A

DISSERTATION REPORT

ON

MODELING AND ANALYSIS OF SINGLE CYLINDER ENGINE CRANKSHAFT USING DYNAMIC STIFFNESS MATRIX METHOD

Submitted in partial fulfilment of the requirements for the award of degree of

MASTER OF TECHNOLOGY

IN

DESIGN ENGINEERING

Submitted By

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Jaipur-302017 (Rajasthan)

CERTIFICATE

This is to certify that the dissertation work entitled "**Modeling and Analysis of Single Cylinder Engine Crankshaft using Dynamic Stiffness Matrix Method**" by **Tarun Agarwal** is a bonafide work completed under my supervision and guidance, and hence approved for submission to the **Department of Mechanical Engineering, Malaviya National Institute of Technology, Jaipur** in partial fulfillment of the requirements for the award of the degree of **Master of Technology** with specialization in **Design Engineering**. The matter embodied in this Seminar Report has not been submitted for the award of any other degree, or diploma.

(Dr. T.C. Gupta)

Department of Mechanical Engineering,

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Place: Jaipur

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Jaipur-302017 (Rajasthan)

Candidate's Declaration

I hereby certify that the work which is being presented in the dissertation entitled "**Modeling and Analysis of Single Cylinder Engine Crankshaft using Dynamic Stiffness Matrix Method**", in partial fulfilment of the requirements for the award of the Degree of **Master of Technology** in **Design Engineering**, submitted in the **Department of Mechanical Engineering, MNIT, Jaipur** is an authentic record of my own work carried out for a period of one year under the supervision **Dr. T.C. Gupta, Associate Professor of Mechanical Engineering Department, MNIT, Jaipur.**

I have not submitted the matter embodied in this dissertation for the award of any other degree.

(Tarun Agarwal)

ID: 2013PDE5011

M. Tech. (Design Engineering)

Place: Jaipur

Date:

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It has been a great experience working on this dissertation project "**Modeling and Analysis of Single Cylinder Engine Crankshaft using Dynamic Stiffness Matrix Method**" towards the partial fulfillment of the requirements for the award of the Degree of **Master of Technology** in **Design Engineering**.

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Abstract

Crankshaft is a sophisticated engine component. Many analysis method has been reported in last few years. FEM is most reliable method used for crankshaft analysis but its main drawback is its complexity and cost for analysis. So in this study for vibration analysis of crankshaft a simpler method is proposed. Here crankshaft is modeled by a set of jointed structures consisting of simple round rods and simple beam blocks of rectangular cross-section. The front pulley, timing gear, and the fly-wheel is idealized by a set of masses and moments of inertia. The main journal bearings is idealized by a set of linear springs and dash-pots. For each constituent member, the dynamic stiffness matrix was derived using the governing differential equation. Then the dynamic stiffness matrix for the total crankshaft system was constructed, and the natural frequencies and mode shapes were calculated.

Keywords: Vibration analysis of crankshaft, Transfer matrix method, Dynamic stiffness matrix method.

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LIST OF SYMBOLS

A	Cross Sectional Area
d	Nodal Displacement Vector
E	Modulus of Elasticity
f	Nodal Force Vector
G	Modulus of Rigidity
i	Imaginary number $\sqrt{-1}$
l	Length
M	Moment about y-axis
M_z	Moment about z-axis
P	Force in x-direction
Q	Shear Force in z-direction
T	Torsion about x-axis
Tr	Transformation Matrix
t	Time Variable
u	Displacement in x-direction
V	Shear Force in y-direction
v	Displacement in y-direction
w	Displacement in z-direction
θ	Slope
ρ	Density of Crankshaft Material
ε	Strain
ω	Natural Frequency
β,c	Constant
K_{Dyn}	Dynamic Stiffness Matrix
f(x,t)	Force which is Function of Space and Time

Special Notations

[ ]	Matrix
{ }	Vector

Abbreviations

DSM	Dynamic Stiffness Matrix
FEM	Finite Element Method
MOI	Moment of Inertia
PDE	Partial Differential Equation
TMM	Transfer Matrix Method

Chapter 1 : Introduction

1.1 Background

Crankshaft of single cylinder engine is used to convert the reciprocating motion of piston into rotational motion. Crankshaft has a crank throw or crank pin, additional bearing surface whose axis is offset from the crank, to which big end of connecting rod is attached.

It is typically connected to a flywheel to reduce the pulsation characteristic of the four-stroke cycle at one end and at other end, pulley and gear (for cam shaft) is mounted. Sometimes a vibrational damper is also used to reduce torsional vibration often caused along the length of the crankshaft.

In recent years, noise vibration and harshness of engine is becoming integral part of engine design process along with the traditional issues of durability and performance. Extensive static and dynamic analysis is performed on engine component as crankshaft and engine block in order to improve their durability and NVH performance.

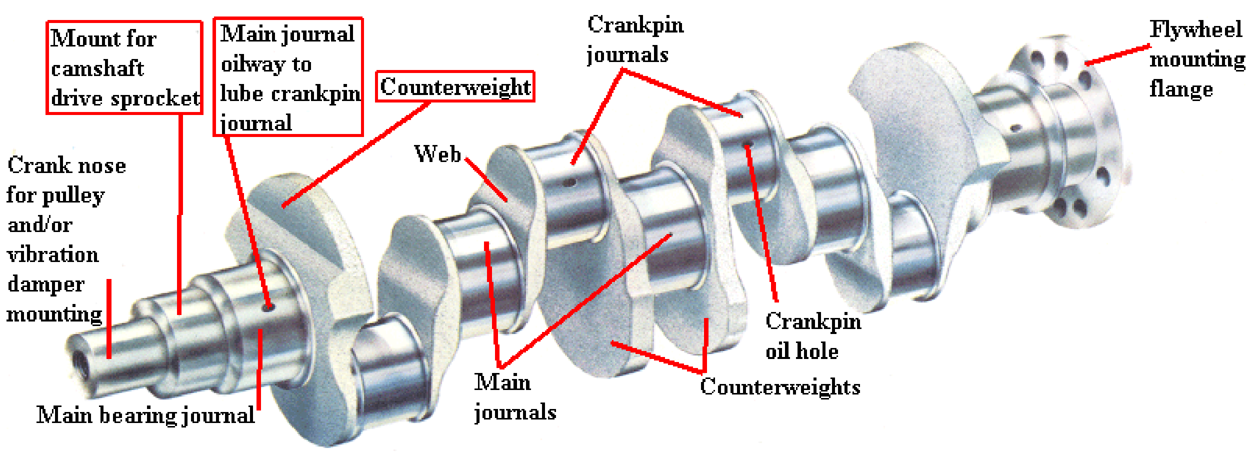

Figure 1.1 Crankshaft

In our study, we present the procedure for measuring the natural frequencies and mode shapes of single cylinder engine crankshaft. MATLAB is used to calculate the natural frequency and mode shape of the crankshaft. FEM (Finite Element Method) has been the only versatile approach for such analysis. However, even for a relatively simple crankshaft, modeling and computation by FEM is very tedious and expensive. To overcome these shortcomings of FEM analysis, Nagamatsu et al., developed the "Reduced Impedance Method" (RIM), in which they modeled the crank journal and crankpin with round bars, and derived impedance matrices, using the "Transfer Matrix Method" (TMM). Thus, they used FEM analysis only for the crank arm and counterweight.

To simplify the analysis by RIM, and to eliminate FEM from analysis, we idealized the crank arm and counterweight as a set of jointed structures consisting simple beam blocks, along with above modeling for the crankpin and crank journal. After idealizing each constituent member of crankshaft as a beam block we derive the dynamic stiffness matrix (DSM) for each member of crankshaft. DSM is a matrix which relate nodal force vector to nodal displacement vector.

The front pulley, crank gear, and the flywheel, were each idealized as a set of masses and moments of inertia. The main journal bearings are idealized by a set of linear springs and dash-pots system. Finally, the dynamic stiffness matrix for the total idealized jointed structure was constructed by the method of superposition, and the natural frequencies and mode shapes were calculated.

The result obtained using DSM method is in good agreement with the results obtain from FEM result with fewer element.

1.2 Thesis Outline

Chapter 1 introduces the background of modelling of crankshaft used in single cylinder engine. Chapter 2 includes literature review, research gaps and also research objectives. Chapter 3 includes modelling of crankshaft and theoretical analysis along with brief discussion of co-ordinate transformation and newton bisection method for finding natural frequencies and mode shapes. Chapter 4 contains the derivations of dynamic stiffness matrix for different type of elements. In chapter 5, mathematical modelling of crankshaft is discussed by using dynamic stiffness matrix method. Chapter 6 contains the results and discussions followed by conclusions and future works in chapter 7.

Chapter 2 : Literature Review

2.1 Literature review

Kang et al. [2], “Modal Analyses and Experiments for Engine Crankshafts”, investigates the coupled modes including coupled torsional flexural vibration and coupled longitudinal flexural vibration for non-rotating crankshafts which are free-free suspended. The finite element models of those are generally used in two categories beam elements and solid elements. By using these two models the natural frequencies and mode shapes of two crankshafts are determined. Results show that the solid element is more appropriate than the beam element in the modal analysis of crankshafts. Solid element modelling in crankshaft analysis produces much better results.

Y.Yu, Feng, L.Yu [11] showed the analysis of the three-dimension vibrations of reciprocating compressor crankshaft system under working conditions using a spatial finite element model based on 3-node Timoshenko beam. The crankshaft was idealized by a set of jointed structures consisting of simple round rods and simple beam blocks, the main journal bearings were idealized by a set of linear springs and dash-pots, and the flywheel and motor were idealized by a set of masses and moments of inertia.

In this study, **Mourelatous [3]** “An efficient Crankshaft Dynamic Analysis using Sub structuring with Ritz vectors” described a structural analysis using dynamic sub structuring with Ritz vectors for predicting the dynamic response of an engine crankshaft, based on the finite-element method (FEM). A two-level dynamic sub-structuring is performed using a set of load-dependent Ritz vectors. So FEM has been the only versatile approach for analysis of crankshaft or structure. However, even for a relatively simple crankshaft, tedious modeling and expensive computation costs are inevitable. To overcome these shortcomings of FEM analysis, Nagamatsu et al., developed the "Reduced Impedance Method" (RIM), in which they modeled the crank journal and crankpin with round bars, and derived impedance matrices, using the "Transfer Matrix Method" (TMM). They used FEM analysis only for the crankarm and counterweight. To simplify the analysis by RIM, and eventually to eliminate FEM from the analysis, the crankarm and counterweight also is idealized as a set of jointed structures consisting of simple beam blocks, along with the above modeling for the crankpin and crankjournal. Then using TMM, they derived the DSM for each constituent member. For the free vibration analysis of beams with various attachments, the lumped-mass (model) transfer matrix method (LTMM) is one of the most popular approaches.

Mourelatos et al. [4], formulate a finite element based modelling for the dynamic behavior of a rotating flexible shaft supported by a flexible support structure. The interaction between the rotating shaft and the flexible support is modelled by either linear or non-linear springs distributed around the circumference of the shaft. The coupling between the flexibility of the shaft and the flexibility of the support structure are considered. He developed a system approach to formulate the dynamic response of a rotating shaft supported by a flexible support structure.

Wu and Chen [5], "A lumped-mass TMM for free vibration analysis of a multi-step Timoshenko beam carrying eccentric lumped masses with rotary inertias" proposed a modified LTMM so that one may easily determine the natural frequencies and the corresponding mode shapes of a multistep Timoshenko beam with various boundary (supporting) conditions and carrying various concentrated elements with eccentricity of each lumped mass considered, by using the same formulation developed from a beam with ''free–free'' boundary conditions. Based on the formulation for a ''free–free'' beam carrying a number of sets of concentrated elements with each set consisting of a lumped mass (with or without rotary inertia and/or eccentricity), a translational spring and a rotational spring, one may easily determine the lowest several natural frequencies.

Wu and Chen [6], "A continuous-mass TMM for free vibration analysis of a non-uniform beam with various boundary conditions and carrying multiple concentrated elements" presented a modified continuous-mass (model) transfer matrix method (CTMM) to determine the natural frequencies and associated mode shapes of a uniform or non-uniform beam with various classical (or non-classical) boundary conditions and carrying multiple sets of concentrated elements with each set consisting of a point mass (with eccentricity and rotary inertia), a translational spring and a rotational spring.

Charles et al. [7], in "Crankshaft Torsional Vibration of Diesel engines" presents an investigation of the diesel engine combustion related fault detection capability of crankshaft torsional vibration. In this paper he has discussed two typical experimental studies on 16- and 20-cylinder engines, with and without faults.

Yasin Yilmaz, Gunay Anlas [8] study the effects of counterweight mass and position on main bearing load and crankshaft bending stress of an in-line six-cylinder diesel engine using Multibody System Simulation Program, ADAMS. In the analysis, rigid, beam and 3D solid crankshaft models are used. The load from gas pressure rather than inertia forces is the parameter with the most important influence on design of the crankshaft.

Zhang et al. [9], describes two system models—a rigid body model and a flexible body model for predicting torsional vibrations of the crankshaft under different engine powers and propeller pitch settings. In the flexible body model, the distributed torsional flexibility and mass moment of inertia of the crankshaft are considered using the finite element method. He proposed theoretical and experimental procedures for investigating the torsional vibration of the crankshaft in an engine propeller dynamical system.

Hoisnard et al. [10], "Model-based diagnosis of large diesel engines based on angular speed variations of the crankshaft" work aims at monitoring large diesel engines by analyzing the crankshaft angular speed variations. It focuses on a powerful 20-cylinder diesel engine with crankshaft natural frequencies within the operating speed range. Due to the crankshaft flexibility, torsional vibrations are superimposed on the rigid rotational motion of the crankshaft and complicate the analysis.

Meng et al. [12], in their study "Finite Element Analysis of 4-Cylinder Diesel Crankshaft" found the stress analysis and modal analysis of a 4-cylinder crankshaft using finite element method. The relationship between the frequency and the vibration modal was explained by the modal analysis of crankshaft. The crankshaft deformation was mainly bending deformation under the lower frequency. And the maximum deformation was located at the link between main bearing journal and crankpin and crank cheeks.

Bulatovic et al. [13], presents the procedures for measuring and analyzing the angular velocity variation of twelve-cylinder diesel engine crankshaft on its free end and on the power output end. In addition, he deals with important aspects of the measurement of crankshaft torsional oscillations. In this paper "Measurement and analysis of angular velocity variations of twelve-cylinder diesel engine crankshaft" original procedure of measuring and analyzing the angular velocities variation of crankshaft of V46-TKA twelve-cylinder diesel engine at two proximities, with two different sensors is presented.

Fonte et al. [14], in his work "Failure mode analysis of two crankshafts of a single cylinder" reports an investigation carried out on two damaged crankshafts of single cylinder diesel engines. He firstly presents a short review on fatigue power shafts for supporting the failure mode analysis to determine the root cause of failure. Finite element analysis was done in order to find the critical zones where high stress concentrations are present. Results showed a clear failure by fatigue under low stress and high cyclic fatigue on crankpins.

