-1-\rho-\rho'-\eta';1,0,\rho_0,\rho_0',0,0),\\ &(-1-\rho-\rho';0,1,\rho_0,\rho_0',0,0)\\ F^* &= (-1-\eta-\eta'-\rho-\rho';0,0,\rho_0,\rho_0',0,0),(-1-\eta-\eta'-\rho;1,0,\rho_0,0,0,0),\\ &(1-d;0,0,1,0,0,2),\\ &(1-d';0,0,0,1,2,0),(-1-\rho-\rho'-\eta';1,0,\rho_0,\rho_0',0,0),\\ &(-1-\eta'-\rho-\rho'-l;1,0,\rho_0,\rho_0',0,0)\\ G^* &= (1-\eta'-l,1);(1-d,1);(1-d',1);-;\\ &(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1)\\ H^* &= (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(0,1);(\sigma,1);(\sigma',1) \end{aligned}\right\}$$

provided the conditions obtainable from **result 1** are satisfied. Again, on taking $f(z)=e^z$ and considering $\Phi^{\lambda_j;\rho_j P}_{\mu_j;\sigma_j Q}(z,s,a)$ and $S^U_V[z]$ equal to unity in **result 2**, we arrive at the following result after a little simplification:

$$\frac{1}{x}\int_0^x \left(\frac{z}{x}\right)^n e^z \left(1-\frac{z}{x}\right)^{\rho'+\rho+1} H^{0,4:1,2;1,1;1,0;1,2;1,2}_{4,3:2,3;1,2;0,1;3,1;3,1}\left[\begin{array}{c|c} \begin{array}{c} w'(1-\frac{z}{x})^{\rho_0'}\\ w(1-\frac{z}{x})^{\rho_0}\\ -(1-\frac{z}{x})\\ \frac{1}{q}\\ \frac{1}{q'} \end{array} & \begin{array}{c} A^{**}:\quad C^{**}\\ \\ \\ \\ B^{**}:\quad D^{**} \end{array} \end{array}\right]$$

$$= H^{0,2:1,2;1,2;1,2;1,2;1,2}_{2,4:2,2;2,2;2,1;3,1;3,1}\left[\begin{array}{c|c} \begin{array}{c} w' \\ w \\ -x \\ \frac{1}{q} \\ \frac{1}{q'} \end{array} & \begin{array}{c} E^{**}: \quad G^{**} \\ \\ \\ \\ F^{**}: \quad H^{**} \end{array} \end{array}\right] \tag{5.2.2}$$

where

$$\left.\begin{array}{r} A^{**} = (1-\vartheta';1,0,0,1,0),(1-\vartheta;0,1,0,0,1),(-\rho;0,\rho_0,1,0,0),(\eta-\eta'-\rho';\rho_0',0,1,0,0) \\ B^{**} = (1-d';2,0,0,1,0),(1-d;0,2,0,0,1),(-1-\rho-\rho';\rho_0',\rho_0,1,0,0) \\ C^{**} = (1-d',1),(-\rho',\rho_0');(1-d,1);-;(1,1),(1-d'+\vartheta'),(\zeta',1); \\ (1,1),(1-d+\vartheta),(\zeta,1) \\ D^{**} = (0,1),(1-\kappa',\delta'),(\eta-\eta'-\rho',\rho_0');(0,1),(1-\kappa,\delta);(0,1);(\rho',1);(\rho,1) \end{array}\right\}$$

$$E^{**} = (1-\vartheta';1,0,0,1,0),(1-\vartheta;0,1,0,0,1)$$

$$F^{**} = (1-d';2,0,0,1,0),(1-d;0,2,0,0,1),(-1-\eta'-\rho';\rho_0',0,1,0,0),$$
$$(-1-\eta-\rho;0,\rho_0,1,0,0)$$

$$G^{**} = (1-d',1),(-\rho',\rho_0');(1-d,1),(-\rho,\rho_0);(-\eta',1),(-\eta,1);$$
$$(1,1),(1-d'+\vartheta'),(\zeta',1);(1,1),(1-d+\vartheta),(\zeta,1)$$

$$H^{**} = (0,1),(1-\kappa',\delta');(0,1),(1-\kappa,\delta);(0,1);(\rho',1);(\rho,1)$$

provided the conditions obtainable from **result 2** are satisfied.

Finally, in **result 2** if we take $f(z) = {}_2F_1\left(a,b;c;\left(\alpha-\frac{z}{x}\right)^m\right)$ and consider both extended Hurwitz-Lerch Zeta functions equal to unity we obtain the following after a little simplification:

$$x^{-\eta'-\rho'-1}\int_0^x\int_0^t t^{\eta'-\eta-\rho-1}(x-t)^{\rho'}(t-z)^{\rho}z^{\eta}E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{z}{t}\right)^{\rho_0};q,\sigma,\zeta\right)S_V^U\left[y\left(\frac{z}{t}\right)^{\eta_1}\left(1-\frac{z}{t}\right)^{\rho_1}\right]$$

$$\cdot E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{x}\right)^{\rho_0'};q',\sigma',\zeta'\right)S_{V'}^{U'}\left[y'\left(\frac{t}{x}\right)^{\eta_1'}\left(1-\frac{t}{x}\right)^{\rho_1'}\right]{}_2F_1\left(a,b;c;\left(\alpha-\frac{z}{x}\right)^m\right)dzdt$$

$$=\frac{\Gamma(\zeta)\Gamma(\zeta')\Gamma(\eta+\eta_1R+1)\Gamma(c)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')\Gamma(a)\Gamma(b)}\sum_{R=0}^{[V/U]}\sum_{R'=0}^{[V'/U']}\frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!}$$

$$y^R(y')^{R'}\left(1-\frac{1}{\alpha}\right)^{\rho+\rho'+\eta_1R+\eta_1'R'+2}H_{7,6:0,1;1,2;1,2;3,1;3,1;2,2;0,1}^{0,7:1,0;1,1;1,1;1,2;1,2;1,2;1,0}\left[\begin{array}{c|cc} -(1-x)\left(1-\frac{1}{\alpha}\right) & M^*: & O^* \\ -w(1-x)^{\rho_0}\left(1-\frac{1}{\alpha}\right)^{\rho_0} & & \\ -w'(1-x)^{\rho_0'}\left(1-\frac{1}{\alpha}\right)^{\rho_0'} & & \\ \frac{1}{q} & & \\ \frac{1}{q'} & & \\ -(\alpha-1)^m & & \\ \frac{1}{\alpha} & N^*: & P^* \end{array}\right] \tag{5.2.3}$$

where

$$\left.\begin{array}{r} M^*=(-\eta+\eta'-\rho-(\rho_1+\eta_1)R+\eta_1'R';1,\rho_0,0,0,0,0,0),(-\rho-\eta_1R;0,\rho_0,0,0,0,0,0),\\ (-\rho'-\eta_1'R';1,0,0,0,,\rho_0',0,0),(1-\vartheta;0,1,0,1,0,0,0),\\ (1-\vartheta';0,0,1,0,1,0,0),(-2-\rho-\rho'-\eta-2\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,m,1),\\ (-1-\rho-\rho'-\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,0,1)\\ N^*=(1-\rho-\rho'-\rho_0'R'-\rho_0R;1,\rho_0,\rho_0',0,0,0,0),\\ (-\eta-\eta'-\rho-\eta_1'R'-(\rho_1+\eta_1)R;0,\rho,0,0,0,0,0),\\ (1-d;0,2,0,1,0,0,0),(1-d';0,0,2,0,1,0,0),\\ (-1-\rho-\rho'-\eta-2\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,0,0),\\ (-2-\rho-\rho'-\eta-2\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,0,1) \end{array}\right\}$$

$$\left.\begin{aligned} O^* = -;(1-d,1);(1-d',1);(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1);\\ (1-a,1),(1-b,1),(0,1);-\\ P^* = (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(\sigma,1);(\sigma',1);(1-c,1);(0,1) \end{aligned}\right\}$$

provided the conditions obtainable from **result 2** are satisfied.

# 6

# SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING HILFER DERIVATIVE OPERATORS AND AN INTEGRAL OPERATOR INVOLVING $\overline{H}$-FUNCTION

The main findings of this chapter have been published as detailed below:

1. **N. JOLLY, P. HARJULE and R. JAIN** (2017). FRACTIONAL DIFFERENTIAL EQUATION ASSOCIATED WITH AN INTEGRAL OPERATOR WITH THE $\overline{H}$-FUNCTION IN THE KERNEL, *Global Journal of Pure and Applied Mathematics*, **13(7)**, 3505–3517.

2. **N. JOLLY, P. HARJULE and R. JAIN** (2018). ON THE SOLUTION OF GENERAL FAMILY OF FRACTIONAL DIFFERENTIAL EQUATION INVOLVING HILFER DERIVATIVE OPERATOR AND $\overline{H}$-FUNCTION, *Int. J. Math. And Appl.*, **6(1-A)**, 155–162.

3. **P. HARJULE, N. JOLLY and R. JAIN** (2017). A SOLUTION OF

## 6. SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING HILFER DERIVATIVE OPERATORS AND AN INTEGRAL OPERATOR INVOLVING $\overline{H}$-FUNCTION

FRACTIONAL DIFFERENTIAL EQUATION INVOLVING HILFER DERIVATIVE OPERATOR, *Journal of Indian Acad. Math.*, **39(2)**, 255–263.

