A

Dissertation Report

On

Vibration Analysis of Rotating Tapered Timoshenko Beam

Submitted in Partial Fulfilment of M.Tech. in Mechanical Engineering

By

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2013PDE5080

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JAIPUR –302017 (RAJASTHAN) INDIA

June-2015

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CERTIFICATE

This is certified that the dissertation entitled **"Vibration analysis of rotating tapered Timoshenko beam"** prepared by **KOMAL** (ID-2013PDE5080), in the partial fulfilment of **Master of Technology** in **MECHANICAL ENGINEERING** of Malaviya National Institute of Technology, Jaipur is found to be satisfactory and is hereby approved for submission. The contents of this dissertation, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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This Dissertation report is hereby approved for submission.

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ACKNOWLEDGEMENT

I am deeply grateful to my mentor **Dr. T.C. Gupta** for his constant encouragement and guidance for completion of this dissertation work. His moral support and valuable feedback have been a great source of inspiration for enhancing my work in this dissertation work.

I am extremely grateful to respected **Dr. Himanshu Chaudhary, Dr. Dinesh Kumar** and **Dr. Amit Singh** for always being a source of inspiration.

I would like to thank **Manoj Gupta**, **Harsh Dixit** (Phd. Research Scholar) and **Ajit Singh** for helping me throughout my dissertation work.

Lastly I would like to pay high regards to my parents and friends for their sincere inspiration and motivation throughout my dissertation work.

Komal

Design Engineering

(2013PDE5080)

Abstract

This thesis studies the vibrational behaviour of a rotating twisted tapered Timoshenko beam following the detailed formulation a FEM code in Matlab was constructed. The analytical results was matched with the published work. The stiffness and mass matrices of a rotating twisted and tapered beam element are derived. The angle of twist, breadth and depth of beam are assumed to vary linearly along the length of the beam. The effect of shear deformation and rotary inertia are also considered in deriving the element matrices. The first nine natural frequencies and mode shapes in bending-bending mode are calculated for free-free beamand first 6 natural frequency and mode shapes in x-z plane and y-z are made using matlab.

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Nomenclature

A	area of cross-section
B	breadth of beam
e	offset
E	Young's modulus
g	acceleration due to gravity
G	shear modulus
h	depth of beam
Ixx, Iyy, Ixy	moment of inertia of beam cross-section about xx-, yy- and xy-axis
[K]	element stiffness matrix
l	length of an element
L ¸	length of total beam
[M]	element mass matrix
t	time parameter
u	nodal degrees of freedom
U	strain energy
V	displacement in xzplane
W	displacement in yzplane
x, y	co-ordinate axes
z	co-ordinate axis and length parameter
z_e	distance of the "first node of the element from the root of the beam
α	depth taper ratio"h1/h0
β	breadth taper ratio"b1/b0
θ	angle of twist
ρ	weight density
k	shear coefficient
Ω	rotational speed of the beam (rad/s)
b	bending
s	shear

Chapter 1 Introduction

1.1Introduction

Rotating beams has been widely used in modelling of several engineering components, such as compressor blades ,turbine blades, propellers, robot manipulators ,helicopter rotor and spinning space structures. Rotating Euler Bernoulli beam model is the the topic of many studies in open literature. yet, application of the Timoshenko Beam Theory that accounts for rotary inertia and shear deformation effects is crucial when the dimensions of the beam cross-section are equivalent to the beam length and when higher modes are required.

It is well-known that beams are very common types of components and can be categorized according to their geometric configuration as tapered or uniform and thick and slender . It is used in many engineering applications and a large number of researches can be found in literature about transverse vibration of uniform isotropic beams. But if practically analysed, the non-uniform beams may provide a improved or more appropriate distribution of strength and mass than uniform beams and as a result can meet special functional requirements of robotics ,turbine blade, architecture, aeronautics, architecture and other new engineering applications and it have been the subject of numerous studies. Non-prismatic members are being increasingly used in diversities as for their aesthetic ,cost-effective, and other considerations.

Design of such to reduce mechanical vibrations, mode localization requires a knowledge of their natural frequencies and the mode shapes of vibration. For the tapered beam vibration analysis Timoshenko beam theory is used. Free vibration analysis that has been done in here is a process of describing a beam in terms of its natural characteristics which are the frequency and mode shapes. Change of modal characteristics directly provides an indication of beam condition based on changes in frequencies of vibration and mode shapes. Many methods have been many methods developed yet now for calculating the frequencies and mode shapes of beam. Due to advancement in computational techniques and accessibility of software, FEA is quite a less cumbersome than the conventional methods.

In this work, the "Finite element technique is applied to "Find the natural frequencies and mode shapes of beams in the bending-bending mode of vibration by taking into account the taper, the pre-twist and the rotation simultaneously. The coupling that exists between the flexural and torsional vibration is not considered. The taper and the angle of twist are assumed to vary linearly along the length of the beam. The element stiffness and mass matrices are derived and the effects of depth and breadth taper ratios, angular velocity and pre-twist on the natural frequencies are studied.

1.2 Thesis outline

This thesis contains six chapter organised as follows

- Chapter 1 gives an introduction of tapered beam and various practical uses of these beam
- Chapter 2 gives literature review which gives the idea about various work that has been already done on these beam
- Chapter 3 gives mathematical formulation of pre-twisted tapered Timoshenko beam
- Chapter 4 gives procedure to find the natural frequency and mode shapes
- Chapter 5 gives values of mode shape and natural frequency and their diagram
- Chapter 6 gives conclusions and future work

Chapter 2 Literature Review

2.1 Literature review

Mabie and Rogers [1] developed the differential equation from the Euler-Bernoulli equation for the free vibration of a tapered cantilever beam. The beam tapers linearly in the horizontal and in the vertical planes simultaneously. The effects of different taper ratio on the vibration frequency have been analysed.

Mabie and Rogers [2] studied the free vibrations of non-uniform cantilever beams with an end support have been investigated, using the equations of Bernoulli- Euler. Two configurations of interest are treated in their analysis (a) constant width and linearly variable thickness and (b) constant thickness and linearly variable width. Charts have been plotted for each case.

Sharp and Cobble [3] derived the equation of motion of beam for a uniform cross- section damped beam elastically restrained against rotation at either ends. This has been solved for the displacement under very general distributed load conditions. The result is based on the properties of Hermitian operator.

Carnegie &Thomas [4] determined the natural frequencies and mode shapes of vibration of tapered ,pre-twisted cantilever blades which is of great importance in the design of many engineering components. These include turbine blading, helicopter rotor blades, compressor blading and aircraft propeller blades.

Chun [5] considered the free vibration of a beam hinged at one end by a rotational spring with a constant spring constant and the other end free. The beam includes the ‘simply supported-free’ beam and the ‘clamped-free’ beam as the limiting cases of the zero spring constant and the infinite spring constant, respectively. Normal functions derived here can be of use to an approximate analysis for rectangular plate vibrations when one pair of the parallel edges is of the spring hinged-free type.

Goel [6] investigated the transverse vibrations of linearly tapered beam, elastically restrained against rotation at one of the end. He studied the vibration characteristics of beam which carries a concentrated mass. He assumed that one end of the beam is free and the other end is hinged by a rotational spring of constant stiffness. Results for the first three Eigen frequencies with different values of stiffness ratios (ratio of spring stiffness and beam stiffness at either end) and taper ratio are shown. Other cases of a tapered cantilever beam with a concentrated mass at the free end and spring hinged at the other end have also been presented.

Hibbeler [8] studied the free vibration analysis of a beam having a combination of clamped or ideally pinned end supports. In many real cases, however, beams are subjected to a certain amount of bending stiffness at their end support. He analysed and considered such a case assuming the beam to be spring-hinged at both ends. The general frequency equation and the normal mode function are derived for the case when the spring stiffness at each support is dissimilar. The first five roots of this equation are computed and presented in a tabulated form so that the roots may be obtained for a variety of spring support and boundary conditions. Tapered beams have been analysed by many investigators using different technique

Gupta [9] developed the stiffness and consistent mass matrices for linearly tapered beam element of any cross-sectional shape. Variation of area and moment of inertia of the cross section along the axis of the element was exactly represented by simple functions involving shape factors

Raju et al. [10] studied the free vibration analysis of beam using the simple finite element mehtod. They applied it to the huge amplitude vibrations for different conditions i.e. clamped tapered beams and simply supported with linearly varying breadth and depth tapers.

M.A De Rosta and **N.M Aucie [13]** studied the dynamic behaviour of beams with linearly varying cross-section is studied, in the presence of axially and rotationally flexible ends. The equation of motion was solved in the form Bessel functions and the boundary conditions lead to the frequency equation which is a function of four flexibility coefficients.

B Posidala [14] studied the problem of free transverse vibrations of Timoshenko beams with attachments like translational and rotational springs concentrated mass including the moment of inertia linear undamped oscillators and additional supports is considered. The frequency equation for the collective system is derived by means of the Lagrange multiplier . The precise solution of the free vibration problem of the beam without attachments is taken into account for the formulation of the free vibration problem of the collective system. Numerical examples show the isolated or coupling influences of the additional elements on the collective system's frequencies. The comparison of results obtained from the present approach with results of the exact solution indicates a good agreement

N M Auciello [15] studied free vibrations of cantilever tapered beams with a mass at the tip. The rotatory inertia of the concentrated mass is considered with its eccentricity of mass. The non-dimensional frequency coefficients are given in tabular form at the end of the paper and comparisons with other results from the literature are presented

Bruce Geist and Joyce [19] studied Asymtopic formulas .Asymptotic formulas are derived for the eigenvalues of a free-ended Timoshenko beam which has variable mass density and constant beam parameters . These asymptotic formulas show how the eigenvalues and hence how the natural frequencies of such a beams depend upon the material and geometric parameters which appear as coefficients in the Timoshenko differential equations.

G.Falsone and D.Settineri [20] studied a new finite element approach for the solution of the Timoshenko beam is shown. Similarly to the Bernoulli-Euler beam theory, it has been assumed a single fourth order differential equation governs the equilibrium of the Timoshenko beam. The results obtained through this approach are very good, both in terms of computation and accuracy effort.

J. R. Banerjee AND A. J. Sobey [21] find The kinetic energy of the rotating Timoshenko beam element from the velocity components

H. Ante [22] studied fundamental solutions for the second order differential equations of Timoshenko_s theory

2.2 Research Gap

Mabie and Rogers studied the free vibrations of non-uniform cantilever beams with an end support have been investigated, using the equations of Bernoulli- Euler. Carnegie &Thomas **[** determined the natural frequencies and mode shapes of vibration of tapered ,pre-twisted cantilever blades. Hibbeler studied the free vibration analysis of a beam having a combination of clamped or ideally pinned end supports. Gupta developed the stiffness and consistent mass matrices for linearly tapered beam element of any cross-sectional shape. B Posidala studied the problem of free transverse vibrations of Timoshenko beams with attachments like translational and rotational springs concentrated mass. G.Falsone and D.Settineri studied a new finite element approach for the solution of the Timoshenko beam. J. R. Banerjee And A. J. Sobey find The kinetic energy of the rotating Timoshenko beam element from the velocity components.

2.3 Research Objective

In this work the finite element method is applied for finding the frequencies of natural vibration of doubly tapered and twisted beams. The stiffness and mass matrices of the beam element are developed by taking bending deflection, bending slope, shear deflection and shear slope in two planes as nodal degrees of freedom. The effects of shear deformation and rotary inertia, which are of significant importance at higher modes of vibration, are considered in the derivation. The natural frequencies of vibration have been calculated for a pre-twisted double tapered beam by using the finite element method . The mass matrix and the stiffness matrix is calculated using Mathmaitica software and various natural frequency are calculated using Matlab software and also mode shapes in both y-z plane and x-z plane are plotted.

3.1 Displacement model

Figure 3.1 (a) shows a doubly tapered, twisted beam element of length *l* with the nodes as *l* and 2. The breadth, depth and the twist of the element are assumed to be linearly varying along its length. The breadth and depth at the two nodal points are shown asb_1, h_1 and b_2, h_2 respectively. The pre-twist at the two nodes is denoted by θ_1 andθ_2. Figure 3.1(b) shows the nodal degrees of freedom of the element where bending deflection, bending slope, shear deflections and shear slope in the two planes are taken as the nodal degrees of freedom. Figure 3.1(c) shows the angle of twist θ at any section *z*. The beam is assumed to rotate about the *x*-*x*-axis at a speed of *Ω* rad/s. The total deflections of the element in the *y* and *x* directions at a distance *z* from node 1, *w*(*z*) and *v*(*z*), are taken as

$$w(z) = w_b\,(z) + \ w_s(z) \quad , v(z) = \ v_b(z) + \ v_s(z), \tag{3.1}$$

Where $w_b\,(z)$ and $v_b(z)$ are the deflections due to bending in the yz and *xz* planes respectively, and $w_s(z)$and $v_s(z)$ are the deflections due to shear in the corresponding planes.