Anyaegbunam, and Osadebe [15], discussed about a simple and direct approach for the formulation of the dynamic stiffness matrix of a beam-column element. They considered the model as a system with distributed mass thereby, treating the system as having an infinite number of degrees of freedom. The differential equation of motion of this system, in which the axial compressive force is accounted for, is derived by applying Newton's second law of motion. By imposing the appropriate boundary conditions, the dynamic stiffness matrix which includes the effect of axial compressive force is synthesized.

2.2 Research Gaps

Meng et al. investigated the stress and modal analysis of 4-Cylinder diesel engine crankshaft by using FEM. Yu et al. studied the vibration analysis of reciprocating compressor crankshaft system using spatial finite element model. Mourelatous calculated the dynamic response of engine crankshaft using sub structuring with Ritz vectors. Wu et al. investigated the vibration analysis of non-uniform beam by using continuous mass TMM and lumped mass TMM. Kang et al. investigated the coupled modes vibration for non-rotating crankshaft with the help of experimental analysis. Yasin et al. studied the effect of counterweight mass and crankshaft bending stress of an inline six cylinder diesel engine. Hoisnard et al. described the effect of angular speed variations of the crankshaft on the natural frequencies. Fonte et al. investigated the failure mode analysis of two crankshafts of a single cylinder.

No one has discussed about the mode shapes and natural frequencies of the crankshaft for In plane and Out of plane mode by assuming the bearings as isotropic spring dashpot system.

2.3 Research Objective

Finite element modelling of crankshaft will be developed and derive the dynamic stiffness matrix for the whole system by considering crankshaft as an assembly of round rods and rectangular cross-section blocks. Natural frequencies and mode shapes of crankshaft will be find out by using the newton bisection methods for In plane and Out of plane mode.

Chapter 3 : Modeling and Analysis

3.1 Modeling

a) Crankshaft: The crankshaft is considered to be a set of rigidly jointed structures consisting of round rods and blocks of rectangular cross-section, instead of assemblage of finite elements as in FEM. Crank journal and crank pin is idealized by rod of length L_j and L_p, and diameter D_j and D_p. Here D_j and D_p are taken as their original diameters, while L_j and L_p measured from the centre plane of crankarm (L_p is taken as equal to the length between the two centre plane of crankarms of single cylinder engine crankshaft). Crankarm and counterweight is idealized by block of rectangular cross section. The dimension of block are determined so as to keep their centres of gravity as well as their original masses and moment of inertia at their original positions.

b) Front Pulley, Crank Gear and Flywheel: The front pulley, crank gear and flywheel are assumed as a idealized set of masses and moments of inertia about three orthogonal axes attached at their original center of gravity.

c) Crankshaft Main Bearing: Main bearing (journal bearing) is idealized by an isotropic oil film which is considered as a set of linear springs and dashpots in vertical and horizontal directions attached at the crank journal axis.

3.2 Analysis

The vibration behavior of idealized jointed structure of a single cylinder crankshaft can be analyzed in different ways. The Dynamic Stiffness Matrix method (DSM) is probably the simplest method and it is well established. We have applied this method to the single cylinder engine crankshaft and obtained sufficient agreement between the calculated and experimental results. Dynamic stiffness matrix can be derived by different methods, but there are two main methods:

a) By transfer matrix method
b) By the governing differential equation of element (discussed in next chapter)

Transfer Matrix Method:

Transfer matrix method is an approach to matrix structural analysis that uses a mixed form of the element force-displacement relationship and transfers the structural behavior parameters the joint forces and displacement from one end of the structures of line element to other. However, that description by TMM is not compatible with description by other methods like FEM because in TMM there are different arrangement of forces and displacement vectors. So, to form the dynamic stiffness matrix we have to rearrange the force and displacement vector of TMM as shown in following,

The displacement and force vector are defined by

$$d = \left[u, \ v, \ w, \ \theta_x, \ \theta_y, \ \theta_z\right]^T$$

$$f = [P, \ V, \ Q, \ T, \ M, \ M_z]^T$$

Let the two end points be: $i-1$ and i. Then, we can derive the following expression

$$\begin{bmatrix} d_i \\ f_i \end{bmatrix} = \begin{bmatrix} [t_{11}] & [t_{12}] \\ [t_{21}] & [t_{22}] \end{bmatrix} \begin{bmatrix} d_{i-1} \\ f_{i-1} \end{bmatrix}$$

$$\begin{bmatrix} f_{i-1} \\ f_i \end{bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{bmatrix} d_{i-1} \\ d_i \end{bmatrix}$$

Here $[d_{i-1,}f_{i-1}]^T$ and $[d_{i,}f_i]^T$ are the state vector at point $i-1$ and i, and $[t_{ij}]$ is element of transfer matrix and $[f_{i-1},f_i]^T$ and $[d_{i-1},d_i]^T$ are the force and displacement vectors, and $[K_{ij}]$ is element of dynamic stiffness matrix.

Where,

$$[k_{11}] = -[t_{12}]^{-1}[t_{11}], [k_{12}] = [t_{12}]^{-1},$$

$$[k_{21}] = [t_{21}] - [t_{22}][t_{12}]^{-1}[t_{11}], [k_{22}] = [t_{22}][t_{12}]^{-1}$$

After the derivation of dynamic stiffness matrix for each constituent members in local coordinate, we transform dynamic stiffness matrix in local coordinate to global coordinate using transformation matrix (coordinate transformation). Using this dynamic stiffness matrix in global coordinate, we construct the dynamic stiffness matrix for total crankshaft by the method of superposition.

Finally, by solving the determinant of the dynamic stiffness matrix of total crankshaft system, we determined the natural frequencies of the system. The mode shapes corresponding to each of the natural frequencies is calculated by substituting each natural frequency back into the dynamic stiffness matrix.

3.2.1 Beam Element

In this study, both the idealized round rods and rectangular blocks treated as homogeneous Euler Bernoulli beam elements of uniform cross section.

Because of symmetry of cross section, we can consider the following vibration mode

1. Axial mode along the x axis
2. Torsional mode about x axis
3. Bending mode in x-z plane
4. Bending mode in x-y plane

Now we know that dynamic stiffness matrix is a matrix which relates the nodal force vector to nodal displacement vector.

1. Axial mode along x-axis

$$f^a = K^a d^a$$

Where, $d^a = \left[u_{1,} u_2\right]^T$, $f^a = [P_1, P_2]^T$ and K^a is a 2 X 2 square matrix.

2. Torsional mode about x axis

$$f^t = K^t d^t$$

Where, $d^t = [\theta_1^x, \theta_2^x]^T$, $f^a = [T_1, T_2]^T$ and K^a is a 2 X 2 square matrix.

3. Bending mode in x-z plane

$$f^{xz} = K^{xz} d^{xz}$$

Where, $d^{xz} = \left[w_{1,} \theta_1^y, w_2, \theta_2^y\right]^T$, $f^{xz} = [Q_1, M_1, Q_2, M_2]^T$ and K^{xz} is a 4 X 4 square matrix.

4. Bending mode in x-y plane

$$f^{xy} = K^{xy} d^{xy}$$

Where, $d = \left[v_{1,} \theta_1^z, v_2, \theta_2^z\right]^T$, $f^{xy} = [V_1, M_1^z, V_2, M_2^z]^T$ and K^{xy} is a 4 X 4 square matrix.

Therefore, for a beam element,

$$\begin{Bmatrix} f_{i-1} \\ f_i \end{Bmatrix} = K^{beam} \begin{Bmatrix} d_{i-1} \\ d_i \end{Bmatrix}$$

$$\begin{bmatrix} P_1 \\ V_1 \\ Q_1 \\ T_1 \\ M_1 \\ M_1^z \\ P_2 \\ V_2 \\ Q_2 \\ T_2 \\ M_2 \\ M_2^z \end{bmatrix} = \begin{bmatrix} k_{11}^a & 0 & 0 & 0 & 0 & 0 & k_{12}^a & 0 & 0 & 0 & 0 & 0 \\ & k_{11}^{xy} & 0 & 0 & 0 & k_{12}^{xy} & 0 & k_{13}^{xy} & 0 & 0 & 0 & k_{14}^{xy} \\ & & k_{11}^{xz} & 0 & k_{12}^{xz} & 0 & 0 & 0 & k_{13}^{xz} & 0 & k_{14}^{xz} & 0 \\ & & & k_{11}^t & 0 & 0 & 0 & 0 & 0 & k_{12}^t & 0 & 0 \\ & & & & k_{22}^{xz} & 0 & 0 & 0 & k_{23}^{xz} & 0 & k_{24}^{xz} & 0 \\ & & & & & k_{22}^{xy} & 0 & k_{23}^{xy} & 0 & 0 & 0 & k_{24}^{xy} \\ & & & Sym. & & & k_{22}^a & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & k_{33}^{xy} & 0 & 0 & 0 & k_{34}^{xy} \\ & & & & & & & & k_{33}^{xz} & 0 & k_{34}^{xz} & 0 \\ & & & & & & & & & k_{22}^t & 0 & 0 \\ & & & & & & & & & & k_{44}^{xz} & 0 \\ & & & & & & & & & & & k_{44}^{xy} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_1^x \\ \theta_1^y \\ \theta_1^z \\ u_2 \\ v_2 \\ w_2 \\ \theta_2^x \\ \theta_2^y \\ \theta_2^z \end{bmatrix}$$

The derivations of k_{ij}^{xz} , k_{ij}^{xy} , k_{ij}^a and k_{ij}^t are given in next chapter.

3.2.2 Lumped Masses and Moment of Inertia

The pulley, crank gear and the flywheel are idealized as a set of lumped masses and MOI attached at their centers of gravity. Since a set of inertia forces and inertia torques induced depending upon their linear and angular acceleration at frequency ω, following is the dynamic stiffness matrix,

$$f = K^M d$$

Where, $d = \left[u, \ v, \ w, \ \theta_x, \ \theta_y, \ \theta_z\right]^T$, $f = [P, \ V, \ Q, \ T, \ M, \ M_z]^T$, and

$K^M = -\omega^2\, diag(m, m, m, J_{xx}, J_{yy}, J_{zz})$ is the set of dynamic stiffness matrix for a set of lumped masses and moment of inertia.

3.2.3 Linear Spring and Dash-Pots

The oil film of crank journal bearing is assumed to be isotropic and idealized by a set of linear springs and dash-pots. Assuming only spring forces and damping forces in y and z directions for each set of spring-dashpot, following is the dynamic stiffness matrix,

$$f = K^s d$$

Where, $d = \left[u,\ v,\ w,\ \theta_x,\ \theta_y,\ \theta_z\right]^T$, $f = [P,\ V,\ Q,\ T,\ M,\ M_z]^T$, and

$K^s = diag(0, K_{yy} + jC_{yy}\omega, K_{zz} + jC_{yy}\omega, 0,0,0)$ is the set of dynamic stiffness matrix for springs and dash-pots.

3.3 Coordinate Transformation and Construction of DSM for Total Crankshaft System

Let us assume a point P whose co-ordinate is (x, z) in x-z coordinate system. If we rotate this coordinate system by an angle α, then the coordinate of same point P with respect to the new coordinate system is $(\bar{x}, \bar{z})$ as shown in fig. 3.1.

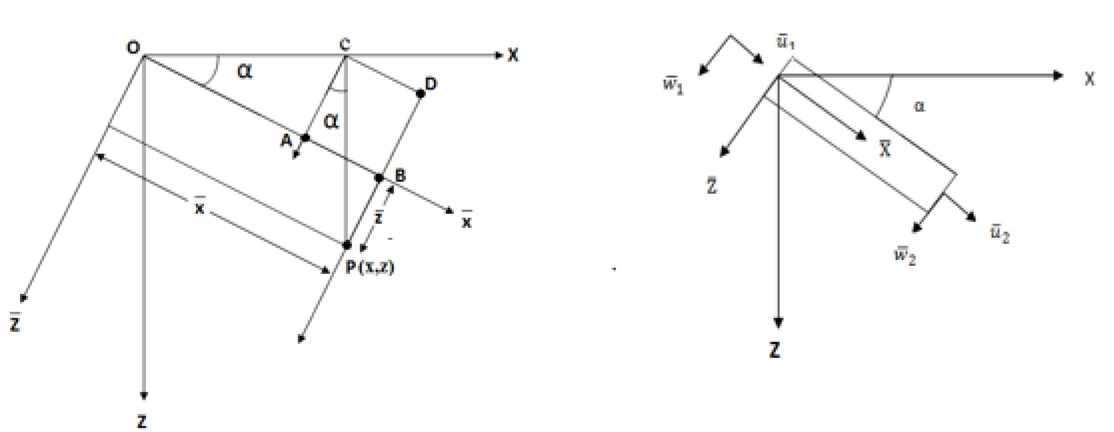

Figure 3.1 Representation of a point in local and global coordinate

$$OB = OA + AB = OA + CD$$

$$\bar{x} = x cos\alpha + z sin\alpha$$

$$PB = PD - BD = PD - AC$$

$$\bar{z} = zcos\alpha - xsin\alpha$$

$$\begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{z} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \\ \theta \end{bmatrix}$$

Then, the displacement is given as

$$\begin{bmatrix} \bar{u} \\ \bar{w} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}$$

$$\begin{bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{w}_2 \\ \bar{\theta}_2 \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha & 0 & 0 & 0 & 0 \\ -sin\alpha & cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\alpha & sin\alpha & 0 \\ 0 & 0 & 0 & -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{bmatrix}$$

$$\{\bar{d}\} = [Tr]\{d\}$$

$$[\bar{K}]\{\bar{d}\} = \{\bar{F}\}$$

$$\{\bar{F}\} = [Tr]\{F\}$$

$$[\bar{K}][Tr]\{d\} = [Tr]\{F\}$$

$$[Tr]^{-1}[\bar{K}][Tr]\{d\} = \{F\}$$

So from local to global coordinate transformation, we get

$$[K] = [Tr]^{-1}[\bar{K}][Tr]$$

After the construction of DSM for each element in global coordinate and stepwise superposition of the DSM for each member, we can finally construct the DSM for total crankshaft system:

$$f = K(\omega)d$$

Where, f is the force vector$[f_1, f_2, \dots \dots \dots f_n]^T$, d is displacement vector$[d_1, d, \dots \dots \dots d_n]^T$ and $K(\omega)$ is dynamic stiffness matrix of total crankshaft system in global coordinate.