In this chapter, first we define the $\overline{H}$-function, the Mittag-Leffler function, it's various generalizations and certain fractional integral operators that we will be using in our study. Next, we define Laplace transform and provide some necessary results required in finding solutions of fractional differential equations involving Hilfer derivative operator and an integral operator involving $\overline{H}$-function.
Further, we define a General Family of fractional differential equations. Then we consider generalized form of aforementioned equations and find it's solution in form of theorem 1. On account of general nature of theorem 1 we can obtain a number of special cases by taking special values of the parameters involved therein. We mention here two believed to be new and three known special cases. Next, we formulate theorem 2 by considering $\alpha_1 = \alpha_2 = \alpha$ and $c = 0$ in theorem 1. Furthermore, we obtain two unknown and three known special cases by specializing parameters of $\overline{H}$-function and various other parameters involved therein. By giving numerical values to the parameters in the functions and the operators involved in theorem 2 we have plotted some graphs with the help of MATLAB SOFTWARE.
Next, we obtain solution of another fractional differential equation and present it in form of theorem 3. Finally, by specializing the parameters occuring therein we can obtain a number of special cases of our theorem. However, here we mention only two known and two unknown special cases.

# 6.1 INTRODUCTION

## 6.1.1 THE $\overline{H}$-FUNCTION

The $\overline{H}$-function was introduced by Inayat Hussain [23] and studied by Bushman and Srivastava [3]. It is defined and represented in the following manner:

$$\overline{H}^{m,n}_{p,q}\left[z \left| \begin{array}{ll} (e_j, E_j; \in_j)_{1,n}, & (e_j, E_j)_{n+1,p} \\ (f_j, F_j)_{1,m}, & (f_j, F_j; \Im_j)_{m+1,q} \end{array} \right.\right] = \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \overline{\Theta}(\xi) z^{\xi} d\xi, \tag{6.1.1}$$

where, $\omega = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi) = \frac{\prod\limits_{j=1}^{m} \Gamma(f_j - F_j\xi) \prod\limits_{j=1}^{n} \{\Gamma(1 - e_j + E_j\xi)\}^{\in_j}}{\prod\limits_{j=m+1}^{q} \{\Gamma(1 - f_j + F_j\xi)\}^{\Im_j} \prod\limits_{j=n+1}^{p} \Gamma(e_j - E_j\xi)}, \tag{6.1.2}$$

and

$$1 \leqq m \leqq q \qquad \text{and} \qquad 0 \leqq n \leqq p \qquad (m, q \in \mathbb{N} = \{1, 2, 3, \cdots\};\ n, p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{6.1.3}$$

The nature of contour $\mathfrak{L}$ in (6.1.1) and various conditions on its parameters can be seen in the paper by Gupta, Jain and Agarwal [17].

## 6.1.2 THE MITTAG-LEFFLER FUNCTION AND IT'S GENERALIZATIONS

The familiar Mittag-Leffler function $E_\delta(z)$ [43] is defined by the following series:

$$E_\delta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)} =: E_{\delta,1}(Z) \qquad (\delta \in \mathbb{C};\ \Re(\delta) > 0). \tag{6.1.4}$$

It's first generalization $E_{\delta,\kappa}(z)$ was introduced by Wiman [81] , defined and represented in the following manner:

$$E_{\delta,\kappa}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n+\kappa)} \qquad (\delta,\kappa \in \mathbb{C};\ \Re(\delta)>0,\ \Re(\kappa)>0). \tag{6.1.5}$$

A further generalization of $E_{\delta,\kappa}(z)$ was given by Prabhakar [51] as follows:

$$E_{\delta,\kappa}^{\vartheta}(z) := \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\delta n+\kappa)} \frac{z^n}{n!} \qquad (\kappa,\delta,\vartheta \in \mathbb{C};\ \Re(\delta)>0, \Re(\kappa)>0, \Re(\vartheta)>0). \tag{6.1.6}$$

### 6.1.3 THE LAPLACE TRANSFORM

The Laplace transform $\mathcal{L}[f(x)](s)$ of the function $f(x)$ is defined as follows:

$$\mathcal{L}[f(x)](s) = \int_0^{\infty} e^{-sx}\, f(x)dx \qquad \big(\Re(s)>0\big), \tag{6.1.7}$$

provided that the integral exists.

The following Laplace transform formula for the generalized Mittag-Leffler function $E_{\delta,\kappa}^{\vartheta}(z)$ was given by Prabhakar [51]:

$$\mathcal{L}[x^{\kappa-1}E_{\delta,\kappa}^{\vartheta}(\omega x^{\delta})](s) = \frac{s^{\vartheta\delta-\kappa}}{(s^{\delta}-\omega)^{\vartheta}} \tag{6.1.8}$$

$$(\delta,\omega,\vartheta \in \mathbb{C}; \Re(\kappa)>0; \Re(s)>0; |\tfrac{\omega}{s^{\delta}}|<1).$$

### 6.1.4 FRACTIONAL INTEGRAL OPERATORS

The Riemann-Liouville fractional integral and derivative operator $I_{a+}^{\mu}$ and $D_{a+}^{\mu}$, which are defined by (see, for details, [29], [42] and [55])

$$(I_{a+}^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt \qquad \big(\Re(\mu) > 0\big) \tag{6.1.9}$$

and

$$(D_{a+}^{\mu} f)(x) = \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\mu} f)(x) \qquad \big(\Re(\mu) > 0;\ n = [\Re(\mu)] + 1\big), \tag{6.1.10}$$

($[x]$ denotes the greatest integer in the real number $x$)

will be required during the course of our study in this chapter.

Hilfer [20] generalized the operator in (6.1.10) and defined a general fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \leqq \nu \leqq 1$ with respect to $x$ as follows:

$$(D_{a+}^{\mu,\nu} f)(x) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dx} \left(I_{a+}^{(1-\nu)(1-\mu)} f\right)\right)(x). \tag{6.1.11}$$

Eq.(6.1.11) yields the classical Riemann-Liouville fractional derivative operator $D_{a+}^{\mu}$ when $\nu = 0$ and for $\nu = 1$ it reduces to the fractional derivative operator introduced by Joseph Liouville (1809-1882) in 1832, which is called the Liouville-Caputo fractional derivative operator (see [13], [29] and [79]).

In our present investigation we make use of an integral operator with $\overline{H}$-function in its kernel defined as follows:

$$\left(\overline{\mathcal{H}}_{a+;p,q;\beta}^{w;m,n;\gamma}\, \varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\, \overline{H}_{p,q}^{m,n}[w(x-t)^{\gamma}]\varphi(t)dt \tag{6.1.12}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C}\backslash\{0\};\ 1 \leqq m \leqq q;\ 0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\right).$$

If we take $w = 1, m = 1$ and $a = 0$ in (6.1.12), we obtain an integral operator introduced by Harjule(see for details [19, p.80, Eq.(5.1.10)]). Next, if we reduce

$\overline{H}$-function to the polylogarithm function of order $\eta$ [8, p.30] in (6.1.12), we obtain the following

$$\left(\mathcal{F}^{w;1,1;\gamma}_{a+;1,2;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\; F[w(x-t)^{\gamma},\eta]\varphi(t)dt \tag{6.1.13}$$

$$\bigg(\Re(\beta) > 0;\; w \in \mathbb{C}\setminus\{0\}\bigg),$$

provided that the integral exists.

Further, if we reduce $\overline{H}$-function to the generalized Wright hypergeometric function [18, p.271, Eq.(7)] in (6.1.12), we get

$$\left(\overline{\psi}^{w;p;\gamma}_{a+;q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\;{}_p\overline{\psi}_q\left[\begin{array}{c}(e_j,E_j;\in_j)_{1,p}\\ \\ (f_j,F_j;\Im_j)_{1,q}\end{array};w(x-t)^{\gamma}\right]\varphi(t)dt \tag{6.1.14}$$

$$\bigg(\Re(\beta) > 0;\; w \in \mathbb{C}\setminus\{0\};\; p \leqq q+1\bigg),$$

provided that the integral exists then the Theorem can be specialized to the following form.

Next, if we reduce $\overline{H}$-function to the generalized Riemann zeta function [17, p.27, section 1.11, Eq.(1)] in (6.1.12), we obtain

$$\left(\phi^{w;1,2;\gamma}_{0+;2,2;\beta}\,\varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\; \phi(w(x-t)^{\gamma},\varrho,\eta)\varphi(t)dt \tag{6.1.15}$$

$$\bigg(\Re(\beta) > 0;\; w \in \mathbb{C}\setminus\{0\}\bigg),$$

provided that the integral exist.

Again, if we reduce $\overline{H}$-function to the generalized Wright Bessel function [18, p.271, Eq.(8)] in (6.1.12), we obtain

$$\left(\overline{J}^{w;1,0;\gamma}_{0+;0,2;\beta}\,\varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\,\overline{J}^{\zeta,\epsilon}_{\vartheta}(w(x-t)^{\gamma})\varphi(t)dt \tag{6.1.16}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\}\right),$$

provided that the integral exist.

### 6.1.5 REQUIRED RESULTS

First, we recall the following known result (see, for details, [78] and [79]):

$$\mathcal{L}[(D^{\mu,\nu}_{0+}f)(x)](s) = s^{\mu}\mathcal{L}[f(x)](s) - s^{-\nu(1-\mu)}\left(I^{(1-\nu)(1-\mu)}_{0+}f\right)(0+) \qquad \left(\Re(s) > 0;\ 0 < \mu < 1\right), \tag{6.1.17}$$

where the initial-value term:

$$\left(I^{(1-\nu)(1-\mu)}_{0+}f\right)(0+),$$

involves the Riemann-Liouville fractional integral (6.1.9) (*with* $a = 0$) of the function $f(t)$ of order

$$\mu \mapsto (1-\nu)(1-\mu), \tag{6.1.18}$$

evaluated in the limit as $x \to 0+$.