The displacement models for $w_b\,(z)$, $v_b(z), w_s(z)$ and $v_s(z)$ are assumed to be polynomials of third degree. They are similar in nature except for the nodal constants. These expressions are given by

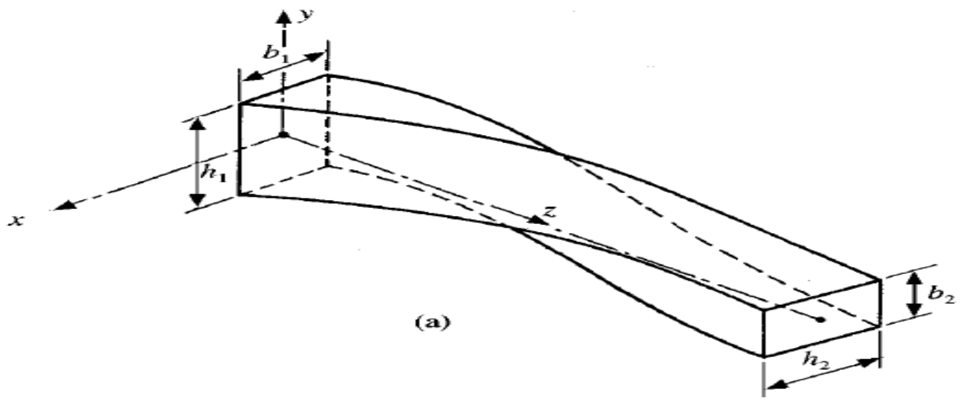
$$w_b = \frac{u_1}{l^3}(2z^3 - 3lz^2 + l^3) + \frac{u_2}{l^3}(3lz^2 - 2z^3) - \frac{u_3}{l^2}(z^3 - 2lz^2 + l^2z) - \frac{u_4}{l^2}(z^3 - lz^2) \tag{3.2}$$

$$w_s = \frac{u_1}{l^3}(2z^3 - 3lz^2 + l^3) + \frac{u_2}{l^3}(3lz^2 - 2z^3) - \frac{u_3}{l^2}(z^3 - 2lz^2 + l^2z) - \frac{u_4}{l^2}(z^3 - lz^2) \tag{3.3}$$

$$v_b = \frac{u_1}{l^3}(2z^3 - 3lz^2 + l^3) + \frac{u_2}{l^3}(3lz^2 - 2z^3) - \frac{u_3}{l^2}(z^3 - 2lz^2 + l^2z) - \frac{u_4}{l^2}(z^3 - lz^2) \tag{3.4}$$

$$v_s = \frac{u_1}{l^3}(2z^3 - 3lz^2 + l^3) + \frac{u_2}{l^3}(3lz^2 - 2z^3) - \frac{u_3}{l^2}(z^3 - 2lz^2 + l^2z) - \frac{u_4}{l^2}(z^3 - lz^2) \tag{3.5}$$

Where u_1, u_2, u_3 and u_4 represent the bending degrees of freedom and u_5, u_6, $u_7$and u_8 are the shear degrees of freedom in the *yz* plane; u_9, u_{10}, u_{11} and u_{12} represent the bending degrees of freedom and u_{13}, u_{14}, u_{15} and u_{16} shear degrees of freedom in the *xz* plane.



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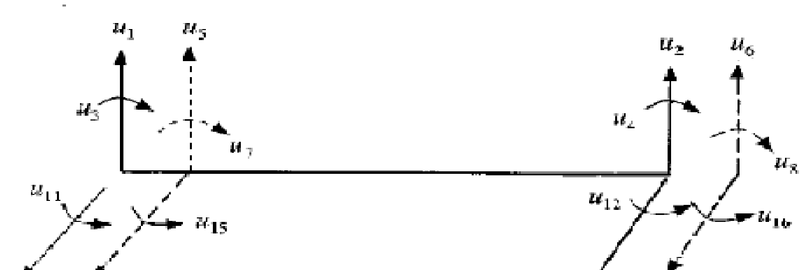

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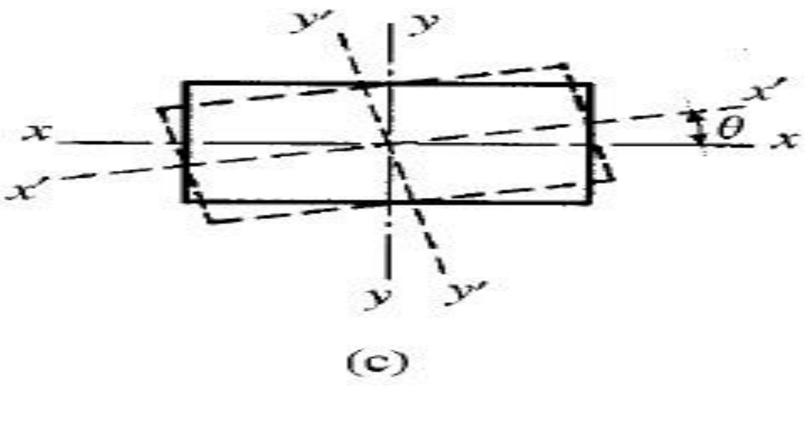

(c)

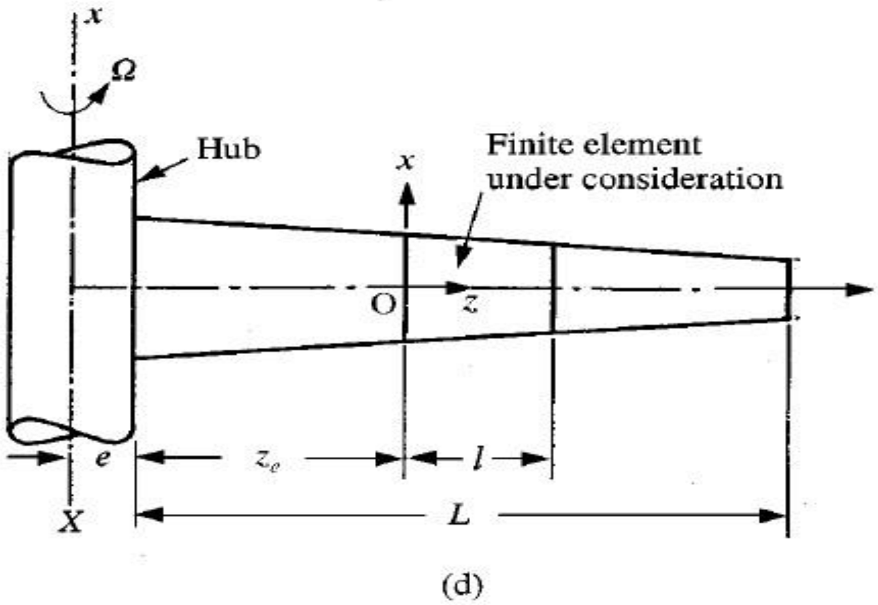

Figure 3-1:(a) An element of tapered and twisted beam, (b) degrees of freedom of an element, (c) angle of twist h,(d) rotation of tapered beam

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3.2 Calculation of shape function

The analysis of two dimensional beams using finite element formulation is identical to matrix analysis of structures. The Bernoulli- Euler beam equation is based on the assumption that the plane normal to the neutral axis before deformation remains normal to the neutral axis after deformation but not in Timoshenko beam. Since there are four nodal variables for the beam element, a cubic polynomial function for y(z), is assumed as

$$Y(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \tag{3.6}$$

From the assumption for the Euler-Bernoulli beam, slope is computed from Eq. (3.6) is

$$\theta(z) = a_1 + 2a_2 z^1 + 3\, a_3\, z^2 \tag{3.7}$$

Where a_0, a_1, a_2, a_3 are the constants. The Eq.(3.6) can be written as

$$Y(z) = \lceil 1 \quad z \quad z^2 z^3 \rceil \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{3.8}$$

$$Y(z) = [C][a]$$

Where

$$[C] = \lceil 1 \quad z \quad z^2 z^3 \rceil$$

$$[a] = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

For convenience local coordinate system is taken z_1=0, z_2=l that leads to

$$u_1 = a_0 \quad u_2 = a_1$$

$u_3 = a_0 + a_1 l + a_2 l^2 + a_3\, l^3$

$u_4 = a_1 + 2a_2 l + 3a_3\, l^2$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$[u] = [A][a][a] = [A]^{-1}[u]$

Eq. (3.8) can be written as

$Y(z) = [C][A]^{-1}[u]$

$Y(z) = [H][u]$

Where

$[H] = [C][A]^{-1}$

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{-1}{l^2} \end{bmatrix}$$

[H] = [H_1(z), H_2(z), H_3(z), H_4(z)]

Where Hi(z) are called as Hermitian shape function whose values are given below

$$H_1 = \frac{1}{l^3}(2z^3 - 3lz^2 + l^3) H_2 = \frac{1}{l^3}(3lz^2 - 2z^3)$$

$$H_3 = \frac{-1}{l^2}(z^3 - 2lz^2 + l^2 z) H_4 = \frac{-1}{l^2}(z^3 - lz^2) \qquad (3.9)$$

3.3 Element stiffness matrix

3.3.1 Expression for strain energy (u)

3.3.1.1 Strain energy due to bending

If the bending deflections in *yz* and *xz* planes of a beam arew_b and v_b , respectively, the axial strain and stress induced due to w_b an d v_b are given by ε_{xx} due to $w_b = y\frac{\partial^2 w_b}{\partial z^2}$ -

ε_{yy} due to $v_b = y\frac{\partial^2 v_b}{\partial z^2}$and σ_{xx} due to$w_b = Ey\frac{\partial^2 w_b}{\partial z^2}$ σ_{xx} due to $v_b = Ey\frac{\partial^2 v_b}{\partial z^2}$ .The strain energy stored in the beam due to bending is given by **U** due to bending

$$= \frac{1}{2}\int_V \varepsilon_{xx}\ \sigma_{xx} dV$$

$$= \frac{1}{2}\int_V (\varepsilon_{xx}\ due\ to\ w_b\ + \varepsilon_{xx}\ due\ to\ v_b)(\sigma_{xx}\ due\ to w_b + \sigma_{xx}\ \ due\ to\ v_b\ )\ dV$$

$$= \frac{E}{2}\int_0^l dz \int_A \left(y\frac{\partial^2 w_b}{\partial z^2} + y\frac{\partial^2 v_b}{\partial z^2}\right)^2$$

$$= \frac{E}{2}\int_0^l dz\ \left[\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 \int_A x^2 dA + \left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 \int_A y^2 dA + 2\frac{\partial^2 w_b}{\partial z^2}\frac{\partial^2 v_b}{\partial z^2}\int_A xydA\ \right]$$

$$= \frac{E}{2}\int_0^l dz\ \left[\frac{EI_{xx}}{2}\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 + \frac{EI_{xy}}{2}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 + EI_{xy}\ \frac{\partial^2 w_b}{\partial z^2}\frac{\partial^2 v_b}{\partial z^2}\ \right] \qquad (3.10)$$

Where **V** is the volume, *l* is the length and *A* is the cross-sectional area of the beam

3.3.1.2 Strain energy due to shearing

Let F_x and F_y be the shear forces that produce the shear deflections $dv_s$and dw_sin an element of length dz respectively. Then the strain energy of the beam due to shearing is given by

$$U_{due\ to\ shearing} = \frac{1}{2}\int_0^l \left(F_x\frac{dv_s}{dz} + F_y\frac{dw_s}{dz}\right) dz$$

Substituting $AG\mu\frac{dv_s}{dz}$ and $AG\mu\frac{dw_s}{dz}$for F_x and F_y respectively, one can obtains

$$U_{due\ to\ shearing} = \int_0^l \frac{\mu AG}{2}\left[\left(\frac{dw_s}{dz}\right)^2 + \left(\frac{dv_s}{dz}\right)^2\right] dz$$

3.3.1.3. Strain energy due to rotation

The rotation of a beam induces an axial force *P* in the beam due to centrifugal action. If the beam is bending in the *yz* plane (Figure 3.2), the change in the horizontal projection of an element of length ds is given by

$$ds - dz = \left\{(dz)^2 + \left(\frac{\partial w}{\partial z}dz\right)^2\right\}^{\frac{1}{2}} - dz \approx \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2$$

Since the axial force P acts against the changes in the horizontal projection, the work done by P is given by

$$U_{duetoPandw} = -\frac{1}{2}\int_0^l \mathrm{P(z)}\left(\frac{\partial w}{\partial z}\right)^2 dz$$