3.4 Natural Frequency and Mode Shape using Newton Bisection Method

We calculate the natural frequencies of total crankshaft system by solving the following equation for ω, by "Newton bisection method"

$$det\ K(\omega) = 0$$

Newton bisection method is a numerical method of finding root or solution of an equation in the form of $f(x) = 0$ in which $f(x)$ is a continuous function defined in interval [a, b]. Where, $f(a)$ and $f(b)$ have opposite signs or we say $f(a)f(b) < 0$, then we say that at least one root "c" must lie between a and b.

$$c \in (a, b)\ such\ that\ f(c) = 0$$

$$c = \frac{a + b}{2}$$

<u>Algorithm:</u>

1. Define a and b such that $f(a)f(b) < 0$.
2. Define $c = \frac{a+b}{2}$.
3. If $b - c \leq$ ε then stop and accept c as the root. ε is error tolerance i.e. the absolute error in calculating the root must be less than ε.
4. If $f(a)f(c) \leq 0$ then set c as the new b. Otherwise set c as the new a. Return to the step1.

4.1 Dynamic Stiffness Matrix for Rod

Consider an elastic rod of length *l* with uniform cross sectional area A, as shown in Fig.4.1. We consider a small element of length *dx*, the force acting on the cross section of that element is given by P and $P + dP$ with

$$P = \sigma_x A = EA\frac{\partial u}{\partial x}$$

Where, σ_x is the axial stress, E is Young's modulus of elasticity, u is the axial displacement (displacement in x-direction), and $\partial u/\partial x$ is axial strain. If an external load $f(x,t)$, per unit length is applied, then the summation of forces in x-direction is

$$(\sigma_x + d\sigma_x)A + f(x,t)dx - \sigma_x A = \rho A dx\frac{\partial^2 u}{\partial t^2}$$

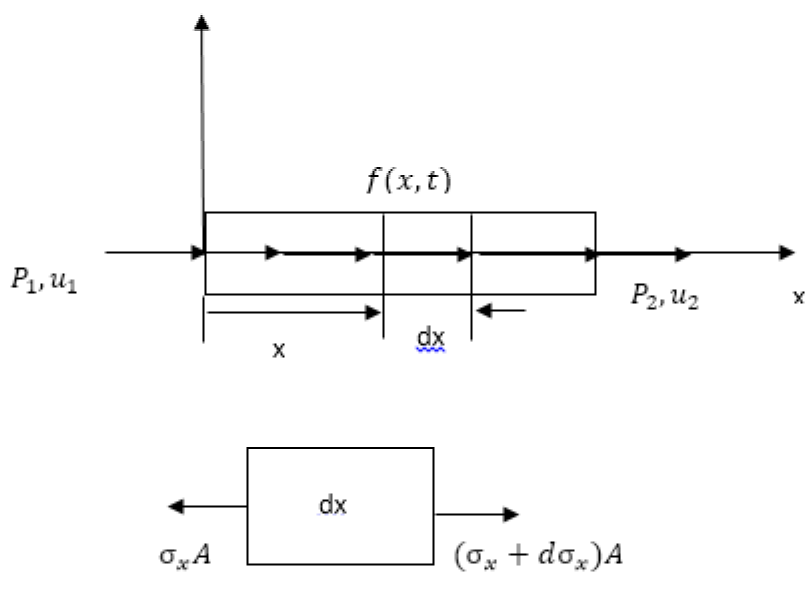

Figure 4.1 Standard rod element with nodal DOF

Where, ρ is the density of the rod. For free vibration of the rod, $f(x,t) = 0$.

So,

$$d\sigma_x \ A = \rho A dx \frac{\partial^2 u}{\partial t^2}$$

$$\sigma_x = E\varepsilon$$

$$\sigma_x = E \frac{\partial u}{\partial x}$$

$$d\sigma_x = E \frac{\partial^2 u}{\partial x^2} \ dx$$

$$EA \frac{\partial^2 u}{\partial x^2} = \rho A dx \frac{\partial^2 u}{\partial t^2}$$

$$EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} = 0$$

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1.1)$$

Where, $c^2 = EA/\rho A$. Equation (4.1.1) is the governing partial differential equation of free vibration of the rod. The governing PDE is separable partial differential equation. The solution of displacement field in separable form is assumed as

$$u(x,t) = U(x)T(t)$$

Where, the function $U(x)$ represents the normal mode and depends only on x and the function $T(t)$ depends only on t. Therefore, substituting this assumed solution of $u(x,t)$ in equation (4.1.1) gives,

$$\frac{c^2}{U(x)} \frac{d^2U(x)}{dx^2} = \frac{1}{T(t)} \frac{d^2T(t)}{dt^2}$$

Now, setting each side of equation equal to an unknown constant gives two ordinary differential equations

$$\frac{c^2}{U(x)} \frac{d^2U(x)}{dx^2} = a \ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1.2)$$

$$\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = a \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4.1.3)$$

First assuming the temporal solution for $T(t)$ and substituting in equation (4.1.3)

$$T(t) = Ge^{i\omega t}$$

$$\frac{d^2T(t)}{dt^2} - T(t)a = 0$$

$$-\omega^2 Ge^{i\omega t} - Ge^{i\omega t}a = 0$$

Simplifying and solving for unknown coefficient *a* gives,

$$a = -\omega^2$$

So,

$$\frac{c^2}{U(x)}\frac{d^2U(x)}{dx^2} = \frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = -\omega^2$$

Similarly, for spatial ordinary differential equation (4.1.2) (substituting the value of unknown coefficient *a* found from temporal solution), we have

$$\frac{d^2U(x)}{dx^2} + \frac{\omega^2}{c^2}U(x) = 0$$

$$\frac{d^2U(x)}{dx^2} + \beta^2 U(x) = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4.1.4)$$

$$\beta^2 = \frac{\omega^2}{c^2}$$

$$\beta^2 = \frac{\omega^2 \rho A}{EA}$$

$$\beta = \omega\sqrt{\frac{\rho}{E}}$$

Assuming the solution for the spatial ordinary differential equation and substituting in equation (4.1.4) gives,

$$U(x) = Ce^{sx}$$

$$Cs^2\, e^{sx} + \beta^2 Ce^{sx} = 0$$

$$s^2 + \beta^2 = 0$$

$$s = \pm i\beta$$

$$U(x) = C_1 e^{-i\beta x} + C_2 e^{i\beta x}$$

$$U(x) = \begin{bmatrix} e^{-i\beta x} & e^{i\beta x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Now, applying the boundary condition,

$$U = U_1 \; at \; x = 0 \; and \; U = U_2 \; at \; x = l$$

$$U(0) = U_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$U(L) = U_2 = \begin{bmatrix} e^{-i\beta l} & e^{i\beta l} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\{d\} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U(0) \\ U(l) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{-i\beta l} & e^{i\beta l} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\{d\} = [D]\{C\}$$

Similarly, evaluating the spatial force and substituting the boundary condition,

$$P(x) = EA\frac{dU}{dx} = EA(-i\beta C_1 e^{-i\beta x} + i\beta C_2 e^{i\beta x})$$

$$P(x) = EA\begin{bmatrix} -i\beta e^{-i\beta x} & i\beta e^{i\beta x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$P(0) = \left(-EA\frac{dU}{dx}\right)_{at\; x=0} = EA\begin{bmatrix} i\beta & -i\beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$P(l) = \left(EA\frac{dU}{dx}\right)_{at\; x=l} = EA\begin{bmatrix} -i\beta e^{-i\beta l} & i\beta e^{i\beta l} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\{f\} = \begin{bmatrix} P(0) \\ P(l) \end{bmatrix} = EA \begin{bmatrix} i\beta & -i\beta \\ -i\beta e^{-i\beta l} & i\beta e^{i\beta l} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\{f\} = [F]\{C\}$$

We know that dynamic stiffness matrix is that which relate nodal force vector to nodal displacement.

$$\{f\} = [K_{dyn}]\{d\}$$

$$[F]\{C\} = [K_{dyn}][D]\{C\}$$

$$[K_{dyn}] = [F][D]^{-1}$$

$$[K_{dyn}] = EA \begin{bmatrix} i\beta & -i\beta \\ -i\beta e^{-i\beta l} & i\beta e^{i\beta l} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ e^{-i\beta l} & e^{i\beta l} \end{bmatrix}^{-1}$$

From Trigonometry,

$$e^{ix} = cosx + isinx$$

$$e^{-ix} = cosx - isinx$$

After using these relations and solving through MATLAB, we get,

$$[K_{dyn}] = EA \begin{bmatrix} \frac{\beta cos\beta l}{sin\beta l} & \frac{-\beta}{sin\beta l} \\ \frac{-\beta}{sin\beta l} & \frac{\beta cos\beta l}{sin\beta l} \end{bmatrix}$$

$$[K_{dyn}] = EA\beta \begin{bmatrix} \frac{cos\beta l}{sin\beta l} & \frac{-1}{sin\beta l} \\ \frac{-1}{sin\beta l} & \frac{cos\beta l}{sin\beta l} \end{bmatrix}$$

$$[K_{dyn}] = EA\omega\sqrt{\frac{\rho}{E}}\begin{bmatrix} \frac{cos\beta l}{sin\beta l} & \frac{-1}{sin\beta l} \\ \frac{-1}{sin\beta l} & \frac{cos\beta l}{sin\beta l} \end{bmatrix} \; where \; \beta = \sqrt{\frac{\rho}{E}}\omega$$

4.2 Dynamic Stiffness Matrix for Torsion

For a shaft of length *l* with a uniform cross sectional area A, let us consider a small element of length dx as shown in fig 4.2. If polar mass moment of inertia of the shaft per unit length is I_o and angular twist is θ, the inertia torque acting on the element of length dx is $I_o dx \frac{\partial^2\theta}{\partial t^2}$ (assuming no external torque is acting). Balancing the torques, we get

$$(T + dT) - T = I_o dx \frac{\partial^2\theta}{\partial t^2}$$

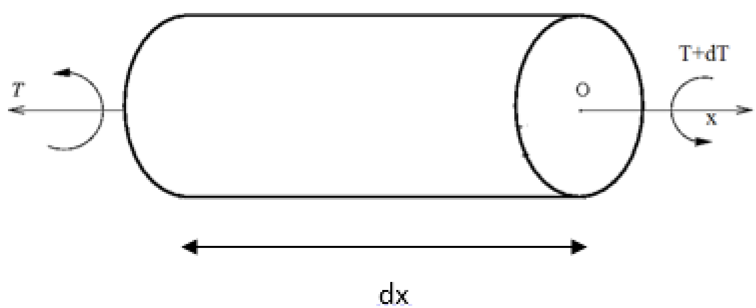

Figure 4.2 Standard torsional element with nodal DOF

$$dT = I_o dx \frac{\partial^2\theta}{\partial t^2}$$

$$dT = \frac{\partial T}{\partial x} dx$$

If polar moment of inertia of area for beam is I_P and modulus of rigidity is G, then

$$T = GI_P \frac{d\theta}{dx}$$

$$\frac{\partial}{\partial x}\left(GI_P \frac{\partial\theta}{\partial x}\right) = I_o \frac{\partial^2\theta}{\partial t^2}$$

$$GI_P \frac{\partial^2\theta}{\partial x^2} = I_o \frac{\partial^2\theta}{\partial t^2}$$

$$c^2 \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial t^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (4.2.1)$$

Where, $= \sqrt{GI_p/I_0}$. Equation (4.2.1) is the governing partial differential equation of free vibration of torsion of shaft and $I_0 = \rho I_p$. Therefore,

$$c = \sqrt{\frac{G}{\rho}}$$

It can be seen that the above PDE in eqn. (4.2.1) is similar to the PDE of axial vibration of rod obtained in eqn. (4.1.1). By analogy,

$$u \to \theta$$

$$E \to G$$

$$A \to I_p$$

So $[K_{Dyn}]$ in case of torsion is

$$[K_{dyn}] = GI_p\omega\sqrt{\frac{\rho}{G}}\begin{bmatrix} \frac{cos\beta l}{sin\beta l} & \frac{-1}{sin\beta l} \\ \frac{-1}{sin\beta l} & \frac{cos\beta l}{sin\beta l} \end{bmatrix}$$

Where, $\beta = \sqrt{\frac{\rho}{G}}\omega$

4.3 Dynamic Stiffness Matrix for Beam

Consider a beam element of length dx of cross sectional area *A*, and mass density ρ, as shown in Fig.4.3, where, *V* is shear force, *M* is bending moment and $f(x,t)$ is external load per unit length applied on beam element. Since the inertia force acting on the beam element of the beam is $\rho A dx \frac{\partial^2 w}{\partial t^2}$

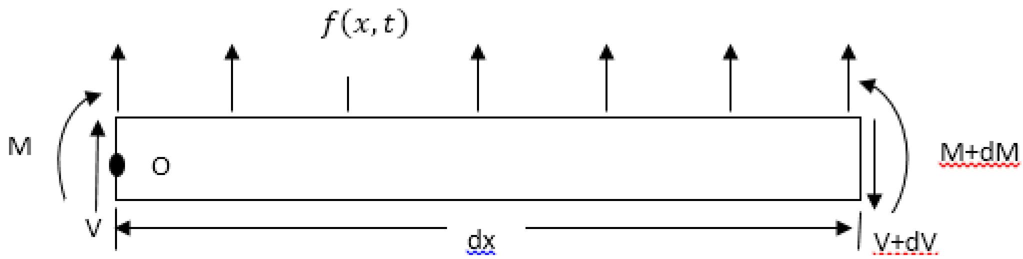

Figure 4.3 Infinitesimal beam element with nodal DOF

Then equating the force, we get

$$V - (V + dV) + f(x,t)dx = \rho A dx \frac{\partial^2 w}{\partial t^2}$$

$$-dV + f(x,t)dx = \rho A dx \frac{\partial^2 w}{\partial t^2}$$

$$dV = \frac{\partial V}{\partial x} dx$$

$$-\frac{\partial V}{\partial x} dx + f(x,t)dx = \rho A dx \frac{\partial^2 w}{\partial t^2}$$

$$-\frac{\partial V}{\partial x} + f(x,t) = \rho A \frac{\partial^2 w}{\partial t^2} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (4.3.1)$$

Equating the moment about point O

$$(M + dM) - (V + dV)dx + f(x,t)dx \frac{dx}{2} - M = 0$$

Eliminating the terms involving second power in dx

$$dM - Vdx = 0$$

$$dM = \frac{\partial M}{\partial x} dx$$

$$\frac{\partial M}{\partial x} dx - Vdx = 0$$

$$V = \frac{\partial M}{\partial x}$$

$$\frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2}$$

Substituting in eqn. (4.3.1),

$$-\frac{\partial^2 M}{\partial x^2} + f(x,t) = \rho A \frac{\partial^2 w}{\partial t^2}$$

$$M = EI \frac{\partial^2 w}{\partial x^2}$$

$$-\frac{\partial^2 (EI \frac{\partial^2 w}{\partial x^2})}{\partial x^2} + f(x,t) = \rho A \frac{\partial^2 w}{\partial t^2}$$

For free vibration

$$f(x,t) = 0$$

$$-\frac{\partial^2 (EI \frac{\partial^2 w}{\partial x^2})}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial^2 (EI \frac{\partial^2 w}{\partial x^2})}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0$$

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0$$

$$c^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3.2)$$

Equation (4.3.2) is the governing partial differential equation of motion of beam element.