Next, for $a = 0$, by using the *Convolution Theorem* for the Laplace Transform in (6.1.7), we find from the definition (6.1.12) that

$$
\begin{aligned}
&\mathcal{L}\left[\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\,\varphi\right)(x)\right](s) \\
&= \mathcal{L}\left[x^{\beta-1}\,\overline{H}_{p,q}^{m,n}[wx^{\gamma}]\right](s)\cdot\mathcal{L}[\varphi(x)](s) \\
&= s^{-\beta}\overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\left|\begin{array}{ll}(1-\beta,\gamma;1), & (e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p} \\ & (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{array}\right.\right]\Phi(s)
\end{aligned} \tag{6.1.19}
$$

$$
\left(\Re(s)>0;\ \gamma>0;\ \Re(\beta)+\min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\}>0\right),
$$

where,

$$
\Phi(s):=\mathcal{L}[\varphi(x)](s)\qquad\big(\Re(s)>0\big).
$$

In its special case when $\varphi(x)\equiv 1$, (6.1.19) immediately yields

$$
\begin{aligned}
&\mathcal{L}\left[\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\,1\right)(x)\right](s) \\
&= s^{-\beta-1}\overline{H}_{p+1,q}^{m,n+1}\left[ws^{-\gamma}\left|\begin{array}{ll}(1-\beta,\gamma;1), & (e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p} \\ & (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{array}\right.\right]
\end{aligned} \tag{6.1.20}
$$

$$
\left(\Re(s)>0;\ \gamma>0;\ \Re(\beta)+\min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\}>0\right).
$$

Further, we state the following formulae [79] that will be used in the proof of theorem 1:

$$\frac{s^{\beta_i(\alpha_i-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c}=\frac{1}{b}\left(\frac{s^{\beta_i(\alpha_i-1)}}{s^{\alpha_2}+\frac{c}{b}}\right)\left(\frac{1}{1+\frac{a}{b}(\frac{s^{\alpha_1}}{s^{\alpha_2}+\frac{c}{b}})}\right)=\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\frac{s^{\alpha_1 r+\beta_i\alpha_i-\beta_i}}{(s^{\alpha_2}+\frac{c}{b})^{r+1}}$$
$$=\mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_i(1-\alpha_i)-1}$$
$$\cdot E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_i(1-\alpha_i)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\Bigg](s)\qquad(i=1,2)\tag{6.1.21}$$

and

$$\frac{F(s)}{as^{\alpha_1}+bs^{\alpha_2}+c}=\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\left(\frac{s^{\alpha_1 r}}{(s^{\alpha_2}+\frac{c}{b})^{r+1}}F(s)\right)=\mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r$$
$$\cdot\left(x^{(\alpha_2-\alpha_1)r+\alpha_2-1}E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2}\left(-\frac{c}{b}x^{\alpha_2}\right)*f(x)\right)\Bigg](s)$$
$$=\mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\left(E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f\right)(x)\Bigg](s).\tag{6.1.22}$$

Now, we shall evaluate an important result which will be required in obtaining our solution of fractional differential equation given by theorem 1. This result is believed to be new and of importance by itself.

$$\frac{\lambda}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{-\beta-1}\,\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\,\middle|\,\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right]$$
$$=\mathcal{L}\Bigg[\frac{\lambda}{b}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\left(-\frac{a}{b}\right)^r(r+1)_j\,x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\frac{1}{j!}\left(-\frac{c}{b}\right)^j$$
$$\cdot\,\overline{H}^{m,n+1}_{p+1,q+1}\left[wx^{\gamma}\,\middle|\,\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta,\gamma;1)\end{matrix}\right]\Bigg]\tag{6.1.23}$$

provided that the conditions stated in (6.1.1) and (6.1.7) are satisfied.

*Proof.* We first express $\overline{H}$-function in the form of contour integral and then interchange the order of summation and integration(which is permissible under the conditions stated) in the left hand side of (6.1.23)

$$= \frac{\lambda}{2\pi i b} \int_{\mathfrak{L}} \sum_{r=0}^{\infty} \left( -\frac{a}{b} \right)^r \frac{s^{\alpha_1 r - \gamma\xi - \beta - 1}}{(s^{\alpha_2} + \frac{c}{b})^{r+1}} \; w^{\xi} \; \Gamma(\beta + \gamma\xi) \; \overline{\Theta}(\xi) \, d\xi \tag{6.1.24}$$

using (6.1.21) we get

$$= \frac{\lambda}{2\pi i b} \int_{\mathfrak{L}} \mathcal{L}\left[ \sum_{r=0}^{\infty} \left( -\frac{a}{b} \right)^r \left( x^{(\alpha_2 - \alpha_1) r + \alpha_2 + \gamma\xi + \beta} \, E^{r+1}_{\alpha_2, (\alpha_2 - \alpha_1) r + \alpha_2 + \gamma\xi + \beta + 1} \left( -\frac{c}{b} \, x^{\alpha_2} \right) \right] (s) \tag{6.1.25}$$

$$\cdot \, \Gamma(\beta + \gamma\xi) \overline{\Theta}(\xi) \, d\xi$$

Further, we express generalization of the Mittag-Leffler function in the series form and reinterpret $\overline{H}$-function in order to obtain the right hand side of (6.1.23). $\square$

## 6.2 A GENERAL FAMILY OF FRACTIONAL DIFFERENTIAL EQUATIONS

The following family of fractional differential equations [79, p.803, Eq.(3.7)] was introduced and studied by several authors [18, 22] on account of their importance in dielectric relaxation in glasses.

$$a \left( D_{0+}^{\alpha_1, \beta_1} \, y \right) (x) + b \left( D_{0+}^{\alpha_2, \beta_2} \, y \right) (x) + c y(x) = g(x), \tag{6.2.1}$$

where

$$\left( 0 < \alpha_1 \leqq \alpha_2 < 1; \; 0 \leqq \beta_1, \beta_2 \leqq 1 \, \text{and} \, a, b, c \in \mathbb{R} \right),$$

in the space of Lebesgue integrable functions ( see [13, 78]) $y \in L(0,\infty)$ with the initial conditions:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}\, y\right)(0+) = C_i \qquad (i=1,2), \tag{6.2.2}$$

where, without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leqq (1-\beta_2)(1-\alpha_2).$$

If $C_1 < \infty$, then $C_2 = 0$ unless $(1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2)$.

In the present chapter, we shall study the following generalized form of fractional differential equation given by (6.2.1) by establishing the following theorem.

# THEOREM 1

*The following fractional differential equation*:

$$a\left(D_{0+}^{\alpha_1,\beta_1}\, y\right)(x) + b\left(D_{0+}^{\alpha_2,\beta_2}\, y\right)(x) + cy(x) = \lambda\left(\overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma}\, 1\right)(x) + f(x) \tag{6.2.3}$$

$$\Bigg(0 < \alpha_1 \leqq \alpha_2 < 1; 0 \leqq \beta_1 \leqq 1;\ 0 \leqq \beta_2 \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\};\ 1 \leqq m \leqq q;$$

$$0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\Bigg),$$

*with the initial condition*:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}\, y\right)(0+) = C_i \qquad (i=1,2), \tag{6.2.4}$$

*has its solution in the space $L(0,\infty)$ given by*

$$\begin{aligned}
y(x) =& \frac{1}{b}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a}{b}\right)^r\Bigg[aC_1x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}\, E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\
&+ b\,C_2x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}\, E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right) + E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f(x)\\
&+\lambda\sum_{j=0}^{\infty}(r+1)_j\; x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\,\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\
&\cdot\overline{H}^{m,n+1}_{p+1,q+1}\left[wx^{\gamma}\,\middle|\,\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\(f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta,\gamma;1)\end{matrix}\right]\Bigg],
\end{aligned} \tag{6.2.5}$$

*where $C_1, C_2$ and $\lambda$ are arbitrary constants and the function $f$ is suitably prescribed.*

*Proof.* We denote by $Y(s)$ the Laplace transform of the function $y(x)$, which is given as in (6.1.7). Then, by applying the Laplace transform operator $\mathcal{L}$ to each side of (6.2.3), and using the formulas (6.1.17) and (6.1.20) and the initial condition (6.2.4), we find that

$$\begin{aligned}
&a(s^{\alpha_1}Y(s)-C_1\,s^{\beta_1(\alpha_1-1)})+b(s^{\alpha_2}Y(s)-C_2\,s^{\beta_2(\alpha_2-1)})+c\,Y(s)\\
&\qquad= F(s)+\lambda s^{-\beta-1}\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\,\middle|\,\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\(f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right]
\end{aligned} \tag{6.2.6}$$

which readily yields

$$Y(s) =\frac{a\,C_1}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{\beta_1(\alpha_1-1)}+\frac{b\,C_2}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{\beta_2(\alpha_2-1)}+\frac{F(s)}{(as^{\alpha_1}+bs^{\alpha_2}+c)}$$

$$+\frac{\lambda}{(as^{\alpha_1}+bs^{\alpha_2}+c)}s^{-\beta-1}\,\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q}\end{matrix}\right.\right]. \tag{6.2.7}$$

Using (6.1.21), (6.1.22) and (6.1.23) we obtain

$$\begin{aligned}Y(s)=&\mathcal{L}\Bigg[\frac{1}{b}\sum_{r=0}^{\infty}\left(-\frac{a}{b}\right)^r\Bigg[aC_1\,x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\ &+bC_2\,x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\ &+\left(\mathbf{E}^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f\right)(x)\\ &+\lambda\sum_{j=0}^{\infty}(r+1)_j\,x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\ \cdot\overline{H}^{m,n+1}_{p+1,q+1}&\left[wx^{\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ \\ (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta,\gamma;1)\end{matrix}\right.\right]\Bigg]\Bigg](s).\end{aligned} \tag{6.2.8}$$

Finally, by applying the inverse of Laplace transform, we get the solution (6.2.5) asserted by the main theorem. □