The work done by the transverse distributed force $p_w(z)$can be written as

$$U_{dueto_{p_w}} = -\frac{1}{2}\int_0^l p_w(z)wdz$$

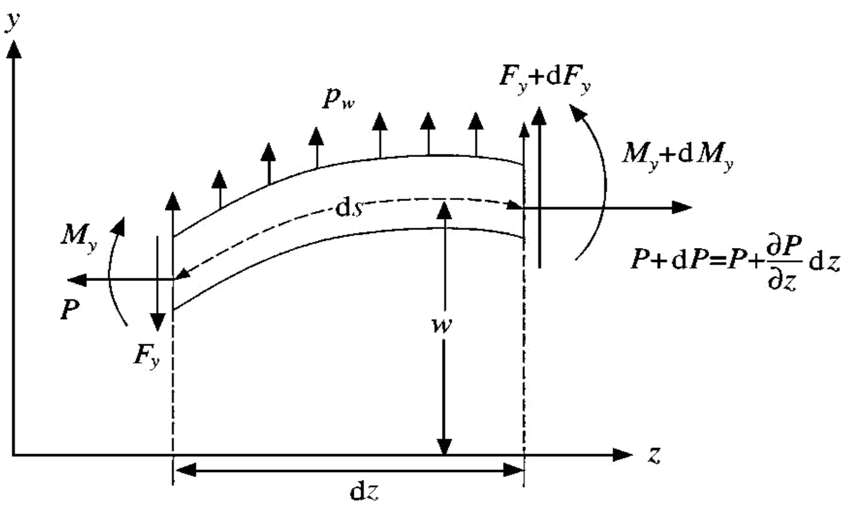

Figure 3-2: An element of the beam in equilibrium

The expressions corresponding to the bending of the beam in the *xz* plane can be obtained similarly as

$$U_{duetoPandv} = -\frac{1}{2}\int_0^l \mathrm{P(z)}\left(\frac{\partial v}{\partial z}\right)^2 dz$$

$$U_{dueto_{p_v}} = -\frac{1}{2}\int_0^l p_v(z)wdz$$

The total strain energy of the beam can be obtained as given in equation (3.11) by combining above equations

3.3.1.4. Expression for kinetic energy

Consider a small element of area dA and length dz at a point in the cross-section having co-ordinates (x, y) with respect to x- and y-axes. The kinetic energy of this element is given by

$$\frac{\rho}{2g}\left[(\dot{w}^2+\dot{v}^2)+\left(y\dot{\phi}_x+x\dot{\phi}_y\right)^2\right]dzdA$$

Where $\dot{\phi}_x$ and $\dot{\phi}_y$ denote the bending slopes, $\partial v_b/\partial z$ and $\partial w_b/\partial z$, respectively, and a dot over a symbol represents derivative with respect to time. Integrating this equation over the beam cross-section, the kinetic energy of an element of length dz can be obtained as

$$dT=\frac{\rho}{2g}\left[\mathrm{A}(\dot{w}^2+\dot{v}^2)+\left(I_{xx}\dot{\phi}_y{}^2+2I_{xy}\dot{\phi}_x\dot{\phi}_y+I_{yy}\dot{\phi}_x{}^2\right)\right]dz$$

The kinetic energy of the entire beam (T) can be expressed as

$$T=\int_0^l\frac{\rho A}{2g}\left[\mathrm{A}(\dot{w}^2+\dot{v}^2)+\left(I_{xx}\dot{\phi}_y{}^2+2I_{xy}\dot{\phi}_x\dot{\phi}_y+I_{yy}\dot{\phi}_x{}^2\right)\right]dz$$

3.3.2 Stiffness matrix

The total strain energy U of a beam of length l, due to bending and shear deformation including rotary inertia and rotation effects is given by

$$\begin{aligned}U=\int_0^l&\left[\left\{\frac{EI_{xx}}{2}\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2+EI_{xy}\frac{\partial^2 w_b}{\partial z^2}\,\frac{\partial^2 v_b}{\partial z^2}+\frac{EI_{yy}}{2}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2\right\}\right.\\&\left.+\frac{\mu AG}{2}\left\{\left(\frac{\partial^2 w_s}{\partial z^2}\right)^2+\left(\frac{\partial^2 v_s}{\partial z^2}\right)^2\right\}\right]dz+\frac{1}{2}\int_0^l P(z)\left(\frac{\partial w_b}{\partial z}+\frac{\partial w_S}{\partial z}\right)^2dz\\&+\frac{1}{2}\int_0^l P(z)\left(\frac{\partial v_b}{\partial z}+\frac{\partial v_S}{\partial z}\right)^2dz-\int_0^l p_w(z)\,(w_b+w_s)\,d\\&-\int_0^l p_v(z)(v_b+v_s)\;dz\end{aligned}\tag{3.11}$$

Where

$$P(z)=\int_{e+z_e+z}^{L+e}m\Omega^2\,\xi d\xi\approx\frac{\rho A\Omega^2}{2g}\left[(L+e)^2-(e+z_e+z)^2\right]$$

$$= \frac{\rho A \Omega^2}{2g}\left[\left(eL + \frac{1}{2}L^2 - ez_e - \frac{1}{2}{z_e}^2\right) - (e + z_e\,)z - \frac{1}{2}z^2\right] \qquad (3.12)$$

$$p_w(z) = \frac{\rho A \Omega^2}{2g}(w_b + w_s) \qquad (3.13)$$

$$p_v(z) = \frac{\rho A \Omega^2}{2g}\ (v_b + v_s) \qquad (3.14)$$

Where e is the offset and z_e is the distance of the first node of the element from the root of the beam as shown in Figure 3.1(d), and *P*(*z*) is the axial force acting at section *z*.

As the cross-section of the element changes with *z* and as the element is twisted, the cross-sectional area *A*, and the moments of inertia I_{xx} , I_{yy} and I_{xy} will be function of z.

$$A(z) = b(z)h(z) = \left\{b_1 + (b_2 - b_1)\frac{z}{l}\right\}\left\{h_1 + (h_2 - h_1)\frac{z}{l}\right\}$$

$$= \frac{1}{l^2}(c_1 z^2 + c_2 l z + c_3 l^2)$$

Where

$$c_1 = (b_2 - b_1)(h_2 - h_1),$$

$$c_2 = b_1(h_2 - h_1) + h_1(b_2 - b_1),$$

$$c_3 = b_1 h_1$$

$$I_{xx}(z) = I_{x'x'}cos^2\theta\ + I_{y'y'}sin^2\theta$$

$$I_{yy}(z) = I_{y'y'}cos^2\theta\ + I_{x'x'}sin^2\theta,$$

$$I_{xy}(z) = (I_{y'y'} - I_{x'x'})\frac{sin2\theta}{2}$$

Where $x'x'$ and $y'y'$ are the axes inclined at an angle θ, the angle of twist, at any point in the element, to the original axis xx and yy as shown in Figure 1(c).The value of $I_{x'y'=0}$ and the value of $I_{x'x'}$ and $I_{y'y'}$ can be computed as

$$I_{x'x'}(z) = \frac{b(z)h^3(z)}{12l^4}[a_1z^4 + a_2lz^3 + a_3l^2z^2 + a_4l^3z + a_5l^4] \tag{3.15}$$

Where

$a_1 = (b_2 - b_1)\,(h_2 - h_1)^3,$

$a_2 = b_1(h_2 - h_1)^3 + 3(b_2 - b_1)(h_2 - h_1)^2h_1\,,$

$a_3 = 3b_1\,h_1\,(h_2 - h_1)^3 + 3(b_2 - b_1)(h_2 - h_1){h_1}^2$

$a_4 = 3b_1\,{h_1}^2(h_2 - h_1) + (b_2 - b_1){h_1}^3,$

$a_5 = b_1\,{h_1}^3$

$$I_{y'y'}(z) = \frac{b(z)h^3(z)}{12l^4}[d_1z^4 + d_2lz^3 + d_3l^2z^2 + d_4l^3z + d_5l^4] \tag{3.16}$$

Where

$d_1 = (h_2 - h_1)(b_2 - b_1)^3,$

$d_2 = h_1\,(b_2 - b_1)^3 + 3(h_2 - h_1)(b_2 - b_1)^2\,b_1\,,$

$d_3 = 3h_1\,b_1(b_2 - b_1)^3 + 3(h_2 - h_1)\,(b_2 - b_1)\,{b_1}^2,$

$d_4 = 3h_1{b_1}^2(b_2 - b_1) + (h_2 - h_1){b_1}^3$

$d_5 = h_1{b_1}^3\,,$

By substituting the expressions of w_b, w_s, v_b, v_s, A, I_{xx}, I_{xy} and I_{yy} from equations (3.2), (3.3),(3.4) ,(3.5),(3.15)and (3.16) into equation (3.11), the strain energy U can be expressed as

$$U = \frac{1}{2}\mathbf{u}^T[K]\mathbf{u} \tag{3.17}$$

Where u is the vector of nodal displacements u_1 , u_2 , ... , u_{16}, and $[K]$ is the elemental sti**ff**ness matrix of order 16. Denoting the integrals

$$\int_0^l EI_{xx}\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 dz = [u_1\,u_2\,u_3\,u_4]^T\;[AK][u_1\,u_2\,u_3\,u_4\,] \tag{3.18}$$

$$\int_0^l EI_{yy}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 dz = [\, u_9\, u_{10}\, u_{11}\, u_{12}]^T[BK][u_9\, u_{10}\, u_{11}\, u_{12}\,] \tag{3.19}$$

$$\int_0^l EI_{yy}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 dz = [\, u_9\, u_{10}\, u_{11}\, u_{12}]^T[BK][u_9\, u_{10}\, u_{11}\, u_{12}\,] \tag{3.20}$$

$$\int_0^l \mu AG\left(\frac{\partial w_S}{\partial z}\right)^2 dz = [u_5\, u_6\, u_7\, u_8\,]^T\, [CK][u_5\, u_6\, u_7\, u_8\,] \tag{3.21}$$

$$\int_0^l EI_{xy}\left(\frac{\partial^2 w_b}{\partial z^2}\right)\left(\frac{\partial^2 v_b}{\partial z^2}\right) dz = [u_1\, u_2\, u_3\, u_4]^T\, [DK][u_9\, u_{10}\, u_{11}\, u_{12}\,] \tag{3.22}$$

$$\int_0^l P(z)\left(\frac{\partial w_b}{\partial z}\right)^2 dz = [u_1\, u_2\, u_3\, u_4]^T\, [EK][u_1\, u_2\, u_3\, u_4] \tag{3.23}$$

And

$$\int_0^l \frac{2\rho A\Omega^2}{g} {w_b}^2\, dz = [u_1\, u_2\, u_3\, u_4]^T\, [FK][u_1\, u_2\, u_3\, u_4] \tag{3.23}$$

The element stiffness matrix can be given such as

$$[K] = \begin{bmatrix} [AK]+[EK]+[FK] & [EK]+[FK] & [DK] & [0] \\ [EK]-[FK] & [CK]+[EK]-[FK] & [0] & [0] \\ [DK] & [0] & [BK]+[EK]-[FK] & [EK]-[FK] \\ [0] & [0] & [EK]-[FK] & [CK]+[EK]-[FK] \end{bmatrix} \tag{3.24}$$

elements are formulated. [0] is a null matrix of order 4.where [*AK*], [*BK*], [*CK*], [*DK*], [*EK*] and [*FK*] are symmetric matrices of order 4 and their

3.4 Element mass matrix

The kinetic energy of the element T considering the effects of shear deformation and rotary inertia is given by

$$T = \int_0^l \frac{\rho A}{2g}\left(\frac{\partial w_b}{\partial z}+\frac{\partial w_s}{\partial z}\right)^2 + \frac{\rho A}{2g}\left(\frac{\partial v_b}{\partial z}+\frac{\partial v_s}{\partial z}\right)^2 + \frac{\rho I_{yy}}{2g}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2 + \frac{\rho I_{xy}}{g}\frac{\partial^2 v_b}{\partial z\partial t}\frac{\partial^2 w_b}{\partial z\partial t}$$

$$+\frac{\rho I_{xy}}{2g}\left(\frac{\partial^2 w_b}{\partial z\partial t}\right)^2 \qquad (3.25)$$

By defining

$$\int_0^l \frac{\rho A}{g}\left(\frac{\partial w_b}{\partial t}\right)^2 dz = [\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ]^T[AM][\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ] \qquad (3.26)$$

$$\int_0^l \frac{\rho I_{xx}}{g}\left(\frac{\partial^2 w_b}{\partial z\partial t}\right)^2 dz = [\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ]^T\ [BM][\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ] \qquad (3.27)$$

$$\int_0^l \frac{\rho I_{yy}}{g}\left(\frac{\partial^2 v_b}{\partial z\partial t}\right)^2 dz = [\dot{u}_9\dot{u}_{10}\dot{u}_{11}\dot{u}_{12}\ ]^T\ [BM][\dot{u}_9\dot{u}_{10}\dot{u}_{11}\dot{u}_{12}\ ] \qquad (3.28)$$

And

$$\int_0^l \frac{\rho I_{xy}}{g}\left(\frac{\partial^2 v_b}{\partial z\partial t}\right)\left(\frac{\partial w_b}{\partial t}\right) dz = [\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ]^T[DM][\dot{u}_9\dot{u}_{10}\dot{u}_{11}\dot{u}_{12}] \qquad (3.29)$$

Where $\dot{u}_i$ denotes the time derivative of the nodal displacement $\dot{u}_i$, i=1, 2,. . ., 16, the kinetic energy of the element can be expressed as

$$U = \frac{1}{2}\dot{\boldsymbol{u}}^{\boldsymbol{T}}[M]\dot{\mathbf{u}}\ , \qquad (3.30)$$

where [*M*] is the mass matrix given by

$$[M] = \begin{bmatrix} [AM+BM & [AM] & [DM] & [0] \\ [AM] & [AM] & [AM] & [0] \\ [DM] & [AM] & [AM]+[CM] & [AM] \\ [0] & [0] & [AM] & [AM] \end{bmatrix} \qquad (3.31)$$

And [*AM*], [*BM*] [*CM*] and [*DM*] are symmetric matrices of order 4 .

