# A STUDY OF SOME ADVANCES IN FRACTIONAL CALCULUS AND SPECIAL FUNCTIONS WITH MODELLING AND APPLICATIONS IN PHYSICAL SYSTEM

by

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under the supervision of

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*Dedicated to*

*My Family*

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# CERTIFICATE

This is to certify that the thesis entitled **'A Study of some Advances in Fractional Calculus and Special Functions with Modelling and Applications in Physical System'** submitted by **Ms. Sonal Jain** to the Malaviya National Institute of Technology Jaipur, Rajasthan, India for the award of the degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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# Declaration

I hereby declare that the thesis entitled **A Study of some Advances in Fractional Calculus and Special Functions with Modelling and Applications in Physical System** is my own work conducted under the supervision of **Dr. Ritu Agarwal**, Assistant Professor, Department of Mathematics, Malaviya National Institute of Technology Jaipur, Rajasthan, India.

I firmly declare that the presented work does not contain any part of any work that has been submitted for the award of any degree either in this Institute or in any other University/Institute without proper citation.

Jaipur **Sonal Jain**
March 2017 **(2013RMA9043)**

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University, Jaipur, *Dr. K.C. Jain*, former Professor and Head, Department of Mathematics, MNIT, Jaipur, *Dr. M.A. Pathan*, Professor and Ex Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh, *Dr. H.M. Srivastava*, Emeritus Professor, Department of Mathematics and Statistics University of Victoria, Victoria, *Dr. Rekha Srivastava*, former Adjunct Professor, Department of Mathematics, University of Victoria, Victoria and *Martin stynes*, Chair Professor, Beijing Computational Science Research Center, Beijing for their all time invaluable support, matchless suggestions, unflinching and timely help that paved the way for completion of my thesis.

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# Abstract

The thesis has been divided into six chapters. Besides the introductory first chapter, second chapter concerns with study the integral transforms and fractional integral and derivative formulas of extended hypergeometric functions and incomplete hypergeomatric function. As an application, a probability density function involving the extended generalized hypergeometric functions is introduced. In third chapter, we establish certain new image formulas of generalized Lommel-Wright function by applying the operators of fractional integration involving Appell's function $F_3(\cdot)$ due to *Marichev-Saigo-Maeda*. Furthermore, by employing some integral transforms on the resulting formulas, we present some more image formulas. All the results derived here are of general character and can yield a number of results in the theory of special functions. In fourth chapter, we investigate the analytic solution of the solutions of time-space fractional advection-dispersion equation involving fractional Laplace operator and analytic solution of the generalized space-time fractional reaction-diffusion equation involving fractional Laplace operator, following with some illustrations and concrete applications. In second last chapter, we find the $P_\alpha$-transform of Caputo fractional derivatives and derive $P_\alpha$-transform for Volterra and Abel integral equations. We find the solution of fractional Volterra integral equation. We discuss its application for solving singular integral equation

having Bessel function in its kernel. The solution of non homogeneous time fractional heat equation in a spherical domain has been discussed. Last chapter deals with the study of certain polynomials and matrix function with Lie algebraic approach. Differential recurrence relations of some well-known orthogonal polynomials using results concerning eigenvector for the product of two operators defined on a Lie algebra of endomorphisms of a vector space have also been discussed.

I have great pleasure in pointing out that during period of research work, visited several institutes and universities. I had useful discussions on topics bearing on the subject matter of the thesis with eminent mathematicians Professor *A. M. Mathai*, Professor *M. A. Pathan*, Professor *H. J. Houbold*, Professor *F. Mainardi*, Professor *M. Stynes*, Professor *K. C. Gupta* and Professor *S. P. Goyal* .

In addition I have also participated in 4 national level workshops, 3 short term training programs, conferences (4 international and 2 national), presented a poster in Bluenose Applied and Computational Math Days workshop at Saint Marys University, Halifax, Nova Scotia, Canada and 2 summer schools, one is 11th SERB School on Matrix Methods and Fractional Calculus held at Centre for Mathematical and Statistical Sciences (CMSS) Peechi Campus, KFRI, Peechi, Trichur, Kerala and second is AARMS-PIMS Summer School in Differential Equations and Numerical Analysis, Dalhousie University, Halifax, Canada, and also received international travel grant from NBHM, to build my networking with other experts in my field and share ideas on recent advances in my field of study. It was an opportunity to expand my knowledge and enhance performance in my research.

# Contents

# 1

# Introduction

## 1.1 Special Functions

### 1.1.1 Growth of Special Functions

A special function is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated. Besides elementary functions such as $x^n$, $e^x$, $\log x$ and $\sin x$, higher functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) come under the category of special functions. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions. The history of special functions is closely tied to the problems of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries, the boundary-value problems of electromagnetism and heat in the nineteenth, and the eigenvalue problems of quantum mechanics in the twentieth centuries respectively. During Seventeenth-century, England was the birthplace of special functions. *John Wallis* at Oxford took first steps towards the theory of the gamma function long before *Euler* reached it. *Wallis* had also the first encounter with elliptic integrals while using Cavalieri's primitive for runner of the calculus. [It is curious that two kinds of special functions encountered in the seventeenth century, *Wallis* elliptic integrals and *Newton's* elementary symmetric functions, belong to the class of hypergeometric functions of several variables, which was not studied systematically nor even defined formally until the end of the nineteenth century]. A more sophisticated calculus, which made possible the real flowering of special functions, was developed by *Newton* and by *Leibnitz* in Germany during the period 1665-1685. *Taylor's* theorem was found by Scottish mathematician *Gregory* in 1670, although it was not published until 1715 after rediscovery by *Taylor*.

In 1703 *James Bernoulli* solved a differential equation by an infinite series which would now be called the series representation of a Bessel function. Although Bessel functions were met by Euler and others in various mechanics problems, no systematic study of the functions was made until 1824, and the principal achievements in the eighteenth century were the gamma function and the theory of elliptic integrals. Euler found most of the major properties of the gamma function around 1730. In 1772 *Euler* evaluated the beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by *Legendre* and *Gauss*, respectively, early in the next century. Other significant developments were the discovery of *Vandermonde's* theorem in 1772 and the definition of Legendre polynomials and the discovery of their addition theorem by *Laplace* and *Legendre* during 1782-1785. In a slightly different form the polynomials had already been met by *Liouville* in 1722. The golden age of special functions, which was centered in nineteenth-century Germany and France, was the result of developments in both mathematics and physics: the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) which required solutions of partial differential equations containing the Laplacian operator. The discovery of elliptic functions (the inverse of elliptic integrals) and their property of double periodicity was published by *Abel* in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century, specially in the hands of *Jacobi* and *Weierstrass.*

Another major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by *Euler* and *Pfaff*) with *Gauss's* memoir on the ${}_2F_1$ series in 1812, a memoir which was a landmark also on the path towards rigor in mathematics. The ${}_3F_2$ series was studied by *Clausen* (1828) and the ${}_1F_1$ series by *Kummer* (1836). The function which *Bessel* considered in his memoir of 1824 are ${}_0F_1$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics. Near the end of the century *Appell* (1880) introduced hypergeometric functions of two variables, and *Lauricella* generalized them to several variables in 1893.

The subject was considered to be part of pure mathematics in 1900, applied mathematics in 1950. In physical science special functions gained added importance as solutions of the Schrödinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907, *Barnes* used gamma function to develop a new theory of Gauss's hypergeometric functions ${}_2F_1$. Various generalizations of ${}_2F_1$ were introduced by *Horn*, *Kampe dé Fériet*, *MacRobert* and *Meijer*. From another new viewpoint, that of a differential difference equation discussed much earlier for polynomials by *Appell* (1880), *Truesdell* (1948) made a partly successful effort at unification by fitting a number of special functions into a single framework.

### 1.1.2 The Gauss Hypergeometric Function and its Generalization

*John Wallis*, in his work Arithmatica Infinitorum in 1655, first used the term *hypergeometric* (from the Greek word $\nu\pi\epsilon\rho$ above or beyond) to denote any series which was beyond the ordinary geometric series $1 + z + z^2 + z^3 + \ldots$. In particular, he studied the series $1 + a + a(a + 1) + a(a + 1)(a + 2) + \ldots$.

Because of the many relations connecting the special functions to each other and to the elementary functions, it is natural to enquire whether more general functions can be developed so that the special functions and elementary functions are merely specializations of these general functions. General functions of this nature have, in fact, been developed and are collectively referred to as functions of the hypergeometric type. There are several varieties of these functions, but the most common are the hypergeometric functions.

Some important results concerning the hypergeometric function had been developed earlier by *Euler* and others, but it was famous German mathematician *Gauss*, who in 1812, studied the following infinite series which is generalization of the elementary geometric series and popularly known as Gauss series or more precisely Gauss hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1) \cdot b(b+1)}{c(c+1)2 \cdot 1} z^2 + \ldots \tag{1.1}$$

Here, and in what follows, $(\lambda)_n$ denotes the Pochhammer symbol (or the *shifted* factorial) defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}) \end{cases} \tag{1.2}$$

The familiar (*Euler's*) gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C}\setminus\mathbb{Z}_0^-$, is given by

$$\Gamma(z) = \begin{cases} \int_0^\infty e^{-t}t^{z-1}dt & (\Re(z)>0) \\ \frac{\Gamma(z+n)}{\prod_{i=0}^{n-1}(z+i)} & (z\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ n\in\mathbb{N}), \end{cases} \tag{1.3}$$

$(\mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\};\ \ \mathbb{Z}^- := \{-1,-2,-3,\dots\};\ \ \mathbb{N} := \{1,2,3,\dots\})$,
Gauss represented the series (1.1) by the symbol ${}_2F_1\,(a,b;c;z)$ and called it the hypergeometric function. Here $z$ is a real or complex variable, $a, b$ and $c$ are parameters having real or complex values and $c \neq 0,-1,-2,...$

$${}_2F_1\,(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n. \tag{1.4}$$

If $c$ is zero or a negative integer, the function ${}_2F_1\,(a,b;c;z)$ is not defined unless one of the parameters $a$ or $b$ is also a negative integer such that $-c < -a$. If either of the parameters $a$ or $b$ is a negative integer, say $-r$, then in this case (1.4) reduces to the hypergeometric polynomial defined by

$${}_2F_1\,(-r,b;c;z) = \sum_{n=0}^{r}\frac{(-r)_n(b)_n}{(c)_n n!}z^n, \quad -\infty < z < \infty$$

The series given by (1.4) is convergent when $|z| < 1$ and when $|z| = 1$, provided that $\Re(c-a-b) > 0$ and also when $z = -1$, provided that $\Re(c-a-b) > -1$.

In (1.4), if we replace $z$ by $\frac{z}{b}$ and let $b \to \infty$, then on taking into account the formula

$$\lim_{b\to\infty} \frac{(b)_n}{b^n} z^n = z^n,$$

we arrive at the following well-known Kummer's series

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n = 1 + \frac{a}{c \cdot 1} z + \frac{a(a+1)}{c(c+1)1 \cdot 2} z^2 + \dots \tag{1.5}$$

The above series is represented by the symbol ${}_1F_1\,(a; c; z)$ and known as confluent hypergeometric function. The series given by (1.5) is absolutely convergent for all values of $a, c$ and $z$, real or complex, excluding $c = 0, -1, -2, \dots$.

Gauss's hypergeometric function ${}_2F_1$ and its confluent form ${}_1F_1$ form the core of special functions and include as its special cases most of the commonly used functions. Thus ${}_2F_1$ includes as its special cases Legendre function, the incomplete beta function, the complete elliptic functions of the first and second kinds and most of the classical orthogonal polynomials. On the other hand, the confluent hypergeometric function ${}_1F_1$ includes as its special cases Bessel functions, parabolic cylinder functions, Coulomb wave functions, etc. Whittaker functions are also a slightly modified form of confluent hypergeometric functions. On account of their usefulness, the functions ${}_2F_1$ and ${}_1F_1$ have already been explored to a considerable extent by a number of eminent mathematicians like *C. F. Gauss*, *E. E. Kummer*, L. J. Slater, R. Mellin and E.W. Barnes.

Hypergeometric function ${}_2F_1$ has been generalized by various mathematicians, mainly in three ways:

(1) increasing the number of parameters,

(ii) increasing the number of variables and

(iii) increasing the number of parameters as well as variables.

The most known generalization of first kind is the generalized hypergeometric function, defined by the series (*Rainville* [146, chapter 5, eq.2]):

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., a_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_q)_n}\frac{z^n}{n!}, \tag{1.6} $$

where $p$ and $q$ are positive integers or zero (interpreting an empty product as 1) and we assume that the variable $z$, the numerator parameters $a_1, a_2, ..., a_p$ and the denominator parameters $b_1, b_2, ..., b_q$ take on complex values, provided that $b_j \neq 0, -1, -2, \ldots$ for $j = 1, 2, \ldots, q$.

An application of the elementary ratio test to the power series on the right hand side of (1.6) shows that

(i) If $p \leq q$; the series converges for all finite $z$,

(ii) If $p = q + 1$, then the series converges for $|z| < 1$ and diverges for $|z| > 1$.

(iii) Furthermore, with $p = q + 1$, the series (1.6) is

   (a) Absolutely convergent on the circle $|z| = 1$, if $\Re(w) > 0$, where

$$ w = \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k $$

   (b) conditionally convergent for $|z| = 1, x \neq 1$ if $-1 < \Re(w) \neq 0$, and

   (c) divergent for $|z| = 1$ if $\Re(w) \leq -1$.

(iv) If $p > q+1$, the series never converges except when $z = 0$, and the function is only defined when the series terminates.

A comprehensive account of ${}_2F_1, {}_1F_1$ and ${}_pF_q$ functions can be found in the standard works by *Exton* [39], *Rainville* [146] and *Slater* [163]. In attempt to give meaning ${}_pF_q$ in the case when $p > q + 1$, *MacRobert* [102] and *Meijer* [113, 114] introduced and studied in detail, the two special functions which are well-known in the literature as the E-function and the G-function, respectively. A detailed account of the

G-function is given in the works by *Luke* [99] and Mathai and *Saxena* [107]. The E-and G-functions include wide variety of special functions as their particular cases. Though E-and G-functions are quite general in character, but still many functions like Wright's generalized hypergeometric function (Wright [186]), Wright's generalized Bessel function (Wright [187]), Mittag-Leffler function (Mittag-Leffler [118]) and several other functions do not form their special cases.

*Wright* [185] has further extended the generalization of the hypergeometric series in the following form:

$$
{}_p\Psi_q\,(z) = {}_p\Psi_q \left[ z \;\middle|\; \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{\Pi_{i=1}^{p} \Gamma(a_i + nA_i)}{\Pi_{j=1}^{q} \Gamma(b_j + nB_j)} \frac{z^n}{n!}, \tag{1.7}
$$

where $a_i, b_j \in \mathbb{C}$ and $A_i, B_j \in \mathbb{R} = (-\infty, \infty)$; $A_i, B_j \neq 0$, $i = 1, 2, ..., p$, $j = 1, 2, ..., q$, $\sum\limits_{j=1}^{q} B_j - \sum\limits_{j=1}^{p} A_j > -1$.

When $A_i$ and $B_j$ are equal to 1, (1.7) differs from the generalized hypergeometric function ${}_pF_q$ by a constant multiplier only.

The currently popular literature on *Special Functions* contains several generalizations of the gamma function $\Gamma(z)$, the beta function $B(\alpha, \beta)$, the hypergeometric functions ${}_1F_1$ and ${}_2F_1$, and the generalized hypergeometric functions ${}_rF_s$ with $r$ numerator and $s$ denominator parameters (see, for details, [16, 17, 92, 166, 172] *and* the references cited in each of these papers). In particular, for an appropriately bounded sequence $\{\kappa_\ell\}_{\ell \in \mathbb{N}_0}$ of essentially arbitrary (real or complex) numbers, *Srivastava et al.* [172, p. 243, Eq.(2.1)] recently considered the function $\Theta\left(\{\kappa_\ell\}_{\ell \in \mathbb{N}_0}; z\right)$ given by

$$
\Theta\left(\{\kappa_\ell\}_{\ell \in \mathbb{N}_0}; z\right) := \begin{cases} \sum\limits_{\ell=0}^{\infty} \kappa_\ell \dfrac{z^\ell}{\ell!} & (|z| < R;\ 0 < R < \infty;\ \kappa_0 := 1) \\ \mathfrak{M}_0\, z^\omega\ \exp(z) \left[1 + O\left(\dfrac{1}{z}\right)\right] & \big(\Re(z) \to \infty;\ \mathfrak{M}_0 > 0;\ \omega \in \mathbb{C}\big) \end{cases} \tag{1.8}
$$

for some suitable constants $\mathfrak{M}_0$ and $\omega$ depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. In terms of the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ defined by (1.8), *Srivastava et al.* [172] introduced and investigated the following remarkably deep generalizations of the extended gamma function, the extended beta function and the extended Gauss type hypergeometric function:

$$\Gamma_{\mathfrak{p}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z) := \int_0^\infty t^{z-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{\mathfrak{p}}{t}\right)\,\mathrm{d}t \tag{1.9}$$

$$\big(\Re(z)>0;\ \Re(\mathfrak{p})\geqq 0\big),$$

$$\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;\mathfrak{p}) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{\mathfrak{p}}{t(1-t)}\right)\mathrm{d}t \tag{1.10}$$

$$\big(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(\mathfrak{p})\geqq 0\big)$$

and

$$\mathfrak{F}_{\mathfrak{p}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b;c;z) := \frac{1}{B(b,c-b)}\sum_{n=0}^\infty (a)_n\,\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n,c-b;\mathfrak{p})\,\frac{z^n}{n!} \tag{1.11}$$

$$\big(|z|<1;\ \Re(c)>\Re(b)>0;\ \Re(\mathfrak{p})\geqq 0\big),$$

respectively, provided that the defining integrals in the equations (1.9) and (1.10) exist.