Where,

$$c = \sqrt{\frac{EI}{\rho A}}$$

Assuming the solution in variable separable form and substituting in equation (4.3.2),

$$w(x,t) = W(x)T(t)$$

$$\frac{c^2}{W(x)}\frac{d^4W}{dx^4} = \frac{-1}{T(t)}\frac{d^2T}{dt^2} = a(assume)$$

Now, setting each side of equation equal to an unknown constant gives two constant coefficient ordinary differential equations

$$\frac{-1}{T(t)}\frac{d^2T(t)}{dt^2} = \ a \ and \ \frac{c^2}{W(x)}\frac{d^4W}{dx^4} = a$$

Assuming temporal solution and substituting in above PDE

$$T(t) = Ge^{i\omega t}$$

$$\frac{d^2T(t)}{dt^2} + T(t)a = 0$$

$$-\omega^2 Ge^{i\omega t} + Ge^{-i\omega t}a = 0$$

$$a = \omega^2$$

$$\frac{c^2}{W(x)}\frac{d^4W}{dx^4} = \frac{-1}{T(t)}\frac{d^2T}{dt^2} = \omega^2$$

Similarly, for spatial ordinary differential equation (substituting the value of unknown coefficient *a* found from temporal solution)

$$\frac{d^4W}{dx^4} - \beta^4 W(x) = 0$$

$$\beta^4 = \frac{\rho A\omega^2}{EI}$$

$$\frac{d^4W}{dx^4} - \beta^4 W(x) = 0$$

Assuming the solution and substituting in the above equation gives,

$$W(x) = Ce^{sx}$$

$$Cs^4e^{sx} - \beta^4Ce^{sx} = 0$$

$$s^4 - \beta^4 = 0$$

$$s^2 = \pm\beta^2$$

$$s_{1,2} = \pm\beta$$

$$s_{3,4} = \pm i\beta$$

Hence the solution is,

$$W(x) = C_1e^{-i\beta x} + C_2e^{-\beta x} + C_3e^{i\beta x} + C_4e^{\beta x}$$

$$W(x) = \begin{bmatrix} e^{-i\beta x} & e^{-\beta x} & e^{i\beta x} & e^{\beta x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\theta(x) = \frac{dW}{dx} = -i\beta C_1e^{-i\beta x} - \beta C_2e^{-\beta x} + i\beta C_3e^{i\beta x} + \beta C_4e^{\beta x}$$

$$\theta(x) = \frac{dW}{dx} = \begin{bmatrix} -i\beta e^{-i\beta x} & -\beta e^{-\beta x} & i\beta e^{i\beta x} & \beta e^{\beta x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

For transverse shear force and bending moment for fig. 4.3

$$M = EI\frac{d^2W}{dx^2}$$

$$V = \frac{dM}{dx}$$

$$V = EI\frac{d^3W}{dx^3}$$

And

$$\frac{dW}{dx} = -i\beta C_1 e^{-i\beta x} - \beta C_2 e^{-\beta x} + i\beta C_3 e^{i\beta x} + \beta C_4 e^{\beta x}$$

$$\frac{d^2W}{dx^2} = i^2\beta^2 C_1 e^{-i\beta x} + \beta^2 C_2 e^{-\beta x} + i^2\beta^2 C_3 e^{i\beta x} + \beta^2 C_4 e^{\beta x}$$

$$\frac{d^3W}{dx^3} = -i^3\beta^3 C_1 e^{-i\beta x} - \beta^3 C_2 e^{-\beta x} + i^3\beta^3 C_3 e^{i\beta x} + \beta^3 C_4 e^{\beta x}$$

$$M(x) = EI\frac{d^2W}{dx^2} = EI(i^2\beta^2 C_1 e^{-i\beta x} + \beta^2 C_2 e^{-\beta x} + i^2\beta^2 C_3 e^{i\beta x} + \beta^2 C_4 e^{\beta x})$$

$$M(x) = EI\frac{d^2W}{dx^2} = EI(-\beta^2 C_1 e^{-i\beta x} + \beta^2 C_2 e^{-\beta x} - \beta^2 C_3 e^{i\beta x} + \beta^2 C_4 e^{\beta x})$$

$$M(x) = EI\begin{bmatrix} -\beta^2 e^{-i\beta x} & \beta^2 e^{-\beta x} & -\beta^2 e^{i\beta x} & \beta^2 e^{\beta x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$V(x) = EI\frac{d^3W}{dx^3} = EI(-i^3\beta^3 C_1 e^{-i\beta x} - \beta^3 C_2 e^{-\beta x} + i^3\beta^3 C_3 e^{i\beta x} + \beta^3 C_4 e^{\beta x})$$

$$V(x) = EI(i\beta^3 C_1 e^{-i\beta x} - \beta^3 C_2 e^{-\beta x} - i\beta^3 C_3 e^{i\beta x} + \beta^3 C_4 e^{\beta x})$$

$$V(x) = EI\begin{bmatrix} i\beta^3 e^{-i\beta x} & -\beta^3 e^{-\beta x} & -i\beta^3 e^{i\beta x} & \beta^3 e^{\beta x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

The equations shown above are the expressions for the displacement (w), slope (θ), bending moment (M), and shear force (V) for the Euler Bernoulli beam element respectively.

4.3.1 Beam Element in x-z plane

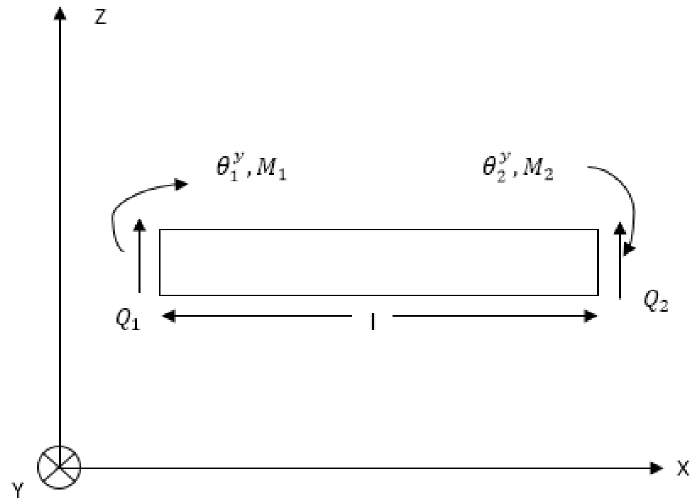

Figure 4.4 Beam element in x-z plane with nodal DOF

$$W(x) = [e^{-i\beta x} \quad e^{-\beta x} \quad e^{i\beta x} \quad e^{\beta x}] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$W_1 = W(0) = [1 \quad 1 \quad 1 \quad 1] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$W_2 = W(l) = [e^{-i\beta l} \quad e^{-\beta l} \quad e^{i\beta l} \quad e^{\beta l}] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\theta^y = \frac{dW}{dx} = [-i\beta e^{-i\beta x} \quad -\beta e^{-\beta x} \quad i\beta e^{i\beta x} \quad \beta e^{\beta x}] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\theta_1^y = -\left(\frac{dW}{dx}\right)_{at\ x=0} = [i\beta \quad \beta \quad -i\beta \quad -\beta] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\theta_2^y = -\left(\frac{dW}{dx}\right)_{at\ x=l} = [i\beta e^{-i\beta l} \quad \beta e^{-\beta l} \quad -i\beta e^{i\beta l} \quad -\beta e^{\beta l}]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

So nodal displacement and slope of Beam element are

$$\{d\} = \begin{bmatrix} W_1 \\ \theta_1^y \\ W_2 \\ \theta_2^y \end{bmatrix}$$

$$\{d\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ i\beta & \beta & -i\beta & -\beta \\ e^{-i\beta l} & e^{-\beta l} & e^{i\beta l} & e^{\beta l} \\ i\beta e^{-i\beta l} & \beta e^{-\beta l} & -i\beta e^{i\beta l} & -\beta e^{\beta l} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\{d\} = [D]\{C\}$$

$$M = EI_{xz}\frac{d^2W}{dx^2} = EI_{xz}[-\beta^2 e^{-i\beta x} \quad \beta^2 e^{-\beta x} \quad -\beta^2 e^{i\beta x} \quad \beta^2 e^{\beta x}]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$M_1 = \left(EI_{xz}\frac{d^2W}{dx^2}\right)_{at\ x=0} = EI_{xz}[-\beta^2 \quad \beta^2 \quad -\beta^2 \quad \beta^2]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$M_2 = -\left(EI_{xz}\frac{d^2W}{dx^2}\right)_{at\ x=l} = -EI_{xz}[-\beta^2 e^{-i\beta l} \quad \beta^2 e^{-\beta l} \quad -\beta^2 e^{i\beta l} \quad \beta^2 e^{\beta l}]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$= EI_{xz}[\beta^2 e^{-i\beta l} \quad -\beta^2 e^{-\beta l} \quad \beta^2 e^{i\beta l} \quad -\beta^2 e^{\beta l}]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$Shear\ force\ Q_1 = \left(EI_{xz}\frac{d^3W}{dx^3}\right)_{at\ x=0} = EI_{xz}[i\beta^3 \quad -\beta^3 \quad -i\beta^3 \quad \beta^3]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$Q_2 = -\left(EI_{xz}\frac{d^3W}{dx^3}\right)_{at\ x=l} = -EI_{xz}\begin{bmatrix} i\beta^3 e^{-i\beta l} & -\beta^3 e^{-\beta l} & -i\beta^3 e^{i\beta l} & \beta^3 e^{\beta l} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$= EI_{xz}\begin{bmatrix} -i\beta^3 e^{-i\beta l} & \beta^3 e^{-\beta l} & i\beta^3 e^{i\beta l} & -\beta^3 e^{\beta l} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\{f\} = \begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix}$$

$$\{f\} = EI_{xz}\begin{bmatrix} i\beta^3 & -\beta^3 & -i\beta^3 & \beta^3 \\ -\beta^2 & \beta^2 & -\beta^2 & \beta^2 \\ -i\beta^3 e^{-i\beta l} & \beta^3 e^{\beta l} & i\beta^3 e^{i\beta l} & -\beta^3 e^{\beta l} \\ \beta^2 e^{-i\beta l} & -\beta^2 e^{-\beta l} & \beta^2 e^{i\beta l} & -\beta^2 e^{\beta l} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\{f\} = [F]\{C\}$$

From the definition, Dynamic Stiffness Matrix is

$$\{f\} = \left[K_{Dyn}\right]\{d\}$$

$$[F]\{C\} = \left[K_{Dyn}\right][D]\{C\}$$

$$\left[K_{Dyn}\right] = [F][D]^{-1}$$

$$\left[K_{Dyn}\right]$$
$$= EI_{xz}\begin{bmatrix} i\beta^3 & -\beta^3 & -i\beta^3 & \beta^3 \\ -\beta^2 & \beta^2 & -\beta^2 & \beta^2 \\ -i\beta^3 e^{-i\beta l} & \beta^3 e^{\beta l} & i\beta^3 e^{i\beta l} & -\beta^3 e^{\beta l} \\ \beta^2 e^{-i\beta l} & -\beta^2 e^{-\beta l} & \beta^2 e^{i\beta l} & -\beta^2 e^{\beta l} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ i\beta & \beta & -i\beta & -\beta \\ e^{-i\beta l} & e^{-\beta l} & e^{i\beta l} & e^{\beta l} \\ i\beta e^{-i\beta l} & \beta e^{-\beta l} & -i\beta e^{i\beta l} & -\beta e^{\beta l} \end{bmatrix}^{-1}$$

From Trigonometry,

$$e^x = coshx + sinhx$$

$$e^{-x} = coshx - sinhx$$

$$e^{ix} = cosx + isinx$$

$$e^{-ix} = cosx - isinx$$

After putting these expression for $[K_{Dyn}]$ and solving through MATLAB

$$[K_{Dyn}] = \begin{bmatrix} k_{11}^{xz} & k_{12}^{xz} & k_{13}^{xz} & k_{14}^{xz} \\ k_{21}^{xz} & k_{22}^{xz} & k_{23}^{xz} & k_{24}^{xz} \\ k_{31}^{xz} & k_{32}^{xz} & k_{33}^{xz} & k_{34}^{xz} \\ k_{41}^{xz} & k_{42}^{xz} & k_{43}^{xz} & k_{44}^{xz} \end{bmatrix}$$

$$k_{11}^{xz} = k_0^{xz}\beta^3(cos\beta l\ sinh\beta l + cosh\beta l\ sin\beta l)$$

$$k_{12}^{xz} = k_{21}^{xz} = -k_0^{xz}\beta^2\ sin\beta l\ sinh\beta l$$

$$k_{13}^{xz} = k_{31}^{xz} = -k_0^{xz}\beta^3(sin\beta l + sinh\beta l)$$

$$k_{14}^{xz} = k_{41}^{xz} = -k_0^{xz}\beta^2(cos\beta l - cosh\beta l)$$

$$k_{22}^{xz} = -k_0^{xz}\beta(cos\beta l\ sinh\beta l - cosh\beta l\ sin\beta l)$$

$$k_{23}^{xz} = k_{32}^{xz} = -k_0^{xz}\beta^2(cos\beta l - cosh\beta l)$$

$$k_{24}^{xz} = k_{42}^{xz} = -k_0^{xz}\beta(sin\beta l - sinh\beta l)$$

$$k_{33}^{xz} = k_0^{xz}\beta^3(cos\beta l\ sinh\beta l + cosh\beta l\ sin\beta l)$$

$$k_{34}^{xz} = k_{43}^{xz} = k_0^{xz}\beta^2\ sin\beta l\ sinh\beta l$$

$$k_{44}^{xz} = -k_0^{xz}\beta(cos\beta l\ sinh\beta l - cosh\beta l\ sin\beta l)$$

Where,

$$k_0^{xz} = \frac{EI_{xz}}{1 - cosh\beta l\ cos\beta l}\ \&\ \beta = \sqrt[4]{\omega^2\rho A/EI_{xz}}$$

4.3.2 Beam Element in x-y plane

Boundary condition of slope, shear force and bending moment in x-y plane as shown in fig.4.5 is different from that of x-z plane.