Chapter 4 Numerical analysis

4.1 Numerical analysis

For numerical analysis a rotating double tapered, twisted-twisted Timoshenko beam is considered with the following properties

Geometrical properties

Width of the beam =b_0 (at root) =2.54e-2m,

Depth of the beam = h_0 (at root) = 0.46e-2m,

Twist angle at root (θ_1) =0

Twist angle at free end (θ_2)=45^0

Offset (e)=0

β =breath taper ratio = $\frac{b_1}{b_0}$=2.56

α =depth taper ratio= $\frac{h_1}{h_0}$==2.29

Length of the beam = 0.1524 m

Material properties

Elastic modulus of the beam (E)=2.07e11 N/m^2

Density = 800 Kg/m^3

Shear modulus of beam(G)=0.796e11 N/m^2

4.2 Calculation of frequency of free-free beam

First each component such as AK,BK,CK,DK,EK,FK for stiffness matrix and AM,BM,CM,DM for mass matrix are find out using the MATHEMATICA program by using integration function for single element of beam. Then a global matrix is made for stiffness and mass matrix. After that these stiffness and mass matrix are used in MATLAB to find out the natural frequency and mode shapes of beam.

4.2.1 Calculating stiffness matrix

$$\int_0^l EI_{xx}\left(\frac{\partial^2 w_b}{\partial z^2}\right)^2 \; dz = [u_1\, u_2\, u_3\, u_4]^T \; [AK][u_1\, u_2\, u_3\, u_4\,]$$

$$= [u_1\, u_2\, u_3\, u_4\,]\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= [u_1\, u_2\, u_3\, u_4\,]\begin{bmatrix} k_{11}u_1 + k_{12}u_2 + k_{13}u_3 + k_{14}u_4 \\ k_{21}u_1 + k_{22}u_2 + k_{23}u_3 + k_{24}u_4 \\ k_{31}u_1 + k_{32}u_2 + k_{33}u_3 + k_{34}u_4 \\ k_{41}u_1 + k_{42}u_2 + k_{43}u_3 + k_{44}u_4 \end{bmatrix}$$

$$= k_{11}u_1\, u_1 + k_{12}u_1\, u_2 + k_{13}u_1\, u_3 + k_{14}u_1\, u_4 + k_{21}u_1u_2 + k_{22}u_2u_2 + k_{23}u_2u_3 + k_{24}u_2u_4 + k_{31}u_1u_3 + k_{32}u_2u_3 + k_{33}u_3u_3 + k_{34}u_3u_4 + k_{41}u_1u_4 + k_{42}u_2u_4 + k_{43}u_3u_4 + k_{44}u_4u_4$$

$$\begin{aligned} = k_{11}u_1\, u_1 &+ (k_{12} + k_{21})u_1\, u_2 + (k_{13} + k_{31})u_1\, u_3 + (k_{14} + k_{41})u_1\, u_4 + k_{22}u_2u_2 \\ &+ (k_{23} + k_{32})u_2u_3 + (k_{24} + k_{42})u_2u_4 + k_{33}u_3u_3 + (k_{34} + k_{43})u_3u_4 \\ &+ k_{44}u_4u_4 \end{aligned}$$

$$[AK] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

When we calculate integral it is found

$$\begin{aligned} = 120676u_1\, u_1 &+ (-241353)u_1\, u_2 + (-19951.7)u_1\, u_3 + (-16830.5)u_1\, u_4 \\ &+ 120676u_2u_2 + (19951.7)u_2u_3 + (16830.5)u_2u_4 + 1104.24u_3u_3 \\ &+ 832.163u_3u_4 + 866.401u_4u_4 \end{aligned}$$

$$= 120676u_1\, u_1 \ + (-120676.5 - 120676.5)u_1\, u_2 + (-9975.85 - 9975.85)u_1\, u_3$$
$$+ (-8415.25 - 8415.25)u_1\, u_4 + 120676u_2u_2$$
$$+ (9975.85 + 9975.85)u_2u_3 + (8415.25 + 8415.25)u_2u_4$$
$$+ 1104.24u_3u_3 + (416.081 + 416.081\ )u_3u_4 + 866.401u_4u_4$$

By putting the various element found from the above eqn are in [AK] is found which is given as

$$\text{AK} = \begin{bmatrix} 120676 & -12067.5 & -9975.85 & -8415.25 \\ -120676.5 & 120676 & 9975.85 & 8415.25 \\ -9975.85 & 9975.85 & 1104.24 & 416.0815 \\ -8415.25 & 8415.25 & 416.081 & 866.401 \end{bmatrix}$$

Similarly [BK] matrix

$$\int_0^l EI_{yy}\left(\frac{\partial^2 v_b}{\partial z^2}\right)^2\ dz = [\,u_9\, u_{10}\, u_{11}\, u_{12}]^T[BK][u_9\, u_{10}\, u_{11}\, u_{12}\,]$$

$$= [u_9\, u_{10}\, u_{11}\, u_{12}\,] \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_9 \\ u_{10} \\ u_{11} \\ u_{14} \end{bmatrix}$$

$$= [u_9\, u_{10}\, u_{11}\, u_{12}\,] \begin{bmatrix} k_{11}u_9 \ + k_{12}u_{10} + k_{13}u_{11} + k_{14}u_{12} \\ k_{21}u_9 + k_{22}u_{10} + k_{23}u_{11} + k_{24}u_{12} \\ k_{31}u_9 + k_{32}u_{10} + k_{33}u_{11} + k_{34}u_{12} \\ k_{41}u_9 + k_{42}u_{10} + k_{43}u_{11} + k_{44}u_{12} \end{bmatrix}$$

$$= k_{11}u_9\, u_9 \ + k_{12}u_9\, u_{10} + k_{13}u_9\, u_{11} + k_{14}u_9\, u_{12} + k_{21}u_9u_{10} + k_{22}u_{10}u_{10} + k_{23}u_{10}u_{11} + k_{24}u_{10}u_{12} + k_{31}u_9u_{11} \ + k_{32}u_{10}u_{11} + k_{33}u_{11}u_{11} + k_{34}u_9u_{12} + k_{41}u_9u_{12} + k_{42}u_{10}u_{12} + k_{43}u_{11}u_{12} + k_{44}u_{12}u_{12}$$

$$= k_{11}u_9\, u_9 \ \ + (k_{12} + k_{21})u_9\, u_{10} + (k_{13} + k_{31})u_9\, u_{11} + (k_{14} + k_{41})u_9\, u_{12} + k_{22}u_{10}u_{10}$$
$$+ (k_{23} + k_{32})u_{10}u_{11} + (k_{24} + k_{42})u_{10}u_{12} + k_{33}u_{11}u_{11}$$
$$+ (k_{34} + k_{43})u_{11}u_{12} \ + k_{44}u_{12}u_{12}$$

$$[BK] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

When we calculate integral it is found

$$\begin{aligned} = (1.7155e6)u_9\, u_9 \ &+ (-1.71555e6 - 1.71555e6)u_9\, u_{10} \\ &+ (-182797.5 - 182797.5)u_9\, u_{11} + (-78652 - 78652)u_9\, u_{12} \\ &+ k_{22}u_{10}u_{10} + (182797.50 + 1827970.5)u_{10}u_{11} \\ &+ (78652 + 78652)u_{10}u_{12} + k_{33}u_{11}u_{11} + (7326.35 + 7326.35)u_{11}u_{12} \\ &+ k_{44}u_{12}u_{12} \end{aligned}$$

$$\mathrm{BK} = \begin{bmatrix} 1.71555e6 & -1.71555e6 & -182797.5 & -78652 \\ -1.71555e6 & 1.71555e6 & 182797.5 & 78652 \\ -182797.5 & 182797.5 & 20532 & 7326.35 \\ -78652 & 78652 & 7326.35 & 4660.27 \end{bmatrix}$$

Others matrix can also find out

$$\mathrm{CK} = \begin{bmatrix} 3.12228e7 & -3.122275e7 & -56179.5 & -0.69888e6 \\ -3.12225e7 & 3.12228e7 & 56179.5 & 0.69888e6 \\ -56179.5 & 56179.5 & 119015 & -22074.05 \\ -0.69888e6 & 0.69888e6 & -22074.5 & 53716.6 \end{bmatrix}$$

$$\mathrm{DK} = \begin{bmatrix} -183790 & 183789.5 & 17001.55 & 11008 \\ 183789.5 & -183790 & -17001.55 & -11008 \\ 17001.55 & -17001.55 & -2043.69 & -547.34 \\ 11008 & -11008 & -547.34 & -1130.28 \end{bmatrix}$$

$$\mathrm{EK} = \begin{bmatrix} 135.576 & -135.5755 & 0.776925 & -3.890015 \\ -135.5755 & 135.576 & -0.776925 & 3.890015 \\ 0.776925 & -0.776925 & 0.623666 & -0.104259 \\ -3.890015 & 3.890015 & -0.104259 & 0.172899 \end{bmatrix}$$

$$\mathrm{FK} = \begin{bmatrix} 322.153 & 75.2695 & -6.1359 & 2.930385 \\ 75.2695 & 132.153 & 2.585675 & 3.378205 \\ -6.1359 & 2.585675 & 0.156434 & -0.096719 \\ 2.930385 & 3.378205 & -0.096719 & 0.1039 \end{bmatrix}$$

After that stiffness matrix is calculated by the relation

$$[K] = \begin{bmatrix} [AK]+[EK]+[FK] & [EK]+[FK] & [DK] & [0] \\ [EK]-[FK] & [CK]+[EK]-[FK] & [0] & [0] \\ [DK] & [0] & [BK]+[EK]-[FK] & [EK]-[FK] \\ [0] & [0] & [EK]-[FK] & [CK]+[EK]-[FK] \end{bmatrix}$$

4.2.2 Calculating mass matrix

$$\int_0^l \frac{\rho A}{g}\left(\frac{\partial w_b}{\partial t}\right)^2 dz = [\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ]^T[AM][\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ]$$

$$= [\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ] \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{bmatrix}$$

$$= [\dot{u}_1\dot{u}_2\dot{u}_3\dot{u}_4\ ] \begin{bmatrix} k_{11}\dot{u}_{1_1} + k_{12}\dot{u}_2 + k_{13}\dot{u}_3 + k_{14}\dot{u}_4 \\ k_{21}\dot{u}_1 + k_{22}\dot{u}_2 + k_{23}\dot{u}_3 + k_{24}\dot{u}_4 \\ k_{31}\dot{u}_1 + k_{32}\dot{u}_2 + k_{33}\dot{u}_3 + k_{34}\dot{u}_4 \\ k_{41}\dot{u}_1 + k_{42}\dot{u}_2 + k_{43}\dot{u}_3 + k_{44}\dot{u}_4 \end{bmatrix}$$

$$= k_{11}\dot{u}_1\dot{u}_1 + k_{12}\dot{u}_1\dot{u}_2 + k_{13}\dot{u}_1\dot{u}_3 + k_{14}\dot{u}_1\dot{u}_4 + k_{21}\dot{u}_1\dot{u}_2 + k_{22}\dot{u}_2\dot{u}_2 + k_{23}\dot{u}_2\dot{u}_3 + k_{24}\dot{u}_2\dot{u}_4 + k_{31}\dot{u}_1\dot{u}_3 + k_{32}\dot{u}_2\dot{u}_3 + k_{33}\dot{u}_3\dot{u}_3 + k_{34}\dot{u}_3\dot{u}_4 + k_{41}\dot{u}_1\dot{u}_4 + k_{42}\dot{u}_2\dot{u}_4 + k_{43}\dot{u}_3\dot{u}_4 + k_{44}\dot{u}_4\dot{u}_4$$

$$= k_{11}u_1\,u_1 + (k_{12}+k_{21})\dot{u}_1\dot{u}_2 + (k_{13}+k_{31})\dot{u}_1\dot{u}_3 + (k_{14}+k_{41})\dot{u}_1\dot{u}_4 + k_{22}\dot{u}_2\dot{u}_2 + (k_{23}+k_{32})\dot{u}_2\dot{u}_3 + (k_{24}+k_{42})\dot{u}_2\dot{u}_4 + k_{33}\dot{u}_3\dot{u}_3 + (k_{34}+k_{43})\dot{u}_3\dot{u}_4 + k_{44}\dot{u}_4\dot{u}_4$$

$$[AM] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

When we calculate integral it is found

$$= 120676u_1\,u_1 + (-241353)u_1\,u_2 + (-19951.7)u_1\,u_3 + (-16830.5)u_1\,u_4 + 120676u_2u_2 + (19951.7)u_2u_3 + (16830.5)u_2u_4 + 1104.24u_3u_3 + 832.163u_3u_4 + 866.401u_4u_4$$

$$\text{AM} = \begin{bmatrix} 0.00408425 & 0.0000954265 & -0.0000077791 & 3.715145e-6 \\ 0.0000954265 & 0.000167544 & 3.278115e-6 & 4.28289e-6 \\ -0.0000077791 & -3.278115e-6 & 1.9832e-7 & -1.226205e-7 \\ 3.715145e-6 & 4.28289e-6 & -1.226205e-7 & 1.31724e-7 \end{bmatrix}$$