Recently, *Lin et al.* [92] introduced and investigated a substantially more general family of the generalized beta function and the *Gauss* type hypergeometric functions, which they defined by

$$\begin{aligned}\mathfrak{B}_{\mathfrak{p};\mu,\nu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta) &= \mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;\mathfrak{p};\mu,\nu)\\ &:= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{\mathfrak{p}}{t^\mu(1-t)^\nu}\right)\mathrm{d}t \end{aligned}\tag{1.12}$$

$$\big(\min\{\Re(\alpha),\Re(\beta),\Re(\mu),\Re(\nu)\}>0;\ \Re(\mathfrak{p})\geqq 0\big)$$

and

$$
{}_{r+q}\mathfrak{F}_{s+q}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu\right)}\left[\begin{array}{r} a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q; \\ \\ z \\ \\ c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q; \end{array}\right]
$$

$$
:= \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{r}(a_j)_n}{\prod\limits_{j=1}^{s}(c_j)_n} \prod_{j=1}^{q} \left( \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha_j+n,\gamma_j-\alpha_j;\mathfrak{p};\mu,\nu)}{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha_j,\gamma_j-\alpha_j;\mathfrak{p};\mu,\nu)} \right) \frac{z^n}{n!} \tag{1.13}
$$

$$
\big(q,r,s\in\mathbb{N}_0;\ |z|<1;\ \Re(\gamma_j)>\Re(\alpha_j)>0\ \ (j=1,\cdots,q);
$$

$$
\min\{\Re(\mu),\Re(\nu)\}>0;\ \Re(\mathfrak{p})\geqq 0\big),
$$

where, as usual, an empty product is interpreted as 1 and the involved parameters and the argument $z$ are *tacitly* assumed to be so constrained that the series on the right-hand side is absolutely convergent. The special case of the definition (1.13) when

$$
\mu=\nu=1 \qquad \text{and} \qquad q=r=s=1 \quad (a_1=1;\ \alpha_1=b;\ \gamma_1=c)
$$

coincides precisely with the definition (1.11). Also, for

$$
\mu=\nu=m \qquad \text{and} \qquad q=r=s=1 \quad (a_1=1;\ \alpha_1=b;\ \gamma_1=c)
$$

and with the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ given by

$$
\kappa_\ell = \frac{(\rho)_\ell}{(\sigma)_\ell} \tag{1.14}
$$

the definition (1.13) would obviously correspond to the Gauss type hypergeometric function introduced by *Parmar* [131, p. 44]:

$$
F_{\mathfrak{p}}^{(\rho,\sigma;m)}(a,b;c;z) := \sum_{n=0}^{\infty}(a)_n\,\frac{B^{(\rho,\sigma;m)}(b+n,\,c-b)}{B(b,\,c-b)}\,\frac{z^n}{n!} \tag{1.15}
$$

$$\big(|z| < 1;\ \Re(\mathfrak{p}) \geqq 0;\ \min\{\Re(\rho), \Re(\sigma), \Re(m)\} > 0;\ \Re(c) > \Re(b) > 0\big),$$

which, in case

$$\mu = \nu = m = 1 \qquad \text{and} \qquad q = r = s = 1 \quad (a_1 = 1;\ \alpha_1 = b;\ \gamma_1 = c)$$

and with the sequence $\{\kappa_\ell\}_{\ell \in \mathbb{N}_0}$ given by

$$\kappa_\ell = 1 \qquad (\ell \in \mathbb{N}_0)\,, \tag{1.16}$$

reduces immediately to the following $\mathfrak{p}$-Gauss hypergeometric function $F_{\mathfrak{p}}(a, b; c; z)$ studied by *Chaudhry et al.* [17]:

$$F_{\mathfrak{p}}(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \,\frac{B(b+n, c-b; \mathfrak{p})}{B(b, c-b)}\, \frac{z^n}{n!} \tag{1.17}$$

$$\big(\mathfrak{p} \geqq 0;\ |z| < 1;\ \Re(c) > \Re(b) > 0\big).$$

By using a very specialized case of the generalized beta function $\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell \in \mathbb{N}_0}\right)}(\alpha, \beta; \mathfrak{p}; \mu, \nu)$ defined by (1.12), with the sequence $\{\kappa_\ell\}_{\ell \in \mathbb{N}_0}$ given in a form as in (1.14), the following *particular* case of the extended generalized hypergeometric function

$${}_{r+q}\mathfrak{F}_{s+q}^{\left(\{\kappa_\ell\}_{\ell \in \mathbb{N}_0};\mathfrak{p};\mu,\nu\right)}$$

in (1.13) was studied by Luo *et al.* [100].

### 1.1.3 Extended Hypergeometric Functions

The extended generalized hypergeometric function is introduced by *Luo* et al. [101] and is defined as

$${}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} z; \mathfrak{p}\right] := \sum_{n=0}^{\infty} \Theta_n^{(p,q)}\, \frac{z^n}{n!} \tag{1.18}$$

$$\big(\Re(\mathfrak{p})>0;\ \min\{\Re(\kappa),\Re(\mu)\}\geqq 0;\ \min\{\Re(\alpha),\Re(\beta)\}>0\big),$$

where the coefficients $\Theta_n^{(p,q)}$ are given by

$$\Theta_n^{(p,q)} := \begin{cases} (a_1)_n \prod\limits_{j=1}^{q} \left( \frac{\mathcal{B}_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}(a_{j+1}+n,b_j-a_{j+1})}{B(a_{j+1},b_j-a_{j+1})} \right) \\ \qquad \big(p=q+1;\ \Re(b_j)>\Re(a_{j+1})>0\ \ (j=1,\cdots,q);\ |z|<1\big) \\ \\ \prod\limits_{j=1}^{q} \left( \frac{\mathcal{B}_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}(a_j+n,b_j-a_j)}{B(a_j,b_j-a_j)} \right) \\ \qquad \big(p=q;\ \Re(b_j)>\Re(a_j)>0\ \ (j=1,\cdots,q);\ z\in\mathbb{C}\big) \\ \\ \dfrac{1}{(b_1)_n\cdots(b_r)_n} \prod\limits_{j=1}^{p} \left( \frac{\mathcal{B}_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}(a_j+n,b_{r+j}-a_j)}{B(a_j,b_{r+j}-a_j)} \right) \\ \qquad \big(r=q-p>0;\ \Re(b_{r+j})>\Re(a_j)>0\ \ (j=1,\cdots,p);\ z\in\mathbb{C}\big), \end{cases} \tag{1.19}$$

and the generalized beta function $\mathcal{B}_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}(x,y)$ is given by the following very specialized case of the definition (1.12):

$$\mathcal{B}_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\ {}_1F_1\left(\alpha;\beta;-\frac{\mathfrak{p}}{t^{\kappa}(1-t)^{\mu}}\right)\,\mathrm{d}t \tag{1.20}$$

$$\big(\min\{\Re(\mathfrak{p}),\Re(\kappa),\Re(\mu)\}\geqq 0;\ \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\}>0\big).$$

In its particular case when $p-1=q=1$, the definition in (1.18) immediately yields the extended Gauss type hypergeometric function given by *Srivastava* et al. (see, e.g. [166])

$${}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a,b; \\ c; \end{array} z;\mathfrak{p}\right] := \sum_{n=0}^{\infty}(a)_n \frac{\mathcal{B}_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\,\frac{z^n}{n!} \tag{1.21}$$

$$\big(\min\{\Re(\mathfrak{p}),\Re(\kappa),\Re(\mu)\}\geqq 0;\ \min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(c)>\Re(b)>0;\ |z|<1\big),$$

which happens to be a very specialized case of the extended Gauss type hypergeometric function

$$\mathfrak{F}_{\mathfrak{p}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b;c;z)$$

defined by (1.11).

### 1.1.4 Incomplete Hypergeometric Functions

The theory of the incomplete gamma functions, as a part of the theory of confluent hypergeometric functions, has received its first systematic exposition by *Tricomi* [178] in the early 1950s. *Musallam* and *Kalla* ([1], [2]) considered a more general incomplete gamma function involving the Gauss hypergeometric function and established a number of analytic properties including recurrence relations, asymptotic expansions and computation for special values of the parameters.

The incomplete gamma type functions like $\gamma(s,x)$ and $\Gamma(s,x)$ (see, equations (1.22) and (1.23) here), both of which are certain generalizations of the classical gamma function $\Gamma(z)$, have been investigated by many authors.

The familiar incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, defined as (see, e.g. *Srivastava* et al. [167])

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt \qquad (\Re(s) > 0;\ x \geq 0) \tag{1.22}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt \qquad (x \geq 0;\ \Re(s) > 0 \quad \text{when} \ \ x = 0) \tag{1.23}$$

respectively, satisfy the following decomposition formula

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s) \qquad (\Re(s) > 0, x \geq 0). \tag{1.24}$$

The function $\Gamma(z)$, and its incomplete versions $\gamma(s,x)$ and $\Gamma(s,x)$, play important roles in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering.

Very recently, *Srivastava* et al. [167] introduced and studied some fundamental properties and characteristics of a family of two potentially useful and generalized incomplete hypergeometric functions defined as follows:

$$ {}_p\gamma_q[z] := {}_p\gamma_q \left[ \begin{array}{cc} (a_1, x), & a_2, ..., a_p; \\ & b_1, ..., b_q; \end{array} z \right] := \sum_{n=0}^{\infty} \frac{(a_1; x)_n \ (a_2)_n ...(a_p)_n}{(b_1)_n ...(b_q)_n} . \frac{z^n}{n!}, \tag{1.25} $$

and

$$ {}_p\Gamma_q[z] :={}_p \Gamma_q \left[ \begin{array}{cc} (a_1, x), & a_2, ..., a_p; \\ & b_1, ..., b_q; \end{array} z \right] := \sum_{n=0}^{\infty} \frac{[a_1; x]_n \ (a_2)_n ...(a_p)_n}{(b_1)_n ...(b_q)_n} . \frac{z^n}{n!} \tag{1.26} $$

where $(a_1; x)_n$ and $[a_1; x]_n$ are interesting generalization of the Pochhammer symbol $(\lambda)_n$, in terms of the incomplete gamma type functions $\gamma(\lambda, x)$ and $\Gamma(\lambda, x)$.

The generalized Pochhammer symbols in terms of incomplete gamma type function are defined as follows

$$ (\lambda; x)_\nu := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \tag{1.27} $$

and

$$ [\lambda; x]_\nu := \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \tag{1.28} $$

$$ (x \geq 0, \quad \lambda \in \mathbb{C}). $$

These incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ satisfy the following decomposition relation:

$$ (\lambda; x)_\nu + [\lambda; x]_\nu = (\lambda)_\nu \qquad (x \geq 0, \ \lambda \in \mathbb{C}). $$

In the definition (1.22), (1.23), (1.25), (1.26), (1.27) and (1.28), the argument $x \geq 0$ is independent of the argument $z \in \mathbb{C}$ which occurs in the definitions (1.18), (1.25) and (1.26).

As already pointed out by *Srivastava* et al. [167, p.675, Remark 7], since

$$|(\lambda;x)_\nu| \leqq |(\lambda)_\nu| \quad \text{and} \quad |[\lambda;x]_\nu| \leqq |(\lambda)_\nu| \quad (x \geq 0, \;\; \lambda \in \mathbb{C}), \tag{1.29}$$

the precise (sufficient) conditions under which the infinite series in the definitions (1.25) and (1.26) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$ (see, for details, *Rainville* [146, p.72-73] and *Srivastava* and *Karlsson* [169, p.20]. Indeed, in their special case when $x=0$, ${}_p\Gamma_q$ $(p,q \in \mathbb{N}_0)$ would reduce immediately to the extensively investigated generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$.

### 1.1.5 Some more Special Functions

(i) **Mittag-Leffler Function**

In 1971, *Prabhakar* [139] introduced the generalization of two parameter Mittag-Leffler function as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \qquad \gamma,\alpha,\beta \in \mathbb{C}. \tag{1.30}$$

Taking $\gamma = 1$, (1.30) reduces to the two parameter Mittag-Leffler function studied by *Wiman* [184] and defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \alpha,\beta \in \mathbb{C}, \; \Re(\beta) > 0. \tag{1.31}$$

As $\gamma \to 0$, then by virtue of the limit formula *Saxena* et al. [157, Eq.24]

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} \tag{1.32}$$

Taking $\beta = 0$, (1.31) reduces to the one parameter Mittag-Leffler function *Mainardi* and *Gorenflo* [103, Eq.2.1]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n)} \quad \alpha \in \mathbb{C}. \tag{1.33}$$

On taking $\alpha = 1$ and $\beta = 1$, (1.31) reduces to the well known exponential function:

$$E_{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \exp(z) \tag{1.34}$$

Some special cases of Mittag-Leffler function $E_\alpha(z)$ (see, e.g. *Haubold* and *Sexena* [61]) is

$$E_0(z) = \frac{1}{1-z}, \quad |z| < 1, \tag{1.35}$$

$$E_1(z) = e^z, \quad z \in \mathbb{C}, \tag{1.36}$$

$$E_2(z) = \cos h(\sqrt{z}), \quad z \in \mathbb{C}, \tag{1.37}$$

$$E_2(-z^2) = \cos(z), \quad z \in \mathbb{C}, \tag{1.38}$$

$$E_{1/2}(\pm z^{1/2}) = e^z[1 + \operatorname{erf}(\pm z^{1/2})] = e^z \operatorname{erfc}(\mp z^{1/2}), \quad z \in \mathbb{C} \tag{1.39}$$

where erfc denotes the complimentary error function and the error function is defined as:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad \operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \quad z \in \mathbb{C} \tag{1.40}$$

The following inverse Fourier transform formula for Generalized Mittag-Leffler function is given by *Haubold et al.* [160, Eq.25]

$$F^{-1}\{E_{\beta,\gamma}(-at^\beta |k|^\alpha); x\} = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[ \frac{|x|}{a^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1, \frac{1}{\alpha}), (\gamma, \frac{\beta}{\alpha}), (1, \frac{1}{2}) \\ (1, \frac{1}{\alpha}), (1, 1), (1, \frac{1}{2}) \end{matrix} \right] \tag{1.41}$$

where $min\{R(\alpha), R(\beta), R(\gamma)\} > 0$ and $\alpha > 0$.

(ii) **Wright Function**

The Wright function $W(\alpha, \beta; z)$ [29, p.300, Eq.6.7.13] defined by series representation

$$W(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)\ k!},\ \alpha > -1, z \in \mathbb{C}. \tag{1.42}$$

The relationship between the Wright function and the exponential function is given by *Gorenflo* and *Luchko* [50, Eq.27]

$$W\left(-\frac{1}{2}, \frac{1}{2}; z\right) = \frac{1}{\sqrt{\pi}} e^{z^2/4} \tag{1.43}$$

(iii) **Generalized Lommel-Wright Function**

The $J_{\nu,\lambda}^{\mu,m}(z)$, the *generalized Lommel-Wright function*, introduced and studied by de *Oteiza*, *Kalla* and *Conde* [32] as a further (4-indices) generalization of the Bessel and Bessel-Maitland (Wright) functions, is defined as

$$\begin{aligned} J_{\nu,\lambda}^{\mu,m}(z) &= (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda + k + 1))^m \Gamma(\nu + k\mu + \lambda + 1)} \\ &= (z/2)^{\nu+2\lambda} {}_1\Psi_{m+1}[(1,1);\ \underbrace{(\lambda+1,1),}_{m-times}\ (\nu+\lambda+1,\mu); -z^2/4] \end{aligned} \tag{1.44}$$

$$(z \in \mathbb{C}\backslash(-\infty, 0],\ \ \mu > 0,\ \ m \in \mathbb{N},\ \ \ \nu, \lambda \in \mathbb{C})$$

where ${}_p\Psi_q$ is the Fox-Wright generalized hypergeometric function as defined in (1.7).

These functions and their special cases $J_{\nu}^{\mu}(z), J_{\nu,\lambda}^{\mu}(z)$, as depending on the arbitrary fractional parameter $\mu > 0$, present a fractional order extension of the Bessel function $J_{\nu}(z)$ and as such, are closely related to fractional order analogues of the Bessel operators and fractional order equations and systems modeling numerous real world phenomena arising in applied science.

*Prieto et al.* [140] obtained some results related to fractional calculus operators of generalized Lommel-Wright function. *Konovska* [130] studied the convergence of series involving generalized Lommel-Wright function.

It may be noted that the special cases of generalized Lommel-Wright function (1.44), when $m = 1$ immediately reduce to a generalization of the Bessel function, introduced by *Pathak* also called generalized Bessel Maitland function [132]:

$$J^{\mu}_{\nu,\lambda}(z) = J^{\mu,1}_{\nu,\lambda}(z) = \left(\frac{z}{2}\right)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)} \tag{1.45}$$

$$z \in \mathbb{C}\backslash(-\infty, 0], \quad \mu > 0, \quad \nu, \lambda \in \mathbb{C}.$$

The $J^{\mu}_{\nu}(z)$ *Bessel Maitland* function defined by [165, p.19, Eq. 2.6.10]

$$J^{\mu}_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\ \Gamma(\nu+\mu k+1)} = {}_0\Psi_1[\text{——}; (\nu+1,\mu); -z] \tag{1.46}$$

$$z, \nu \in \mathbb{C}, \quad z \neq 0, \quad \Re(\nu) > -1, \quad \mu > 0.$$

which, in view of the definition (1.44), yields the following relationship with the generalized Lommel-Wright function $J^{\mu,m}_{\nu,\lambda}(z)$

$$J^{\mu}_{\nu}(z) = z^{\frac{-\nu}{2}} J^{\mu,1}_{\nu,0}(z)\ (2\sqrt{z}) \tag{1.47}$$

If we take $m = 1, \mu = 1$ and $\lambda = \frac{1}{2}$ in (1.44), then we obtain the *Struve function* $H_{\nu}(\cdot)$ (see, e.g. *Mathai* et al. [110, p.28, Eq.(1.170)])

$$H_{\nu}(z) = J^{1,1}_{\nu,1/2} = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})} \quad z, \nu \in \mathbb{C}. \tag{1.48}$$

If we take $m = 1$, $\mu = 1$ and $\lambda = 0$ in (1.44), then it give the following relationship with the classical *Bessel function* (see, e.g. *Mathai* et al. [110, p.27, Eq.(1.161)]):

$$J_\nu(z) = J^{1,1}_{\nu,0}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(\nu+k+1)k!} \quad z,\nu \in \mathbb{C}, \quad z \neq 0, \quad \Re(\nu) > -1. \tag{1.49}$$

The Bessel function of the second kind or the *Neumann function* $Y_\nu(z)$ defined as (see, e.g. [38, Chapter 7]):

$$Y_\nu(z) = \frac{J_{\nu(x)} \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi} \tag{1.50}$$

$$\big(z \in \mathbb{C} \setminus \{0\};\ \nu \notin \{-1,-2,-3,\cdots\}\big).$$

The $K_\nu(z)$ is the modified Bessel function of the third kind (or the Macdonald function) defined by (see, for example, [38, Chapter 7])

$$K_\nu(z) := \frac{\pi}{2\sin(\nu\pi)} \; [I_{-\nu}(z) - I_\nu(z)]\,, \tag{1.51}$$

where

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2n}}{n!\ \Gamma(\nu+n+1)} \tag{1.52}$$

$$\big(z \in \mathbb{C} \setminus \{0\};\ \nu \notin \{-1,-2,-3,\cdots\}\big).$$

(iv) **Whittaker Function**

The Whittaker function (see, e.g. *Mathai et al.* [110, p. 22]) is introduced by *Whittaker*(1904):

$$W_{\sigma,\eta}(z) = \frac{\Gamma(-2\eta)}{\Gamma(\frac{1}{2}-\sigma-\eta)} M_{\sigma,\eta}(z) + \frac{\Gamma(2\eta)}{\Gamma(\frac{1}{2}-\sigma+\eta)} M_{\sigma,-\eta}(z) = W_{\sigma,-\eta}(z) \tag{1.53}$$

$$\sigma \in \mathbb{C}, \;\; \Re(1/2 + \eta \pm \sigma) > 0$$

where

$$M_{\sigma,\eta}(z) = z^{\eta+\frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(\frac{1}{2} - \sigma + \eta; 2\eta + 1; z\right), \;\; \Re(1/2{+}\eta{\pm}\sigma) > 0, \;\; |\arg z| < \pi \tag{1.54}$$

The following integral formula involving the Whittaker function (see *Mathai et al.* [110, p. 56]) is used in finding the image formulae:

$$\int_0^\infty t^{\xi-1} e^{-\frac{t}{2}} W_{\sigma,\eta}(t)\, dt = \frac{\Gamma(\xi + \eta + \frac{1}{2})\Gamma(\xi - \eta + \frac{1}{2})}{\Gamma(\xi - \sigma + \frac{1}{2})} \tag{1.55}$$

$$(\sigma \in \mathbb{C}, \;\; \Re(\xi \pm \eta) > -1/2)$$

.