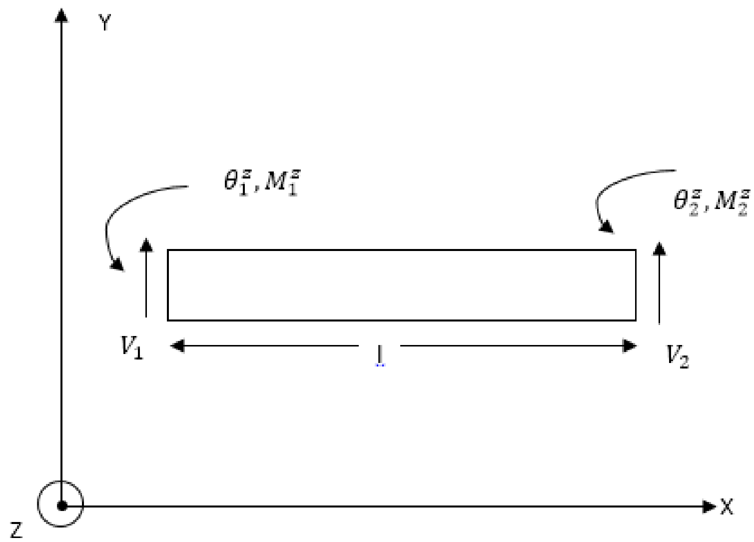

Figure 4.5 Beam element in x-y plane with nodal DOF

Therefore,

$$v(x) = \begin{bmatrix} e^{-i\beta x} & e^{-\beta x} & e^{i\beta x} & e^{\beta x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$v_1 = v(0) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$v_2 = v(l) = \begin{bmatrix} e^{-i\beta l} & e^{-\beta l} & e^{i\beta l} & e^{\beta l} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\theta_1^z = \left(\frac{dv}{dx}\right)_{at\ x=0} = \begin{bmatrix} -i\beta & -\beta & i\beta & \beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\theta_2^z = \left(\frac{dv}{dx}\right)_{at\ x=l} = \begin{bmatrix} -i\beta e^{-i\beta l} & -\beta e^{-\beta l} & i\beta e^{i\beta l} & \beta e^{\beta l} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$M_1^z = -\left(EI_{xy}\frac{d^2v}{dx^2}\right)_{at\ x=0} = EI_{xy}[\beta^2 \quad -\beta^2 \quad \beta^2 \quad -\beta^2]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$M_2^z = \left(EI_{xy}\frac{d^2v}{dx^2}\right)_{at\ x=l} = EI_{xy}[-\beta^2 e^{-i\beta l} \quad \beta^2 e^{-\beta l} \quad -\beta^2 e^{i\beta l} \quad \beta^2 e^{\beta l}]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$Shear\ force\ V_1 = \left(EI_{xy}\frac{d^3v}{dx^3}\right)_{at\ x=0} = EI_{xy}[\ i\beta^3 \quad -\beta^3 \quad -i\beta^3 \quad \beta^3]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$V_2 = -\left(EI_{xy}\frac{d^3v}{dx^3}\right)_{at\ x=l} = EI_{xy}[-i\beta^3 e^{-i\beta l} \quad \beta^3 e^{-\beta l} \quad i\beta^3 e^{i\beta l} \quad -\beta^3 e^{\beta l}]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

$$\left[K_{Dyn}\right] = [F][D]^{-1}$$

$$\left[K_{Dyn}\right] = EI_{xy}\begin{bmatrix} i\beta^3 & -\beta^3 & -i\beta^3 & \beta^3 \\ \beta^2 & -\beta^2 & \beta^2 & -\beta^2 \\ -i\beta^3 e^{-i\beta l} & \beta^3 e^{\beta l} & i\beta^3 e^{i\beta l} & -\beta^3 e^{\beta l} \\ -\beta^2 e^{-i\beta l} & \beta^2 e^{-\beta l} & -\beta^2 e^{i\beta l} & \beta^2 e^{\beta l} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -i\beta & -\beta & i\beta & \beta \\ e^{-i\beta l} & e^{-\beta l} & e^{i\beta l} & e^{\beta l} \\ -i\beta e^{-i\beta l} & -\beta e^{-\beta l} & i\beta e^{i\beta l} & \beta e^{\beta l} \end{bmatrix}^{-1}$$

After solving this we see that each elements of K_{xy} is same as elements of K_{xz}, only I_{xz} is replaced by I_{xy} (moment of inertia of area about z-axis), and reversing the sign of the k_{12}^{xz}, k_{14}^{xz}, k_{23}^{xz}, k_{34}^{xz} and these symmetrical elements.

Chapter 5 : Mathematical Modeling for Crankshaft

Mathematical modeling of single cylinder engine crankshaft using Dynamic Stiffness Matrix Method and to determine the natural frequency and mode shapes of the crankshaft using MATLAB. Material for crankshaft is alloy steel and free vibration analysis is performed.

Material:

Material used for crankshaft is AISI 4340 alloy steel. The mechanical property of AISI 4340 alloy steel is given below:

Table 5.1 Properties of Alloy steel

Properties	Value
Density(ρ)	7800 kg/m^3
Elastic Modulus(E)	210 Gpa
Shear Modulus(G)	77 Gpa

Table 5.2 Chemical composition for AISI 4340 Alloy steel

Element	Content (%)
Iron, Fe	95.195-96.33
Nickel, Ni	1.65-2.0
Chromium, Cr	0.700-0.900
Manganese, Mn	0.600-0.800
Carbon, C	0.370-0.430
Molybdenum, Mo	0.200-0.300
Silicon, Si	0.150-0.300
Sulfur, S	0.0400

Table 5.3 Dimension of member elements of a single cylinder engine crankshaft as shown in fig.5.1 [1]

Element	Dimension(mm)
1	φ 20 x 50
2	φ 25 x 59
3, 5	33 x 17 x 28.5
4	φ 26 x 52
6	φ 25 x 39
7	φ 22 x 15
8	φ 18 x 25
9	φ 13 x 18
10, 11	33 x 17 x 15
12, 13	70 x 13 x 35

Here, φ denotes the diameter of element

Table 5.4 Mass and Moment of inertia for front pulley and flywheel

Parts	Mass (kg)	Moment of inertia(kg m^2)		
		I_x	I_y	I_z
Front pulley	2.01	6.38 x 10^{-3}	3.65 x 10^{-3}	3.65 x 10^{-3}
Flywheel	10.15	0.15	.075	.075

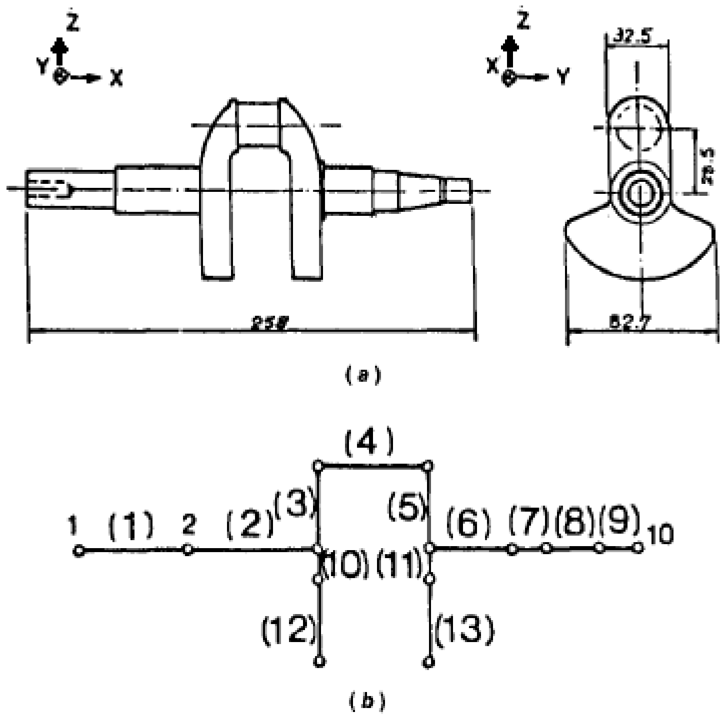

Figure 5.1 Rough sketch and idealized model of single cylinder engine crankshaft [1]

Above figure (5.1) shows a sketch of single cylinder engine crankshaft and its model idealized with 13 constituent member element. The dimension of each element are also shown in Table 5.3.

In the planar-structure crankshaft, two kinds of coupled vibration are induced independently.

1) In plane mode: In this mode, coupled vibration of the bending mode in the crank throw plane(x-z plane) and the axial mode along the x-axis.
2) Out of plane mode: In this mode, coupled vibration of the bending mode in the plane orthogonal to the crank throw plane (x-y plane) and the torsional mode about x-axis.

<u>In plane Mode:</u>

For in plane mode, the dynamic stiffness matrix for coupled vibration of bending mode in x-z plane and axial mode along x-axis is given as follows

For axial mode

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} k_{11}^a & k_{12}^a \\ k_{21}^a & k_{22}^a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

For bending mode in x-z plane

$$\begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} k_{11}^{xz} & k_{12}^{xz} & k_{13}^{xz} & k_{14}^{xz} \\ k_{21}^{xz} & k_{22}^{xz} & k_{23}^{xz} & k_{24}^{xz} \\ k_{31}^{xz} & k_{32}^{xz} & k_{33}^{xz} & k_{34}^{xz} \\ k_{41}^{xz} & k_{42}^{xz} & k_{43}^{xz} & k_{44}^{xz} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1^y \\ w_2 \\ \theta_2^y \end{bmatrix}$$

So, *K* matrix for coupled vibration of bending mode in x-z plane and axial mode is

$$\begin{bmatrix} P_1 \\ Q_1 \\ M_1 \\ P_2 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} k_{11}^a & 0 & 0 & k_{12}^a & 0 & 0 \\ 0 & k_{11}^{xz} & k_{12}^{xz} & 0 & k_{13}^{xz} & k_{14}^{xz} \\ 0 & k_{21}^{xz} & k_{22}^{xz} & 0 & k_{23}^{xz} & k_{24}^{xz} \\ k_{21}^a & 0 & 0 & k_{22}^a & 0 & 0 \\ 0 & k_{31}^{xz} & k_{32}^{xz} & 0 & k_{33}^{xz} & k_{34}^{xz} \\ 0 & k_{41}^{xz} & k_{42}^{xz} & 0 & k_{43}^{xz} & k_{44}^{xz} \end{bmatrix} \begin{bmatrix} u_1 \\ w_1 \\ \theta_1^y \\ u_2 \\ w_2 \\ \theta_2^y \end{bmatrix}$$

The values of all the element of K^a and K^{xz} have been given in chapter 4.

Out of plane mode:

Similarly, the dynamic stiffness matrix for out of plane mode is,

For torsional mode,

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} k_{11}^t & k_{12}^t \\ k_{21}^t & k_{22}^t \end{bmatrix} \begin{bmatrix} \theta_1^x \\ \theta_2^x \end{bmatrix}$$

For bending mode in x-y plane

$$\begin{bmatrix} V_1 \\ M_1^z \\ V_2 \\ M_2^z \end{bmatrix} = \begin{bmatrix} k_{11}^{xz} & k_{12}^{xz} & k_{13}^{xz} & k_{14}^{xz} \\ k_{21}^{xz} & k_{22}^{xz} & k_{23}^{xz} & k_{24}^{xz} \\ k_{31}^{xz} & k_{32}^{xz} & k_{33}^{xz} & k_{34}^{xz} \\ k_{41}^{xz} & k_{42}^{xz} & k_{43}^{xz} & k_{44}^{xz} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1^z \\ v_2 \\ \theta_2^z \end{bmatrix}$$

So, *K* matrix for coupled vibration of the bending mode in x-y plane and the torsional mode is

$$\begin{bmatrix} T_1 \\ V_1 \\ M_1^z \\ T_2 \\ V_2 \\ M_2^z \end{bmatrix} = \begin{bmatrix} k_{11}^t & 0 & 0 & k_{12}^t & 0 & 0 \\ 0 & k_{11}^{xy} & k_{12}^{xy} & 0 & k_{13}^{xy} & k_{14}^{xy} \\ 0 & k_{21}^{xy} & k_{22}^{xy} & 0 & k_{23}^{xy} & k_{24}^{xy} \\ k_{21}^t & 0 & 0 & k_{22}^t & 0 & 0 \\ 0 & k_{31}^{xy} & k_{32}^{xy} & 0 & k_{33}^{xy} & k_{34}^{xy} \\ 0 & k_{41}^{xy} & k_{42}^{xy} & 0 & k_{43}^{xy} & k_{44}^{xy} \end{bmatrix} \begin{bmatrix} \theta_1^x \\ v_1 \\ \theta_1^z \\ \theta_2^x \\ v_2 \\ \theta_2^z \end{bmatrix}$$

Procedure:

Step1: Derive the dynamic stiffness matrix for each element for in plane mode and out of plane mode in local coordinate.

Step 2: Transform the dynamic stiffness matrix from local coordinate to global coordinate using coordinate transformation (transformation matrix) as described in chapter 3.

Step 3: Assemble the dynamic stiffness matrix for each element to make the dynamic stiffness matrix for whole crankshaft.

For example for in plane mode,

Dynamic stiffness matrix for element 1

$$\begin{bmatrix} P_1^{(1)} \\ Q_1^{(1)} \\ M_1^{(1)} \\ P_2^{(1)} \\ Q_2^{(1)} \\ M_2^{(1)} \end{bmatrix} = \begin{bmatrix} k_{11}^{a(1)} & 0 & 0 & k_{12}^{a(1)} & 0 & 0 \\ 0 & k_{11}^{xz(1)} & k_{12}^{xz(1)} & 0 & k_{13}^{xz(1)} & k_{14}^{xz(1)} \\ 0 & k_{21}^{xz(1)} & k_{22}^{xz(1)} & 0 & k_{23}^{xz(1)} & k_{24}^{xz(1)} \\ k_{21}^{a(1)} & 0 & 0 & k_{22}^{a(1)} & 0 & 0 \\ 0 & k_{31}^{xz(1)} & k_{32}^{xz(1)} & 0 & k_{33}^{xz(1)} & k_{34}^{xz(1)} \\ 0 & k_{41}^{xz(1)} & k_{42}^{xz(1)} & 0 & k_{43}^{xz(1)} & k_{44}^{xz(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ w_1^{(1)} \\ \theta_1^{y(1)} \\ u_2^{(1)} \\ w_2^{(1)} \\ \theta_2^{y(1)} \end{bmatrix}$$

Superscripts represent the element number

Dynamic stiffness matrix for element 2

$$\begin{bmatrix} P_2^{(2)} \\ Q_2^{(2)} \\ M_2^{(2)} \\ P_3^{(2)} \\ Q_3^{(2)} \\ M_3^{(2)} \end{bmatrix} = \begin{bmatrix} k_{11}^{a(2)} & 0 & 0 & k_{12}^{a(2)} & 0 & 0 \\ 0 & k_{11}^{xz(2)} & k_{12}^{xz(2)} & 0 & k_{13}^{xz(2)} & k_{14}^{xz(2)} \\ 0 & k_{21}^{xz(2)} & k_{22}^{xz(2)} & 0 & k_{23}^{xz(2)} & k_{24}^{xz(2)} \\ k_{21}^{a(2)} & 0 & 0 & k_{22}^{a(2)} & 0 & 0 \\ 0 & k_{31}^{xz(2)} & k_{32}^{xz(2)} & 0 & k_{33}^{xz(2)} & k_{34}^{xz(2)} \\ 0 & k_{41}^{xz(2)} & k_{42}^{xz(2)} & 0 & k_{43}^{xz(2)} & k_{44}^{xz(2)} \end{bmatrix} \begin{bmatrix} u_2^{(2)} \\ w_2^{(2)} \\ \theta_2^{y(2)} \\ u_3^{(2)} \\ w_3^{(2)} \\ \theta_3^{y(2)} \end{bmatrix}$$