**Corollary 1.** *If we reduce $\overline{H}$-function occurring in the integral operator on the right hand side of (6.2.3) to the function associated with Gaussian Model free energy([1, p.4126, 4127, Eq.(23),(28)] and [23, p.98, Eq.(1.4)]), we observe that the following fractional differential equation*:

$$a\left(D^{\alpha_1,\beta_1}_{0+}\,y\right)(x)+b\left(D^{\alpha_2,\beta_2}_{0+}\,y\right)(x)=\lambda\left(\mathfrak{F}^{w;1,1;\gamma}_{0+;1,2;\beta}\,1\right)(x)+f(x) \tag{6.2.9}$$

$$\left(0<\alpha_1\leqq\alpha_2<1;0\leqq\beta_1\leqq1;\;0\leqq\beta_2\leqq1;\;\Re(\beta)>0;\;w\in\mathbb{C}\setminus\{0\}\right)$$

*with the initial condition (6.2.4) has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned}y(x) =&\frac{1}{b}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a}{b}\right)^r\Bigg[aC_1x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\&+b\,C_2x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\&+E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f(x)-\frac{\lambda}{4\pi^{\frac{d}{2}}}\sum_{j=0}^{\infty}(r+1)_j\;x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta}\,\frac{1}{j!}\left(-\frac{c}{b}\right)^j\\&\cdot\overline{H}^{1,3}_{3,3}\left[-x^{\gamma}\left|\begin{matrix}(1-\beta-\gamma,\gamma;1),(0,1;2),(-\frac{1}{2},1;d)\\ \\(0,1),(-1,1;1+d),(-\alpha_2 j-(\alpha_2-\alpha_1)r-\alpha_2-\beta-\gamma,\gamma;1)\end{matrix}\right.\right]\Bigg],\end{aligned}\tag{6.2.10}$$

*where* $C_1, C_2$ *and* $\lambda$ *are arbitrary constants and the function* $f$ *is suitably prescribed.*

**Corollary 2.** *If we reduce* $\overline{H}$*-function to the Polylogarithm function of order* $p$ *[8, p.30] in the integral operator on the right-hand side of (6.2.3), we obtain the following fractional differential equation:*

$$a\left(D^{\alpha_1,\beta_1}_{0+}\,y\right)(x)+b\left(D^{\alpha_2,\beta_2}_{0+}\,y\right)(x)=\lambda\left(\mathbb{F}^{w;1,1;\gamma}_{0+;1,2;\beta}\,1\right)(x)+f(x)\tag{6.2.11}$$

$$\left(0<\alpha_1\leqq\alpha_2<1;\;0\leqq\beta_1\leqq1;\;0\leqq\beta_2\leqq1;\;\Re(\beta)>0;\;w\in\mathbb{C}\setminus\{0\};\;p\leqq q+1\right),$$

*with the initial condition (6.2.4) has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned}y(x) =&\frac{1}{b}\sum_{r=0}^{\infty}(-1)^r\left(\frac{a}{b}\right)^r\Bigg[aC_1x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)\\&+b\,C_2x^{(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)-1}\,E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}\,x^{\alpha_2}\right)+E^{r+1}_{\alpha_2,(\alpha_2-\alpha_1)r+\alpha_2,-\frac{c}{b};0+}f(x)\end{aligned}$$

$$- \lambda \sum_{j=0}^{\infty} (r+1)_j \ x^{(\alpha_2-\alpha_1)r+\alpha_2(j+1)+\beta} \, \frac{1}{j!} \left( -\frac{c}{b} \right)^j$$
$$\cdot \overline{H}_{2,3}^{1,2} \left[ -wx^{\gamma} \left| \begin{array}{c} (1-\beta,\gamma;1),(1,1;p+1) \\ \\ (1,1),(0,1;p),(-\alpha_2 j - (\alpha_2-\alpha_1)r - \beta, \gamma; 1) \end{array} \right. \right] \Bigg], \qquad (6.2.12)$$

*where $C_1, C_2$ and $\lambda$ are arbitrary constants and the function $f$ is suitably prescribed.*

## KNOWN SPECIAL CASES OF THEOREM 1

If we consider $\lambda = 0$ in the right hand side of (6.2.3), we get the result obtained by Tomovski et al. [79, p.803, theorem 5]. Again, if we take a=1, b=c=0 and reduce $\overline{H}$-function to the Mittag-Leffler function(see[68] and [78]) in the integral operator on the right hand side of (6.2.3), we get the result obtained by Srivastava and Tomovski [78, p.207,Theorem 8]. Further, if we take a=1, b=c=0 and reduce $\overline{H}$-function to the H-function [19, p.10, Eq.(1.1.42)], we get the result obtained by Srivastava et al.[69, p.115, Theorem 2].

**If in theorem 1, we take $\alpha_1 = \alpha_2 = \alpha$ and $c = 0$, we get the following result which is of interest by itself and of great practical importance. We shall denote it by a theorem and not a corollary.**

# THEOREM 2

*The following fractional differential equation*:

$$a \left( D_{0+}^{\alpha,\beta_1} \, y \right)(x) + b \left( D_{0+}^{\alpha,\beta_2} \, y \right)(x) = \lambda \left( \overline{\mathcal{H}}_{0+;p,q;\beta}^{w;m,n;\gamma} \, 1 \right)(x) + f(x) \qquad (6.2.13)$$

$$\Bigg(0 < \alpha < 1; 0 \leqq \beta_1, \beta_2 \leqq 1; \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ 1 \leqq m \leqq q;$$

$$0 \leqq n \leqq p;\ \Re(\beta) + \min_{1 \leqq j \leqq m} \left\{ \Re\left(\frac{\gamma f_j}{F_j}\right) \right\} > 0 \Bigg),$$

*with the initial condition*:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha)}\, y\right)(0+) = C_i \qquad (i = 1, 2), \tag{6.2.14}$$

*has its solution in the space* $L(0, \infty)$ *given by*

$$\begin{aligned} y(x) =& \frac{a\, C_1}{(a+b)} \frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)} + \frac{b\, C_2}{(a+b)} \frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)} \\ &+ \frac{\lambda}{(a+b)} x^{\alpha+\beta}\, \overline{H}_{p+1,q+1}^{m,n+1} \left[ wx^{\gamma} \left| \begin{matrix} A^* \\ \\ B^* \end{matrix} \right. \right] + \frac{1}{(a+b)\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, f(t) dt, \end{aligned} \tag{6.2.15}$$

*where* $A^* = (1-\beta, \gamma; 1), (e_j, E_j; \in_j)_{1,n}, (e_j, E_j)_{n+1,p}$,
$B^* = (f_j, F_j)_{1,m}, (f_j, F_j; \Im_j)_{m+1,q}, (-\beta-\alpha, \gamma; 1)$ *and* $C_1, C_2$, $\lambda$ *are arbitrary constants and the function* $f$ *is suitably prescribed.*

**Corollary 1.** *If we reduce* $\overline{H}-$*function occuring in the right hand side of (6.2.13) to the polylogarithm function of order* $\eta$ *[8, p.30], we obtain the following fractional differential equation*:

$$a\left(D_{0+}^{\alpha,\beta_1}\, y\right)(x) + b\left(D_{0+}^{\alpha,\beta_2}\, y\right)(x) = \lambda\left(\mathfrak{F}_{0+;1,2;\beta}^{w;1,1;\gamma}\, 1\right)(x) + f(x) \tag{6.2.16}$$

$$\left(0 < \alpha < 1; 0 \leqq \beta_1; \beta_2 \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\}\right),$$

*with the initial condition (6.2.14) has its solution in the space $L(0,\infty)$ given by*

$$\begin{aligned}y(x) =&\frac{a\,C_1}{(a+b)}\ \frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)}+\frac{b\,C_2}{(a+b)}\ \frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)}\\ &-\frac{\lambda}{(a+b)}x^{\alpha+\beta}\overline{H}^{1,2}_{2,3}\left[-wx^{\gamma}\left|\begin{matrix}(1-\beta,\gamma;1),(1,1;\eta+1)\\ \\ (1,1),(0,1;\eta),(-\beta-\alpha,\gamma;1)\end{matrix}\right.\right]\\ &+\frac{1}{(a+b)\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\ f(t)dt,\end{aligned}\tag{6.2.17}$$

*where $C_1, C_2$ and $\lambda$ are arbitrary constants and the function $f$ is suitably prescribed.*

**Corollary 2.** *If we reduce $\overline{H}-$function occuring in the right hand side of (6.2.13) to the generalised Wright hypergeometric function [18, p.271, Eq.(7)], we obtain the following fractional differential equation:*

$$a\left(D^{\alpha,\beta_1}_{0+}\ y\right)(x)+b\left(D^{\alpha,\beta_2}_{0+}\ y\right)(x)=\lambda\left(\overline{\psi}^{w;p;\gamma}_{0+;q;\beta}\ 1\right)(x)+f(x)\tag{6.2.18}$$

$$\left(0<\alpha<1;0\leqq\beta_1;\beta_2\leqq 1;\ \Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\};\ p\leqq q+1\right),$$

*with the initial condition (6.2.14) has its solution in the space $L(0,\infty)$ given by*

$$\begin{aligned}y(x) =&\frac{a\,C_1}{(a+b)}\ \frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_1(1-\alpha)\big)}+\frac{b\,C_2}{(a+b)}\ \frac{x^{\alpha+\beta_2(1-\alpha)-1}}{\Gamma\big(\alpha+\beta_2(1-\alpha)\big)}\\ &+\frac{\lambda}{(a+b)}x^{\alpha+\beta}\ {}_{p+1}\overline{\psi}_{q+1}\left[\begin{matrix}(1-e_j,E_j;\in_j)_{1,p},(\beta,\gamma;1)\\ \\ (1-f_j,F_j;\Im_j)_{1,q},(1+\beta+\alpha,\gamma;1)\end{matrix};\omega x^{\gamma}\right]\\ &+\frac{1}{(a+b)\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\ f(t)dt,\end{aligned}\tag{6.2.19}$$

*where $C_1, C_2$ and $\lambda$ are arbitrary constants and the function $f$ is suitably prescribed.*

## KNOWN SPECIAL CASES OF THEOREM 2

If we consider $a = 1, b = 0$ and reduce $\overline{H}$-function to the Mittag-Leffler function (see [68] and [78]) in the integral operator on the right-hand side of (6.2.13), we get the result obtained by Srivastava and Tomovski [78, p.207, Theorem 8]. Again, if we take $a = 1, b = 0$ and reduce $\overline{H}$-function to the Fox's H-function [19, p.10, Eq.(1.1.42)] in the integral operator on the right-hand side of (6.2.13), we get a result obtained by Srivastava.et.al [69, p.115, Theorem 2].