(v) **Krätzel Function** The Krätzel Function introduced by Krätzel [84] is defined for $x > 0$ by the integral

$$Z_\rho^\nu(z) = \int_0^\infty t^{\nu-1} e^{-t^\rho - x/t} dt \tag{1.56}$$

where $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$, such that $\Re(\nu) < 0$ for $\rho \leq 0$.

(vi) **H-Function**

The H-function (see, e.g. *Mathai et al.* [110]) introduced by *Charles Fox*(1961) by means of a Mellin-Barnes type integral in the following manner

$$\begin{aligned} H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix}\right] &= H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_q, B_q) \end{matrix}\right] \\ &= \frac{1}{2\pi i} \int_\Omega \Theta(\xi) z^{-\xi}\, d\xi \end{aligned} \tag{1.57}$$

where $i = \sqrt{(-1)}$ and

$$\Theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + sB_j) \prod_{j=1}^{n} \Gamma(1 - a_j - sA_j)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - sB_j) \prod_{j=n+1}^{p} \Gamma(a_j + sA_j)} \tag{1.58}$$

and an empty product is interpreted as unity; $m, n, p, q \in \mathbb{N}_0$ with $0 \leq n \leq p$, $1 \leq m \leq q$, $A_i, B_j \in \mathbb{R}_+, a_j, b_j \in \mathbb{C}, i = 1, 2, ..., p; j = 1, 2, ..., q$ such that

$$A_i(b_j + k) \neq B_j(a_i - \lambda - 1), k, \lambda \in \mathbb{N}_0; i = 1, 2, ..., n; j = 1, 2, ..., m, \tag{1.59}$$

The contour $\Omega$ is the infinite contour which separates all the poles of $\Gamma(b_j + sB_j),\ j = 1, 2, ..., m$ from all the poles of $\Gamma(1 - a_i + sA_i),\ i = 1, 2, ..., n$.

We note that most of the elementary and special functions are the special cases of The $H-$function. Few of them are mentioned here

(a) **Lorenzo-Hartley G-function** [54, p. 64, Eq. (2.3)]

$$H_{1,2}^{1,1}\left[-az^q \middle| \begin{matrix} (1-r,1) & \\ (0,1), & (1+\nu-rq,q) \end{matrix}\right] = \frac{\Gamma(r)}{z^{rq-\nu-1}} G_{q,\nu,r}[a,z]. \tag{1.60}$$

Here $G_{q,\nu,r}$ is the Lorenzo-Hartley G-function [98, Eq. 7].

(b) **Generalized Mittag-Leffler function** [109, p. 25, Eq. (1.137)].

$$H_{1,2}^{1,1}\left[-z \middle| \begin{matrix} (1-\gamma,1) & \\ (0,1), & (1-\beta,\alpha) \end{matrix}\right] = \Gamma(\gamma) E_{\alpha,\beta}^{\gamma}(z), \tag{1.61}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}; \qquad \Re(\alpha, \beta, \gamma) > 0)$$

where $E_{\alpha,\beta}^{\gamma}$ is the generalized Mittag-Leffler function as defined in (1.30) also follows as special cases of $H-$ function.

(c) **Generalized Hypergeometric function** as defined in (1.6) also follows as special cases of $H-$ function [165, p. 18, Eq. (2.6.3)]

$$H^{1,p}_{p,q+1}\left[z\left|\begin{array}{c}(1-a_j,1)_{1,p}\\(0,1),(1-b_j,1)_{1,q}\end{array}\right.\right]=\frac{\prod\limits_{j=1}^{p}\Gamma(a_j)}{\prod\limits_{j=1}^{q}\Gamma(b_j)}{}_pF_q[(a_p);(b_q);-z] \tag{1.62}$$

(d) **Generalized Bessel Maitland Function** [110, p. 25, Eq. (1.139)]

$$H^{1,1}_{1,3}\left[\frac{z^2}{4}\left|\begin{array}{c}(\lambda+\frac{\nu}{2},1)\\(\lambda+\frac{\nu}{2},1),(\frac{\nu}{2},1),(\mu(\lambda+\frac{\nu}{2})-\lambda-\nu,\mu)\end{array}\right.\right]=J^{\mu}_{\nu,\lambda}(z) \tag{1.63}$$

where $J^{\mu}_{\nu,\lambda}$ is the generalized Bessel Maitland function (1.45).

(e) **Wright's Generalized Bessel Function**[165, p. 19, Eq. (2.6.10)]

$$H^{1,0}_{0,2}\left[z\left|\begin{array}{c}--\\(0,1),(-\lambda,\nu)\end{array}\right.\right]=J^{\nu}_{\lambda}(z) \tag{1.64}$$

where $J^{\nu}_{\lambda}(z)$ is the Bessel maitland Function (1.46).

(f) **Krätzel Function** [110, p. 25, Eq. (1.141)]

$$H^{2,0}_{0,2}\left[z\left|\begin{array}{c}--\\(0,1),(\frac{\nu}{\rho},\frac{1}{\rho})\end{array}\right.\right]=Z^{\nu}_{\rho}(z) \qquad z,\nu\in\mathbb{C},\rho>0 \tag{1.65}$$

where $Z^{\nu}_{\rho}$ is the Krätzel function given in the eq. (1.56).

(g) **Modified Bessel function of the third kind** [49, p. 155, Eq. (2.6)]

$$H^{2,0}_{1,2}\left[z\left|\begin{array}{c}\left(1-\frac{\sigma+1}{\beta},\frac{1}{\beta}\right)\\(0,1),\qquad\left(-\gamma-\frac{\sigma}{\beta},\frac{1}{\beta}\right)\end{array}\right.\right]=K_{\nu}(z) \tag{1.66}$$

where $K_{\nu}(z)$ is the modified Bessel function of the third kind given in the eq.

(1.51).

(vii) **Appell function**

The 3rd Appell function (also known as Horn function) (see, e.g. [169, p.23]).

$$F_3(\alpha, \alpha^{'}, \beta, \beta^{'}; \eta; x; y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha^{'})_n(\beta)_m(\beta^{'})_n}{(\eta)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\max\{|x|, |y|\} < 1). \tag{1.67}$$

The properties of this function are discussed in detail by *Srivastava* and *Karlsson* [169]. The Appell function $F_3$ satisfies a system of two linear partial differential equations of the second order and reduces to the Gauss hypergeomatric function ${}_2F_1$ as follows (see, e.g. [169, p. 301, Eq. 9.4])

$$F_3(\alpha, \eta - \alpha, \beta, \eta - \beta; \eta; x; y) = {}_2F_1(\alpha, \beta; \eta; x + y - xy). \tag{1.68}$$

Further, it is easy to see that

$$F_3(\alpha, 0, \beta, \beta^{'}, \eta; x, y) = {}_2F_1(\alpha, \beta; \eta; x) \tag{1.69}$$

and

$$F_3(0, \alpha^{'}, \beta, \beta^{'}, \eta; x, y) = {}_2F_1(\alpha^{'}, \beta^{'}; \eta; y). \tag{1.70}$$

It is known that the 3rd Appell function cannot be expressed as a product of two ${}_2F_1$ functions that satisfy the system of two linear partial differential equations of the second order.

## 1.2 Integral Transform

If $f(x)$ denotes of a prescribed class of functions defined on a given interval $[a, b]$ and $K(x, s)$ denotes a definite function of $x$ in that interval for each value of $s$, a parameter whose domain is prescribed, then the linear integral transform $T[f(x); s]$

of the function $f(x)$ is defined in the following manner:

$$T[f(x); s] = \int_a^b K(x,s)f(x)dx \tag{1.71}$$

wherein the class of functions and the domain of parameter $s$ are so prescribed that the above integral exists. In (1.71), $K(x,s)$ is known as the kernel of the transform, $T[f(x); s]$ is the image of $f(x)$ in the said transform; and $f(x)$ is the original of $T[f(x); s]$.

If an integral equation can be determined that

$$f(x) = \int_\alpha^\beta \phi(s,x)T[f(x); s]ds \tag{1.72}$$

then (1.72) is termed as the inversion formula of (1.71)

### 1.2.1 $P_\delta$ - Transform

Pathway model is based on the principle of switching among three different families of functions, say generalized extended type-1 beta family, type-2 beta family and gamma family. This type of switching property can be used in practical situations where one needs to fit a parametric family of distributions to experimental data or to switch among three different functional forms. When the pathway parameter is allowed to vary, we get three different forms. In real scaler case, the pathway model is defined as

$$\begin{aligned} f(x) =& c_1|x|^\gamma[1-a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}}, \quad 1-a(1-\alpha)|x|^\delta > 0, \alpha < 1 \\ =& c_2|x|^\gamma[1+a(\alpha-1)|x|^\delta]^{-\frac{\eta}{\alpha-1}}, \quad -\infty < x < \infty, \alpha > 1 \\ =& c_3|x|^\gamma e^{-a\eta|x|^\delta}, \quad -\infty < x < \infty, \alpha \to 1 \end{aligned} \tag{1.73}$$

where $a > 0$, $\delta > 0$, $\gamma > 0$, $\eta > 0$. $c_1, c_2$ and $c_3$ are the normalizing constants if we consider each of them as statistical density. Three different functional forms are, generalized form extended type-1 beta, type-2 beta form respectively. The Tsallis statistics [179, 180], and superstatistics are covered by the pathway model. In recent years, the pathway model and Tsallis statistics have been applied in many areas like thermonuclear reaction rate theory in astrophysics [59, 60, 88] and in applied analysis [77, 86, 89] by Kumar and co-workers. In 2011, the Kumar introduced a fractional type integral transform called $P$- transform or pathway transform defined by

$$(P_{\nu}^{\rho,\beta,\alpha} f)(x) = \int_0^{\infty} D_{\rho,\beta}^{\nu,\alpha}(xt) f(t) dt, \quad x > 0, \tag{1.74}$$

where $D_{\rho,\beta}^{\nu,\alpha}(x)$ denotes the function

$$D_{\rho,\beta}^{\nu,\alpha}(x) = \int_0^{[\frac{1}{a(1-\alpha)}]^{\frac{1}{\rho}}} y^{\nu-1}[1 - a(1-\alpha)y^{\rho}]^{\frac{1}{1-\alpha}} e^{-xy^{-\beta}} dy, \quad x > 0 \tag{1.75}$$

with $\nu \in C, \beta > 0, \rho > 0, a > 0, \alpha < 1$ or

$$D_{\rho,\beta}^{\nu,\alpha}(x) = \int_0^{\infty} y^{\nu-1}[1 + a(\alpha - 1)y^{\rho}]^{-\frac{1}{1-\alpha}} e^{-xy^{-\beta}} dy, x > 0 \tag{1.76}$$

for $\nu \in C, \beta > 0, a > 0, \rho \in R, \alpha > 1$. When $D_{\rho,\beta}^{\nu,\alpha}(x)$ takes the from (1.75) or (1.76), the transform will be called as type-1 or type-2 P- transform, respectively, which are defined in the space $L_{\nu,r}(0, \infty)$ consisting of the Lebesgue measurable complex valued functions $f$ for which

$$||f||_{\nu,r} = \left\{ \int_0^{\infty} |t^{\nu} f(t)|^r \frac{dt}{t} \right\}^{\frac{1}{r}} < \infty \tag{1.77}$$

for $1 \leq r < \infty, \nu \in \mathbb{R}$. The $P$-transform and the $P_{\alpha}$-transform both are based on pathway idea but the $P_{\alpha}$-transform deals the problem with much easy compared to $P$-transform.

**Corollary 1.2.1.** *If conditions of Theorem 1.2.6 are satisfied and $\alpha \to 1$, then the Laplace transform obtained as $\lim_{\alpha\to 1} P_\alpha[f(t); s] = L[f(t); s]$ defined in (1.81) converges absolutely if $\Re(s) > c$. Moreover instead of condition (iii) if $f(t) = O(t^\gamma)$, $\Re(\gamma+1) > 0$ as $t \to \infty$, then the Laplace transform obtained as $\lim_{\alpha\to 1}[P_\alpha; s] = L[f(t); s]$ converges absolutely for $\Re(s) > 0$.*

**Theorem 1.2.2.** *[87, Theorem 3] (Convolution Theorem for $P_\alpha$-transform) If $F(s)$ and $G(s)$ are the $P_\alpha$- transform of the functions $f(t)$ and $g(t)$, respectively, then the product $F(s)G(s)$ is the $P_\alpha$- transform of the function $\int_0^t f(t-\tau)g(\tau)d\tau$. That is,*

$$F(s)G(s) = P_\alpha\left[\int_0^t f(t-\tau)g(\tau)d\tau; s\right] = P_\alpha[f(t); s]P_\alpha[g(t); s]. \tag{1.78}$$

**Lemma 1.2.3.** *[87] For $\nu \in \mathbb{C}$, $\Re(\nu) > 0$ and for $\alpha > 1$, we have*

$$P_\alpha\left[{}_0\mathbb{D}_t{}^{-\nu} f(t); s\right] = \left\{\frac{\alpha-1}{\ln[1+(\alpha-1)s]}\right\}^\nu P_\alpha[f(t); s] \tag{1.79}$$

*where ${}_0\mathbb{D}_t{}^{-\nu}$ is Riemann-Liouville fractional integral defined in equation (1.87).*

**Theorem 1.2.4.** *[87] If $f(t)$ and its derivatives up to order $n$ are of exponential order and are $P_\alpha$-transformable and if $f(t)$ and its derivatives up to $(n-1)$th order are continuous with the exception of the origin and if $n$th derivative $f^{(n)}(t)$ is at least piecewise continuous and if $P_\alpha[f(t); s] = F(s)$ then*

$$P_\alpha[f^{(n)}(t); s] = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^n F(s) - \sum_{m=1}^{n}\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{n-m} f^{(m-1)}(0+), \tag{1.80}$$

*where $f(0+) = \lim_{\epsilon\to 0} f(0+\epsilon)$.*

The $P_\delta$ - *transform* of a complex valued function $f(z)$ of a real variable $z$ denoted by $P_\delta[f(z); s]$ is a function $F(s)$ of a complex variable $s$, valid under certain conditions on $f(z)$, (given below in Lemma 1.2.5) is defined as (Kumar [85])

$$P_\delta[f(z); s] = F(s) = \int_0^\infty [1 + (\delta - 1)s]^{-\frac{z}{\delta-1}} f(z)dz, \quad \delta > 1 \tag{1.81}$$

Here $\lim\limits_{\delta\to 1+} [1+(\delta-1)s]^{-\frac{t}{\delta-1}} = e^{-st}$ defines a class of transforms. All these transforms are the paths going from the binomial form $[1 + (\delta - 1)s]^{-\frac{t}{\delta-1}}$ to the exponential from $e^{-st}$. In $P_\delta-$ transform the variable $t$ is shifted from the binomial factor $[1+(\delta-1)s]^{-\frac{t}{\delta-1}}$ to the exponent and hence this form is more suitable for obtaining translation, convolution etc.

The $P_\delta$ - transform of power function $t^{\rho-1}$ is given by Kumar [87, Eq. 32]

$$P_\delta[t^{\rho-1}; s] = \left\{\frac{\delta - 1}{\ln[1 + (\delta - 1)s]}\right\}^\rho \Gamma(\rho) \quad (\Re(\rho) > 0;\ \delta > 1). \tag{1.82}$$

The convergence conditions for the $P_\delta-$ transform of a function $f(t)$ to exist are given by the following results.

**Lemma 1.2.5.** *[85] If $f(z)$ is integrable over any finite interval $(a, b)$, $0 < a < z < b$, there exists a real number $c$ such that,*
*(i) for any arbitrary $b > 0$, $\int_b^\varrho e^{-cz}f(z)dz$ tends to a finite limit as $\varrho \to \infty$*
*(ii) for any arbitrary $a > 0$, $\int_\nu^a |f(z)dz|$ tends to a finite limit as $\nu \to 0+$,*
*then the $P_\delta$-transform $P_\delta[f(z); s]$ exists for $\Re\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) > c$ for $s \in \mathbb{C}$.*

**Theorem 1.2.6.** *[87, Theorem 1] If*
*(i) f(t) is integrable over a finite limit $(a, b)$, $0 < a < t < b$,*
*(ii) for arbitary positive a, the integral $\int_\nu^a |f(t)|dt$ tends to a finite limit as $\nu \to 0+$*
*(iii) $f(t) = O(e^{ct})$, $c > 0$ as $t \to \infty$ where $O(\cdot)$ is the standard big O notation which means $f(t)$ is of order not exceeding $e^{ct}$,*
*then the $P_\alpha$-transform defined in (1.81) converges absolutely if $\Re\left(\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right) > c$, $\alpha > 1$.*
*If instead of condition $(iii)$, we have the condition $f(t) = O(t^\gamma)$, $\Re(\gamma+1) > 0$ as $t \to$*

$\infty$, *then the pathway-Laplace transform converges absolutely for* $\Re\left(\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right) > 0$.

Furthermore, upon letting $\delta \to 1$ in the definition (1.81), the $\mathcal{P}_\delta$-transform is reduced to the classical Laplace transform:

$$\mathbb{L}[f(t); s] := \int_0^\infty e^{-st}\, f(t)\mathrm{d}t, \quad \Re(s) > 0, \tag{1.83}$$

provided that the integral exists.
By closely comparing the definitions in (1.81) and (1.83), it is easily observed that the $\mathcal{P}_\delta$-transform is essentially the same as the classical Laplace transform *with* the following parameter change:

$$s \;\mapsto\; \frac{\ln[1+(\delta-1)s]}{\delta-1} \qquad (\delta > 1). \tag{1.84}$$

In view of the Eq. (1.84), the following relationship holds true between the $\mathcal{P}_\delta$-transform defined by (1.81) and the classical Laplace transform:

$$\mathcal{P}_\delta[f(t) : s] = \mathbb{L}\left[f(t) : \frac{\ln[1+(\delta-1)s]}{\delta-1}\right] \qquad (\delta > 1) \tag{1.85}$$

or, equivalently,

$$\mathfrak{L}[f(t) : s] = \mathcal{P}_\delta\left[f(t) : \frac{e^{(\delta-1)s}-1}{\delta-1}\right] \qquad (\delta > 1) \tag{1.86}$$

which can indeed be applied reasonably simply to convert the table of Laplace transforms into the corresponding table of $\mathcal{P}_\delta$-transforms and *vice versa*.