Dynamic stiffness matrix for element 1 & 2

$$\begin{bmatrix} P_1 = P_1^{(1)} \\ Q_1 = Q_1^{(1)} \\ M_1 = M_1^{(1)} \\ P_2 = P_2^{(1)} + P_2^{(2)} \\ Q_2 = Q_2^{(1)} + Q_2^{(2)} \\ M_2 = M_2^{(1)} + M_2^{(2)} \\ P_3 = P_3^{(2)} \\ Q_3 = Q_3^{(2)} \\ M_3 = M_3^{(2)} \end{bmatrix} = \left[K_{Dyn}\right]_{1\&2} \begin{bmatrix} u_1 = u_1^{(1)} \\ w_1 = w_1^{(1)} \\ \theta_1^y = \theta_1^{y(1)} \\ u_2 = u_2^{(1)} + u_2^{(2)} \\ w_2 = w_2^{(1)} + w_2^{(2)} \\ \theta_2^y = \theta_2^{y(1)} + \theta_2^{y(2)} \\ u_3 = u_3^{(2)} \\ w_3 = w_3^{(2)} \\ \theta_3^y = \theta_3^{y(2)} \end{bmatrix}$$

$$\left[K_{Dyn}\right]_{1\&2}$$

$$= \begin{bmatrix} k_{11}^{a(1)} & 0 & 0 & k_{12}^{a(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{11}^{xz(1)} & k_{12}^{xz(1)} & 0 & k_{13}^{xz(1)} & k_{14}^{xz(1)} & 0 & 0 & 0 \\ 0 & k_{21}^{xz(1)} & k_{22}^{xz(1)} & 0 & k_{23}^{xz(1)} & k_{24}^{xz(1)} & 0 & 0 & 0 \\ k_{21}^{a(1)} & 0 & 0 & k_{22}^{a(1)} + k_{11}^{a(2)} & 0 & 0 & k_{12}^{a(2)} & 0 & 0 \\ 0 & k_{31}^{xz(1)} & k_{32}^{xz(1)} & 0 & k_{33}^{xz(1)} + k_{11}^{xz(2)} & k_{34}^{xz(1)} + k_{12}^{xz(2)} & 0 & k_{13}^{xz(2)} & k_{14}^{xz(2)} \\ 0 & k_{41}^{xz(1)} & k_{42}^{xz(1)} & 0 & k_{43}^{xz(1)} + k_{21}^{xz(2)} & k_{44}^{xz(1)} + k_{22}^{xz(2)} & 0 & k_{23}^{xz(2)} & k_{24}^{xz(2)} \\ 0 & 0 & 0 & k_{21}^{a(2)} & 0 & 0 & k_{22}^{a(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{31}^{xz(2)} & k_{32}^{xz(2)} & 0 & k_{33}^{xz(2)} & k_{34}^{xz(2)} \\ 0 & 0 & 0 & 0 & k_{41}^{xz(2)} & k_{42}^{xz(2)} & 0 & k_{43}^{xz(2)} & k_{44}^{xz(2)} \end{bmatrix}$$

Using the same procedure, assembling is done for whole crankshaft system, for both in plane mode and out of plane mode. MATLAB is used for assembling, which is in the form,

$$f = K(\omega)d$$

Where, f is force vector $[f_1, f_2, \dots \dots \dots f_n]^T$, q is displacement vector$[d_1, d, \dots \dots \dots d_n]^T$ and $K(\omega)$ is the dynamic stiffness matrix of total crankshaft system in global coordinate.

Step 4: Put $K(\omega) = 0$ and find the natural frequencies for both in plane mode and out of plane mode. Newton Bisection method is used for solving $K(\omega) = 0$.

Step 5: The mode shapes can be calculated by substituting the ω back into the equation $K(\omega)d = 0$ and solving for d.

Table 6.1 Natural frequencies of a single cylinder engine crankshaft In Plane Mode (free-free)

Mode	Measured(Hz)	Calculated(Hz)	Calculated(Using Matlab in Hz)
1st	1250	1259	1128.5
2nd	2500	2567	1512.5
3rd	2950	3528	2755.5
4th	3950	3842	3224.5
5th	5200	4500	3504.5
			3851.5
			4070.5
			4373.5
			4408.5

Obtained natural frequencies for In plane mode are closely matched with the calculated and measured natural frequencies [1]. However, first natural frequency 1128.5 Hz is less than the measured natural frequency of 1250 Hz. But second and third natural frequencies 2755.5 Hz and 3504.5 Hz are greater than the measured natural frequencies 2500 Hz and 2950 Hz respectively. Now the fourth and fifth natural frequencies 3504.5 Hz and 4408.5 Hz are less than the natural frequencies of 3950 Hz and 5200 Hz respectively.

Mode shapes for In Plane Mode:

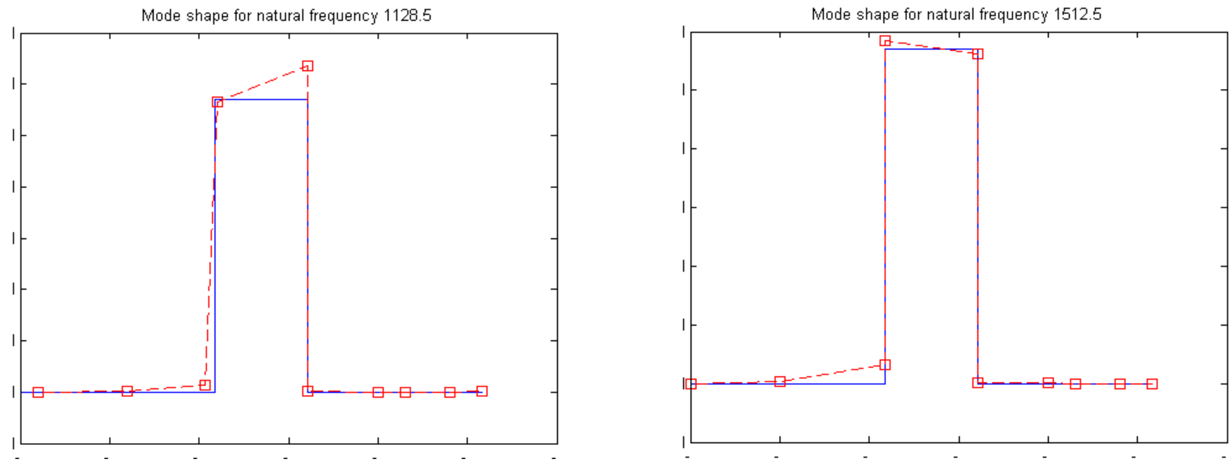

Figure 6.1 Mode shapes for natural frequencies 1128.5 & 1512.5 Hz

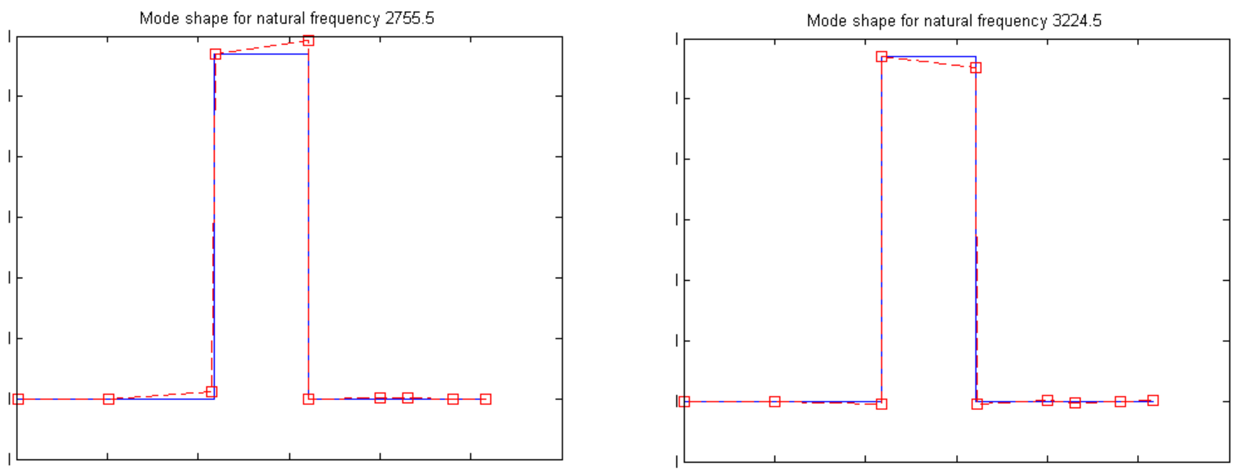

Figure 6.2 Mode shapes for natural frequencies 2755.5 & 3224.5 Hz

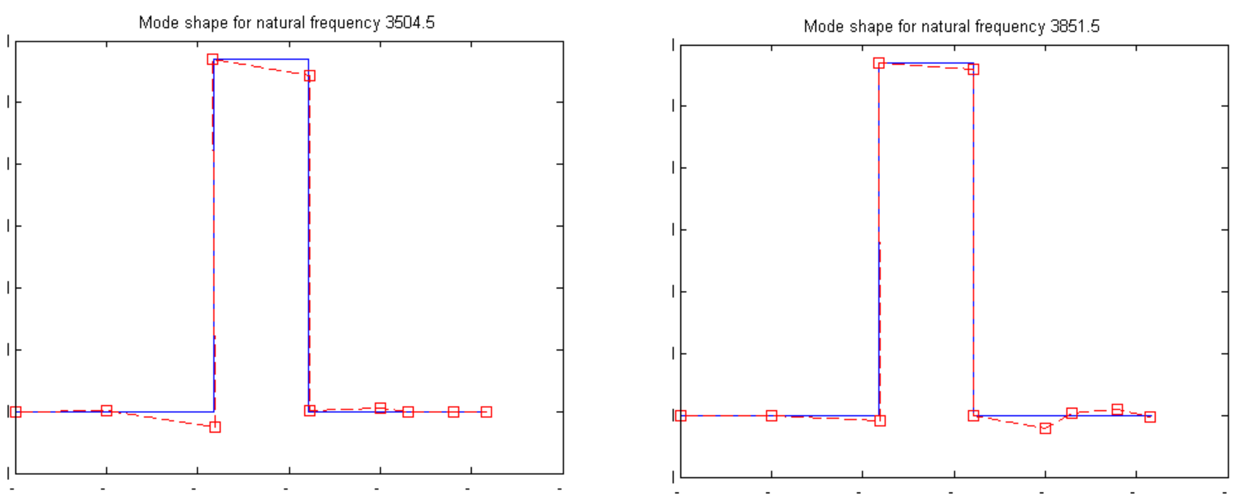

Figure 6.3 Mode shapes for natural frequencies 3504.5 & 3851.5 Hz

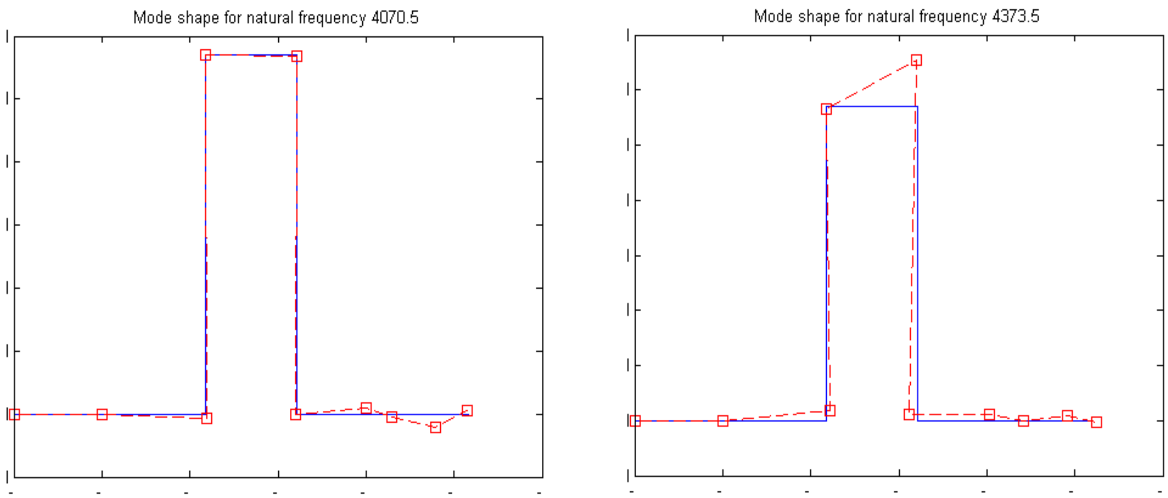

Figure 6.4 Mode shapes for natural frequencies 4070.5 & 4373.5 Hz

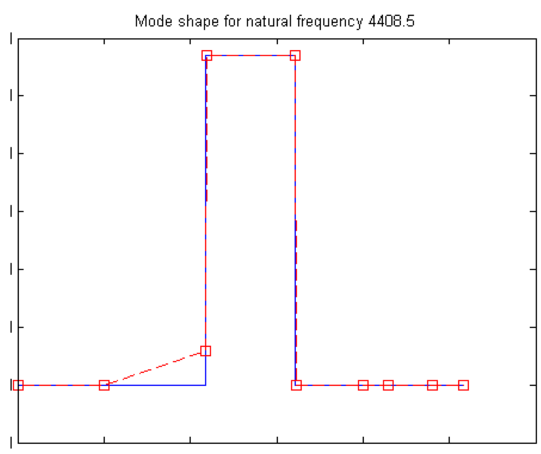

Figure 6.5 Mode shapes for natural frequency 4408.5 Hz

The above figures show the mode shapes of crankshaft for In plane mode. The mode shapes obtained at one natural frequency is different from the other mode shape. However, the mode shape for natural frequencies 1128.5, 2755.5, 4373.5 Hz are looks similar and the mode shape for natural frequencies 3224.5, 3504.5, 3851.5 are looks similar and the remaining mode shapes are different. It means the movement of each element of crankshaft is in different manner.

Table 6.2 Natural frequencies of a single cylinder engine crankshaft Out of Plane Mode (free-free)

Mode	Measured(Hz)	Calculated(Hz)	Calculated(Using Matlab in Hz)
1st	1650	1444	1314.5
2nd	3050	2847	1819.5
3rd	3900	3812	2122.5
4th	5050	4840	2303.5
5th	5700	5810	3877.5
			4024.5
			4305.5
			4525.5
			4810.5
			6002.5

Obtained natural frequencies for Out of plane mode are closely matched with the calculated and measured natural frequencies [1]. However, first, second, third and fourth natural frequencies 1314.5, 2303.5, 3877.5, 4810.5 Hz are less than the measured natural frequencies of 1650, 3050, 3900, 5050 Hz respectively. But the fifth natural frequency 6002.5 Hz greater than the measured natural frequencies of 5700 Hz.