## NUMERICAL EXAMPLES AND GRAPHICAL REPRESENTATIONS

First of all, by letting $\omega \to 0$ in Corollary 2, the generalized Wright hypergeometric function occuring in (6.2.19) reduces to 1. Further, in order to obtain numerical examples from Corollary 2 we consider $f(x) = x^{\rho}$ where $\Re(\rho) > -1$.

**Example (a).** If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.5$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (6.2.19), we easily arrive at the following result

$$y_{0.5}(x) = \frac{C_1}{2\Gamma(0.625)x^{0.375}} + \frac{C_2}{2\Gamma(0.75)x^{0.25}} + \frac{\lambda x}{2} + \frac{\Gamma(2)}{2\Gamma(2.5)}x^{1.5}. \tag{6.2.20}$$

**Example (b).** If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.6$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (6.2.19), we easily arrive at the following result

$$y_{0.6}(x) = \frac{C_1}{2\Gamma(0.7)x^{0.3}} + \frac{C_2}{2\Gamma(0.8)x^{0.2}} + \frac{\lambda x^{1.1}}{2} + \frac{\Gamma(2)}{2\Gamma(2.6)}x^{1.6}. \tag{6.2.21}$$

**Example (c).** If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.7$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and

$\rho = 1$ in equation (6.2.19), we easily arrive at the following result

$$y_{0.7}(x) = \frac{C_1}{2\Gamma(0.775)x^{0.225}} + \frac{C_2}{2\Gamma(0.85)x^{0.15}} + \frac{\lambda x^{1.2}}{2} + \frac{\Gamma(2)}{2\Gamma(2.7)}x^{1.7}. \tag{6.2.22}$$

**Example (d).** If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 0.9$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (6.2.19), we easily arrive at the following result

$$y_{0.9}(x) = \frac{C_1}{2\Gamma(0.925)x^{0.075}} + \frac{C_2}{2\Gamma(0.95)x^{0.05}} + \frac{\lambda x^{1.4}}{2} + \frac{\Gamma(2)}{2\Gamma(2.9)}x^{1.9}. \tag{6.2.23}$$

**Example (e).** If we take $\beta = 0.5$, $a = 1$, $b = 1$, $\alpha = 1$, $\beta_1 = 0.25$, $\beta_2 = 0.5$ and $\rho = 1$ in equation (6.2.19), we easily arrive at the following result

$$y_1(x) = \frac{1}{2}(C_1 + C_2) + \frac{\lambda x^{1.5}}{2} + \frac{\Gamma(2)}{2\Gamma(3)}x^2. \tag{6.2.24}$$

**Example (f).** If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.5$ and $\rho = 0$ in equation (6.2.19), we easily arrive at the following result

$$y^*{}_{0.5}(x) = \frac{C_2}{\sqrt{\pi x}} + \lambda x^{0.25} + 1. \tag{6.2.25}$$

**Example (g).** If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.65$ and $\rho = 0$ in equation (6.2.19), we easily arrive at the following result

$$y^*{}_{0.65}(x) = \frac{C_2}{\Gamma(0.65)x^{0.35}} + \lambda x^{0.25} + 1. \tag{6.2.26}$$

**Example (h).** If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.9$ and $\rho = 0$ in equation (6.2.19), we easily arrive at the following result

$$y^{*}{}_{0.9}(x) = \frac{C_2}{\Gamma(0.9)x^{0.1}} + \lambda x^{0.25} + 1. \tag{6.2.27}$$

**Example (i).** If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0.95$ and $\rho = 0$ in equation (6.2.19), we easily arrive at the following result

$$y^{*}{}_{0.95}(x) = \frac{C_2}{\Gamma(0.95)x^{0.05}} + \lambda x^{0.25} + 1. \tag{6.2.28}$$

**Example (j).** If we take $\beta = 0.25$, $a = 0$, $b = 1$, $\alpha = 0$, $\beta_1 = 0$, $\beta_2 \to 1$ and $\rho = 0$ in equation (6.2.19), we easily arrive at the following result

$$y^{*}{}_{1}(x) = C_2 + \lambda x^{0.25} + 1. \tag{6.2.29}$$

**Example (k).** If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.5$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (6.2.19), we easily arrive at the following result

$$y^{**}{}_{0.5}(x) = \frac{C_1}{\sqrt{\pi x}} + \lambda x^{0.5} + x^2. \tag{6.2.30}$$

**Example (l).** If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.7$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (6.2.19), we easily arrive at the following result

$$y^{**}_{0.7}(x) = \frac{C_1}{\Gamma(0.7)x^{0.3}} + \lambda x^{0.5} + x^2. \tag{6.2.31}$$

**Example (m).** If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.9$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (6.2.19), we easily arrive at the following result

$$y^{**}_{0.9}(x) = \frac{C_1}{\Gamma(0.9)x^{0.1}} + \lambda x^{0.5} + x^2. \tag{6.2.32}$$

**Example (n).** If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 = 0.95$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (6.2.19), we easily arrive at the following result

$$y^{**}_{0.95}(x) = \frac{C_1}{\Gamma(0.95)x^{0.05}} + \lambda x^{0.5} + x^2. \tag{6.2.33}$$

**Example (o).** If we take $\beta = 0.5$, $a = 1$, $b = 0$, $\alpha = 0$, $\beta_1 \to 1$, $\beta_2 = 0.5$ and $\rho = 2$ in equation (6.2.19), we easily arrive at the following result

$$y^{**}_{1}(x) = C_1 + \lambda x^{0.5} + x^2. \tag{6.2.34}$$

The following graphs (see Figure 6.1, Figure 6.2 and Figure 6.3) are obtained by using MATLAB. Figure 6.1 exhibits a comparison between the behaviours of the solutions $y_\alpha(x)$ given by Eqs.(6.2.20), (6.2.21), (6.2.22), (6.2.23) and (6.2.24) for different values of the parameter $\alpha$. On the other hand, Figure 6.2 illustrates a comparison between the behaviours of the solutions $y^*_{\beta_2}(x)$ given by Eqs.(6.2.25), (6.2.26), (6.2.27), (6.2.28) and (6.2.29) for different values of the parameter $\beta_2$. Similarly, Figure 6.3 illustrates a comparison between the behaviours of the solutions $y^{**}_{\beta_1}(x)$ given by Eqs.(6.2.30), (6.2.31), (6.2.32), (6.2.33) and (6.2.34) for different values of the parameter $\beta_1$.

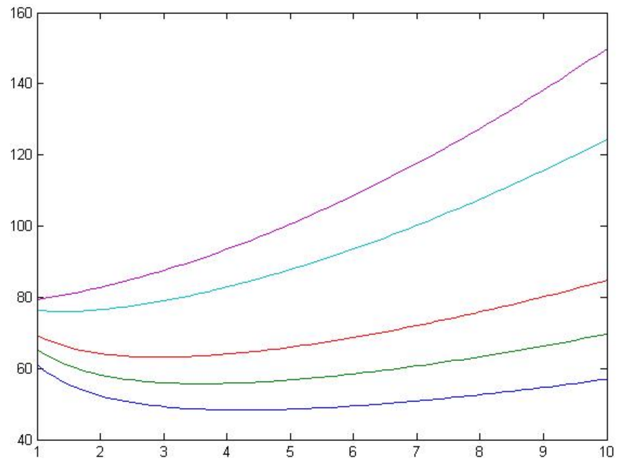


**Figure 6.1:** Solutions $y_\alpha(x)$ for different values of $\alpha$ when $C_1 = 88.4$, $C_2 = 66.6$ and $\lambda = 3$
[Here $y_{0.5}(x)$ is the lowermost graph]

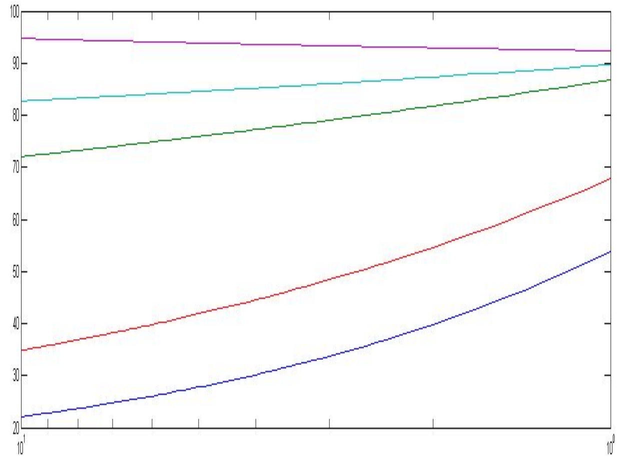

**Figure 6.2:** Solutions $y^*_{\beta_2}(x)$ for different values of $\beta_2$ when $C_2 = 88.4$ and $\lambda = 3$
[Here $y^*_1(x)$ is the uppermost graph and $y^*_{\beta_2}(x)$ is approaching $y^*_1(x)$ as $\beta_2 \to 1$]