Similarly other element of mass matrix can be calculated

$$\text{BM} = \begin{bmatrix} 1.52696e-7 & 1.5269e-7 & -1.9986e-9 & -2.651385e-9 \\ -1.52696e-7 & 1.52696e-7 & 1.99863e-9 & 2.651385e-9 \\ -1.99863e-9 & 1.99863e-9 & 3.37804e-10 & -0.59319e-10 \\ -2.651385e-9 & 2.651385e-9 & -0.59319e-10 & 2.55935e-10 \end{bmatrix}$$

$$\text{CM} = \begin{bmatrix} 1.01363e-6 & 1.013625e-6 & 1.47481e-8 & -3.160565e-8 \\ -1.013625e-6 & 1.01363e-6 & -1.47481e-8 & 3.160565e-8 \\ 1.47481e-8 & -1.47481e-8 & 0.47706e-9 & -1.105485e-9 \\ -3.160565e-8 & 3.160565e-8 & -1.105485e-9 & 1.4392e-9 \end{bmatrix}$$

$$\text{DM} = \begin{bmatrix} -2.99862e-7 & 2.9986e-7 & 2.68616e-9 & 0.687685e-8 \\ 2.99862e-7 & -2.99862e-7 & -2.68616e-9 & -0.687685e-8 \\ 2.68616e-9 & -2.68616e-9 & -4.76806e-10 & 0.94576e-10 \\ 0.687685e-8 & -0.687685e-8 & 0.94576e-10 & -4.18988e-10 \end{bmatrix}$$

Mass matrix is obtain by various element in below eqn

$$[M] = \begin{bmatrix} [AM+BM & [AM] & [DM] & [0] \\ [AM] & [AM] & [AM] & [0] \\ [DM] & [AM] & [AM]+[CM] & [AM] \\ [0] & [0] & [AM] & [AM] \end{bmatrix}$$

4.2.3: Calculation for natural frequency and mode shape

As we know

$$\big[[K] - [M][\Omega^2]\big][U] = 0 \quad (4.1)$$

$$[K] = [M][\Omega^2]$$

$$[\Omega^2] = \frac{K}{M}$$

$$\Omega = \sqrt{\frac{K}{M}}$$

From the above eqn we can find the all the natural frequency

Now we will find the mode shape by putting the value of Ω in eqn (4.1)

5.1Free-Free Beam

5.1.1 Natural Frequency

Table 5-1: Nine natural frequency of rotating Timoshenko beam

Natural Frequency of double tapered pre-twisted rotating twisting Timoshenko beam									
Number	1	2	3	4	5	6	7	8	9
Frequency(Hz)	60	240	9340	35680	53090	126900	161360	286270	501300

5.1.2 Eigen Vector

Table 5-2: Eigenvector corresponding to Eigen values

Mode Shape or Eigen vector corresponding natural frequency									
	60	240	9340	35680	53090	126900	161360	286270	501300
u1	-0.9327	0.0097	-0.0113	-0.0116	0.0066	-0.0185	-0.0018	-0.0047	-0.0059
u2	-0.9327	0.0180	-0.0288	0.0019	-0.0167	0.0413	0.0026	-0.0159	0.0090
u3	0.0011	0.1806	-0.4785	-0.8278	0.6244	-0.0778	-0.2579	-0.1348	-0.4821
u4	0.0016	-0.1828	1	-0.4748	1	-0.2995	-0.3283	0.0353	-0.5412
u5	0.9329	-0.0001	0	0	0.0001	0.0178	0.0002	0.0048	0.0060
u6	0.9329	-0.0001	-0.0001	0.0004	0.0001	-0.0398	0.0004	0.0159	-0.0092
u7	0	0	-0.0009	-0.0031	0.0148	0.0031	0.1110	0.1410	0.4882
u8	0	0	0.0030	-0.0212	0.0134	0.1926	0.1289	-0.6328	0.6002
u9	0.8473	-0.0562	-0.0042	-0.0062	0.0030	0.0195	0.0094	0.0028	0.0038
u10	0.8861	0.0961	-0.0148	0.0255	0.0046	-0.0385	-0.0143	0.0101	-0.0055

u11	-0.2546	-0.9988	-0.1426	0.1666	0.1542	0.6932	0.8116	0.0722	0.3060
u12	-0.2543	-1	0.5564	-1	-0.1302	1	1	-0.4083	0.3653
u13	-1	0.0023	0	0.0001	0.0001	-0.0276	0.0026	0.0083	-0.0105
u14	-1	0.0023	0	0.0008	0.0004	0.0618	-0.0057	0.0255	0.0151
u15	0	0	0.0002	0.0054	0.0052	0.1794	0.1476	0.2632	-0.8411
u16	0	0	0.0010	-0.0247	-0.0124	-0.0432	0.2293	1	-1