Mode shapes for out of plane mode:

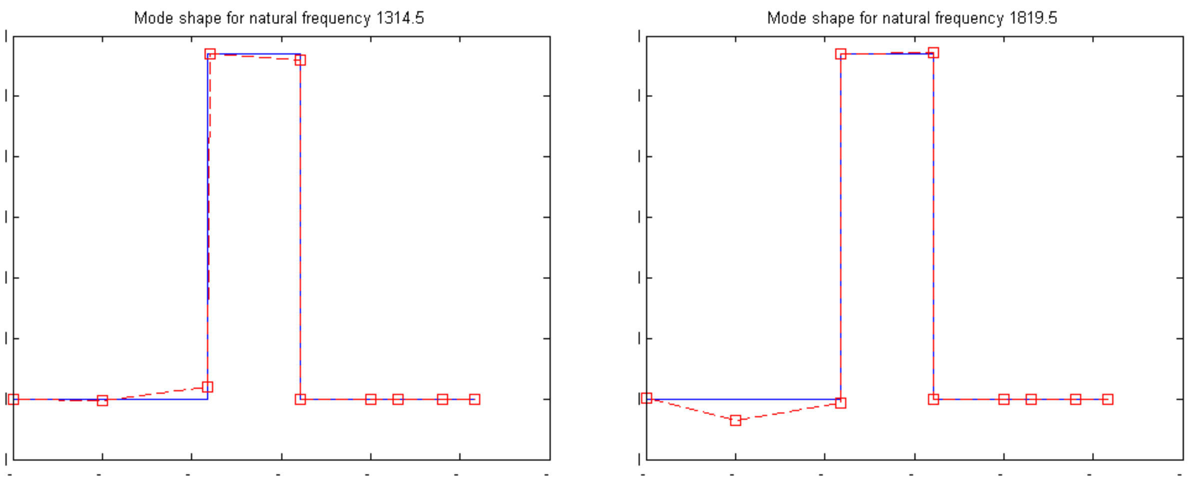

Figure 6.6 Mode shapes for natural frquencies 1314.5 & 1819.5 Hz

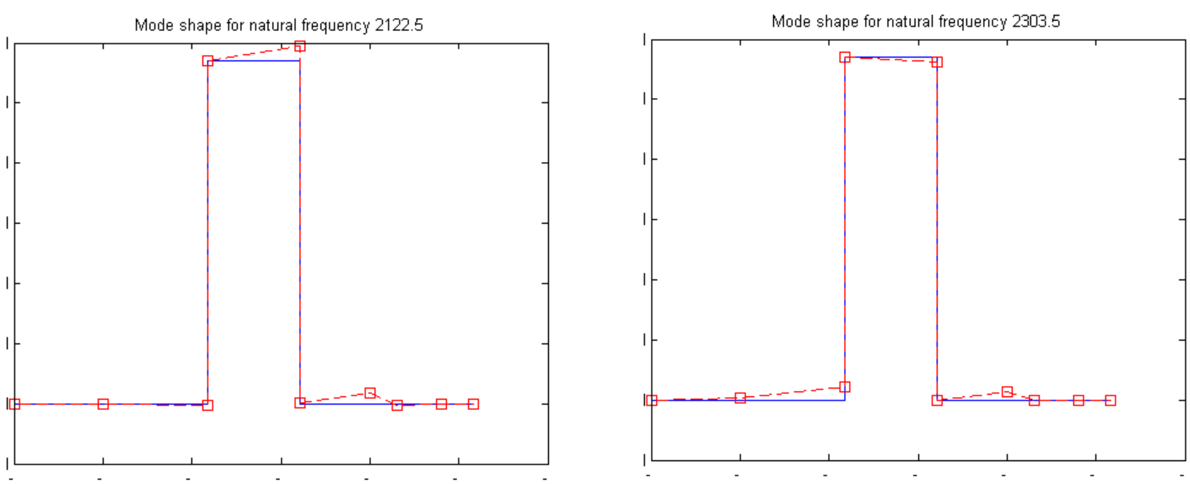

Figure 6.7 Mode shapes for natural frquencies 2122.5 & 2303.5 Hz

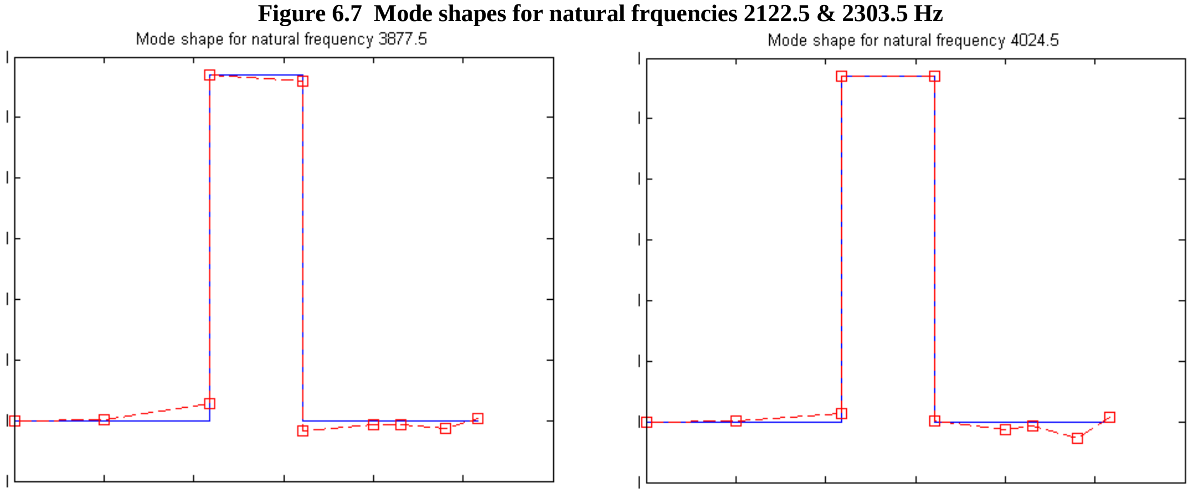

Figure 6.8 Mode shapes for natural frequencies 3877.5 & 4024.5 Hz

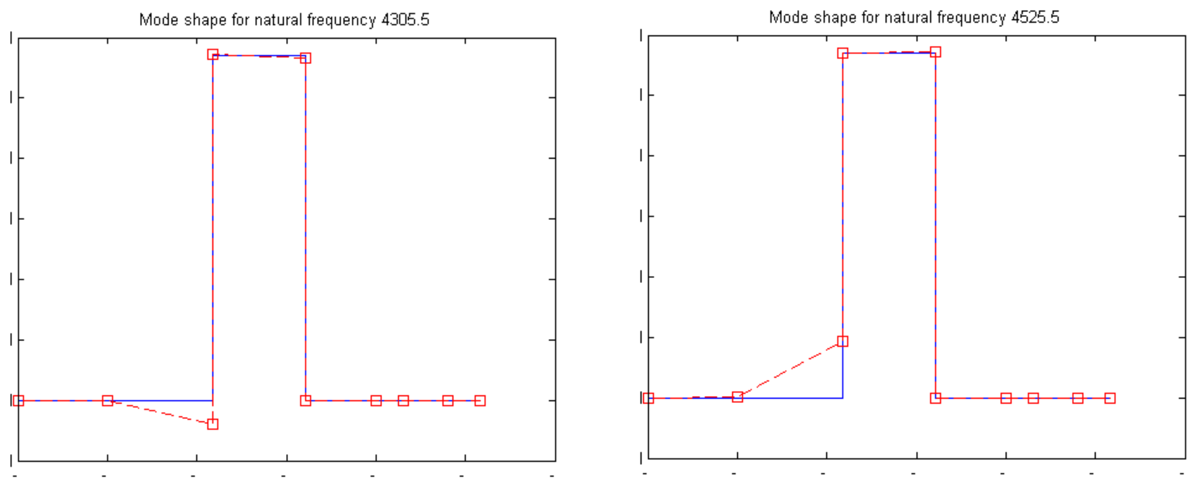

Figure 6.9 Mode shapes for natural frequencies 4305.5 & 4525.5 Hz

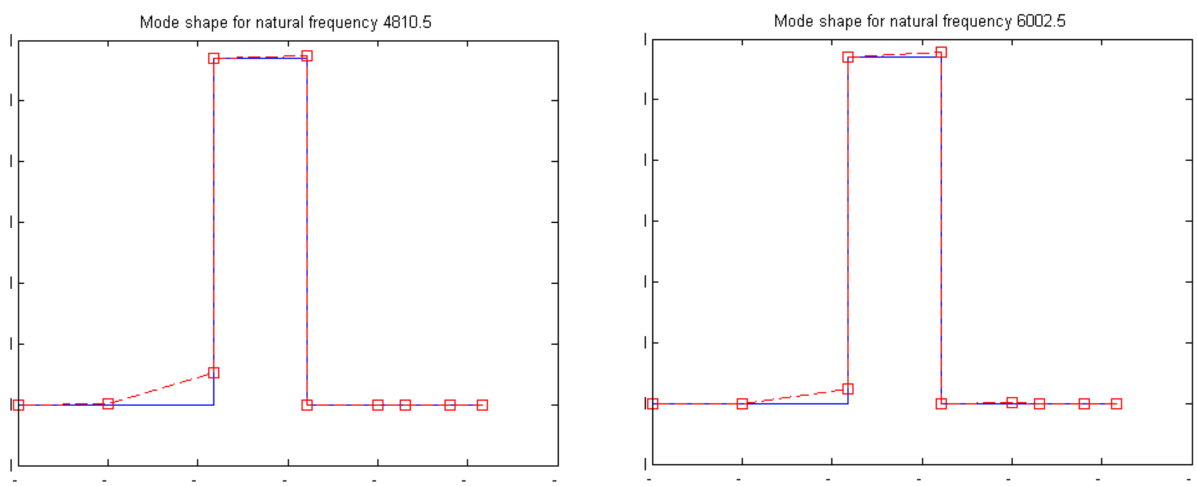

Figure 6.10 Mode shapes for natural frequencies 4810.5 & 6002.5 Hz

The above figure shows the mode shapes of crankshaft for Out of plane mode. The mode shapes obtained at one natural frequency is different from the other mode shape. However, the mode shape for natural frequencies 1314.5, 2303.5, 3877.5 Hz are looks similar except the right hand side element of crankshaft and the mode shape for natural frequencies 2122.5, 4810.5 and 6002.5 Hz are looks similar and the remaining mode shapes are different. It means the movement of each element of crankshaft is in different manner.

Chapter 7 : Conclusions and Future Works

7.1 Conclusions

1) A simple modeling and analysis procedure is proposed for three dimensional free vibrations analysis of single cylinder engine crankshaft system.
2) The DSM for beam element can be derived either from TMM or by the governing differential equation of beam in the form of $K(\omega)$ and also derived from FEM model in the form of $[K - \omega^2 M]$. However, the latter might have more merits in that one can examine the overall distribution of masses and stiffness in the system but the computational cost of analysis by FEM is high.
3) There are good agreement between calculated and experimental results for the natural frequencies.
4) The natural frequency for both in plane mode and out of plane mode is above 1000 Hz and the natural frequency of a running engine is very low <100 Hz. So crankshaft and engine is never in resonant condition
5) We hope that this simple modeling and analysis procedure can be readily used for the practical studies of actual engine crankshafts.

7.2 Future Works

This method can be used for the forced vibration analysis of single cylinder, inline four cylinder engine & V-engine crankshaft. The crankshafts which are used in compressor and other application are also be analyzed by this method. FEM method may also be used for modeling for crank arm, counterweight or whole crankshaft.

References

1) H. Okamura, A. Shinno, T. Yamanaka, A. Suzuki, "Simple Modeling and Analysis for Crankshaft Three-Dimensional Vibrations, Part 1: Background and Application to Free Vibrations", Journal of Vibration and Acoustics (1995) Vol. 117/79.
2) Y. Kang, G.-J. Sheen and M.-H.Tseng, "Modal Analyses and Experiments for Engine Crankshafts", Journal of Sound and Vibration (1998) 214(3), 413-430.
3) Z. P. Mourelatos, "An efficient Crankshaft Dynamic Analysis using Sub structuring with Ritz vectors", Journal of Sound and vibration (2000) 238(3), 495-527.
4) K. Hu, Z.P. Mourelatos, N. Vlahopoulos, "Computational analysis for dynamic response of a rotating shaft on flexible support structure with clearances", Journal of Sound and Vibration 267 (2003) 1–28.
5) Jong-Shyong Wu, Chin-Tzu Chen, "A lumped-mass TMM for free vibration analysis of a multi-step Timoshenko beam carrying eccentric lumped masses with rotary inertias", Journal of Sound and Vibration 301 (2007) 878–897.
6) Jong-Shyong Wu, Chin-Tzu Chen, "A continuous-mass TMM for free vibration analysis of a non-uniform beam with various boundary conditions and carrying multiple concentrated elements", Journal of Sound and Vibration 311 (2008) 1420–1430.
7) P.Charles, Jyoti K. Sinha, F. Gu, L.Lidstone, A.D. Ball, "Detecting the Crankshaft Torsional vibration of Diesel engines for Combustion related Diagnosis", Journal of Sound and Vibration 321 (2009) 1171–1185.
8) Yasin Yilmaz, Gunay Anlas, "An investigation of the effect of counterweight configuration on main bearing load and crankshaft bending stress", Advances in Engineering Software 40 (2009) 95–104.
9) X. Zhang, S.D. Yu, "Torsional vibration of crankshaft in an engine–propeller nonlinear dynamical system", Journal of Sound and Vibration 319 (2009) 491–514.
10) M. Desbazeille, R.B. Randall, F. Guillet, M. El Badaoui, C. Hoisnard, "Model-based diagnosis of large diesel engines based on angular speed variations of the crankshaft", Mechanical Systems and Signal Processing 24 (2010) 1529–1541.

11) B.Y. Yu, Q.K. Feng, X.L. Yu, “Modal and vibration analysis of reciprocating compressor crankshaft system”, 7th International Conference on Compressors and their Systems 2011, 295–303.
12) Jian Meng, Yongqi Liu, Ruixiang Liu, “Finite Element Analysis of 4-Cylinder Diesel Crankshaft”, I.J. Image, Graphics and Signal Processing, 2011, 5, 22-29.
13) Z.M. Bulatovic, M.S. Stavljanin, M.V. Tomic, D.M. Knezevic, S.Lj. Biocanin, “Measurement and analysis of angular velocity variations of twelve-cylinder diesel engine crankshaft”, Mechanical Systems and Signal Processing 25 (2011) 3043–3061.
14) M. Fonte, P. Duart, L. Reis, M. Freitas, V. Infante, “Failure mode analysis of two crankshafts of a single cylinder diesel engine”, Article Engineering Failure Analysis (2015).
15) A.J. Anyaegbunam, “The Dynamic Stiffness Matrix of a Beam-column Element”, Ph.D.Thesis, University of Nigeria, Nsukka.
16) J.N. Reddy, “An Introduction To The Finite Element Method”, 2nd edition, McGraw-Hill, Inc., 1993.
17) P. Seshu, “Textbook of Finite Element Analysis”, PHI Learning Private Limited, New Delhi, 2003
18) A.J.M. Ferreira, “MATLAB Codes for Finite Element Analysis”, Springer Science+Business Media B.V., 2009.
19) Singiresu S. Rao, “Mechanical Vibrations”, 5th edition, Pearson Education, Inc., 2010.
20) Rudra Pratap, “Getting Started With MATLAB”, Oxford University Press, Inc., 2010.