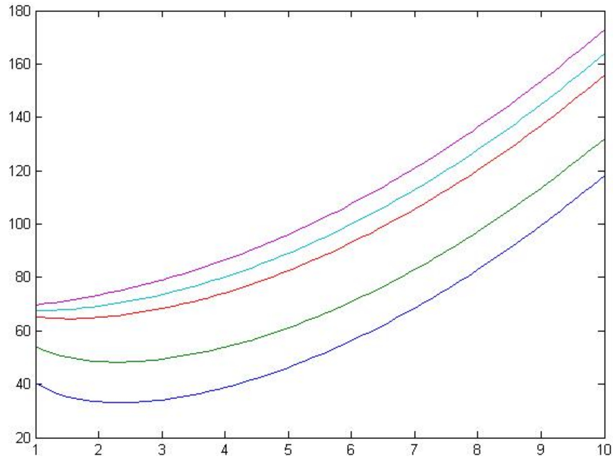


**Figure 6.3:** Solutions $y^{**}_{\beta_1}(x)$ for different values of $\beta_1$ when $C_1 = 66.4$ and $\lambda = 2$ [Here $y^{**}_1(x)$ is the lowermost graph and $y^{**}_{\beta_1}(x)$ is approaching $y^{**}_1(x)$ as $\beta_1 \to 1$]

### 6.2.1 IMPORTANCE OF GRAPHS

It is found that the graphs (see Figure 6.2 and Figure 6.3 ) given here are quite comparable to the corresponding physical phenomena involving ordinary calculus, especially when the parameters $\beta_1 > 0$, $\beta_2 > 0$ get closer and closer to an integer. It can be concluded from the graphs that fractional calculus approach leads us to study a broader spectrum of area in any physical phenomenon as compared to the corresponding physical processes in ordinary calculus.

# THEOREM 3

*The following fractional differential equation*:

$$x\left(D_{0+}^{\alpha,\beta_1}\, y\right)(x) = \lambda\left(\overline{\mathcal{H}}_{0+;p,q,\beta}^{w;m,n;\gamma}\, 1\right)(x) \tag{6.2.35}$$

$$\Bigg(0 < \alpha < 1;\ 0 \leqq \beta_1 \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\};\ 1 \leqq m \leqq q;$$

$$0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\gamma f_j}{F_j}\right)\right\} > 0\Bigg),$$

*with the initial condition*:

$$\left(I_{0+}^{(1-\beta_1)(1-\alpha)}\, y\right)(0+) = C, \tag{6.2.36}$$

*has its solution in the space $L(0,\infty)$ given by*

$$y(x) = C\,\frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma(\alpha+\beta_1(1-\alpha))} + C^*\,\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda x^{\beta+\alpha-1}\,\overline{H}_{p+2,q+2}^{m,n+2}\left[wx^\gamma \,\middle|\, \begin{matrix} E^* \\ \\ F^* \end{matrix}\right], \tag{6.2.37}$$

*where*

$$E^* = (1-\beta,\gamma;1),(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}$$

$$F^* = (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\beta,\gamma;1),(1-\beta-\alpha,\gamma;1),$$

*and $C$, $C^*$ and $\lambda$ are arbitrary constants.*

*Proof.* Since

$$\frac{d^n}{ds^n}\left\{\mathcal{L}\left[f(x)\right](s)\right\} = (-1)^n\,\mathcal{L}\left[x^n\, f(x)\right](s), \tag{6.2.38}$$

if we denote by $Y(s)$ the Laplace transform of the function $y(x)$, which is given as in (6.1.7). Then, by applying the Laplace transform operator $\mathcal{L}$ to each side of (6.2.35), and using the formulas (6.1.17) and (6.1.20) and the initial condition (6.2.36), we find that

$$\frac{d}{ds}\left\{s^{\alpha}Y(s) - Cs^{-\beta_1(1-\alpha)}\right\} = -\lambda s^{-\beta-1}\overline{H}^{m,n+1}_{p+1,q}\left[ws^{-\gamma}\middle|\begin{matrix}A^*\\ \\ B^*\end{matrix}\right]. \tag{6.2.39}$$

Upon integrating both sides, this last equation (6.2.39) yields

$$\begin{aligned}Y(s) =& Cs^{-\alpha-\beta_1(1-\alpha)} + C^*s^{-\alpha}\\ &+ \lambda s^{-\beta-\alpha}\ \overline{H}^{m,n+2}_{p+2,q+1}\left[ws^{-\gamma}\middle|\begin{matrix}E^*\\ \\ G^*\end{matrix}\right],\end{aligned} \tag{6.2.40}$$

where

$$\begin{aligned}E^* &= (1-\beta,\gamma;1),(1-\beta,\gamma;1),(e_j,E_j;\in_j)_{1,n},(e_j,E_j)_{n+1,p}\\ G^* &= (f_j,F_j)_{1,m},(f_j,F_j;\Im_j)_{m+1,q},(-\beta,\gamma;1)\end{aligned}$$

and $C^*$ is a constant of integration. The solution (6.2.37) asserted by (6.2.35) would now follow when we take the inverse Laplace transform of each term in (6.2.40). □

**Remark 6.2.1.** *If we reduce $\overline{H}$-function to the Mittag-Leffler function (see [68, 78] ) in the fractional integral operator on the right-hand side of (6.2.35), we get the result obtained by Srivastava and Tomovski [78, p.208, Theorem 9]. Again, if we reduce $\overline{H}$-function to H-function [68, p.10] in the fractional integral operator on the right-hand side of (6.2.35), we get a result obtained by Srivastava et al. [69, p.120, Theorem 3].*

**Corollary 1.** *If we reduce $\overline{H}$-function in the right hand side of (6.2.35) to the generalized Riemann zeta function [8, p.27, section 1.11, Eq.(1)], we obtain the* *following fractional differential equation*:

$$x\left(D_{0+}^{\alpha,\beta_1}\, y\right)(x) = \lambda\left(\phi_{0+;2,2;\beta}^{w;1,2;\gamma}\, 1\right)(x) \tag{6.2.41}$$

$$\left(0<\alpha<1; 0 \leqq \beta_1 \leqq 1;\ \Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\}\right),$$

*with the initial condition (6.2.36) has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) = C\, \frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma(\alpha+\beta_1(1-\alpha))} + C^*\, \frac{x^{\alpha-1}}{\Gamma(\alpha)} \\ + \lambda x^{\beta+\alpha-1}\, \overline{H}_{4,4}^{1,4}\left[wx^{\gamma}\ \middle|\ \begin{matrix} I^* \\ \\ J^* \end{matrix}\right], \end{aligned} \tag{6.2.42}$$

*where*

$$\begin{aligned} I^* &= (1-\beta,\gamma;1),(1-\beta,\gamma;1),(0,1;1),(1-\eta,1;\varrho) \\ J^* &= (0,1),(-\eta,1;\varrho),(-\beta,\gamma;1),(1-\beta-\alpha,\gamma;1) \end{aligned}$$

*and* $C$, $C^*$ *and* $\lambda$ *are arbitrary constants.*

**Corollary 2.** *If we reduce* $\overline{H}-$*function in the right hand side of (6.2.35) to the generalized Wright Bessel function [18, p.271, Eq.(8)],we observe that the following fractional differential equation*:

$$x\left(D_{0+}^{\alpha,\beta_1}\, y\right)(x) = \lambda\left(\overline{J}_{0+;0,2;\beta}^{w;1,0;\gamma}\, 1\right)(x) \tag{6.2.43}$$

$$\left(0<\alpha<1; 0 \leqq \beta_1 \leqq 1;\ \Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\}\right),$$

*with the initial condition (6.2.36) has its solution in the space* $L(0,\infty)$ *given by*

$$y(x) = C\,\frac{x^{\alpha+\beta_1(1-\alpha)-1}}{\Gamma(\alpha+\beta_1(1-\alpha))} + C^*\,\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda x^{\beta+\alpha-1}\,\overline{H}^{1,2}_{2,4}\left[ wx^{\gamma} \,\middle|\, \begin{matrix} K^* \\ \\ L^* \end{matrix} \right], \quad (6.2.44)$$

*where*

$$K^* = (1-\beta, \gamma; 1), (1-\beta, \gamma; 1)$$
$$L^* = (0,1), (-\kappa, \nu; \mu), (-\beta, \gamma; 1), (1-\beta-\alpha, \gamma; 1)$$

*and* $C$, $C^*$ *and* $\lambda$ *are arbitrary constants.*

# BIBLIOGRAPHY


[1] **Aggarwal R**, A Study of Unified Special Function and Generalized Fractional Integrals with Applications, *Ph.D thesis, MNIT Jaipur*, India (2007).

[2] **Benson DA, Wheatcraft SW and Meerschaert MM**, Application of a fractional advection–dispersion equation. *Water Resour. Res.*, **36(6)** (2000) 1403–1412.

[3] **Buschman RG and Srivastava HM**, The $\overline{H}$-function associated with a certain class of Feynman integrals, *J. Phys. A: Math. Gen.*, **23** (1990) 4707-4710.

[4] **Caputo M**, Linear Model of Dissipation whose Q is almost Frequency Independent - II. *Geophys. J. R. Astr. Soc.*, **13** (1967) 529–539.

[5] **Chaudhry MA, Qadir A, Srivastava HM and Paris RB**, Extended hypergeometric and confluent hypergeometric functions. *Appl. Math. Comput.*, **159** (2004) 589–602.

[6] **Debnath L and Bhatta D**, Integral Transform and Their Applications, (Third edition), *Chapman and Hall (CRC Press), Taylor and Francis group, London and New York* (2014).

[7] **Doetsch G**, Handbuch der Laplace Transformation, Vol I–III, *Birkhauser Stuttgart* (1950–56).

[8] **Erdélyi A, Magnus W, Oberhettinger F and Tricomi FG**, Higher Transcendental Functions, Vol. I, *McGraw Hill Book Co. New York-Toronto-London* (1953).

[9] **Exton H**, Multiple Hypergeometric Functions and Applications. *Halsted Press (Elis Horwood, Chichester), John Wiley and Sons, New York, London Sydney and Toronto* (1976).