5.1.3 Mode Shape Diagram

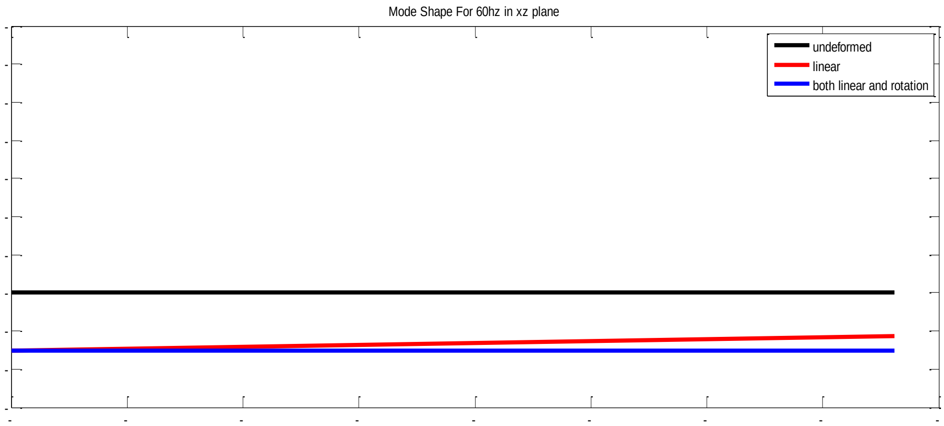

Figure 5-1:Mode shape for 60 Hz in x-z plane

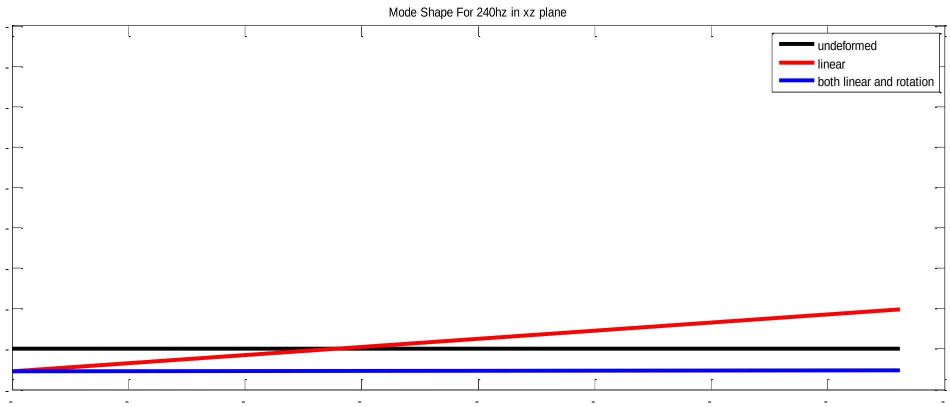

Figure 5-2:Mode shape for 240 Hz in xz plane

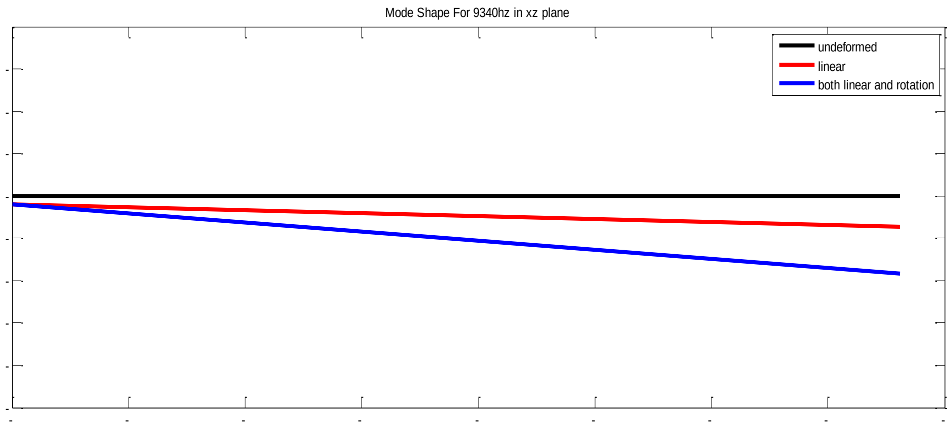

Figure 5-3:Mode shape for 9340 Hz in xz plane

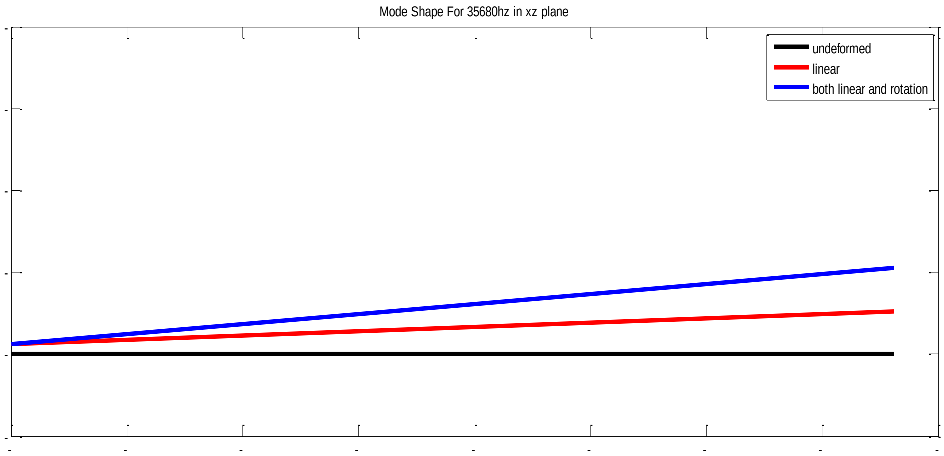

Figure 5-4:Mode shape for 35680 Hz in xz plane

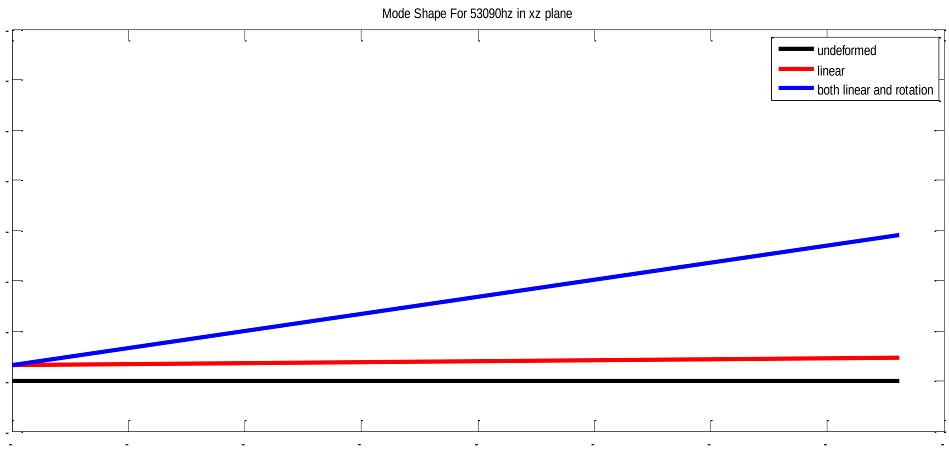

Figure 5-5:Mode shape for 53090 Hz in xz plane

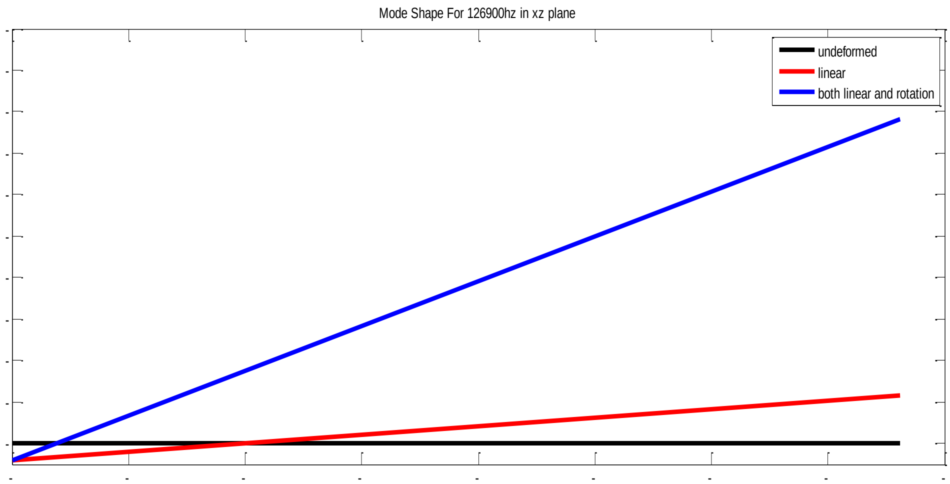

Figure 5-6:Mode shape for 126900 Hz in xz plane

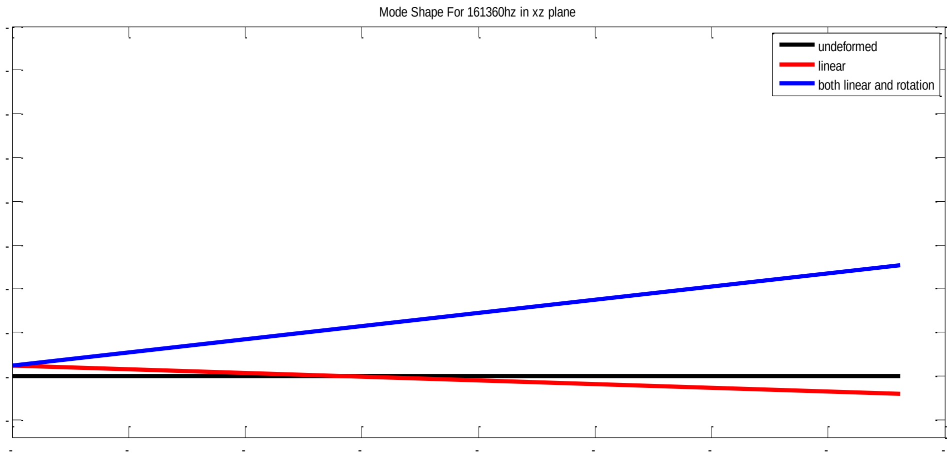

Figure 5-7:Mode shape for 161360 Hz in xz plane

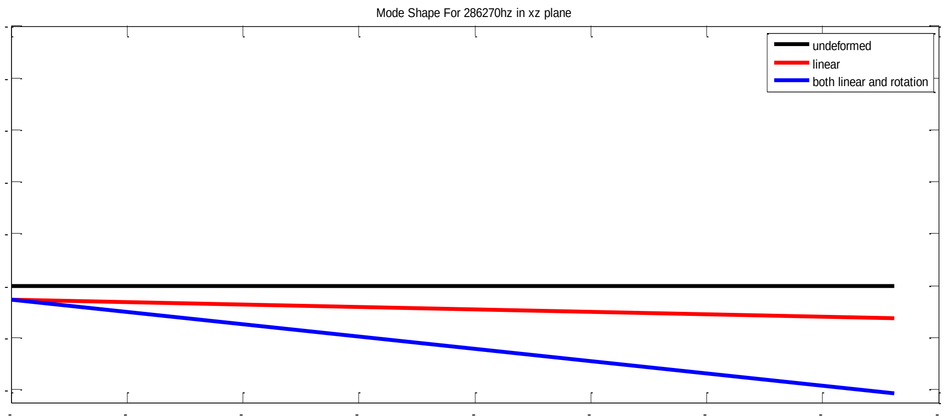

Figure 5-8:Mode shape for 286270 Hz in xz plane

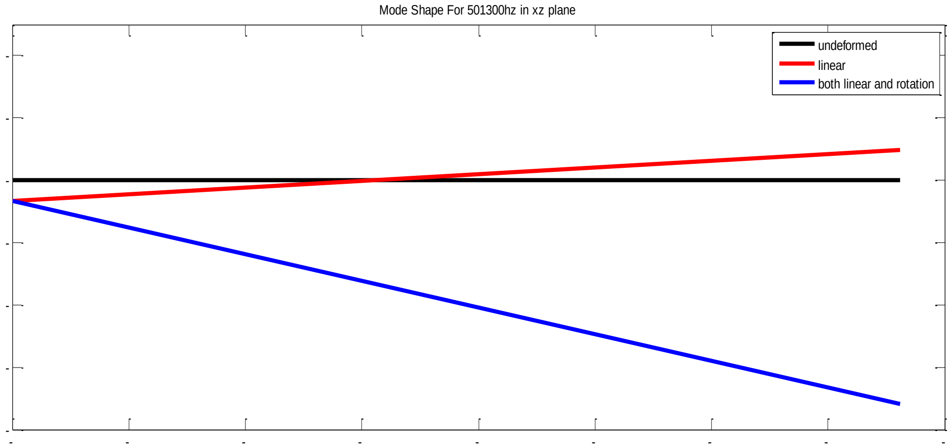

Figure 5-9:Mode shape for 501300 Hz in xz plane

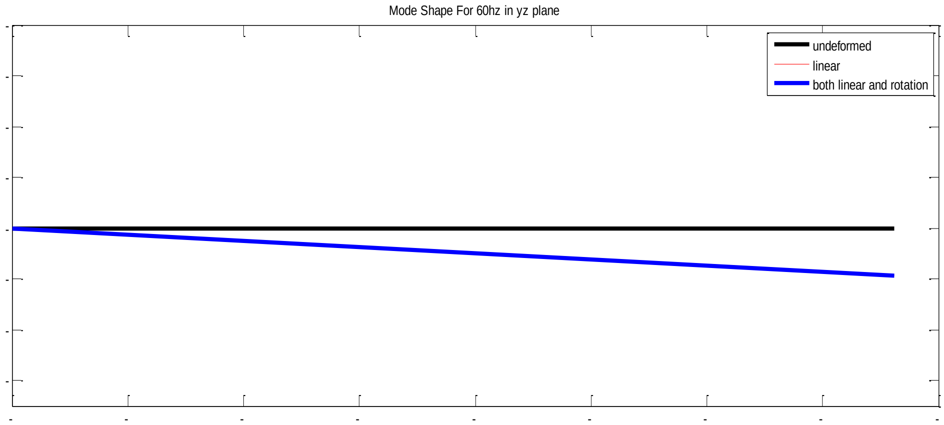

Figure 5-10:Mode shape for 60 Hz in yz plane

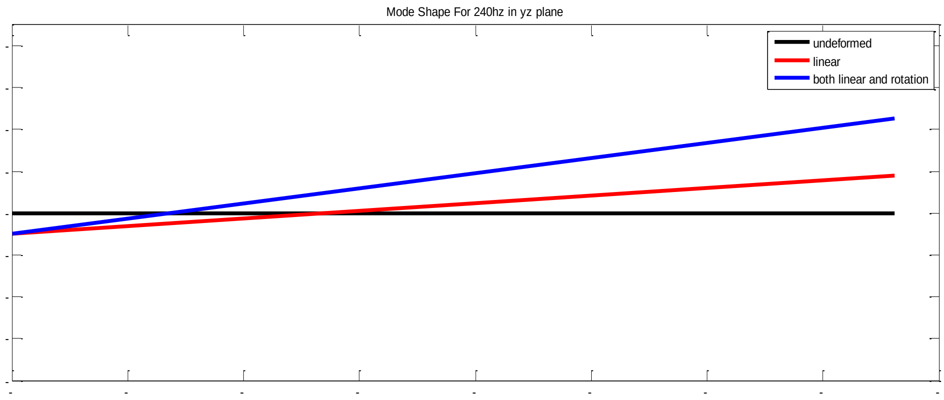

Figure 5-11:Mode shape for 240 Hz in yz plane

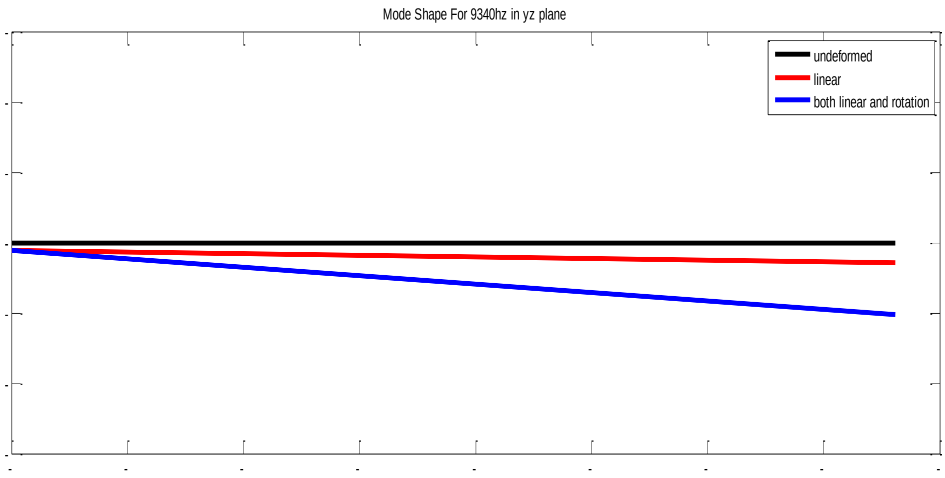

Figure 5-12:Mode shape for 9340 Hz in yz plane

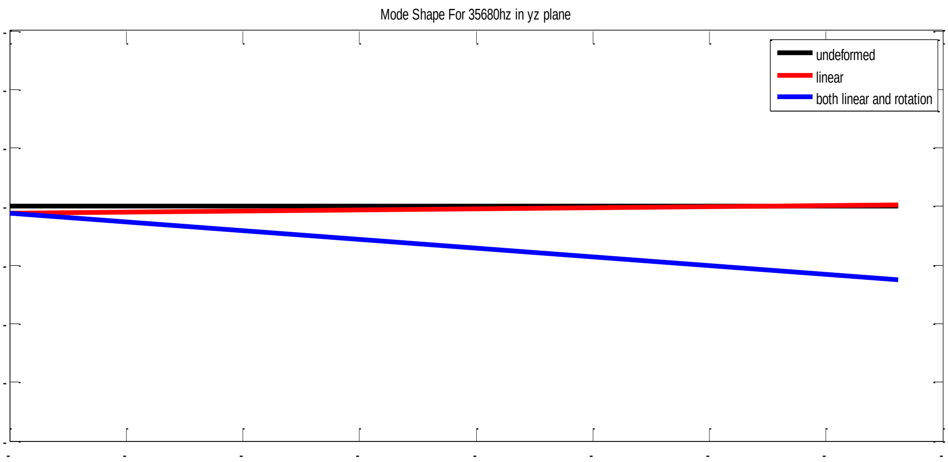

Figure 5-13:Mode shape for 35680 Hz in yz plane

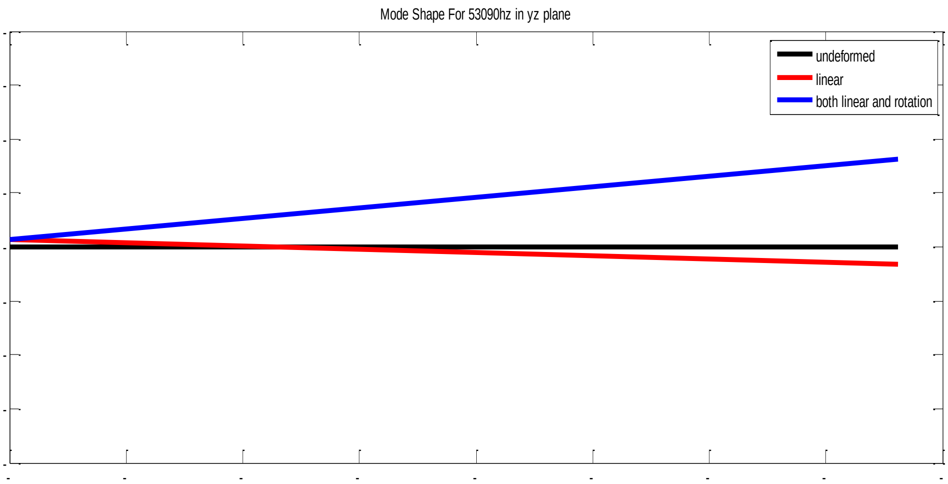