# 1.3 Fractional Calculus

## 1.3.1 Historical Development

The fractional calculus, like many other mathematical disciplines and ideas, has its origin in the striving for extension of meaning. Well known examples are the extensions of the integers to the rational numbers, of the real numbers to the complex numbers, of the factorials of integers to the notation of the gamma function and many such others. The original question that led to the name of fractional calculus was: Can the meaning of derivative of integer order $\frac{d^n y}{dx^n}$ be extended to have a meaning when $n$ is friction? Later the question became : Can $n$ be any number, fractional, irrational or complex? Because this question was answered affirmatively, the name fractional calculus has become a misnomer and might better be called integration and differentiation to an arbitrary order.

In a letter dated 30th September 1695, *L'Hospital* wrote to *Leibnitz* asking him a particular notation that he had used in his publication for the nth derivative of a function $\frac{d^n f(x)}{dx^n}$ i.e. what would the result be if $n = \frac{1}{2}$? *Leibnitz's* response an apparent paradox from which one day useful consequences will be drawn. In these words, fractional calculus was born.

The earliest systematic studies seem to have been made in the beginning and middle of the 19th century by *Liouville* [93], *Riemann* [147] and *Holmgren* [65]. The list of mathematicians who provided important contributions up to the middle of 20th century, includes *Fourier* [42], *Weyl* [183], *Davis* [25–27], *Zygmund* [190], *Erdélyi* [35–37], *Kober* [82], *Hardy and Littlewood* [55], *Grünwald* [53], *Letnikov* [90, 91], *Riesz* [148] and several others.

In 1974 the first international conference on fractional calculus was held at the University of New Haven, Connecticut, U.S.A. The proceedings of the conference were published by Springer-Verlag [149]. Again in 1984 and 1989, the second and third international conferences were held at University of *Starthclyde*, *Glasgow*, *Scotland* [111] and at Nihon University, Tokyo, Japan [121] respectively. Many distinguished mathematicians attended these conferences. These luminaries included *R. Askey, M. Mikolas, M.Al-Bassam, P.Heywood, W. Lamb, R. Bagley, Y.A. Brychkov, R. Gorenflo, S.L. Kalla, E.R. Love, K.Nishimoto, S. Owa, A.P. Prudnikov, B. Ross, S. Samko, H.M. Srivastava, J.M.C. Joshi* and many others. The papers on the fractional calculus and generalized functions, inequalities obtained by use of the fractional calculus and applications of the fractional calculus to probability theory presented in the conference were quite electric.

A systematic (and historical) account of investigations carried out by various authors in the field of fractional calculus and its applications can be found in the paper by *Srivastava* and *Saxena* [174] wherein extensive bibliography on the subject has been given. One can also refer to the research papers by *Pandey* and *Srivastava* [129], *Duren et al.* [34], *Galue et al.* [44], *Nishimoto* and *Srivastava* [122], *Srivastava et al.* [173], *Saigoet al.* [154], *Srivastava* and *Owa* [171], *Manocha* [105], *Srivastava* and *Goyel* [168], *Saigo* and *Raina* [153] and several others. The excellent research monographs by *Oldham and Spanier [123], Nishimoto [120], Miller and Ross [117]* and *Samko et al. [155]* contain extensive and useful literature concerning the fractional calculus.

### 1.3.2 Applications of Fractional Calculus

The basic mathematical ideas of fractional calculus (integral and differential operations of noninteger order)were developed long ago by the mathematicians *Leibniz* (1695), *Liouville* (1834), *Riemann* (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by *Oldham* and *Spanier*. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetic.

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials as well as in the description of rheological properties of rocks and in many other fields. The subject of fractional calculus deals with the investigations of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integer-order derivative and n-fold integral. It has gained importance and popularity during the last four decades or so, mainly due to its vast potential of demonstrated applications in various seemingly diversified fields of science and engineering, such as fluid flow, rheology, diffusion, relaxation, oscillation, anomalous, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric network, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and others. The mathematical modeling and simulation of systems and processes based on the description of their properties in terms of fractional derivatives naturally leads to differential equations of fractional order and to the necessity to solve such equations.

As to the mathematical theory of the differential equations of fractional order, the current situation for the ordinary differential equations is different from the one for the partial differential equations. Whereas it is more or less complete for the ordinary differential equations of fractional order (*Kilbas et al.* [80], *Metzler* and *Klafter* [116]).

Many applications of fractional calculus can be found in other diverse fields in Carpinteri and Mainardi [15], *Jeses* and *Machado* [70], *Podulbuny* [137], *Hilfer* [64]. *Bagley* and *Torvik* [5] found the application of fractional calculus in visco elasticity and electrochemistry of corrosion. *Oldham* and *Spanier* ([124, 125]) explained its applications in electrochemistry and general transport problem. Virtually no area of classical analysis is left untouched by fractional calculus.

The NASA STI (National Aeronautics and Space Administration Scientific and Technical Information) USA, report series TP (Technical Publication) and TM (Technical Memorandum) contain a huge amount of useful research material on practical applications of fractional calculus (*Hartely* and *Lorenzo* [56–58], *Lorenzo* and *Hartley* [97]). Mathematical models, using ordinary differential equations with integer order, have been proven valuable in understanding the dynamics of physical systems. (*Atanackovic* [4], *Baleanu* et al. [6–8], *Herrmann* [62], *Ortigueira* [126], *Uchaikin* [181]). The modeling of these systems by fractional order differential equations has more advantages than classical integer-order mathematical modeling.

### 1.3.3 Operators of Fractional Calculus

The right-sided Riemann-Liouville fractional integral of order $\alpha, (\Re(\alpha) > 0)$ (*Samko et al.* [155]) is defined as:

$$\mathbb{I}_0^{\alpha}(u(t)) = {}_0\mathbb{D}_t{}^{-\alpha}(u(t)) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau,\ t > a \tag{1.87}$$

The right-sided Riemann-Liouville fractional derivative of order $\alpha, (\Re(\alpha) > 0)$ can be defined as:

$${}_0\mathbb{D}_t^{\alpha}(u(t)) = \left(\frac{d}{dt}\right)^n (\mathbb{I}_0^{n-\alpha} u(t)), \qquad n = [\Re(\alpha)] + 1, \tag{1.88}$$

where $[x]$ represents the integral part of the number $x$.
The power function formula involving the Riemann-Liouville operator (see, e.g. *Mathai* et al. [110]) is given by:

$${}_0\mathbb{D}_t^{\alpha} t^{\rho-1}(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)} x^{\rho+\alpha-1} \quad (\Re(\alpha) > 0, \Re(\rho) > 0). \tag{1.89}$$

The Laplace transform of the Riemann-Liouville fractional derivative in equation (2.43) is given by (see, e.g, *Oldham* and *Spanier* [125, p.134]) as

$$L[{}_0\mathbb{D}_t^{\alpha}u(t):s]=s^{\alpha}f(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(0), \tag{1.90}$$

where $(n-1)<\alpha\leq n$.

The Weyl fractional integral operator are defined as follows (see, e.g,*Oldham* and *Spanier* [125, p.53]):

$$\left(\mathbb{W}_{\infty-}^{-\alpha}u\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{\infty}^{x}(t-x)^{\alpha-1}u(t)dt \quad (\Re(\alpha)>0), \tag{1.91}$$

provided both the integrals converge.

The power function formulas involving the Weyl type fractional integral operators (see, e.g. *Mathai* et al. [110, p.107, Eq. 3.123]) are as follows:

$$\left(\mathbb{W}_{\infty-}^{-\alpha}t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)}x^{\alpha+\rho-1} \quad (\Re(\rho)>-1,\ \Re(\alpha)>-1). \tag{1.92}$$

The Erdélyi-Kober type fractional integral operators are defined as follows (see, e.g. *Kober* [82]):

$$\left(\mathbb{E}_{x+}^{\alpha,\eta}u\right)(x)=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}t^{\eta}u(t)dt \quad (\Re(\alpha)>0) \tag{1.93}$$

and

$$\left(\mathbb{K}_{\infty-}^{\alpha,\eta}u\right)(x)=\frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}u(t)dt \quad (\Re(\alpha)>0), \tag{1.94}$$

The function $u(t)$ is so constrained so that both the defining integrals (1.93) and (1.94) converge.

The power function formulas involving the Erdélyi-Kober type fractional integral operators (see, e.g. *Mathai* et al. [110, p.107]) are given by:

$$\left(\mathbb{E}_{x+}^{\alpha,\eta}t^{\rho-1}\right)(x) = \frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\alpha+\eta)}x^{\rho-1} \quad (\Re(\rho+\eta)>0), \tag{1.95}$$

and

$$\left(\mathbb{K}_{\infty-}^{\alpha,\eta}t^{\rho-1}\right)(x) = \frac{\Gamma(1+\eta-\rho)}{\Gamma(1+\alpha+\eta-\rho)}x^{\rho-1} \quad (\Re(\eta)>\Re(\rho)>-1), \tag{1.96}$$

$\alpha, \rho, \eta \in \mathbb{C}$.

The following fractional derivative of order $\alpha, \Re(\alpha) > 0$ is introduced by *Caputo* [14] as

$$ {}_0^C\mathbb{D}_t^{\alpha}(u(t)) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\int\limits_0^t \dfrac{u^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau, & m-1<\alpha\leq m \\ \dfrac{\partial^m}{\partial t^m}u(x,t), & \text{if } \alpha = m \end{cases} \tag{1.97}$$

where $u^{(m)}(t) = \frac{\partial^m}{\partial t^m}u(t), m \in N$ is the $m$-th derivative of the function $u(x,t)$ with respect to $t$.

The power function formula involving the Caputo fractional derivative is as follows (see, e.g. *Ishteva* et al. [67, Theorem 4, p.5]):

$$ {}_0^C\mathbb{D}_t^{\alpha}(t^{\rho}) = \begin{cases} \frac{\Gamma(\rho+1)}{\Gamma\rho-\alpha+1}t^{\rho-\alpha}, & n-1<\alpha<n, \quad \rho>n-1, \quad \rho\in\mathbb{R}, \\ 0, & n-1<\alpha<n, \quad \rho\leq n-1, \quad \rho\in\mathbb{N}. \end{cases} \tag{1.98}$$

A generalization of the Riemann-Liouville fractional derivative operator (1.88) and Caputo fractional derivative operator (1.97) is given by *Hilfer* [64], by introducing a fractional derivative operator of two parameters.

**Definition 1.3.1.** *(Hilfer derivative). Let* $0<\mu<1$ *and type* $0\leq\nu\leq1$, $u\in L^1[a,b]$, $-\infty\leq a<t<b\leq\infty$, $u*K_{(1-\nu)(1-\mu)}\in AC^1[0,b]$. *Then Hilfer derivative is defined as*

$$ {}_0\mathbb{D}_{a+}^{\mu,\nu}(u(t)) = \mathbb{I}_t^{\nu(1-\mu)}\frac{\partial}{\partial t}\left(\mathbb{I}_{a+}^{(1-\nu)(1-\mu)}u(t)\right) \tag{1.99}$$

It is interesting to observe that for $\nu = 0$, Eq. (1.99) reduces to the classical Riemann-Liouville fractional derivative operator (1.88). On the other hand, for $\nu = 1$, it gives the Caputo fractional derivative operator defined by (1.97).
The Laplace transform (see, e.g. *Sneddon* [164, Chapter 1]) for this operator is given by *Hilfer* [63]. Hereafter and without loss of generality, we set $a = 0$ in (1.99).

$$L\{\mathbb{D}_{0+}^{\mu,\nu}u(t);s\} = s^{\mu}L\{u(t)\} - s^{\nu(\mu-1)}\mathbb{I}_{0+}^{(1-\nu)(1-\mu)}u(0^{+}), \quad 0<\mu<1 \tag{1.100}$$

where the initial value term $\mathbb{I}_{0+}^{(1-\nu)(1-\mu)}\ u(0^{+})$ involves the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0^{+}$.

**Definition 1.3.2.** *[46, Eq. 12] (Prabhakar integral). Let* $u \in L^{1}[0,b],\ \ 0 < t < b \leq \infty$. *The Prabhakar integral can be written as*

$$\mathbb{P}_{\rho,\mu,\omega,0^{+}}^{\gamma}u(t) = \int_{0}^{t}(t-y)^{\mu-1}E_{\rho,\mu}^{\gamma}[\omega(t-y)^{\rho}]u(y)dy = \left(u * e_{\rho,\mu,\omega}^{\gamma}\right)(t), \tag{1.101}$$

*where* $\rho,\mu,\omega,\gamma \in \mathbb{C}$, $t \in \mathbb{R}$ *with* $\Re(\rho),\Re(\mu) > 0$ *and the kernel is given by*

$$e_{\rho,\mu,\omega}^{\gamma}(t) = t^{\mu-1}E_{\rho,\mu}^{\gamma}(\omega t^{\rho}),$$

*where* $E_{\rho,\mu}^{\gamma}(\cdot)$ *is the generlized Mittag-Leffler function given in* (1.30).

The fractional Prabhakar derivative was introduced and studied by *Ovidio* and *Polito* [127] as follows.

**Definition 1.3.3.** *[46, Eq. 13] (Prabhakar derivative). Let* $u \in L^{1}[0,b],\ \ 0 < t < b \leq \infty$ *and* $u * e_{\rho,m-\mu,\omega}^{-\gamma}(\cdot) \in W^{m,1}[0,b], m = \lceil \mu \rceil$. *The Prabhakar derivative of the function* $u$ *is given by*

$$\mathbb{D}_{\rho,\mu,\omega,0^{+}}^{\gamma}u(t) = \frac{d^{m}}{dx^{m}}\mathbb{P}_{\rho,m-\mu,\omega,0^{+}}^{-\gamma}u(t) \tag{1.102}$$

*where* $t \in \mathbb{R},\ \ \rho,\mu,\omega,\gamma \in \mathbb{C},\ \ \Re(\rho),\ \ \Re(\mu) > 0.$

A generalization of the Hilfer derivative operator (1.99), was given by *Garra et al.* [46, Def.4.3, Eq. 19] as

**Definition 1.3.4.** *(Hilfer-Prabhakar derivative). Let* $\mu \in (0,1),\ \nu \in [0,1]$, $u \in L^1[0,b],\ 0 < t < b \leq \infty,\ \ u * e^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega}(\cdot) \in AC^1[0,b]$.
*Then Hilfer-Prabhakar derivative is defined by*

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(t) = \left(\mathbb{P}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+}\frac{d}{dt}\left\{\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u\right\}\right)(t) \tag{1.103}$$

*where* $\gamma, \omega \in \mathbb{C}, \rho > 0$

It is interesting to observe that for $\gamma = 0$, Eq. (1.103) reduces to the Hilfer derivative (1.99) and for $\gamma = 0, \nu = 0$, Eq.(1.103) reduces to the classical Riemann-Liouville fractional derivative operator (1.88). On the other hand, for $\gamma = 0, v = 1$, it gives the Caputo fractional derivative operator (1.97), respectively (see, e.g. [46]). The Laplace Transform of Hilfer-Prabhakar derivative (1.103) is given by [46, Eq. 20]

$$\begin{aligned} L\left\{\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(t);s\right\} &= L\left\{\mathbb{P}^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+}\frac{d}{dt}\left(\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(t)\right)\right\}(s) \\ &= s^{\mu}[1-\omega s^{-\rho}]^{\gamma}L[u](s) - s^{-\nu(1-\mu)}[1-\omega s^{-\rho}]^{\gamma\nu}\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(t)\right]_{t=0^+} \end{aligned} \tag{1.104}$$

The Saigo operators $\mathbb{I}^{\alpha,\beta,\eta}_{0,x}$ and $\mathbb{I}^{\alpha,\beta,\eta}_{\infty-}$ are defined as follows (see, e.g. *Mathai* et al. [110, p.104]):

$$\left(\mathbb{I}^{\alpha,\beta,\eta}_{0+}u\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)u(t)dt \tag{1.105}$$

$$(\Re(\alpha) > 0)$$

$$\left(\mathbb{I}^{\alpha,\beta,\eta}_{\infty-}u\right)(x) = \int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)u(t)dt \tag{1.106}$$

$$(\Re(\alpha) > 0).$$

where the function $f(t)$ is so constrained that the defining integrals in (1.105) and (1.106) exist.

The power function formulas involving the Saigo operators are as follows:

Let $\alpha, \beta, \eta, \rho \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then the following formulas hold true:

$$\left(\mathbb{I}_{0+}^{\alpha,\beta,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\eta+\alpha)} x^{\rho-\beta-1} \tag{1.107}$$

$$\Re(\rho) > \max\{0, \Re(\beta-\eta)\}$$

and

$$\left(\mathbb{I}_{\infty-}^{\alpha,\beta,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\beta)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\beta+\eta)} x^{\rho-\beta-1} \tag{1.108}$$

$$(\Re(\rho) < 1 + \min\{\Re(\beta), \Re(\eta)\}).$$

In terms of the Gauss hypergeometric function ${}_2F_1$, the left-sided hypergeometric fractional integral operator $\mathbb{I}_{0+}^{\alpha,\beta,\eta}$ given in the Eq. (1.105), the corresponding left-sided hypergeometric fractional derivative operator (see, for details, [151]) $\mathbb{D}_{0+}^{\alpha,\beta,\eta}$ are defined, for $x > 0$ and $\alpha, \beta, \eta \in \mathbb{C}$, by and

$$\begin{aligned}\left(\mathbb{D}_{0+}^{\alpha,\beta,\eta} f\right)(x) &= \left(\mathbb{I}_{0+}^{-\alpha,-\beta,\alpha+\eta} u\right)(x) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\left(\mathbb{I}_{0+}^{-\alpha+\eta,-\beta-\eta,\alpha+\eta-n} u\right)(x)\right\}\end{aligned} \tag{1.109}$$

$$\big(\Re(\alpha) \geqq 0;\ n = [\Re(\alpha)] + 1\big).$$

The left-sided hypergeometric fractional derivative operator $\mathbb{D}_{0+}^{\alpha,\beta,\eta}$ unifies both the Riemann-Liouville fractional derivative operator ${}_0\mathbb{D}_t^{\alpha}$ and the left-sided Erdélyi-Kober fractional derivative operator ${}^{\mathrm{EK}}\mathbb{D}_{0+}^{\alpha,\eta}$. In fact, we have the following relationships:

$${}_0\mathbb{D}_t^{\alpha} = \mathbb{D}_{0+}^{\alpha,-\alpha,\eta} \qquad \text{and} \qquad {}^{\mathrm{EK}}\mathbb{D}_{0+}^{\alpha,\eta} = \mathbb{D}_{0+}^{\alpha,0,\eta}, \tag{1.110}$$

where (see, for details, [38, Chapter 13])

$$\left({}_0\mathbb{D}_t^\alpha u\right)(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u(t)}{(x-t)^{\alpha-n+1}}\,\mathrm{d}t\right\} \tag{1.111}$$

$$\big(x > 0;\ n = [\Re(\alpha)] + 1;\ \Re(\alpha) \geqq 0\big)$$

and

$$\left({}^{\mathrm{EK}}\mathbb{D}_{0+}^{\alpha,\eta} u\right)(x) := x^\eta \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{t^{\alpha+\eta}\, f(t)}{(x-t)^{\alpha-n+1}}\,\mathrm{d}t\right\} \tag{1.112}$$