[10] **Garg M and Mishra R**, On Product of Hypergeometric Functions, General Class of Multivariable Polynomials and a Generalized Hypergeometric Series associated with Feynman Integrals. *Bull. Cal. Math. Soc.*, **95**(4) (2003), 313–324.

[11] **Glaeske HJ, Kilbas AA and Saigo M**, A modified Bessel–type integral transform and its compositions with fractional calculus operators on spaces $F_{p,\mu}$ and $F'_{p,\mu}$. *Journal of Computational and Applied Mathematics*, **118** (2000) 151–168.

[12] **Gorenflo R and Mainardi F**, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds), Fractals

and Fractional Calculus in Continuum Mechanics, *Springer Series on CSM Courses and Lectures*, **378** (1997) 223-276.

[13] **Gorenflo R, Mainardi F and Srivastava HM**, Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Proceedings of the Eighth International Colloquium on Dif ferential Equations (Plovdiv, Bulgaria; August 18-23, 1997)* (D. Bainov, Editor), VSP Publishers, Utrecht and Tokyo, (1998) 195-202.

[14] **Gradshteyn IS and Ryzhik IM**, Table of Integrals, Series and Products, (Sixth edition), *Academic press* (2000).

[15] **Gupta KC**, New releationship of the H-function with functions of practical utility in fractional calculus. *Ganita Sandesh*, **15(2)** (2001) 63–66.

[16] **Gupta KC and Soni RC**, New Properties of Generalization of Hypergeometric Series Associated with Feynman Integrals. *Kyungpook Math J.*, **41(1)** (2001) 97–104.

[17] **Gupta KC, Jain R and Agrawal R**, On Existence Conditions for Generalized Mellin–Barnes Type Integral. *Nat. Acad. Sci. Lett.*, **30(5&6)** (2007) 169–172.

[18] **Gupta KC, Jain R and Sharma A**, A study of Unified Finite Integral Transforms with Applications. *J. Raj. Aca. Phy. Sci.*, **2(4)** (2003) 269–282.

[19] **Harjule P**, On recent advances in special functions and fractional calculus with applications, *PhD Thesis, MNIT Jaipur*, India (2016).

[20] **Hilfer R(Ed.)**, Applications of Fractional Calculus in Physics, *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong* (2000).

[21] **Hilfer R**, Fractional time evolution, in: Hilfer, R.(Ed.), *Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong*, 87 -130 (2000).

[22] **Hilfer R** ,Experimental evidence for fractional time evolution in glass forming materials, *J. Chem. Phys.* , **284** (2002) 399-408.

[23] **Inayat-Hussain AA**, New properties of hypergeometric series derivable from Feynman Integrals II. A generalization of the H-function, *J. Phys. A: Math. Gen.*, **20** (1987) 4119–4128.

[24] **Jain K**, A Study of Multivariable Polynomials Mellin-Barnes Type Integrals and Integral Transform With Applications. *Ph.D. Thesis, University of Rajasthan*(2008).

[25] **Jain R**, A study of special functions, A General Polynomial, Integral Equations and Fractional Calculus with Applications. *Ph.D. thesis. Univ. of Rajasthan, India* (1992).

[26] **Jain R and Sharma A**, On Unified Integral Formula Involving $H$ -function and General Class of Polynomials. *Aligarh Bull. of Maths.*, **21** ( 2002) 7–12.

[27] **Kilbas AA**, Saigo M, On Mittag-Leffler type function, fractional calculus operators and solutions of integral equations, *Integral Transform. Spec. Funct.*, **4**(1996) 355-370.

[28] **Kilbas AA**, Saigo M and Saxena RK, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integral Transforms Spec. Funct.*, **15** (2004) 31-49.

[29] **Kilbas AA, Srivastava HM and Trujillo JJ**, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, *Elsevier (North-Holland) Science Publishers, Amsterdam*, **204**(2006).

[30] **Kiryakova VS**, Generalized Fractional calculus and Applications. *Longman Pitman Res Notes in Math. Ser.*, **301** (1994).

[31] **Krätzel E**, Integral transformations of Bessel type. In Generalized functions & operational calculus. *(Proc Conf Verna, 1975), Bulg Acad Sci, Sofia*, (1979) 148-165.

[32] **Lerch M**, Sur un Point de la thèorie des Fonctions génératrices d'Abel,*Acta Math*, **27** (1903) 339-351.

[33] **Lorenzo CF and Hartley TT**, Generalized functions for the fractional calculus. *NAZA/TP-209424* (1999) 1–17.

[34] **Luke YL**, The Special functions and Their Approximations. Vols I and II, *Academic Press, New York and London* (1969).

[35] **Marichev OI** , Handbook of integral transforms of higher transcendental functions: theory and algorithmic tables. *Ellis Horwood, Chichester and Wiley, New York* (1983).

[36] **Mathai AM and Haubold HJ**, Special Functions for Applied Sciences, Springer, Berlin (2010).

[37] **Mathai AM and Saxena RK**, Generalized Hypergeometric Functions with Applications in Statistical and Physical Sciences, *Springer–Verlag Berlin, Heidelberg and New-York*(1973).

[38] **Mathai AM, Saxena RK and Haubold HJ**, The H-Function: Theory and Applications, *Publication No. 37 of center for Mathematical Sciences, Pala Campus*(2008).

[39] **McBridge AC and Roach GF**, Fractional Calculus. *University of Stratcheltde, Glasgow, Pitam Advanced Publishing Program* (1985).

[40] **Meerschaert MM, Benson DA, Scheffler HP and Baeumer B**, Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E*, **65**(4) (2002) 1103–1106.

[41] **Metzler R and Klafter J**, The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.*, **339(1)** (2000) 1–77.

[42] **Miller KS and Ross B**, An Introduction to the Fractional Calculus and Fractional Differential Equations ,*A Wiley-Interscience Publication, John*

*Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore* (2009).

[43] **Mittag-Leffler GM**, Sur la nouvelle fonction $E_\alpha(x)$,*C.R. Acad. Sci. Pari*, **137** (1903) 554-558.

[44] **Nishimoto K**, Fractional Calculus, Vol. I(1984), Vol. II(1987), Vol. III(1989), Vol. IV(1991), Vol. V(1996), Descartes Press, Koriyama, Japan.

[45] **Oldham KB and Spanier J**, The fractional calculus. *Academic Press, New York and London* (1974).

[46] **Özarslan MA and Ylmaz B**, The extended Mittag Leffler function and its properties,*J. Inequal. Appl.*, **85** (2014) 1-10.

[47] *Ö***zergin E**, Some properties of hypergeometric functions. *Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, Turkey* (2011).

[48] **Özergin E, Özarslan MA and Altin A**,Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.*,**235**(2011) 4601-4610.

[49] **Parmar RK**, A new generalization of gamma, beta, hypergeometric and confluent hypergeometric functions. *Le Matematiche*, **68** (2013) 33–42.

[50] **Podlubny I**, Fractional Differential Equations. *California,USA: Academic Press* (1999).

[51] **Prabhakar TR**, A singular integral equation with a generalized Mittag-Leffler function in the kernel,*Yokohama Math. J.*, **19** (1971) 7-15.

[52] **Rahman G, Agarwal P, Mubeen S and Arshad M**, Fractional integral operators involving extended Mittag-Leffler function as its Kernel, *Sociedad Matemática Mexican*, DOI 10.1007/s40590-017-0167-5 (2017).

[53] **Rainville ED**, Special Functions, Macmillan, New York (1960).

[54] **Rathie AK**, A New Generalization of Generalized Hypergeometric Functions. *Le Matematiche Fasc. II*, **52** (1997) 297–310.

[55] **Samko SG, Kilbas AA and Marichev OI**, Fractional integrals and derivatives, Theory and Applications, *Gordon and Breach Science Publishers, Yverdon(Switzerland)* (1993).

[56] **Saxena RK**, Functional Relations Involving Generalized H-function. *Le Mathematiche* **53** (1998) 123–131.

[57] **Saxena RK, Chena R and Kalla SL**, Application of generalized H-function in bivariate distribution. *Rev. Acad. Can. Cienc.*, **XIV(1–2)**(2002) 111–120.

[58] **Saxena RK and Gupta N**, Some Abelian Theorems for Distributional H -function Transformation. *Indian J. Pure Appl. Math.*, **25** (1994) 869–879.

[59] **Shukla AK and Prajapati JC**, On a generalization of Mittag-Leffler function and its properties, *J. Math. Anal. Appl.*,**336** (2007) 797-811.

[60] **Slater LJ**, Generalized Hypergeometric Functions, *Cambridge Univ. Press, Cambridge, London and New York* (1966).

[61] **Srivastava HM**, A Contour Integral Involving Fox H-function. *Indian J.Math*, **14** (1972), 1-6.

[62] **Srivastava HM**, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inf. Sci.*,**5** (2011) 390-444.

[63] **Srivastava HM**, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, *SpringerPlus*, **2** (2013) 1-14.

[64] **Srivastava HM**, A new family of the $\lambda-$generalized Hurwitz-Lerch zeta functions with applications, Appl. Math. Inform. Sci., **8** (2014) 1485-1500.

[65] **Srivastava HM, Agarwal P and Jain S**, Generating functions for the generalized Gauss hypergeometric functions,*Appl. Math. and Comput.*, **247** (2014) 348-352.

[66] **Srivastava HM and Choi J**, Series Associated with the Zeta and Related Functions, *Kluwer Academic Publishers, Dordrecht, Boston and London* (2001).

[67] **Srivastava HM and Choi J**, Zeta nad q-Zeta Functions and Associated Series and Integrals, *Elsevier Science Publishers, Amsterdam, London, New York* (2012).