Figure 5-14:Mode shape for 53090 Hz in yz plane

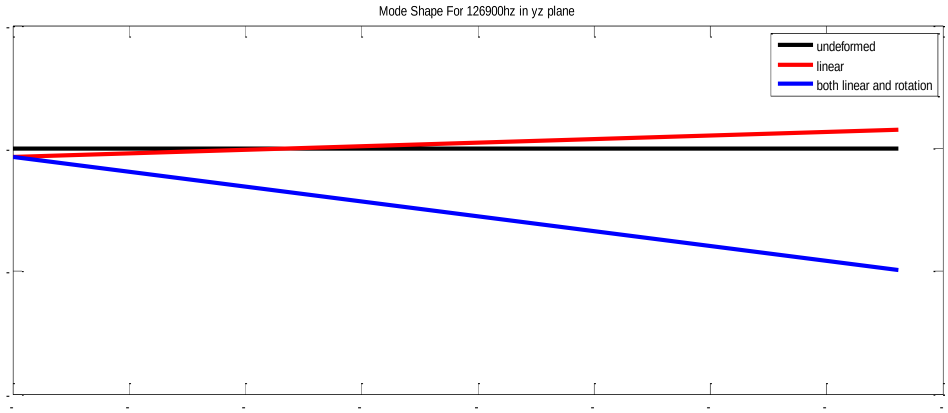

Figure 5-15:Mode shape for 126900 Hz in yz plane

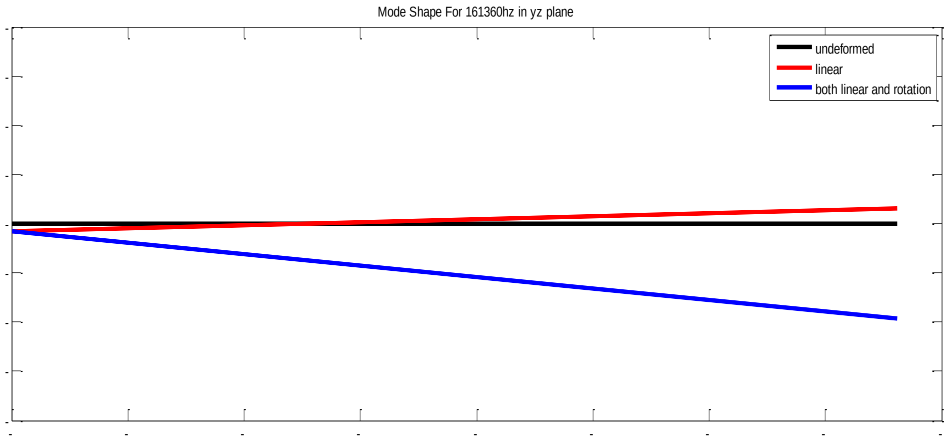

Figure 5-16:Mode shape for 161360 Hz in yz plane

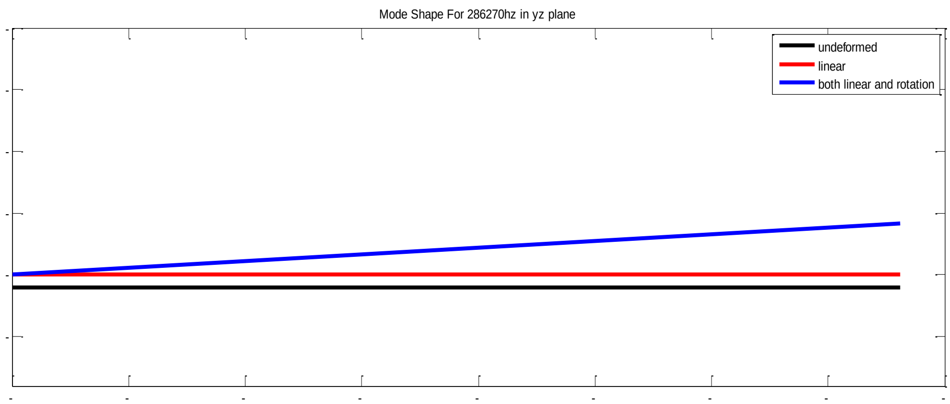

Figure 5-17:Mode shape for 286270 Hz in yz plane

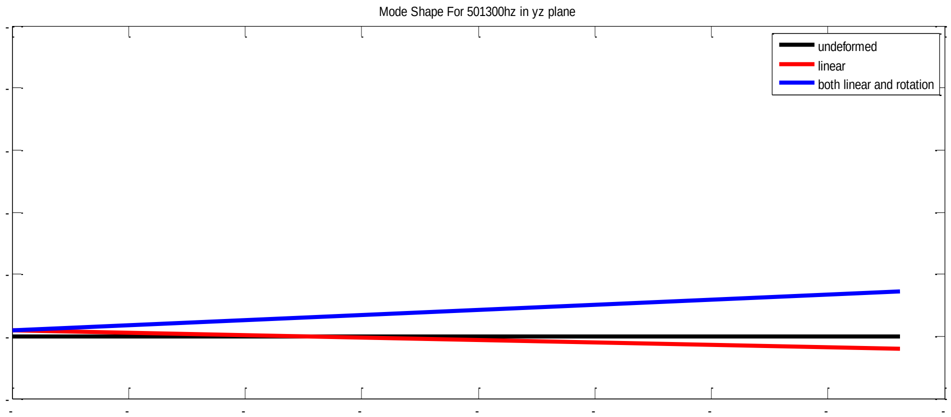

Figure 5-18:Mode shape for 501300 Hz yz plane

5.2 Fixed-Free beam

5.2.1 Natural Frequency

Table 5-3:Natural frequency of fixed-free beam

Natural Frequency of double tapered pre-twisted rotating twisting fixed- free Timoshenko beam						
Number	1	2	3	4	5	6
Frequency(Hz)	2340	9690	17310	46960	87410	26080

5.2.2:Eigen vector

Table 5-4:Eigenvector corresponding to Eigen values

Mode Shape or Eigen vector corresponding natural frequency						
	2340	9690	17310	46960	87410	260800
u2	-0.0823	0.0179	0.0250	-0.0120	0.1338	0.0160
u4	1	-1	-1	0.5063	0.8886	-0.6187
u6	-0.0001	0.0004	-0.0003	0.0004	-0.1427	-0.0161
u8	0	-0.0034	-0.0104	-0.0270	-0.5119	0.6200

u10	-0.0210	0.0443	0.0007	0.0245	-0.0703	-0.0099
u12	0.4013	-0.8869	-0.3003	-1	-1	0.3819
u14	0	0.0005	-0.0004	0.0001	0.2235	0.0259
u16	0	-0.0008	-0.0033	-0.0567	0.5653	-1