$$\big(x > 0;\ n = [\Re(\alpha)] + 1;\ \Re(\alpha) \geqq 0\big).$$

In terms of the Gauss hypergeometric function ${}_2F_1$, the right-sided hypergeometric fractional integral operator $\mathbb{I}_{\infty-}^{\alpha,\beta,\eta}$ given in the Eq. (1.106), the corresponding right-sided hypergeometric fractional derivative operator (see, for details, [151] and [79]) $\mathbb{D}_{\infty-}^{\alpha,\beta,\eta}$ are defined, for $x > 0$ and $\alpha, \beta, \eta \in \mathbb{C}$, by

$$\begin{aligned}\left(\mathbb{D}_{\infty-}^{\alpha,\beta,\eta} u\right)(x) &= \left(\mathbb{I}_{\infty}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\left(\mathbb{I}_{\infty-}^{-\alpha+\eta,-\beta-\eta,\alpha+\eta-n} u\right)(x)\right\}\end{aligned} \tag{1.113}$$

$$\big(\Re(\alpha) \geqq 0;\ n = [\Re(\alpha)] + 1\big).$$

The right-sided hypergeometric fractional derivative operator $\mathbb{D}_{\infty-}^{\alpha,\beta,\eta}$ unifies both the Weyl fractional derivative operator $\mathbb{W}_{\infty-}^\alpha$ and the right-sided Erdélyi-Kober fractional derivative operator ${}^{\mathrm{EK}}\mathbb{D}_{\infty-}^{\alpha,\eta}$. In fact, we have the following relationships:

$$\mathbb{W}_{\infty-}^\alpha = \mathbb{D}_{\infty-}^{\alpha,-\alpha,\eta} \qquad \text{and} \qquad {}^{\mathrm{EK}}\mathbb{D}_{\infty-}^{\alpha,\eta} = \mathbb{D}_{\infty-}^{\alpha,0,\eta}, \tag{1.114}$$

where (see, for details, [38, Chapter 13])

$$\left(\mathbb{W}_{\infty-}^\alpha u\right)(x) := \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u(t)}{(t-x)^{\alpha-n+1}}\,\mathrm{d}t\right\} \tag{1.115}$$

$$\big(x > 0;\ \Re(\alpha) \geqq 0;\ n = [\Re(\alpha)] + 1\big).$$

and

$$\left({}^{\mathrm{EK}}\mathbb{D}_{\infty-}^{\alpha}u\right)(x) := x^{\alpha+\eta}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left\{\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{t^{-\eta}\,u(t)}{(t-x)^{\alpha-n+1}}\,\mathrm{d}t\right\} \tag{1.116}$$

$$\big(x > 0;\ \Re(\alpha) \geqq 0;\ n = [\Re(\alpha)] + 1\big).$$

**Lemma 1.3.5.** (see, for example, [79, pp. 327–328]) *Each of the hypergeometric fractional derivative formulas holds true*:

$$\left(\mathbb{D}_{0+}^{\alpha,\beta,\eta}\,t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\eta)}{\Gamma(\rho+\beta)\Gamma(\rho+\eta)}\,x^{\rho+\beta-1} \tag{1.117}$$

$$\big(x > 0;\ \Re(\alpha) \geqq 0;\ \Re(\rho) > -\min\{0, \Re(\alpha+\beta+\eta)\}\big)$$

*and*

$$\left(\mathbb{D}_{\infty-}^{\alpha,\beta,\eta}\,t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\beta)(1-\rho+\alpha+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\beta)}\,x^{\rho+\beta-1} \tag{1.118}$$

$$\big(x > 0;\ \Re(\alpha) \geqq 0;\ \Re(\rho) < 1 + \min\{\Re(-\beta-\eta), \Re(\alpha+\eta)\}\big).$$

A useful generalization of the hypergeometric fractional integral, including the Saigo operators [150, 151] has been introduced by *Marichev* [106] (see details in Samko et al. [155, p.194, Eq.(10.47)] and whole Section 10.3) and later extended and studied by *Saigo* and *Maeda* [152, p.393, Eq.(4.12) and Eq.(4.13)] in term of any complex order with Appell function $F_3(\cdot)$ in the kernel and Saigo-Maeda [152] introduced the fractional integral operators.

Let $\alpha, \alpha^{'}, \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional integral operators (Marichev-Saigo-Maeda fractional integral operators) involving the Appell's function are defined as follows:

$$\left(\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)$$
$$=\frac{x^{-\alpha}}{\Gamma\gamma}\int_0^x(x-t)^{\gamma-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt,\quad(\Re(\gamma)>0),\tag{1.119}$$

and

$$\left(\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)$$
$$=\frac{x^{-\alpha}}{\Gamma\gamma}\int_x^\infty(t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt,\quad(\Re(\gamma)>0).\tag{1.120}$$

respectively. These operators (integral transforms) were introduced by *Marichev* [106] as Mellin type convolution operators with a special function $F_3(\cdot)$ in the kernel. These operators were rediscovered and studied by *Saigo* in [151] as generalization of the so-called Saigo fractional integral operators (see [81, p.160]).
In (1.119) and (1.120) the symbol $F_3(\cdot)$ denotes so-called 3rd Appell function given in (1.68).
Following *Saigo* et al. [158], the left-hand sided and right-hand sided generalized integration of the type (1.119) and (1.120) for a power function are given by:

Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$, $x>0$ and if $\Re(\gamma)>0,\Re(\rho)>\max\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\}$, then

$$\left(\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x)$$
$$=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}x^{\rho+\gamma-\alpha-\alpha'-1}\tag{1.121}$$

Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$ be such that $\Re(\gamma)>0$ and $\Re(\rho)<1+\min\{\Re(-\beta),\Re(\alpha+$

$\alpha^{'} - \gamma), \Re(\alpha + \beta^{'} - \gamma)\}$ then there exist the relation

$$\begin{aligned}&\left(\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x)\\ &= \frac{\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho+\alpha+\beta'-\gamma)\Gamma(1-\rho-\beta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\alpha'+\beta+\beta'-\gamma)\Gamma(1-\rho+\alpha-\beta)} x^{\rho-\alpha-\alpha'+\gamma-1}\end{aligned} \tag{1.122}$$

If we set $\alpha^{'} = 0$ then in view of the reduction formula (1.69), *Saxena* and *Saigo* [158, p. 93, Eqs. (2.15)] found the following relationship between the Marichev-Saigo-Maeda and the Saigo fractional integral operators:

$$\left(\mathbb{I}_{x+}^{\alpha,0,\beta,\beta',\eta} u\right)(x) = \left(\mathbb{I}_{x+}^{\eta,\alpha-\eta,-\beta} u\right)(x) \quad (\Re(\eta) > 0) \tag{1.123}$$

and

$$\left(\mathbb{I}_{\infty-}^{\alpha,0,\beta,\beta',\eta} u\right)(x) = \left(\mathbb{I}_{x,\infty}^{\eta,\alpha-\eta,-\beta} u\right)(x) \quad (\Re(\eta) > 0) \tag{1.124}$$

where the general operators $\mathbb{I}_{x+}^{\alpha,0,\beta,\beta',\eta}$ and $\mathbb{I}_{x+}^{\alpha,0,\beta,\beta',\eta}$ reduce respectively to the aforementioned Saigo operators $\mathbb{I}_{0,x}^{\alpha,\beta,\eta}$ and $\mathbb{I}_{\infty-}^{\alpha,\beta,\eta}$ (see for details, [151] and the references cited therein).

A symmetric fractional Laplace operator of order $\lambda$ is defined by *Brockmann and Sokolov* [12, Eq. A.7-A.9] as

$$\Delta^{\frac{\lambda}{2}} \equiv \frac{1}{2\cos\left(\frac{\pi\lambda}{2}\right)} \{ {}_{-\infty}D_x^{\lambda} + {}_xD_{\infty}^{\lambda} \}, \qquad 0 < \lambda \leq 2 \tag{1.125}$$

where

$${}_{-\infty}D_x^{\lambda}(u(x)) = \frac{1}{k-\lambda} \int\limits_{-\infty}^{x} \frac{u^{(k)}(t)}{(x-t)^{\lambda+1-k}}\, dt, \qquad k = [\lambda] + 1$$

and

$${}_xD_{\infty}^{\lambda}(u(x)) = \frac{1}{k-\lambda} \int\limits_{x}^{\infty} \frac{u^{(k)}(t)}{(x-t)^{\lambda+1-k}}\, dt, \qquad k = [\lambda] + 1$$

The Fourier transform of fractional Laplace operator (1.125) [12, Eq. A.19] is given by

$$F\{\Delta^{\frac{\lambda}{2}}(u(x,t));\eta\} = -|\eta|^{\lambda} F\{u(x,t);\eta\}, \quad 0 < \lambda \leq 2 \tag{1.126}$$

## 1.4 Brief Chapter wise Summary of the Thesis

We present a brief summary to the work carried out in chapter 2 to 6.

In **Chapter-2**, we establish several formulas involving integral transforms, fractional derivatives of a certain family of extended generalized hypergeometric functions and a family of the incomplete hypergeometric functions also we find probability distributions as a application of generalized hypergeometric function.

We divide this chapter into two parts. In part 'A' first we prove three theorems, which exhibit the connection between the Jacobi, Gegenbauer and Legendre transforms of the extended generalized hypergeometric type function. Next we develop $\mathcal{P}_\delta$-Transforms, Laplace transform, $\mathfrak{K}_\nu$-Transforms and the Hankel $\mathfrak{H}_\nu$-Transforms involving the extended generalized hypergeometric functions. Further, we establish several fractional derivative formulas for the extended generalized hypergeometric type. As on application, a probability density function involving the extended generalized hypergeometric functions is introduced.

In part 'B' we prove three theorems, which exhibit the connection between the Jacobi, Gegenbauer and Legendre transforms of the incomplete hypergeometric functions. Next we develop $\mathcal{P}_\delta$-Transforms, Laplace transform involving the incomplete hypergeometric functions. Further, we establish several fractional derivative formulas for the incomplete hypergeometric functions.

In **Chapter-3**, we establish image formulas for the generalized Lommel-Wright function of first kind involving Saigo-Meada fractional integral operators, in term of the generalized Wright function. Further, we obtain certain theorems involving the results obtained in previous section associated with the integral transforms like beta transforms, pathway transforms, Laplace transforms and Whittaker transforms. Many interesting results are shown to follow from our main results.

In **Chapter-4**, we divide this chapter into two parts. In part 'A' we investigate the analytic solution of the time-space fractional advection-dispersion equation involving fractional Laplace operator, following with some illustrations and application. In part 'B' we investigate the analytic solution of the solutions of generalized space-time fractional reaction-diffusion equation involving fractional Laplace operator, following with some illustrations and concrete applications.

In **Chapter-5**, we find the $P_\alpha$-transform of Caputo fractional derivatives and derive $P_\alpha$-transform for Volterra and Abel integral equation. Further, in Section 3 we find the solution of fractional Volterra integral equation. We discuss its application for solving singular integral equation having Bessel function in its kernel. The solution of non homogeneous time fractional heat equation in a spherical domain has been discussed.

In **Chapter-6**, we obtain a theorem using some operators defined on a Lie algebra of endomorphisms of a vector space which generalizes some results of researchers on the families of special functions and orthogonal polynomials. In particular, we present examples, how the Lie algebraic approach can be used to derive the differential recurrence relations, differential equation for extended Jacobi polynomials and Gegenbauer polynomials.

The method developed in this chapter can also be used to study some other special functions of mathematical physics. Certain properties of some special matrix functions via Lie Algebra are studied in the section. We have established a general theorem concerning eigenvector for the product of two operators defined on a Lie algebra of endomorphisms of a vector space.

A comprehensive list of references has been provided at the end of the thesis.

# 2

# Integral Transform and Fractional Derivative Formulas with Applications

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The main findings of this chapter have been published as detailed below:

1. **H. M. Srivastava, R. Agarwal, S. Jain**. Integral Transform and Fractional Derivative Formulas Involving the Extended Generalized Hypergeometric Functions and Probability Distributions, *Mathematical Methods in the Applied Sciences*, **40**, 255-273.

2. **R. Srivastava, R. Agarwal and S. Jain** (2015). A Family of the Incomplete Hypergeometric Functions and Associated Integral Transform and Fractional Derivative Formulas, *Filomat*, **31(1)**, 125-140.

In several areas of mathematical, physical and engineering sciences, integral transforms and fractional calculus operators play an important rôle from the application viewpoint (see, for details, [80]). A remarkably large number of integral transforms as well as fractional integral and derivative formulas involving various special functions have been investigated by many authors. For example, *Choi and Agarwal* [19] derived some integral transforms and fractional integral formulas involving the generalized hypergeometric function $F_{\mathfrak{p}}^{(\rho,\sigma;m)}(a,b;c;z)$ defined by (1.15). In the present sequel to some of the aforementioned works, we propose to establish several (presumably new) integral transform and fractional derivative formulas involving the extended generalized hypergeometric type functions:

$$
{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} z;\mathfrak{p}\right]
$$

the generalized hypergeometric is given by (1.6). The particular case of this function is given in the Eq. (1.21). We also investigate some families of probability distributions and probability density functions associated with these extended generalized hypergeometric type functions.

## 2.1 Introduction

The theory of various families of *Special Functions* has been one of the most rapidly growing research subjects in *Mathematical Analysis* due mainly to their applications in many different areas of mathematical, physical, statistical and engineering sciences. Recently, a function is drawing attention of many researchers chiefly because of its diverse applications. Popularly known as the extended generalized hypergeometric function, it is more general than the classical and generalized Gauss hypergeometric functions or the confluent hypergeometric functions. Therefore, the corresponding extensions of several other familiar special functions are expected to

be useful and need to be investigated (see, e.g. [16], [17], [131]). For example, *Srivastava et al.* [167] introduced and studied the generalized incomplete Pochhammer symbols and their applications to hypergeometric and related functions.
Definition 2.1.1 below makes use of the classical orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ defined by (see, for details, [38, Chapter 10])

$$P_n^{(\alpha,\beta)}(z) = (-1)^n\, P_n^{(\beta,\alpha)}(-z) = \binom{\alpha+n}{n}\,{}_2F_1\left[\begin{array}{r} -n, \alpha+\beta+n+1; \\ \alpha+1; \end{array}\ \frac{1-z}{2}\right] \tag{2.1}$$

in terms of the familiar Gauss hypergeometric function ${}_2F_1$.

**Definition 2.1.1.** (see, e.g. [29, p. 501]) The Jacobi transform of a function $f(z)$ is defined as follows:

$$\mathbb{J}^{(\alpha,\beta)}[f(z);n] := \int_{-1}^{1} (1-z)^{\alpha}(1+z)^{\beta} P_n^{(\alpha,\beta)}(z)\ f(z)\mathrm{d}z \tag{2.2}$$

$$\big(\min\{\Re(\alpha),\Re(\beta)\} > -1;\ n \in \mathbb{N}_0\big),$$

provided that the function $f(z)$ is so constrained that the integral in (2.2) exists.

Now, if we apply the definition (2.1), it is easily seen that

$$\begin{aligned}&\int_{-1}^{1} (1-z)^{\xi-1}(1+z)^{\eta-1} P_n^{(\alpha,\beta)}(z)\ \mathrm{d}z \\ &\quad = 2^{\xi+\eta-1}\binom{\alpha+n}{n} B(\xi,\eta)\,{}_3F_2\left[\begin{array}{r} -n, \alpha+\beta+n+1, \xi; \\ \alpha+1, \xi+\eta; \end{array}\ 1\right]\end{aligned} \tag{2.3}$$

$$\big(\min\{\Re(\xi),\Re(\eta)\} > 0\big).$$

Since

$$z^{\rho-1} = [1-(1-z)]^{\rho-1} = \sum_{k=0}^{\infty} \frac{(1-\rho)_k}{k!}\ (1-z)^k \qquad (|1-z| < 1;\ \rho \in \mathbb{C}),$$

it follows easily from the integral formula (2.3) that

$$\begin{aligned}
&\int_{-1}^{1}(1-z)^{\xi-1}(1+z)^{\eta-1}P_n^{(\alpha,\beta)}(z)\; z^{\rho-1}\,\mathrm{d}z \\
&\quad = 2^{\xi+\eta-1}\binom{\alpha+n}{n}B(\xi,\eta) \\
&\qquad \cdot F^{1:2;1}_{1:1:0}\left[\begin{array}{rr} \xi: & -n,\alpha+\beta+n+1;1-\rho; \\ \xi+\eta: & \alpha+1;\text{——}\;; \end{array}\; 1,2\right]
\end{aligned} \tag{2.4}$$

$$\big(\min\{\Re(\xi),\Re(\eta)\}>0;\;\rho\in\mathbb{C};\;n\in\mathbb{N}_0\big)$$

in terms of the familiar Kampé de Fériet function (see, for details, [169, p. 27 *et seq.*]). In particular, upon setting $\xi=\alpha+1$ and $\eta=\beta+1$, this last integral formula (2.4) would reduce immediately to the following form:

$$\begin{aligned}
&\mathbb{J}^{(\alpha,\beta)}[z^{\rho-1};n] \\
&\quad := \int_{-1}^{1}(1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)\; z^{\rho-1}\,\mathrm{d}z \\
&\quad = 2^{\alpha+\beta+1}\binom{\alpha+n}{n}B(\alpha+1,\beta+1) \\
&\qquad \cdot F^{1:2;1}_{1:1:0}\left[\begin{array}{rr} \alpha+1: & -n,\alpha+\beta+n+1;1-\rho; \\ \alpha+\beta+2: & \alpha+1;\text{——}\;; \end{array}\; 1,2\right]
\end{aligned} \tag{2.5}$$

$$\big(\min\{\Re(\alpha),\Re(\beta)\}>-1;\;\rho\in\mathbb{C};\;n\in\mathbb{N}_0\big).$$

In its *further* special case when $\rho=m+1\quad(m\in\mathbb{N}_0)$, (2.5) yields the following well-known result for the Jacobi transform of $z^m\;\;(m\in\mathbb{N}_0)$, which is given by (see, e.g. [146, p. 261, Eqs. (14) and (15)])

$$\begin{aligned}
&\mathbb{J}^{(\alpha,\beta)}[z^m;n] \\
&\quad := \int_{-1}^{1} (1-z)^{\alpha}(1+z)^{\beta} P_n^{(\alpha,\beta)}(z)\ z^m\ \mathrm{d}z \\
&\quad = \begin{cases} 0 & (m = 0,1,2,\cdots,n-1) \\ 2^{\alpha+\beta+n+1}\ B(\alpha+n+1,\beta+n+1) & (m=n) \\ 2^{\alpha+\beta+n+1}\binom{m}{n}\ B(\alpha+n+1,\beta+n+1) & \\ \qquad \cdot\ {}_2F_1\left[\begin{matrix} n-m,\alpha+n+1; \\ \alpha+\beta+2n+2; \end{matrix}\ 2\right] & (m = n+1, n+2, n+3, \cdots) \end{cases}
\end{aligned} \tag{2.6}$$

$$\big(\min\{\Re(\alpha),\Re(\beta)\} > -1;\ m,n \in \mathbb{N}_0\big).$$

**Remark 2.1.2.** *The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ contain, as their special cases, such other classical orthogonal polynomials as (for example) the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(z)$, the Legendre (or spherical) polynomials $P_n(z)$, and the Tchebycheff polynomials $T_n(z)$ and $U_n(z)$ of the first and second kind. (see, for details, [170]). In fact, we have the following relationships with the Gegenbauer polynomials $C_n^{\nu}(z)$ and the Legendre polynomials $P_n(z)$:*

$$C_n^{\nu}(z) = \binom{\nu+n-\frac{1}{2}}{n}^{-1}\binom{2\nu+n-1}{n} P_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z) \tag{2.7}$$

*and*

$$P_n(z) = C_n^{\frac{1}{2}}(z) = P_n^{(0,0)}(z), \tag{2.8}$$

*respectively. Thus, by applying the relationships in* (2.7) *and* (2.8) *and ignoring altogether the constant binomial coefficients occurring in* (2.7)*, the parameters $\alpha$*

*and $\beta$ in Definition 2.1.1 above can be suitably specialized to define the corresponding Gegenbauer transform $\mathbb{G}^{(\nu)}[f(z);n]$ and the corresponding Legendre transform $\mathbb{L}[f(z);n]$ as follows:*

$$\begin{aligned}
&\mathbb{G}^{(\nu)}[f(z);n] \\
&\quad = \binom{\nu+n-\frac{1}{2}}{n}^{-1}\binom{2\nu+n-1}{n}\,\mathbb{J}^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}[f(z);n] \\
&\quad := \int_{-1}^{1}\left(1-z^2\right)^{\nu-\frac{1}{2}}\;C_n^{\nu}(z)\;f(z)\mathrm{d}z \qquad \left(\Re(\nu)>-\frac{1}{2};\;n\in\mathbb{N}_0\right)
\end{aligned} \tag{2.9}$$

*and*

$$\mathbb{L}[f(z);n]=\mathbb{G}^{(\frac{1}{2})}[f(z);n]:=\int_{-1}^{1}P_n(z)\;f(z)\mathrm{d}z \qquad (n\in\mathbb{N}_0)\,. \tag{2.10}$$

*Other familiar integral transforms, which will be needed in our present investigation, include the $\mathfrak{K}_\nu$-transform and the Hankel $\mathfrak{H}_\nu$-transform given by Definitions 2.1.3 and 2.1.4 below.*

**Definition 2.1.3.** (see, e.g. [38, Chapter 10] and [110, p. 53, Eq. (2.33)]) The $\mathfrak{K}$-transform of $f(z)$ is defined by

$$\mathfrak{K}_\nu[f(t);s]:=\int_0^\infty (st)^{\frac{1}{2}}\;K_\nu(st)f(t)\mathrm{d}t \qquad (s\in\mathbb{C}), \tag{2.11}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind (or the Macdonald function) given in Eq. (1.51).