[68] **Srivastava HM, Gupta KC and Goyal SP**, The H-functions of one and two variables with Applications. *South Asian Publishers, New Delhi & Madras*(1989).

[69] **Srivastava HM, Harjule P and Jain R**, A General Fractional Differential Equation Associated With an Integral Operator With the H-Function in the Kernel,*Russian Journal of Mathematical Physics*, **22(1)**(2015) 112-126.

[70] **Srivastava HM, Jankov D, Pogány TK and Saxena RK**, Two-sided inequalities for the extended Hurwitz-Lerch Zeta function, *Comput. Math. Appl.*, **62** (2011) 516-522.

[71] **Srivastava HM and Karlsson PW**, Multiple Gaussian Hypergeometric Serie. *Halsted Press (Ellis Horwood Limited), John Wiley and Sons, New York, Chichester, Brisbane, Toronto* (1985).

[72] **Srivastava HM and Manocha HL**, A Treatise on Generating Functions.*Ellis Hortwood Ltd. Chichester, John Wiley and sons, New York* (1984).

[73] **Srivastava HM and Panda R**, Expansion Theorems for the $H$-function of Several Complex Variables. *J. Reine Angew. Math.*, **288** (1976) 129–145.

[74] **Srivastava HM and Panda R**, Some bilateral generating functions for a class of generalized hypergeometric polynomials. *J. Reine Angew. Math.*, (1976) 265–274.

[75] **Srivastava HM and Saxena RK**, Operators of Fractional Integration and Their Applications, *Elsevier, Appl. Math. Comput.*, **118** (2001) 1–52.

[76] **Srivastava HM, Saxena Rk and Parmar Rk**, Some families of the incomplete $H-$functions and the incomplete $\overline{H}-$functions and associated integral transforms and operators of fractional calculus with applications, *Russian J. Math. Phys.*, **25**(2018) 116-138.

[77] **Srivastava HM, Saxena RK, Pogány TK and Saxena R**,Integral and Computational representations of the extended Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.*, **22** (2011) 487-506.

[78] **Srivastava HM and Tomovski Ž**, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel,*Appl. Math. Comput.*, **211** (2009) 198-210.

[79] **Tomovski Ž, Hilfer R and Srivastava HM**, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Integral Transforms Spec. Funct.*, **21** (2010) 797–814.

[80] **Whittaker ET and Watson GN**, A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions;With an Account of the Principal Transcendental Functions, 4th ed., *Cambridge University Press, Cambridge, London and NewYork* (1927).

[81] **Wiman A**, Über den fundamental Satz in der Theorie der Funktionen $E_\alpha(x)$, *Acta Math.*, **29** (1905) 191-201.

[82] **Zaslavsky GM**, Chaos, Fractional kinetics, and anomalous transport. *Phys. Rep.*, **371(6)**(2002) 461–580.

# BIOGRAPHICAL PROFILE OF THE CANDIDATE



| | | |
|---|---|---|
| Full Name | : | NIDHI JOLLY |
| Permanent Address | : | H. No. 270/4, Govindpuri, |
| | | Kalkaji, New Delhi |
| | | Delhi-110019, India |
| Mob. No. | : | +91-8076699264 |
| e-mail | : | nidhinj6@gmail.com |



EDUCATION

| | | |
|---|---|---|
| January, 2016 – July, 2019 | : | Research Scholar, Department of Mathematics |
| | | Malaviya National Institute of Technology, Jaipur. |
| July, 2010 – May, 2012 | : | Master of Science in Mathematics |
| | | (secured 76.8%) |
| | | Ramjas College |
| | | Delhi University , Delhi |

## 6. BIOGRAPHICAL PROFILE OF THE CANDIDATE

| | | |
|---|---|---|
| May, 2008 – July, 2010 | : | Bachelor of Scienece<br>(secured 72.4% )<br>Acharya Narendra Dev College<br>Delhi University , Delhi. |
| June, 2014 –May, 2015 | : | Bachelor of Education<br>(secured 66.9%)<br>Oxford College<br>Maharishi Dayanand University , Rohtak |

ACHIEVEMENTS AND AWARDS

1. Awarded **'Certificate of Merit'** during B.Sc (Hons.) Mathematics Part-I for securing First position in college during academic year 2007-08.

2. Recieved **'AWARD fOR EXCELLENCE'** (Second prize) by DELHI UNIVERSITY STUDENTS UNION.

RESEARCH CONTRIBUTIONS HAVING A BEARING ON THE SUBJECT MATTER OF THE THESIS

1. ON THE SOLUTION OF GENERAL FAMILY OF FRACTIONAL DIFFERENTIAL EQUATION INVOLVING HILFER DERIVATIVE OPERATOR AND $\overline{H}$-FUNCTION, *Int. J. Math. And Appl.*, **6(1-A)** (2018), 155–162, (coauthors Priyanka Harjule and Rashmi Jain).

2. A STUDY OF NEW INTEGRAL TRANSFORM ASSOCIATED WITH $\lambda-$EXTENDED HURWITZ-LERCH ZETA FUNCTION, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales*, https://doi.org/10.1007/s13398-018-0570-4 (2018), (coauthors H.M. Srivastava, Manish Kumar Bansal and Rashmi Jain)

3. AN INTEGRAL OPERATOR INVOLVING GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND ASSOCIATED FRACTIONAL CALCULUS, *The Journal of Analysis*, https://doi.org/10.1007/s41478-018-0119-0 (2018),(coauthors Manish Kumar Bansal and Rashmi Jain).

4. STUDY OF GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND IT'S PROPERTIES, *South East Asian J. of Math. & Math. Sci.*, **14(1)** (2017), 47-58,(coauthor Rashmi Jain).

5. FRACTIONAL DIFFERENTIAL EQUATION ASSOCIATED WITH AN INTEGRAL OPERATOR WITH THE $\overline{H}$-FUNCTION IN THE KERNEL, *Global Journal of Pure and Applied Mathematics*, **13(7)** (2017), 3505–3517, (coauthors Priyanka Harjule and Rashmi Jain).

6. A SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION INVOLVING HILFER DERIVATIVE OPERATOR, *Journal of Indian Acad. Math.*, **39(2)** (2017), 255–263, (coauthors Priyanka Harjule and Rashmi Jain).

7. AN INVESTIGATION OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS, *Palestine Journal of Mathematics*,

(Accepted) (coauthor Rashmi Jain).

PRESENTATION OF RESEARCH PAPERS IN THE CONFERENCES

1. **N. JOLLY and R. JAIN,** ON UNIFIED INTEGRALS INVOLVING $H-$FUNCTION and $S_U^V$ POLYNOMIAL, International Conference on Special Functions and its Applications-2016,Jamia Millia Islamia (New Delhi), India, during September 9-11, 2016

2. **N. JOLLY and R. JAIN,** STUDY OF GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND IT'S PROPERTIES, International Conference On Mathematical Modelling and Computing 2018,JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE (Jaipur) (Raj), India, during April 6-7, 2018

3. **N. JOLLY and R. JAIN,** AN INVESTIGATION OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS, International Conference on Mathematical Modelling, Applied Analysis and Computation, JECRC University, Jaipur (Raj), India, during July 6-8, 2018

WORKSHOPS ATTENDED

1. One week **Short Term Course** on **LaTeX : A Scientific Writing Tool** at *Malaviya National Institute of Technology Jaipur, India* during May 23-27, 2016.

2. One week **Short Term Course** on **Modeling and Simulation Tools** at *Malaviya National Institute of Technology Jaipur, India* during July 03-07, 2017.

3. One week **Short Term Course** on **Software tools for scientific research: Mathematica and LaTeX** at *Malaviya National Institute of Technology Jaipur, India* during January 9-13, 2018.

# LIST OF RESEARCH PAPERS CONTRIBUTED BY THE CANDIDATE HAVING A BEARING ON SUBJECT MATTER OF THE THESIS

1. **N. JOLLY, P. HARJULE and R. JAIN** (2018). ON THE SOLUTION OF GENERAL FAMILY OF FRACTIONAL DIFFERENTIAL EQUATION INVOLVING HILFER DERIVATIVE OPERATOR AND $\overline{H}$-FUNCTION, *Int. J. Math. And Appl.*, **6(1-A)**, 155–162.

2. **H.M. SRIVASTAVA, N. JOLLY, M.K. BANSAL and R. JAIN** (2018). A STUDY OF NEW INTEGRAL TRANSFORM ASSOCIATED WITH $\lambda-$EXTENDED HURWITZ-LERCH ZETA FUNCTION, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales*, https://doi.org/10.1007/s13398-018-0570-4.

3. **M. K. BANSAL, N. JOLLY, R. JAIN and D. KUMAR** (2018). AN INTEGRAL OPERATOR INVOLVING GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND ASSOCIATED FRACTIONAL CALCULUS,

*The Journal of Analysis*, https://doi.org/10.1007/s41478-018-0119-0 .

4. **N. JOLLY and R. JAIN** (2017). STUDY OF GENERALIZED EXTENDED MITTAG-LEFFLER FUNCTION AND IT'S PROPERTIES, *South East Asian J. of Math. & Math. Sci.*, **14(1)**, 47-58

5. **N. JOLLY, P. HARJULE and R. JAIN** (2017). FRACTIONAL DIFFERENTIAL EQUATION ASSOCIATED WITH AN INTEGRAL OPERATOR WITH THE $\overline{H}$-FUNCTION IN THE KERNEL, *Global Journal of Pure and Applied Mathematics*, **13(7)**, 3505–3517.

6. **P. HARJULE, N. JOLLY and R. JAIN** (2017). A SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION INVOLVING HILFER DERIVATIVE OPERATOR, *Journal of Indian Acad. Math.*, **39(2)**, 255–263.

7. **N. JOLLY and R. JAIN**. AN INVESTIGATION OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS, *Palestine Journal of Mathematics*, (Accepted).