5.2.3 Mode Shape Diagram

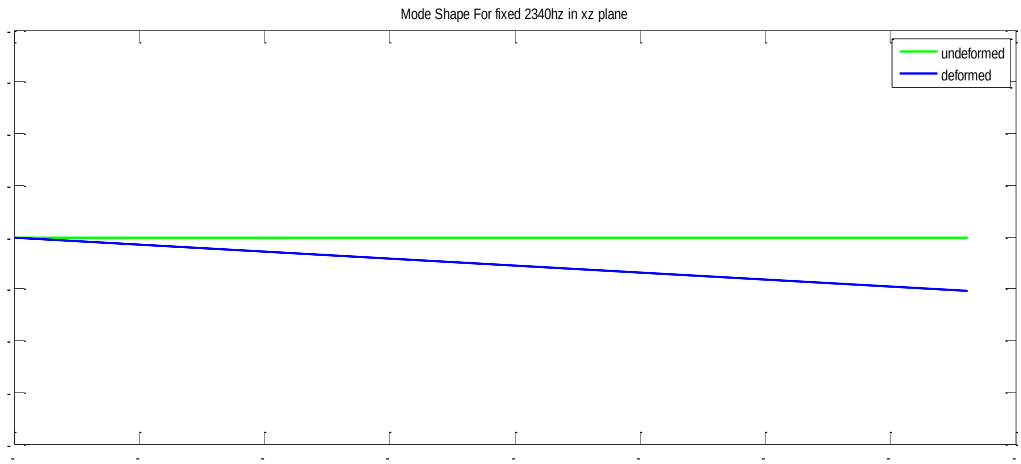

Figure 5-19:Mode shape for 2340 Hz in x-z plane

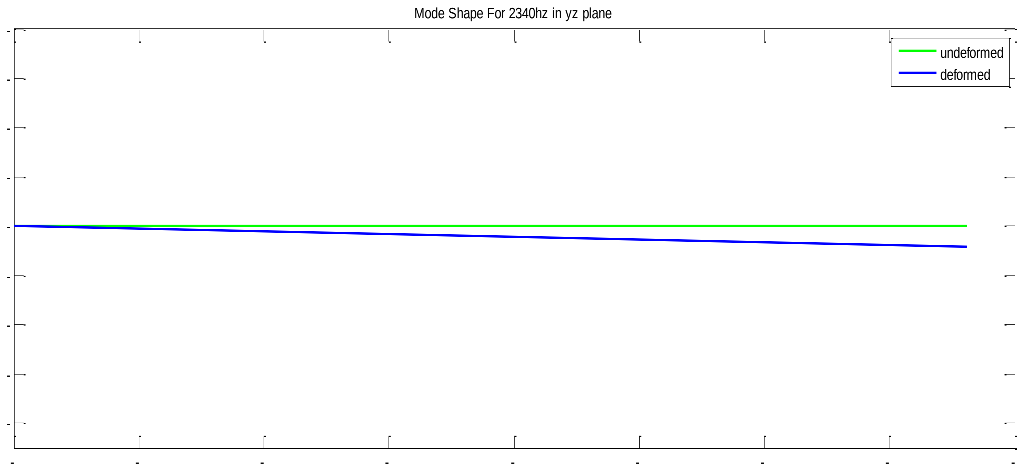

Figure 5-20:Mode shape for 2340 Hz in y-z plane

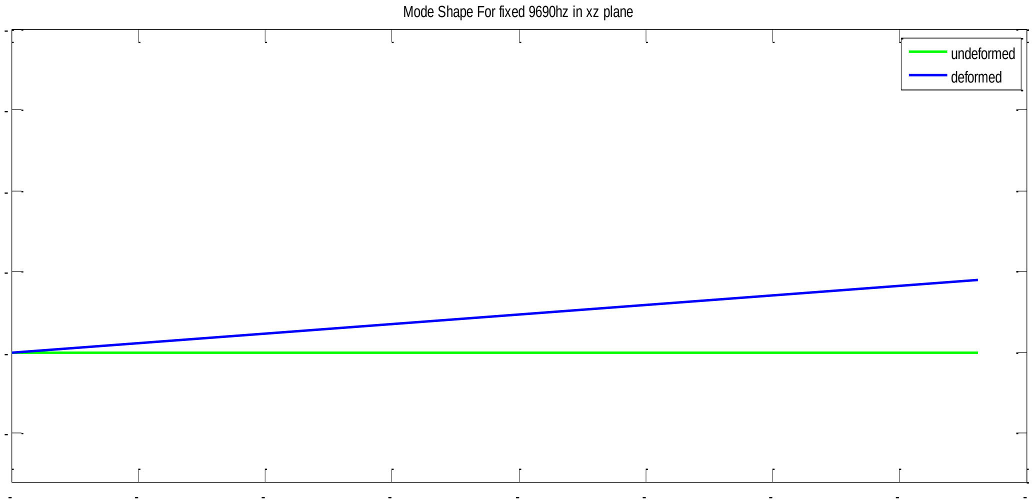

Figure 5-21:Mode shape for 9690 Hz in x-z plane

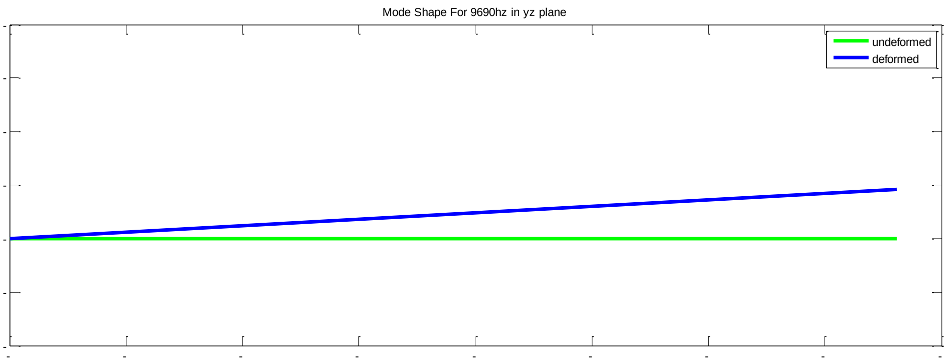

Figure 5-22:Mode shape for 9690 Hz in yz plane

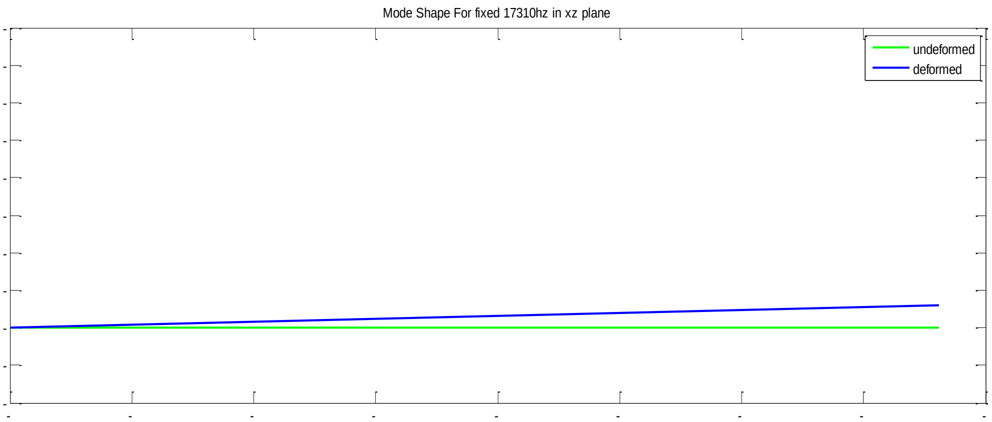

Figure 5-23:Mode shape for 17310 Hz in x-z plane

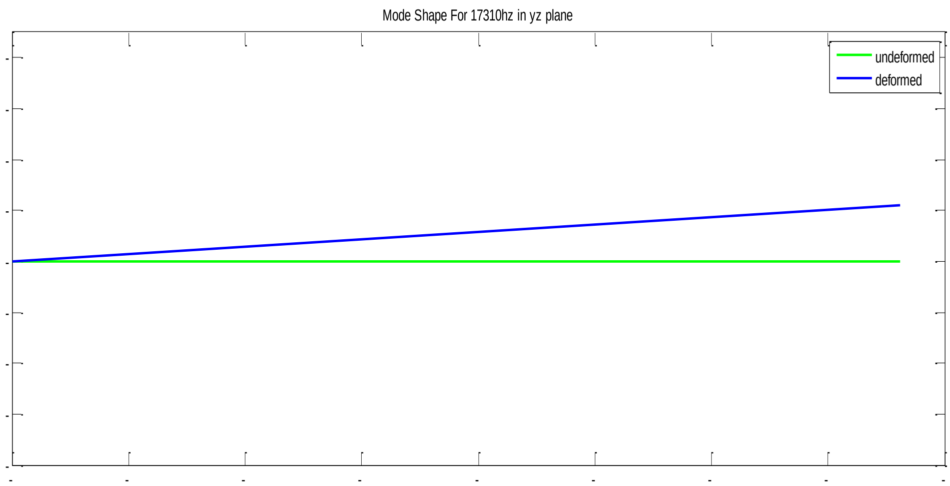

Figure 5-24:Mode shape for 17310 Hz in y-z plane

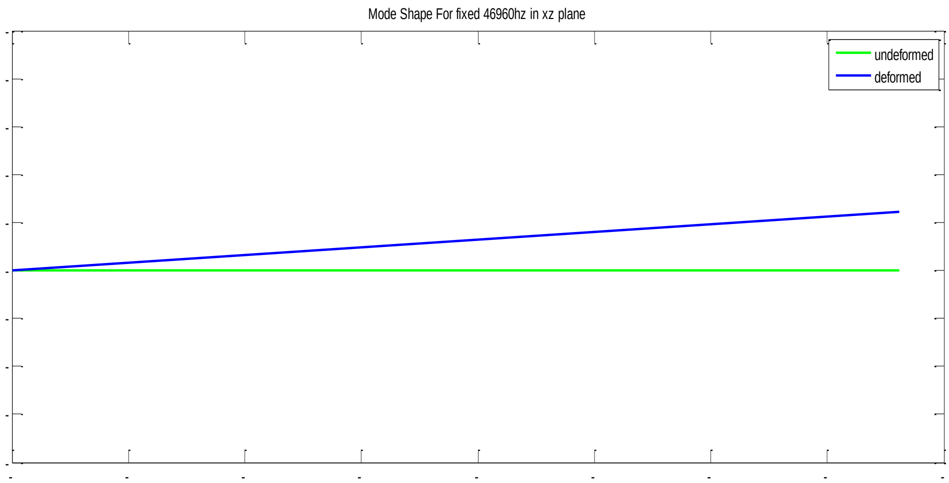

Figure 5-25:Mode shape for 46960 Hz in x-z plane

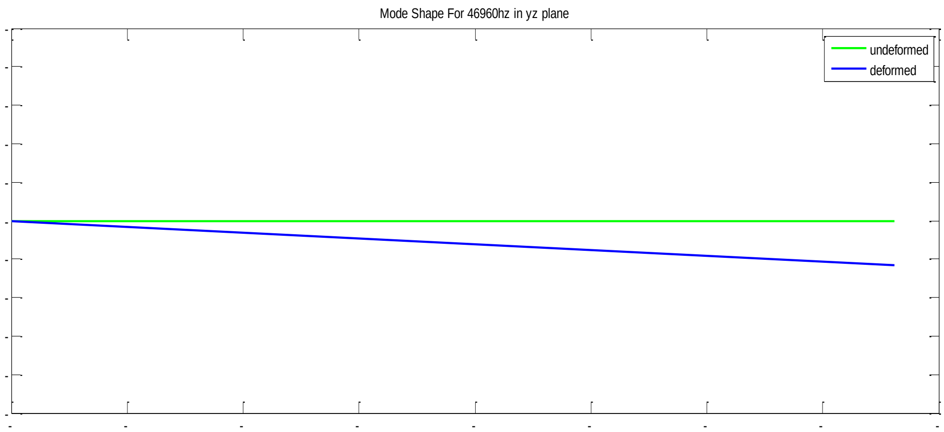

Figure 5-26:Mode shape for 46960 Hz in y-z plane

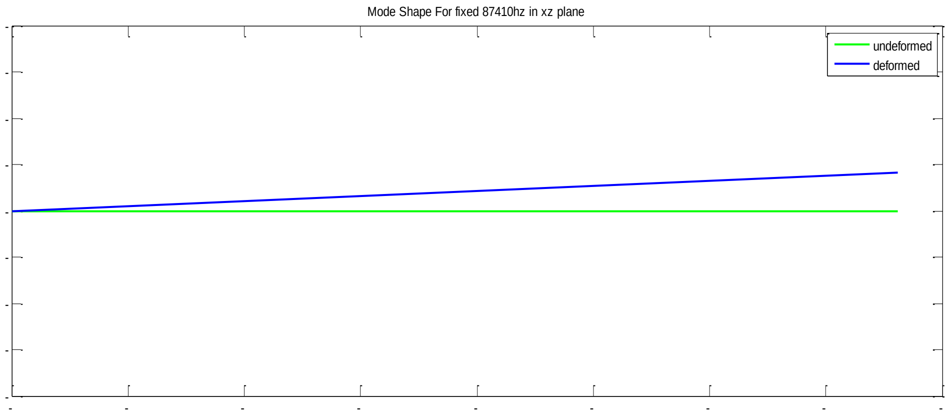

Figure 5-27:Mode shape for 87410 Hz in x-z plane

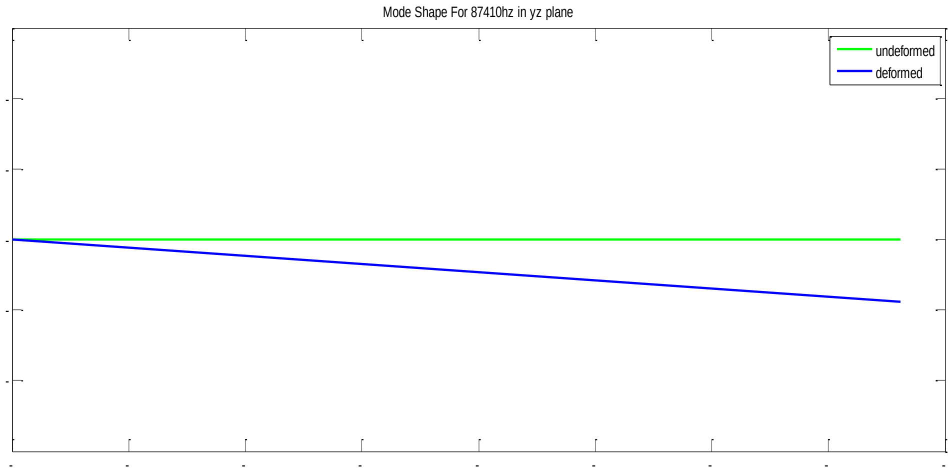

Figure 5-28:Mode shape for 87410 Hz in y-z plane

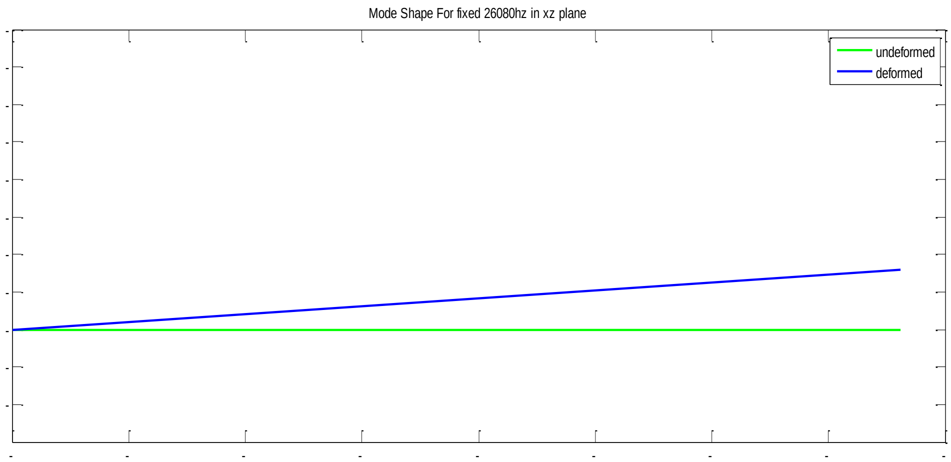

Figure 5-29:Mode shape for 26080 Hz in x-z plane

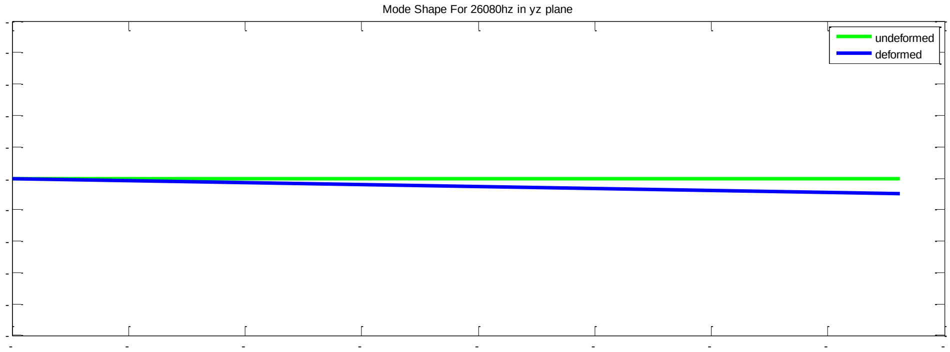

Figure 5-30:Mode shape for 26080 Hz in y-z plane

Chapter 6 Conclusions and Future Work

6.1 Conclusions

(1) The eight-degree-of-freedom pre-twisted Timoshenko beam element is made for the vibration analysis of rotating tapered Timoshenko beam.

(2) The finite element model developed is based on two displacement fields that couple the transverse and angular displacements in two planes. The rotary inertia terms are included.

(3) The functions of breadth and depth of the beam of equal strength have been taken into account in the derivation of the displacement functions of the finite element model.

(4) A computer program in Mathematica is developed and then used to evaluate the mass and stiffness matrix.

(5) The model has the considerable advantage of using few variables as nodal freedoms. Also, nodal variables facilitate the use of this element in the analysis of general structure involving other types of finite element having the same nodal freedoms.

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6.2 Future work

For future work, it is recommended to apply the proposed numerical method to study free vibration problem of a Timoshenko beam with different end condition such as fixed- fixed, fixed-hinge , simple supported internal cracks and cantilever and also it can with damaged boundaries

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