The $\mathfrak{K}_\nu$-transform of the power function $t^{\rho-1}$ is given by (see, e.g. [38, p. 127, Eq. (1)] and [110, p. 54, Eq. (2.37)])

$$\begin{aligned}
\mathfrak{K}_\nu\left[t^{\rho-1};s\right] &= s^{\frac{1}{2}}\int_0^\infty t^{\left(\rho+\frac{1}{2}\right)-1}\;K_\nu(st)\mathrm{d}t \\
&= \frac{2^{\rho-\frac{3}{2}}}{s^\rho}\,\Gamma\!\left(\frac{\rho+\nu+\frac{1}{2}}{2}\right)\Gamma\!\left(\frac{\rho-\nu+\frac{1}{2}}{2}\right)
\end{aligned} \tag{2.12}$$

$$\left(\Re(s) > 0;\ \Re(\rho) > |\Re(\nu)| - \frac{1}{2}\right).$$

**Definition 2.1.4.** (see, e.g. [38, Chapter 8] and [110, p. 56, Eq. (2.43)]) The Hankel $\mathfrak{H}_\nu$-transform of $f(t)$ is defined by

$$\mathfrak{H}_\nu[f(t); s] = \int_0^\infty (st)^{\frac{1}{2}}\ J_\nu(st) f(t) \mathrm{d}t \qquad \big(\Re(s) > 0\big), \tag{2.13}$$

where $J_\nu(z)$ is the Bessel function of the first kind defined in the Eq. (1.49).

The Hankel $\mathfrak{H}_\nu$-transform of the power function $t^{\rho-1}$ is given by (see, e.g. [38, p. 22, Eq. (7)] and [110, p. 57, Eq. (2.46)])

$$\mathfrak{H}_\nu\left[t^{\rho-1}; s\right] = s^{\frac{1}{2}} \int_0^\infty t^{\left(\rho+\frac{1}{2}\right)-1}\ J_\nu(st)\mathrm{d}t = \frac{2^{\rho-\frac{1}{2}}}{s^\rho}\ \frac{\Gamma\left(\dfrac{\nu+\rho+\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{\nu-\rho+\frac{3}{2}}{2}\right)} \tag{2.14}$$

$$\left(\Re(s) > 0;\ -\Re(\nu) - \frac{1}{2} < \Re(\rho) < 1\right).$$

The constraint $\Re(\rho) < 1$ in (3.34) invalidates the term-by-term integration of hypergeometric power series as such, so *instead* we recall here the following known Hankel $\mathfrak{H}_\nu$-transform for $t^{\rho-1}\ e^{-\sigma t}$ (see, e.g. [38, p. 29, Eq. (7)]):

$$\begin{aligned}
\mathfrak{H}_\nu\left[t^{\rho-1}\ e^{-\sigma t}; s\right] &= s^{\frac{1}{2}} \int_0^\infty t^{\left(\rho+\frac{1}{2}\right)-1}\ e^{-\sigma t}\ J_\nu(st)\mathrm{d}t \\
&= \frac{s^{\nu+\frac{1}{2}}}{2^\nu\ \sigma^{\rho+\nu+\frac{1}{2}}}\ \frac{\Gamma\left(\rho+\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} \\
&\quad \cdot\ {}_2F_1\left[\begin{array}{c} \frac{1}{2}\left(\rho+\nu+\frac{1}{2}\right), \frac{1}{2}\left(\rho+\nu+\frac{3}{2}\right); \\ \nu+1; \end{array} -\frac{s^2}{\sigma^2}\right]
\end{aligned} \tag{2.15}$$

$$\left(\Re(s) > 0;\ \Re(\rho+\nu) > -\frac{1}{2};\ \Re(\sigma) > |\Im(s)|\right).$$

**Remark 2.1.5.** *The concept of the Hadamard product (or the Hadamard composition) is also useful in our investigation. It can help us in decomposing a newly-emerged function into two known functions.*

**Definition 2.1.6.** (see, e.g. [138, Chapter 3]) Let

$$f(z) := \sum_{n=0}^{\infty} \mathfrak{a}_n z^n \quad (|z| < R_f) \qquad \text{and} \qquad g(z) := \sum_{n=0}^{\infty} \mathfrak{b}_n z^n \quad (|z| < R_g)$$

be two given power series whose radii of convergence are denoted by $R_f$ and $R_g$, respectively. Then their Hadamard product $(f*g)(z)$ is given by the following power series:

$$(f * g)(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n \mathfrak{b}_n z^n = (g * f)(z) \qquad (|z| < R),$$

where

$$R := \lim_{n\to\infty} \left| \frac{\mathfrak{a}_n \mathfrak{b}_n}{\mathfrak{a}_{n+1} \mathfrak{b}_{n+1}} \right| = \left( \lim_{n\to\infty} \left| \frac{\mathfrak{a}_n}{\mathfrak{a}_{n+1}} \right| \right) \cdot \left( \lim_{n\to\infty} \left| \frac{\mathfrak{b}_n}{\mathfrak{b}_{n+1}} \right| \right) =: R_f \cdot R_g,$$

so that, in general, we have (see, e.g. [138, p. 35])

$$R \geqq R_f \cdot R_g.$$

If, in particular, one of the power series in Definition 2.1.6 defines an entire function, then the Hadamard product defines an entire function, too.

The following decomposition formula provides an illustrative example of the usage of Definition 2.1.6:

$$\begin{aligned} &{}_pF_{r+p}^{(\alpha,\beta;\kappa,\mu)} \left[ \begin{array}{r} a_1, \cdots, a_p; \\ b_1, \cdots, b_{r+p}; \end{array} z; \mathfrak{p} \right] \\ &= {}_1F_r \left[ \begin{array}{r} 1; \\ b_1, \cdots, b_r; \end{array} z \right] * {}_pF_p{}^{(\alpha,\beta;\kappa,\mu)} \left[ \begin{array}{r} a_1, \cdots, a_p; \\ b_{r+1}, \cdots, b_{r+p}; \end{array} z; \mathfrak{p} \right] \quad (|z| < \infty). \quad (2.16) \end{aligned}$$

**Part A**

## 2.2 Jacobi and Related Transforms of the Extended Generalized Hypergeometric Type Functions

In this section, we prove three theorems, which exhibit the connection between the Jacobi, Gegenbauer and Legendre transforms with the following extended generalized hypergeometric type function:

$$
{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} z;\mathfrak{p}\right],
$$

which is given by (1.18).

**Theorem 2.2.1.** *Under the conditions stated already with* (1.18), *the following Jacobi transform formula holds true*:

$$
\begin{aligned}
&\mathbb{J}^{(\alpha',\beta')}\left[z^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} xz;\mathfrak{p}\right];n\right] \\
&\quad= 2^{\alpha'+\beta'+1}\binom{\alpha'+n}{n}B(\alpha'+1,\beta'+1)\sum_{k=0}^{\infty}\Theta_k^{(p,q)} \\
&\qquad\cdot F_{1:1:0}^{1:2;1}\left[\begin{array}{rr} \alpha'+1: & -n,\alpha'+\beta'+n+1;1-\rho-k; \\ \alpha'+\beta'+2: & \alpha'+1;\underline{\hspace{1.5em}}; \end{array} 1,2\right]\frac{x^k}{k!}
\end{aligned}
\tag{2.17}
$$

$$
\big(|x|<1;\ n\in\mathbb{N}_0;\ \min\{\Re(\alpha'),\Re(\beta')\}>-1;\ \rho\in\mathbb{C}\big),
$$

*where the coefficients* $\Theta_k^{(p,q)}$ *are given by* (1.19) *and it is assumed that the Jacobi transform in* (2.17) *exists.*

proof: By applying the definition (2.2) in conjunction with (1.18), we have

$$\begin{aligned}
&\mathbb{J}^{(\alpha',\beta')}\left[z^{\rho-1}\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} xz;\mathfrak{p}\right];n\right] \\
&= \int_{-1}^{1} z^{\rho-1}(1-z)^{\alpha'}(1+z)^{\beta'}P_n^{(\alpha',\beta')}(z)\ \ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} xz;\mathfrak{p}\right]\mathrm{d}z \\
&= \int_{-1}^{1} z^{\rho-1}(1-z)^{\alpha'}(1+z)^{\beta'}\ P_n^{(\alpha',\beta')}(z)\ \ \left(\sum_{k=0}^{\infty}\Theta_k^{(p,q)}\ \frac{(xz)^k}{k!}\right)\mathrm{d}z, \qquad (2.18)
\end{aligned}$$

where the coefficients $\Theta_k^{(p,q)}$ are given by (1.19). Upon changing the order of integration and summation, if we apply the Jacobi transform formula (2.5) *with* the parameter $\rho$ replaced by $\rho+k$ $(\rho\in\mathbb{C};\ k\in\mathbb{N}_0)$, we find from (2.18) that

$$\begin{aligned}
&\mathbb{J}^{(\alpha',\beta')}\left[z^{\rho-1}\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} xz;\mathfrak{p}\right];n\right] \\
&= 2^{\alpha'+\beta'+1}\binom{\alpha'+n}{n}B(\alpha'+1,\beta'+1)\sum_{k=0}^{\infty}\Theta_k^{(p,q)} \\
&\quad\cdot F_{1:1:0}^{1:2;1}\left[\begin{array}{rr} \alpha'+1: & -n,\alpha'+\beta'+n+1;1-\rho-k; \\ \alpha'+\beta'+2: & \alpha'+1;\underline{\qquad};\end{array} 1,2\right]\frac{x^k}{k!} \qquad (2.19)
\end{aligned}$$

under the hypotheses of Theorem 2.2.1. Equation (2.19) is precisely the result (2.17) asserted by Theorem 2.2.1. In view of the definition (2.9), Theorem 2.2.1 yields Corollary 2.2.2 below by setting $\alpha'=\beta'=\nu-\frac{1}{2}$.

**Corollary 2.2.2.** *Under the conditions stated already with* (1.18), *the following*

*Gegenbauer transform formula holds true*:

$$\mathbb{G}^{(\nu)}\left[z^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} xz;\mathfrak{p}\right];n\right]$$
$$= 2^{2\nu}\binom{2\nu+n-1}{n} B\left(\nu+\frac{1}{2},\nu+\frac{1}{2}\right)\sum_{k=0}^{\infty}\Theta_k^{(p,q)}$$
$$\cdot F_{1:1:0}^{1:2;1}\left[\begin{array}{lr} \nu+\frac{1}{2}: & -n,2\nu+n;1-\rho-k; \\ 2\nu+1: & \nu+\frac{1}{2};\underline{\qquad};\end{array} 1,2\right]\frac{x^k}{k!} \tag{2.20}$$

$$\left(|x|<1;\ n\in\mathbb{N}_0;\ \Re(\nu)>-\frac{1}{2};\ \rho\in\mathbb{C}\right),$$

*where the coefficients* $\Theta_k^{(p,q)}$ *are given by* (1.19) *and it is assumed that the Gegenbauer transform in* (2.20) *exists.*

For the Legendre transform defined by (2.10), a special case of Theorem 2.2.1 when $\alpha'=\beta'=0$ (or, alternatively, Corollary 2.2.2 with $\nu=\frac{1}{2}$) yields the following result.

**Corollary 2.2.3.** *Under the conditions stated already with* (1.18), *the following Legendre transform formula holds true*:

$$\mathbb{L}\left[z^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} xz;\mathfrak{p}\right];n\right]$$
$$= 2\sum_{k=0}^{\infty}\Theta_k^{(p,q)}\,F_{1:1:0}^{1:2;1}\left[\begin{array}{lr} 1: & -n,n+1;1-\rho-k; \\ 2: & 1;\underline{\qquad};\end{array} 1,2\right]\frac{x^k}{k!} \tag{2.21}$$

$$\left(|x|<1;\ n\in\mathbb{N}_0;\ \rho\in\mathbb{C}\right),$$

*where the coefficients* $\Theta_k^{(p,q)}$ *are given by* (1.19) *and it is assumed that the Legendre transform in* (2.21) *exists.*

## 2.3 $\mathcal{P}_\delta$-Transforms of the Extended Generalized Hypergeometric Functions

**Theorem 2.3.1.** *Under the conditions stated already with* (1.18), *the following $\mathcal{P}_\delta$-transform formula holds true*:

$$\mathcal{P}_\delta\left[t^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c}a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array} zt;\mathfrak{p}\right];s\right]$$

$$=\frac{\Gamma(\rho)}{[\Lambda(\delta;s)]^\rho}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c}a_1,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array}\frac{z}{\Lambda(\delta;s)};\mathfrak{p}\right]*\,{}_1F_0\left[\begin{array}{c}\rho;\\ -\,;\end{array}\frac{z}{\Lambda(\delta;s)}\right] \tag{2.22}$$

$$\big(|z|<1;\ \min\{\Re(s),\Re(\rho)\}>0;\ \delta>1\big),$$

*where*

$$\Lambda(\delta;s):=\frac{\ln[1+(\delta-1)s]}{\delta-1}\qquad\big(\min\{\Re(s),\Re(\rho)\}>0;\ \delta>1\big) \tag{2.23}$$

*and it is assumed that the $\mathcal{P}_\delta$-transform in* (2.22) *exists.*

Proof: Applying the definitions (1.81) and (1.18), we have

$$\mathcal{P}_\delta\left[t^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c}a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array} zt;\mathfrak{p}\right];s\right]$$

$$=\int_0^\infty t^{\rho-1}[1+(\delta-1)s]^{-\frac{t}{\delta-1}}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c}a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array} zt;\mathfrak{p}\right]\mathrm{d}t$$

$$=\int_0^\infty t^{\rho-1}[1+(\delta-1)s]^{-\frac{t}{\delta-1}}\sum_{n=0}^\infty\Theta_n^{(p,q)}\,\frac{(zt)^n}{n!}\,\mathrm{d}t, \tag{2.24}$$

where the coefficients $\Theta_n^{(p,q)}$ are given by (1.19).

By changing the order of integration and summation in Eq. (2.24) and using Eq.

(1.82), we obtain

$$\begin{aligned}
&\mathcal{P}_\delta\left[t^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt;\mathfrak{p}\right];s\right] \\
&= \sum_{n=0}^{\infty}\Theta_n^{(p,q)}\,\frac{z^n}{n!}\int_0^\infty t^{\rho+n-1}[1+(\delta-1)s]^{-\frac{t}{\delta-1}}\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\Theta_n^{(p,q)}\,\frac{z^n}{n!}\,\frac{\Gamma(\rho+n)}{[\Lambda(\delta;s)]^{\rho+n}},
\end{aligned} \tag{2.25}$$

where $\Lambda(\delta;s)$ is given by (2.23). Finally, by means of the Hadamard product, the assertion (2.22) of Theorem 2.3.1 follows from (2.25) if we make use of the equations (1.18) and (2.16) once again.

A limit case of Theorem 2.3.1 when $\delta\to1+$ yields the following corollary.

**Corollary 2.3.2.** *Under the conditions stated already with* (1.18), *the following Laplace transform formula holds true*:

$$\begin{aligned}
&\mathbb{L}\left[t^{\rho-1}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt;\mathfrak{p}\right];s\right] \\
&= \frac{\Gamma(\rho)}{s^\rho}\,{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} \frac{z}{s};\mathfrak{p}\right] * {}_1F_0\left[\begin{array}{c} \rho; \\ -; \end{array} \frac{z}{s}\right]
\end{aligned} \tag{2.26}$$

$$\big(|z|<1;\ \min\{\Re(s),\Re(\rho)\}>0\big),$$

*where it is assumed that the Laplace transform in* (2.26) *exists.*

**Remark 2.3.3.** *By appealing to the relationship* (1.85)*, it is fairly straightforward to deduce the assertion* (2.22) *of Theorem 2.3.1 from the Laplace transform formula* (2.26) *if we set* $s\mapsto\Lambda(\delta;s)$ *for* $\Lambda(\delta;s)$ *given by* (2.23).