Appendix

MATLAB PROGRAME

APPENDIX 1

```
clc
% Derivation of element of k matrix for xz plane
syms b l E I
a1 = cos(b*l);
a2 = sin(b*l);
a3 = sinh(b*l);
a4 = cosh(b*l);
D = [1 1 1 1;
    1i*b b -1i*b -b;
    a1-1i*a2 a4-a3 a1+1i*a2 a4+a3;
    1i*b*(a1-1i*a2) b*(a4-a3) -1i*b*(a1+1i*a2) -b*(a4+a3)];
F = E*I*[1i*b^3 -b^3 -1i*b^3 b^3;
    -b^2 b^2 -b^2 b^2;
    -1i*b^3*(a1-1i*a2) b^3*(a4-a3) 1i*b^3*(a1+1i*a2) -b^3*(a3+a4);
    b^2*(a1-1i*a2) -b^2*(a4-a3) b^2*(a1+1i*a2) -b^2*(a3+a4)];
sw1 = F/(D)
```

APPENDIX 2

```
clc
% Derivation of element of k matrix for xy plane
syms b l E I
a1 = cos(b*l);
a2 = sin(b*l);
a3 = sinh(b*l);
a4 = cosh(b*l);
D = [1 1 1 1;
    -1i*b -b 1i*b b;
    a1-1i*a2 a4-a3 a1+1i*a2 a4+a3;
    -1i*b*(a1-1i*a2) -b*(a4-a3) 1i*b*(a1+1i*a2) b*(a4+a3)];
F = E*I*[1i*b^3 -b^3 -1i*b^3 b^3;
    b^2 -b^2 b^2 -b^2;
    -1i*b^3*(a1-1i*a2) b^3*(a4-a3) 1i*b^3*(a1+1i*a2) -b^3*(a3+a4);
    -b^2*(a1-1i*a2) b^2*(a4-a3) -b^2*(a1+1i*a2) b^2*(a3+a4)];
sw1 = F/(D)
```

APPENDIX 3

```
clc
% Derivation of K matrix of whole crankshaft for in plane mode
function [stiffness,k1] = Axial_final1(w)
B = [1 2 3 4 5 6;4 5 6 7 8 9;7 8 9 10 11 12;10 11 12 13 14 15; 13 14 15 16
17 18; 16 17 18 19 20 21; 19 20 21 22 23 24; 22 23 24 25 26 27; 25 26 27
28 29 30;7 8 9 31 32 33; 16 17 18 34 35 36;31 32 33 37 38 39; 34 35 36 40
41 42];
stiffness = zeros(42);
rho = 7800*10^-9;
```

```
E = 210000;
% For element 1
d = 20;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 50;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
% Derive the element of k_a matrix
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
k1 = k_ele;
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(1,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 2
d = 25;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 59;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
```

```
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(2,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 3
b = 33;
h = 17;
Ixz = b*h^3/12;
A = b*h;
l = 28.5;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
```

```
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 270;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(3,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 4
d = 26;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 52;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(4,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 5
b = 33;
h = 17;
```

```
Ixz = b*h^3/12;
A = b*h;
l = 28.5;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(5,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 6
d = 25;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 39;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
```

```
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(6,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 7
d = 22;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 15;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
```

```
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(7,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 8
d = 18;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 25;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(8,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 9
d = 13;
Ixz = pi*d^4/64;
A = pi*d^2/4;
l = 18;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
```

```
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(9,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 10
b = 33;
h = 17;
Ixz = b*h^3/12;
A = b*h;
l = 15;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));

k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
```

```
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(10,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 11
b = 33;
h = 17;
Ixz = b*h^3/12;
A = b*h;
l = 15;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
```

```
k_g = inv(trans)*k_ele * trans;
indice=B(11,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 12
b = 70;
h = 13;
Ixz = b*h^3/12;
A = b*h;
l = 15;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(12,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 13
b = 70;
h = 13;
Ixz = b*h^3/12;
A = b*h;
l = 35;
beta_xz = (w^2*rho*A/(E*Ixz))^.25;
gama_xz = beta_xz * l;
k = E * Ixz * beta_xz^3 / (1 - (cosh(gama_xz) * cos(gama_xz)));
k_11 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz));
k_12 = -k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_21 = k_12;
```

```
k_13 = -k * (sinh(gama_xz) + sin(gama_xz));
k_31= k_13;
k_14 = -k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_41 = k_14;
k_22 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
k_23 = k * (cosh(gama_xz) - cos(gama_xz)) / beta_xz;
k_32 = k_23;
k_24 = k * (sinh(gama_xz) - sin(gama_xz)) / beta_xz^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xz) * cos(gama_xz) + cosh(gama_xz) * sin(gama_xz)) /
beta_xz;
k_34 = k * sinh(gama_xz) * sin(gama_xz) / beta_xz;
k_43 = k_34;
k_44 = -k * (sinh(gama_xz) * cos(gama_xz) - cosh(gama_xz) * sin(gama_xz))
/ beta_xz^2;
P = (rho / E) ^ 0.5 * w * l;
k_a = A * (rho * E) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_a(1,1) 0 0 k_a(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_a(2,1) 0 0 k_a(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(13,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For Front Pulley & Flywheel
    indice = [1 2 3 28 29 30];
    load =  -w^2 * blkdiag(2,2,2*3650,10.15,10.15,2*75000);

    stiffness(indice,indice)=stiffness(indice,indice)+load;
% For Spring & Dash Pot
    spring = blkdiag(0,10^5,0,0,10^5,0);
        stiffness(indice,indice)=stiffness(indice,indice)+spring;
```

APPENDIX 4

```
clc
% Eigen Vector & Mode shape for in plane mode
l = [50 59 28.5 52 28.5 39 15 25 18 15 35];
x = [0 50 109 109 161 161 200 215 240 258];
y = [0 0 0 28.5 28.5 0 0 0 0 0];
plot(x,y)
set(gca,'XTickLabel','-');
set(gca,'YTickLabel','I');
hold on
[stiffness,k1] = Axial_final1(1128.5);
f = det(stiffness);
a = stiffness(2:42,2:42);
b = [0;0; stiffness(4,1); 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0;
0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
c1 = inv(a)*b;
```

```
c1 = 10*[1 c1']';
x_new = [x(1)+c1(1) x(2)+c1(4) x(3)+c1(7) x(4)+c1(10) x(5)+c1(13)
x(6)+c1(16) x(7)+c1(19) x(8)+c1(22) x(9)+c1(25) x(10)+c1(28)];
y_new = [y(1)+c1(2) y(2)+c1(5) y(3)+c1(8) y(4)+c1(11) y(5)+c1(14)
y(6)+c1(17) y(7)+c1(20) y(8)+c1(23) y(9)+c1(26) y(10)+c1(29)];
%Add effect of theta
x_new = [x_new(1) x_new(1)+ l(1)*cos(c1(3)) x_new(2)+ l(2)*cos(c1(6))
x_new(4)+ l(3)*sin(c1(9)) x_new(4)+ l(4)*cos(c1(12)) x_new(6)-
l(5)*sin(c1(15)) x_new(6)+ l(6)*cos(c1(18)) x_new(7)+ l(7)*cos(c1(21))
x_new(8)+ l(8)*cos(c1(24)) x_new(9)+ l(9)*cos(c1(27))];
y_new = [y_new(1) y_new(2)- l(1)*sin(c1(3)) y_new(3)- l(2)*sin(c1(6))
y_new(3)+ l(3)*cos(c1(9)) y_new(5)- l(4)*sin(c1(12)) y_new(5)-
l(5)*cos(c1(15)) y_new(7)- l(6)*sin(c1(18)) y_new(8)- l(7)*sin(c1(21))
y_new(9)- l(8)*sin(c1(24)) y_new(10)- l(9)*sin(c1(27))];
plot(x_new,y_new,'--rs')
title('Mode shape for natural frequency 1128.5')
```

APPENDIX 5

```
clc
% Derivation of K matrix of whole crankshaft for out of plane mode
function [stiffness,k1] = Torsional_final1(w)
B = [1 2 3 4 5 6;4 5 6 7 8 9;7 8 9 10 11 12;10 11 12 13 14 15; 13 14 15 16
17 18; 16 17 18 19 20 21; 19 20 21 22 23 24; 22 23 24 25 26 27; 25 26 27
28 29 30;7 8 9 31 32 33; 16 17 18 34 35 36;31 32 33 37 38 39; 34 35 36 40
41 42];
stiffness = zeros(42);
rho = 7800*10^-9;
E = 210000;
G = 77000;
% For element 1
d = 20;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 50;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
```

```
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
% Derive the element of k_t matrix
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
k1 = k_ele;
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(1,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 2
d = 25;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 59;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
```

```
k_g = inv(trans)*k_ele * trans;
indice=B(2,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 3
b = 33;
h = 17;
Ixy = h*b^3/12;
A = b*h;
l = 28.5;
Ip = (h*b^3/12) + (h^3*b/12);
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(3,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 4
d = 26;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 52;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
```

```
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(4,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 5
b = 33;
h = 17;
Ixy = h*b^3/12;
A = b*h;
l = 28.5;
Ip = (h*b^3/12) + (h^3*b/12);
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
```

```
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(5,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 6
d = 25;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 39;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
```

```
indice=B(6,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 7
d = 22;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 15;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(7,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 8
d = 18;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 25;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
```

```
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(8,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 9
d = 13;
Ixy = pi*d^4/64;
A = pi*d^2/4;
l = 18;
Ip = pi*d^4/32;
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
```

```
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 0;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(9,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 10
b = 33;
h = 17;
Ixy = h*b^3/12;
A = b*h;
l = 15;
Ip = (h*b^3/12) + (h*b^3/12);
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(10,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
```

```
% For element 11
b = 33;
h = 17;
Ixy = h*b^3/12;
A = b*h;
l = 15;
Ip = (h*b^3/12) + (h*b^3/12);
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(11,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 12
b = 70;
h = 13;
Ixy = h*b^3/12;
A = b*h;
l = 15;
Ip = (h*b^3/12) + (h*b^3/12);
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
```

```
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(12,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For element 13
b = 70;
h = 13;
Ixy = h*b^3/12;
A = b*h;
l = 35;
Ip = (h*b^3/12) + (h*b^3/12);
beta_xy = (w^2*rho*A/(E*Ixy))^.25;
gama_xy = beta_xy * l;
k = E * Ixy * beta_xy^3 / (1 - (cosh(gama_xy) * cos(gama_xy)));
k_11 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy));
k_12 = k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_21 = k_12;
k_13 = -k * (sinh(gama_xy) + sin(gama_xy));
k_31= k_13;
k_14 = k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_41 = k_14;
k_22 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
k_23 = -k * (cosh(gama_xy) - cos(gama_xy)) / beta_xy;
k_32 = k_23;
k_24 = k * (sinh(gama_xy) - sin(gama_xy)) / beta_xy^2;
k_42 = k_24;
k_33 = k * (sinh(gama_xy) * cos(gama_xy) + cosh(gama_xy) * sin(gama_xy)) /
beta_xy;
k_34 = -k * sinh(gama_xy) * sin(gama_xy) / beta_xy;
k_43 = k_34;
```

```
k_44 = -k * (sinh(gama_xy) * cos(gama_xy) - cosh(gama_xy) * sin(gama_xy))
/ beta_xy^2;
P = (rho / G) ^ 0.5 * w * l;
k_t = Ip * (rho * G) ^ 0.5 * w * [cos(P)/sin(P) -1/sin(P) ; -1/sin(P)
cos(P)/sin(P)];
k_ele = [k_t(1,1) 0 0 k_t(1,2) 0 0 ; 0 k_11 k_12 0 k_13 k_14; 0 k_21 k_22
0 k_23 k_24; k_t(2,1) 0 0 k_t(2,2) 0 0; 0 k_31 k_32 0 k_33 k_34; 0 k_41
k_42 0 k_43 k_44];
theta = 90;
a=cos(theta*pi/180);
b=sin(theta*pi/180);
trans = [a b 0 0 0 0;-b a 0 0 0 0;0 0 1 0 0 0;0 0 0 a b 0;0 0 0 -b a 0;0 0
0 0 0 1];
k_g = inv(trans)*k_ele * trans;
indice=B(13,:);
    stiffness(indice,indice)=stiffness(indice,indice)+k_g;
% For Front Pulley & Flywheel
    indice = [1 2 3 28 29 30];
    load =  -w^2 * blkdiag(6380*2,2.01,3650*2,150*2,10.15,75*2);
    stiffness(indice,indice)=stiffness(indice,indice)+load;
% For Spring & Dash Pot
        spring = blkdiag(0,10^5,0,0,10^5,0);
        stiffness(indice,indice)=stiffness(indice,indice)+spring;
```

APPENDIX 6

```
clc
% Eigen Vector & Mode shape for out of plane mode
l = [50 59 28.5 52 28.5 39 15 25 18 15 35];
x = [0 50 109 109 161 161 200 215 240 258];
y = [0 0 0 28.5 28.5 0 0 0 0 0];
plot(x,y)
set(gca,'XTickLabel','-');
set(gca,'YTickLabel','I');
hold on
[stiffness,k1] = Torsional_final1(2122.5);
f = det(stiffness);
a = stiffness(2:42,2:42);
b = [0;0; stiffness(4,1); 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0;
0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
c1 = inv(a)*b;
c1 = .1 * [1 c1']';
y_new = [y(1)+c1(2) y(2)+c1(5) y(3)+c1(8) y(4)+c1(11) y(5)+c1(14)
y(6)+c1(17) y(7)+c1(20) y(8)+c1(23) y(9)+c1(26) y(10)+c1(29)];
%Add effect of theta
x_new = [x(1) x(1)+ l(1)*cos(c1(3)) x(2)+ l(2)*cos(c1(6)) x(4)+
l(3)*sin(c1(9)) x(4)+ l(4)*cos(c1(12)) x(6)- l(5)*sin(c1(15)) x(6)+
l(6)*cos(c1(18)) x(7)+ l(7)*cos(c1(21)) x(8)+ l(8)*cos(c1(24)) x(9)+
l(9)*cos(c1(27))];
y_new = [y_new(1) y_new(2)- l(1)*sin(c1(3)) y_new(3)- l(2)*sin(c1(6))
y_new(3)+ l(3)*cos(c1(9)) y_new(5)- l(4)*sin(c1(12)) y_new(5)-
l(5)*cos(c1(15)) y_new(7)- l(6)*sin(c1(18)) y_new(8)- l(7)*sin(c1(21))
y_new(9)- l(8)*sin(c1(24)) y_new(10)- l(9)*sin(c1(27))];
plot(x,y_new,'--rs')
title('Mode shape for natural frequency 2122.5')
```