## 2.4 $\mathfrak{K}_\nu$-Transforms and the Hankel $\mathfrak{H}_\nu$-Transforms Involving the Extended Generalized Hypergeometric Functions

**Theorem 2.4.1.** *Under the conditions stated already with* (1.18), *the following* $\mathfrak{K}_\nu$-*transform formula holds true*:

$$\mathfrak{K}_\nu\left[t^{\rho-1}\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt^2;\mathfrak{p}\right];s\right]=\frac{2^{\rho-\frac{3}{2}}}{s^\rho}\,\Gamma\left(\frac{\rho+\nu+\frac{1}{2}}{2}\right)\Gamma\left(\frac{\rho-\nu+\frac{1}{2}}{2}\right)$$

$$\cdot\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} \frac{4z}{s^2};\mathfrak{p}\right] * {}_2F_0\left[\begin{array}{c} \frac{1}{2}\left(\rho+\nu+\frac{1}{2}\right),\frac{1}{2}\left(\rho-\nu+\frac{1}{2}\right); \\ \overline{\qquad\qquad\qquad\qquad}\ ; \end{array} \frac{4z}{s^2}\right] \tag{2.27}$$

$$\left(|z|<1;\ \Re(s)>0;\ \Re(\rho)>|\Re(\nu)|-\frac{1}{2}\right),$$

*where it is assumed that the* $\mathfrak{K}_\nu$-*transform in* (2.27) *exists.*

Proof: By using the definitions in (1.82) and (1.18), we get

$$\begin{aligned}
&\mathfrak{K}_\nu\left[t^{\rho-1}\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt^2;\mathfrak{p}\right];s\right] \\
&= s^{\frac{1}{2}}\int_0^\infty t^{\rho-\frac{1}{2}}\ K_\nu(st)\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt^2;\mathfrak{p}\right]\mathrm{d}t \\
&= s^{\frac{1}{2}}\int_0^\infty t^{\rho-\frac{1}{2}}\sum_{n=0}^\infty \Theta_n^{(p,q)}\ \frac{(zt^2)^n}{n!}\ K_\nu(st)\mathrm{d}t,
\end{aligned} \tag{2.28}$$

where the coefficients $\Theta_n^{(p,q)}$ are given by (1.19). If, upon changing the order of integration and summation in Eq. (5.43), we first apply the integral formula (2.12) and then interpret the resulting expression as a Hadamard product, we are led easily to the assertion (2.27) of Theorem 2.4.1.

Our demonstration of Theorem 2.4.2 below is much akin to that of Theorem 2.4.1. It makes use of the definitions in (2.13) and (1.18) as well as the Hankel $\mathfrak{H}_\nu$-transform formula (2.15). The details involved are being omitted.

**Theorem 2.4.2.** *Under the conditions stated already with* (1.18), *the following Hankel $\mathfrak{H}_\nu$-transform formula holds true*:

$$\mathfrak{H}_\nu\left[t^{\rho-1}\,e^{-\sigma t}\;{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\;zt^2;\mathfrak{p}\right];s\right]=\frac{s^{\nu+\frac{1}{2}}}{2^\nu\;\sigma^{\rho+\nu+\frac{1}{2}}}\;\frac{\Gamma\left(\rho+\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)}$$

$$\cdot\sum_{k=0}^{\infty}\frac{\left(\dfrac{\rho+\nu+\frac{1}{2}}{2}\right)_k\left(\dfrac{\rho+\nu+\frac{3}{2}}{2}\right)_k}{k!\;(\nu+1)_k}\left(-\frac{s^2}{\sigma^2}\right)^k$$

$$\cdot\;{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a_1,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{matrix}\;\frac{4z}{\sigma^2};\mathfrak{p}\right]*\;{}_2F_0\left[\begin{matrix}\frac{1}{2}\left(\rho+\nu+\frac{1}{2}\right)+k,\frac{1}{2}\left(\rho+\nu+\frac{3}{2}\right)+k;\\ \overline{\qquad\qquad\qquad\qquad}\;;\end{matrix}\;\frac{4z}{\sigma^2}\right] \tag{2.29}$$

$$\left(\Re(s)>0;\;\Re(\rho+\nu)>-\frac{1}{2};\;\Re(\sigma)>|\Im(s)|\right),$$

*where it is assumed that the Hankel $\mathfrak{H}_\nu$-transform in* (2.29) *exists.*

**Remark 2.4.3.** *Since (see, e.g. [38, p. 79, Eqs. (14) and (15)])*

$$J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\;\cos z\qquad \textit{and}\qquad J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\;\sin z,$$

*it is fairly straighforward to deduce from the Hankel $\mathfrak{H}_\nu$-transform formula* (2.29) *the corresponding Cosine and Sine transform formulas by trivially setting $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$, respectively.*

## 2.5 Fractional Derivative Formulas for the Extended Generalized Hypergeometric Functions

Here, in this section, we establish several fractional derivative formulas for the following extended generalized hypergeometric type function:

$$
{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} z;\mathfrak{p}\right],
$$

which is given by (1.18). For this purpose, we need the pairs of hypergeometric fractional derivative operators $\mathbb{D}_{0+}^{\omega,\nu,\eta}$ and $\mathbb{D}_{\infty-}^{\omega,\nu,\eta}$, which are defined in the section 2.5.1 and 2.5.2 in terms of the corresponding pairs of hypergeometric fractional integral operators $\mathbb{I}_{0+}^{\omega,\nu,\eta}$ and $\mathbb{I}_{\infty-}^{\omega,-\nu,\eta}$, respectively.
By appealing appropriately to the assertions (1.117) and (1.118), we are led fairly easily to the following results.

**Theorem 2.5.1.** *Under the conditions stated already with* (1.18), *the following left-sided hypergeometric fractional derivative formula holds true*:

$$
\left(\mathbb{D}_{0+}^{\omega,\nu,\eta}\, t^{\rho-1}\, {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt;\mathfrak{p}\right]\right)(x) = x^{\rho+\nu-1}\,\frac{\Gamma(\rho)\Gamma(\rho+\omega+\nu+\eta)}{\Gamma(\rho+\nu)\Gamma(\rho+\eta)}
$$
$$
\cdot\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zx;\mathfrak{p}\right] *\; {}_2F_2\left[\begin{array}{c} \rho,\rho+\omega+\nu+\eta; \\ \rho+\nu,\rho+\eta; \end{array} zx\right] \tag{2.30}
$$

$$
\big(x>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)>-\min\{0,\Re(\omega+\nu+\eta)\}\big),
$$

*where it is assumed that the left-sided hypergeometric fractional derivative in* (2.30) *exists.*

Proof: Our demonstration of the hypergeometric fractional derivative formula (2.30) is based upon the known result (1.117). The details involved are being left as and exercise for the interested reader.

Just as in the proof of Theorem 2.5.1 above, Theorem 2.5.2 below would follow when we apply the hypergeometric fractional derivative formula (1.118).

**Theorem 2.5.2.** *Under the conditions stated already with* (1.18), *the following right-sided hypergeometric fractional derivative formula holds true*:

$$\begin{aligned}&\left(\mathbb{D}_{\infty-}^{\omega,\nu,\eta}\, t^{\rho-1}\, {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix}\ \frac{z}{t};\mathfrak{p}\right]\right)(x)\\ &= x^{\rho+\nu-1}\,\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho-\nu+\eta)}\\ &\quad\cdot\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1,\cdots,a_p; \\ \\ b_1,\cdots,b_q; \end{matrix}\ \frac{z}{x};\mathfrak{p}\right] * {}_2F_2\left[\begin{matrix} 1-\rho-\nu,1-\rho+\omega+\eta; \\ 1-\rho,1-\rho-\nu+\eta; \end{matrix}\ \frac{z}{x}\right] \end{aligned} \tag{2.31}$$

$$\big(x>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\min\{\Re(-\nu-\eta),\Re(\omega+\eta)\}\big),$$

*where it is assumed that the right-sided hypergeometric fractional derivative in* (2.31) *exists.*

Upon setting $\nu=-\omega$ and $\nu=0$ in Theorem 2.5.1, if we use the relationships in (1.110), we can deduce the following corollaries.

**Corollary 2.5.3.** *Under the conditions stated already with* (1.18), *the following Riemann-Liouville fractional derivative formula holds true*:

$$\begin{aligned}&\left({}_0^{RL}\mathbb{D}_t^{\omega}\, t^{\rho-1}\, {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix}\ zt;\mathfrak{p}\right]\right)(x) = x^{\rho-\omega-1}\,\frac{\Gamma(\rho)}{\Gamma(\rho-\omega)}\\ &\quad\cdot\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix}\ zx;\mathfrak{p}\right] * {}_1F_1\left[\begin{matrix} \rho; \\ \rho-\omega; \end{matrix}\ zx\right]\end{aligned} \tag{2.32}$$

$$\big(x>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)>0\big),$$

*where it is assumed that the Riemann-Liouville fractional derivative in* (2.32) *exists.*

**Corollary 2.5.4.** *Under the conditions stated already with* (1.18), *the following left-sided Erdélyi-Kober fractional derivative formula holds true*:

$$\left({}^{\mathrm{EK}}\mathbb{D}_{0+}^{\omega,\eta}\; t^{\rho-1}\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zt;\mathfrak{p}\right]\right)(x) = x^{\rho-1}\,\frac{\Gamma(\rho+\omega+\eta)}{\Gamma(\rho+\eta)}$$
$$\cdot\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} zx;\mathfrak{p}\right] *\; {}_1F_1\left[\begin{array}{c} \rho+\omega+\eta; \\ \rho+\eta; \end{array} zx\right] \tag{2.33}$$

$$\big(x>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)>-\min\{0,\Re(\eta)\}\big),$$

*where it is assumed that the left-sided Erdélyi-Kober fractional derivative in* (2.33) *exists.*

Corollary 2.5.5 and Corollary 2.5.6 below would follow from Theorem 2.5.2 when we set $\nu=-\omega$ and $\nu=0$ in Theorem 2.5.2 and make use of the relationships in (1.114).

**Corollary 2.5.5.** *Under the conditions stated already with* (1.18), *the following Weyl fractional derivative formula holds true*:

$$\left(\mathbb{W}_{\infty-}^{\omega}\; t^{\rho-1}\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} \frac{z}{t};\mathfrak{p}\right]\right)(x)$$
$$= x^{\rho-\omega-1}\,\frac{\Gamma(1-\rho+\omega)}{\Gamma(1-\rho)}$$
$$\cdot\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{array} \frac{z}{x};\mathfrak{p}\right] *\; {}_1F_1\left[\begin{array}{c} 1-\rho+\omega; \\ 1-\rho; \end{array} \frac{z}{x}\right] \tag{2.34}$$

$$\big(x>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\Re(\omega)\big),$$

*where it is assumed that the Weyl fractional derivative in* (2.34) *exists.*

**Corollary 2.5.6.** *Under the conditions stated already with* (1.18), *the following right-sided Erdélyi-Kober fractional derivative formula holds true*:

$$\begin{aligned}
&\left({}^{\mathrm{EK}}\mathbb{D}^{\omega,\eta}_{\infty-}\; t^{\rho-1}\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array} \frac{z}{t};\mathfrak{p}\right]\right)(x)\\
&\quad = x^{\rho-1}\,\frac{\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho+\eta)}\\
&\quad\quad \cdot\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array} \frac{z}{x};\mathfrak{p}\right] *\; {}_1F_1\left[\begin{array}{r} 1-\rho+\omega+\eta;\\ 1-\rho+\eta;\end{array} \frac{z}{x}\right]
\end{aligned} \tag{2.35}$$

$$\big(x>0;\ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\min\{\Re(-\eta),\Re(\omega+\eta)\}\big),$$

*where it is assumed that the right-sided Erdélyi-Kober fractional derivative in* (2.35) *exists.*

## 2.6 An Extended Generalized Hypergeometric Distribution

In this section, we introduce a general family of statistical probability distributions from which certain classical probability distributions can be obtained as special cases (see also the earlier works [108] on this subject).

We begin by recalling the celebrated *Ramanujan's Master Theorem* which was widely used by *Srinivasa Ramanujan Iyengar* (1887-1920) in order to evaluate definite integrals and infinite series. The proof of Ramanujan's Master Theorem was provided by Godfrey Harold Hardy (1877-1947) by making use of Cauchy's Residue Theorem as well as the well-known Mellin Inversion Theorem.

**Theorem 2.6.1.** (Ramanujan's Master Theorem) *Assume that the function $f(x)$ has a power-series expansion in the following form*:

$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(n)}{n!}\,(-x)^n. \tag{2.36}$$

*Then the Mellin transform of the* $f(x)$ *is given by*

$$\mathfrak{M}\left[f(t);s\right] := \int_0^\infty t^{s-1}\, f(t)\mathrm{d}t = \Gamma(s)\varphi(-s), \tag{2.37}$$

*provided that the integral in* (2.37) *exists.*

By applying Theorem 2.6.1 to the extended generalized hypergeometric function defined by (1.18), we can deduce the following corollary.

**Corollary 2.6.2.** *Under the conditions stated already with* (1.18), *the following Mellin transform formula holds true*:

$$\mathfrak{M}\left[{}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix}\ -t;\mathfrak{p}\right];s\right] = \Gamma(s)\Theta_{-s}^{(p,q)}, \tag{2.38}$$

*where*

$$\Theta_{-s}^{(p,q)} := \Theta_n^{(p,q)}\Big|_{n=-s} \tag{2.39}$$

*in terms of the coefficients* $\Theta_n^{(p,q)}$ *given by* (1.19) *and it is assumed that both members of the assertion* (2.38) *exist.*

A special case of the Mellin transform formula (2.38) when $p = q+1$ would correspond to a result derived recently by *Luo et al.* [100]. The general result (2.38) leads us to Definition 2.6.3 below.

**Definition 2.6.3.** For a statistical probability distribution of a random variable $X$, let the probability density function be given by

$$\mathfrak{P}_X(x) := \begin{cases} \left(\dfrac{1}{\Gamma(s)\Theta_{-s}^{(p,q)}}\right) & \\ \quad \cdot\, x^{s-1}\ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1,\cdots,a_p; \\ b_1,\cdots,b_q; \end{matrix}\ -x;\mathfrak{p}\right] & (x>0) \\ 0 & (\text{otherwise}), \end{cases} \tag{2.40}$$

where $\Theta_{-s}^{(p,q)}$ is given by (2.39) and it is *tacitly* assumed that the various arguments parameters involved in the definitions (1.18) and (1.19) are so restricted that

$$\mathfrak{P}_X(x) > 0 \qquad (x > 0). \tag{2.41}$$

Clearly, it follows from Corollary 2.6.2 that

$$\int_0^\infty \mathfrak{P}_X(x)\mathrm{d}x = 1, \tag{2.42}$$

provided that the integral in (2.42) exists.

### 2.6.1 Properties of the Random Variable $X$

In this subsection, we present several interesting properties of the random variable $X$ which is distributed as per the probability density function $\mathfrak{P}_X(x)$ given by (2.40).

(a) **The $k$th Moment:** The $k$th moment $\mathbb{E}[X^k]$ of the random variable $X$ is given by

$$\begin{aligned}\mathbb{E}[X^k] &:= \int_0^\infty t^k \,\mathfrak{P}_X(t)\mathrm{d}t \\ &= \left(\frac{1}{\Gamma(s)\Theta_{-s}^{(p,q)}}\right)\int_0^\infty t^{s+k-1}\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\cdots,a_p; \\ b_1,\cdots,b_q;\end{array} -t;\mathfrak{p}\right]\mathrm{d}t \\ &= \frac{\Gamma(s+k)\Theta_{-s-k}^{(p,q)}}{\Gamma(s)\Theta_{-s}^{(p,q)}},\end{aligned} \tag{2.43}$$

where we have made use of Definition 2.6.3 *and* the Mellin transform formula (2.38) and it is assumed that the integral involved exists.

(b) **The Mean and the Expected Value:** Since the mean $\mu_X$ is the *first* moment, the expected value of the random variable $X$ is a special case of the $k$th moment in (2.43) when $k = 1$. We thus find that

$$\mu_X := \mathbb{E}[X] = \frac{s\Theta_{-s-1}^{(p,q)}}{\Theta_{-s}^{(p,q)}}. \tag{2.44}$$

By setting $k = 2$, the formula (2.43) yields the *second* moment as follows:

$$\mathbb{E}[X^2] = \frac{s(s+1)\Theta_{-s-2}^{(p,q)}}{\Theta_{-s}^{(p,q)}}. \tag{2.45}$$

(c) **The Variance:** Making use of (2.44) and (2.45), the variance $\sigma_X$ of the random variable $X$ can easily be calculated as follows:

$$\begin{aligned}\sigma_X^2 &:= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{s(s+1)\Theta_{-s-2}^{(p,q)}}{\Theta_{-s}^{(p,q)}} - \left(\frac{s\Theta_{-s-1}^{(p,q)}}{\Theta_{-s}^{(p,q)}}\right)^2.\end{aligned} \tag{2.46}$$

### 2.6.2 The Characteristic Function

The characteristic function $\phi_X(t)$ of $\mathfrak{P}_X(x)$ associated with the random variable $X$ is given by

$$\phi_X(t) := \mathbb{E}\left[e^{itX}\right] = \int_0^\infty e^{itx}\, \mathfrak{P}_X(x)\, \mathrm{d}x, \tag{2.47}$$

where $\mathbb{E}$ denotes the *Mathematical Expectation* and $i = \sqrt{-1}$.

We first prove Lemma 2.6.4 below, which provides a *mild* generalization of the Laplace transform formula (2.26) asserted by Corollary 2.3.2.

**Lemma 2.6.4.** *Under the conditions stated already with* (1.18), *the following integral formula holds true*:

$$\begin{aligned}&\int_0^\infty t^{\rho-1}\, e^{-st}\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; zt^\sigma; \mathfrak{p}\right] \mathrm{d}t \\ &\qquad = s^{-\rho}\; {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; \frac{z}{s^\sigma}; \mathfrak{p}\right] * \; {}_1\Psi_0\left[\begin{matrix} (\rho,\sigma); \\ \text{---}\,; \end{matrix}\; \frac{z}{s^\sigma}\right]\end{aligned} \tag{2.48}$$

$$\big(|z| < 1;\ \min\{\Re(s), \Re(\rho), \Re(\sigma)\} > 0\big),$$

*where* ${}_p\Psi_q$ *denotes the Fox-Wright generalized hypergeometric function defined by* (1.7) *and it is assumed that each member of* (2.48) *exists.*

Proof: The demonstration of Lemma 2.6.4 is based upon the following well-known result *Srivastava* and *Manocha* [170, p. 219, Eq. 4.1(3)]:

$$\int_0^\infty t^{\rho-1}\, e^{-st}\, \mathrm{d}t = \frac{\Gamma(\rho)}{s^\rho} \qquad \big(\min\{\Re(s), \Re(\rho)\} > 0\big) \tag{2.49}$$

in conjunction with the definitions (1.18) and (1.7). The details involved are being omitted here.

Obviously, the Laplace transform formula (2.26) asserted by Corollary 2.3.2 follows readily from (2.48) upon setting $\sigma = 1$. More importantly, if we make use of the assertion (2.48) of Lemma 2.6.4 in the equation (2.47), we obtain the characteristic function $\phi_X(t)$ of $\mathfrak{P}_X(x)$ associated with the random variable $X$ as follows:

$$\phi_X(t) = \left(\frac{(-it)^{-s}}{\Gamma(s)\Theta_{-s}^{(p,q)}}\right) \cdot {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1, \cdots, a_p;\ \dfrac{1}{it}; \mathfrak{p} \\ b_1, \cdots, b_q; \end{array}\right] * {}_1\Psi_0\left[\begin{array}{l} (s,1);\ \dfrac{1}{it} \\ \text{---}\ ; \end{array}\right]$$

$$\big(\Re(s) > 0\big).$$

### 2.6.3 Moment Generating Function

The moment generating function $\mathcal{M}_X(t)$ of the random variable $X$ is defined by

$$\mathcal{M}_X(t) := \mathbb{E}\left[e^{tX}\right] = \int_0^\infty e^{tx}\, \mathfrak{P}_X(x)\mathrm{d}x = \phi_X(-it), \tag{2.50}$$

where $\phi_X(t)$ is the characteristic function of $\mathfrak{P}_X(x)$ associated with the random variable $X$, which is given by (2.47). Formally, therefore, we find from (2.50) that

$$\mathcal{M}_X(t) = \left(\frac{(-t)^{-s}}{\Gamma(s)\Theta_{-s}^{(p,q)}}\right)$$
$$\cdot \ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} \frac{1}{t}; \mathfrak{p}\right] * \ {}_1\Psi_0\left[\begin{array}{c} (s,1); \\ \text{---} \ ; \end{array} \frac{1}{t}\right] \tag{2.51}$$
$$\big(\Re(s) > 0\big).$$

Alternatively, we have

$$\begin{aligned}
\mathcal{M}_X(t) &:= \mathbb{E}\left[e^{tX}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{t^k \ X^k}{k!}\right] \\
&= 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \ \mathbb{E}\left[X^k\right] \\
&= 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \ \left(\frac{\Gamma(s+k)\Theta_{-s-k}^{(p,q)}}{\Gamma(s)\Theta_{-s}^{(p,q)}}\right),
\end{aligned} \tag{2.52}$$

where we have applied the $k$th moment formula (2.43).

### 2.6.4 The Cummulative Distribution Function

For $x > 0$, the cummulative distribution function $\mathcal{F}_X(x)$ is given by

$$\mathcal{F}_X(x) := \int_0^x \mathfrak{P}_X(t)\mathrm{d}t = \left(\frac{x^s}{\Gamma(s)\Theta_{-s}^{(p,q)}}\right)$$
$$\cdot \ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{l} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{array} -x; \mathfrak{p}\right] * \ {}_1\Psi_1\left[\begin{array}{r} (s,1); \\ (s+1,1); \end{array} -x\right] \tag{2.53}$$
$$\big(\Re(s) > 0\big),$$

which follows readily from the definition (2.40).

It should be observed that each of the Fox-Wright $\Psi$-functions occurring on the right-hand sides of (2.50), (2.51) and (2.53) can be rewritten in the relatively more familiar $F$-notation.

**Part B**

## 2.7 Jacobi and Related Transforms of the Incomplete Hypergeometric Functions

In this section, we prove three theorems, which exhibit the connection between the Jacobi, Gegenbauer and Legendre transforms with the following incomplete hypergeometric functions:

$$ {}_p\gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} z\right] \quad \text{and} \quad {}_p\Gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} z\right] $$

given by equations (1.25) and (1.26), respectively.

**Theorem 2.7.1.** *Under the conditions stated already with (1.25) and (1.26), the following Jacobi transform formula holds true:*

$$ \begin{aligned} &\mathbb{J}^{(\alpha,\beta)}\left[z^{\rho-1}{}_p\gamma_q[yz];n\right] \\ &= 2^{\alpha+\beta+1}\binom{\alpha+n}{n}B(\alpha+1,\beta+1)\sum_{k=0}^{\infty}\frac{(a_1;x)_k\ (a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \\ &\times\ F^{1:2;1}_{1:1;0}\left[\begin{array}{ccc} \alpha+1: & -n,\alpha+\beta+n+1; & 1-\rho-k; \\ \alpha+\beta+2: & \alpha+1; & \text{---}; \end{array} 1,2\right].\frac{y^k}{k!} \end{aligned} \tag{2.54} $$

$$ (x\geq 0;\ \ n\in\mathbb{N}_0;\ \ \min\{\Re(\alpha),\Re(\beta)\}>-1;\ \ \rho\in\mathbb{C};\ \ p,q\in\mathbb{N}_0), $$

*and*

$$ \begin{aligned} &\mathbb{J}^{(\alpha,\beta)}\left[z^{\rho-1}{}_p\Gamma_q[yz];n\right] \\ &= 2^{\alpha+\beta+1}\binom{\alpha+n}{n}B(\alpha+1,\beta+1)\sum_{k=0}^{\infty}\frac{[a_1;x]_k\ (a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \\ &\times\ F^{1:2;1}_{1:1;0}\left[\begin{array}{ccc} \alpha+1: & -n,\alpha+\beta+n+1; & 1-\rho-k; \\ \alpha+\beta+2: & \alpha+1; & \text{---}; \end{array} 1,2\right].\frac{y^k}{k!} \end{aligned} \tag{2.55} $$

$$(x \geq 0; \;\; n \in \mathbb{N}_0; \;\; \min\{\Re(\alpha), \Re(\beta)\} > -1; \;\; \rho \in \mathbb{C}; \;\; p, q \in \mathbb{N}_0),$$

*where the coefficients ${}_p\gamma_q$ and ${}_p\Gamma_q$ are given by (1.25) and (1.26) and it is assumed that the Jacobi transform in (2.54) and (2.55) exists.*

*Proof*: By applying the definition (2.2) in conjunction with (1.25), we have

$$\begin{aligned}
&\mathbb{J}^{(\alpha,\beta)}\left[z^{\rho-1}{}_p\gamma_q[yz]; n\right] \\
&\int_{-1}^{1}(1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)z^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc} (a_1,x), \;\; a_2,...,a_p; & \\ b_1,...,b_q; & yz \end{array}\right]dz \\
&= \int_{-1}^{1}(1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)z^{\rho-1}\sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k}.\frac{(yz)^k}{k!}dz
\end{aligned} \tag{2.56}$$

Upon changing the order of integration and summation, if we apply the Jacobi transform formula (2.5) with the parameter $\rho$ replaced by $\rho + k$ $(\rho \in \mathbb{C}; k \in \mathbb{N}_0)$, we find from (2.56) that

$$\begin{aligned}
&\mathbb{J}^{(\alpha,\beta)}\left[z^{\rho-1}{}_p\gamma_q[yz]; n\right] \\
&= 2^{\alpha+\beta+1}\binom{\alpha+n}{n}B(\alpha+1,\beta+1)\;\sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \\
&\times\; F_{1:1:0}^{1:2;1}\left[\begin{array}{lccc} \alpha+1: & -n, \alpha+\beta+n+1; & 1-\rho-k; & \\ \alpha+\beta+2: & \alpha+1; & __; & 1,2 \end{array}\right].\frac{y^k}{k!}
\end{aligned}$$

which is precisely the result (2.54) asserted by Theorem 2.7.1.

It is easy to see that a similar argument for equation (1.26) will establish the result (2.55). This completes the proof of the Theorem 2.7.1.

By setting $\alpha = \beta = \nu - \frac{1}{2}$ in view of the definition (2.9), Theorem 2.7.1 yields the following corollary:

**Corollary 2.7.2.** *Under the conditions stated already with (1.25) and (1.26), the* *following Gegenbauer transform formula holds true:*

$$\begin{aligned}&\mathbb{G}^{(\nu)}\left[z^{\rho-1}{}_p\gamma_q[yz];n\right]\\&=2^{2\nu}\binom{2\nu+n-1}{n}B\left(\nu+\frac{1}{2},\nu+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\\&\times\ F^{1:2;1}_{1:1;0}\left[\begin{array}{lll}\nu+\frac{1}{2}: & -n,2\nu+n; & 1-\rho-k;\\ 2\nu+1: & \nu+\frac{1}{2}; & \text{—};\end{array}\ 1,2\right].\frac{y^k}{k!}\end{aligned}\tag{2.57}$$

$$(x\geq 0;\ \ n\in\mathbb{N}_0;\ \ \rho\in\mathbb{C};\ \ p,q\in\mathbb{N}_0),$$

*and*

$$\begin{aligned}&\mathbb{G}\left[z^{\rho-1}{}_p\Gamma_q[yz];n\right]\\&=2^{2\nu}\binom{2\nu+n-1}{n}B\left(\nu+\frac{1}{2},\nu+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{[a_1;x]_k(a_2)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\\&\times\ F^{1:2;1}_{1:1;0}\left[\begin{array}{lll}\nu+\frac{1}{2}: & -n,2\nu+n; & 1-\rho-k;\\ 2\nu+1: & \nu+\frac{1}{2}; & \text{—};\end{array}\ 1,2\right].\frac{y^k}{k!}\end{aligned}\tag{2.58}$$

$$(x\geq 0;\ \ n\in\mathbb{N}_0;\ \ \min\{\Re(\nu)\}>-1/2;\ \ \rho\in\mathbb{C};\ \ p,q\in\mathbb{N}_0),$$

and it is assumed that the Gegenbauer transforms in (2.57) and (2.58) exist.

For the Legendre transform defined by (2.10), a special case of Theorem 2.7.1 when $\alpha=\beta=0$ (or, alternatively, Corollary 4.32 with $\nu=\frac{1}{2}$) yields the following result.

**Corollary 2.7.3.** *Under the conditions stated already with (1.25) and (1.26), the following Legendre transform formulas holds true:*

$$\begin{aligned}&\mathbb{L}\left[z^{\rho-1}{}_p\gamma_q[yz];n\right]\\&=2\sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\ F^{1:2;1}_{1:1;0}\left[\begin{array}{lll}1: & -n,n+1; & 1-\rho-k;\\ 2: & 1; & \text{—};\end{array}\ 1,2\right].\frac{y^k}{k!}\end{aligned}\tag{2.59}$$

$$(x \geq 0; \quad n \in \mathbb{N}_0; \quad \rho \in \mathbb{C}; \quad p, q \in \mathbb{N}_0),$$

*and*

$$\mathbb{L}\left[z^{\rho-1}{}_p\Gamma_q[yz]; n\right]$$
$$= 2\sum_{k=0}^{\infty} \frac{[a_1; x]_k (a_2)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} F_{1:1;0}^{1:2;1}\left[\begin{array}{lccc} 1: & -n, n+1; & 1-\rho-k; & \\ & & & 1,2 \\ 2: & 1; & \text{---}; & \end{array}\right] \cdot \frac{y^k}{k!} \tag{2.60}$$

$$(x \geq 0; \quad n \in \mathbb{N}_0; \quad \rho \in \mathbb{C}; \quad p, q \in \mathbb{N}_0),$$

*it is assumed that the Legendre transforms in (2.59) and (2.60) exist.*

## 2.8 $P_\delta$ - Transform of the Incomplete Hypergeometric Functions

**Theorem 2.8.1.** *Under the conditions stated already with (1.25) and (1.26), the following $P_\delta$ - transform formula holds true:*

$$P_\delta\left[z^{\rho-1}{}_p\gamma_q[yz]; s\right] = \frac{\Gamma(\rho)}{[\wedge(\delta; s)]^\rho}\, {}_{p+1}\gamma_q\left[\begin{array}{ccc} \rho, & (a_1, x), & a_2, \ldots, a_p; \\ & & b_1, \ldots, b_q; \end{array} z\right] \tag{2.61}$$

$$(|z| < 1; \quad \min\{\Re(s), \Re(\rho)\} > 0; \quad \delta > 1, \quad p, q \in \mathbb{N}_0)$$

,

*and*

$$P_\delta\left[z^{\rho-1}{}_p\Gamma_q[yz]; s\right] = \frac{\Gamma(\rho)}{[\wedge(\delta; s)]^\rho}\, {}_{p+1}\Gamma_q\left[\begin{array}{ccc} \rho, & (a_1, x), & a_2, \ldots, a_p; \\ & & b_1, \ldots, b_q; \end{array} z\right] \tag{2.62}$$

$$(|z| < 1; \quad \min\{\Re(s), \Re(\rho)\} > 0; \quad \delta > 1, \quad p, q \in \mathbb{N}_0)$$

*where*

$$\wedge(\delta; s) := \left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) \tag{2.63}$$

*and it is assumed that the $P_\delta$-transform in (2.61) and (2.62) exists.*

*Proof*: Applying the definitions (1.81) and (1.25), we have

$$\begin{aligned} &P_\delta\left[t^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} zt\right];s\right] \\ &= \int_0^\infty t^{\rho-1}[1+(\alpha-1)s]^{-\frac{t}{\alpha-1}}\,{}_p\gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} zt\right]dt \\ &= \int_0^\infty t^{\rho-1}[1+(\alpha-1)s]^{-\frac{t}{\alpha-1}}\sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k}.\frac{(zt)^k}{k!}dt \end{aligned} \tag{2.64}$$

By changing the order of integration and summation in Eq. (2.64) and using Eq. (2.24) therein, we obtain

$$\begin{aligned} &P_\delta\left[t^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} zt\right];s\right] \\ &= \sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k}\frac{(z)^k}{k!}\int_0^\infty t^{\rho+k-1}[1+(\alpha-1)s]^{-\frac{t}{\alpha-1}}dt \\ &= \sum_{k=0}^{\infty}\frac{(a_1;x)_k(a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k}\frac{(z)^k}{k!}\frac{\Gamma(\rho+k)}{[\wedge(\delta;s)]^{\rho+k}}, \end{aligned} \tag{2.65}$$

where $[\wedge(\delta; s)]$ is given by (2.63).

It is easy to see that a similar argument as in the proof of (2.61) will establish the result (2.62). This completes the proof of Theorem 2.8.1.

A limit case of Theorem 2.8.1, when $\delta \to 1+$, yields the following corollary.

**Corollary 2.8.2.** *Under the conditions stated already with (1.25) and (1.26), the following Laplace transform formula holds true:*

$$\mathbb{L}\left\{t^{\alpha-1}{}_p\gamma_q[zt];s\right\} = \frac{\Gamma(\rho)}{s^\rho}{}_{p+1}\gamma_q\left[\begin{array}{lllc} \rho, & (a_1,x), & a_2,...,a_p; & \dfrac{z}{s} \\ & & b_1,...,b_q; & \end{array}\right] \tag{2.66}$$

*and*

$$\mathbb{L}\left\{t^{\alpha-1}{}_p\Gamma_q[zt];s\right\} = \frac{\Gamma(\rho)}{s^\rho}{}_{p+1}\Gamma_q\left[\begin{array}{lllc} \rho, & (a_1,x), & a_2,...,a_p; & \dfrac{z}{s} \\ & & b_1,...,b_q; & \end{array}\right] \tag{2.67}$$

$$(|z|<1;\;\; \min\{\Re(s),\Re(\rho)\}>0,\;\; p,q\in\mathbb{N}_0),$$

*where it is assumed that the Laplace transform in (2.66) and (2.67) exists.*

## 2.9 Fractional Derivative Formulas for Incomplete Hypergeometric Functions

Here, in this section, we establish several fractional derivative formulas for the following Incomplete Hypergeometric Functions:

$${}_p\gamma_q\left[\begin{array}{llc} (a_1,x), & a_2,...,a_p; & z \\ & b_1,...,b_q; & \end{array}\right] \quad \text{and} \quad {}_p\Gamma_q\left[\begin{array}{llc} (a_1,x), & a_2,...,a_p; & z \\ & b_1,...,b_q; & \end{array}\right]$$

which are given by (1.25) and (1.26).

**Theorem 2.9.1.** *Under the conditions stated with (1.25) and (1.26), the following* *left-sided hypergeometric fractional derivative formulas hold true:*

$$\mathbb{D}_{0+}^{\omega,\nu,\eta}\left(t^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc}(a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q;\end{array} zt\right]\right)(x)=x^{\rho+\nu-1}\frac{\Gamma(\rho)\Gamma(\rho+\omega+\nu+\eta)}{\Gamma(\rho+\nu)\Gamma(\rho+\eta)}$$
$$\cdot\ {}_{p+2}\gamma_{q+2}\left[\begin{array}{ccc}\rho, & \rho+\omega+\nu+\eta, & (a_1,x),\ a_2,...,a_p; \\ \rho+\nu, & \rho+\eta, & b_1,...,b_q;\end{array} zx\right] \tag{2.68}$$

$$(x>0;\ \ \Re(\omega)\geqq 0;\ \Re(\rho)>-\min\{0,\omega+\nu+\eta\}),$$

*and*

$$\mathbb{D}_{0+}^{\omega,\nu,\eta}\left(t^{\rho-1}{}_p\Gamma_q\left[\begin{array}{cc}(a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q;\end{array} zt\right]\right)(x)=x^{\rho+\nu-1}\frac{\Gamma(\rho)\Gamma(\rho+\omega+\nu+\eta)}{\Gamma(\rho+\nu)\Gamma(\rho+\eta)}$$
$$\cdot\ {}_{p+2}\Gamma_{q+2}\left[\begin{array}{ccc}\rho, & \rho+\omega+\nu+\eta, & (a_1,x),\ a_2,...,a_p; \\ \rho+\nu, & \rho+\eta, & b_1,...,b_q;\end{array} zx\right] \tag{2.69}$$

$$(x>0;\ \ \Re(\omega)\geqq 0;\ \Re(\rho)>-\min\{0,\omega+\nu+\eta\}),$$

*where it is assumed that the left-sided hypergeometric fractional derivatives in (2.68) and (2.69) exist.*

Proof: Our demonstration of the hypergeometric fractional derivative formula (2.68) and (2.69) is based upon the known result (1.117). The details are omitted as the result are obvious.

Parallel to the proof of Theorem 2.9.1 above, Theorem 2.9.2 below would follow when we apply the hypergeometric fractional derivative formula (1.118).

**Theorem 2.9.2.** *Under the conditions stated already with (1.25) and (1.26), the* *following right-sided hypergeometric fractional derivative formula holds true:*

$$\mathbb{D}_{\infty-}^{\omega,\nu,\eta}\left(t^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} \frac{z}{t}\right]\right)(x) = x^{\rho+\nu-1}\frac{\Gamma(1-\rho-\nu)1-\rho+\omega+\eta}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\nu)}$$

$${}_{p+2}\gamma_{q+2}\left[\begin{array}{cccc} 1-\rho-\nu, & 1-\rho+\omega+\eta, & (a_1,x), & a_2,...,a_p; \\ 1-\rho, & 1-\rho+\eta-\nu, & & b_1,...,b_q; \end{array} \frac{z}{x}\right] \tag{2.70}$$

$$(x>0;\ \ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\min\{\Re(-\nu-\eta),\Re(\omega+\eta)\}),$$

*and*

$$\mathbb{D}_{\infty-}^{\omega,\nu,\eta}\left(t^{\rho-1}{}_p\Gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} \frac{z}{t}\right]\right)(x) = x^{\rho+\nu-1}\frac{\Gamma(1-\rho-\nu)(1-\rho+\omega+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\nu)}$$

$${}_{p+2}\Gamma_{q+2}\left[\begin{array}{cccc} 1-\rho-\nu, & 1-\rho+\omega+\eta, & (a_1,x), & a_2,...,a_p; \\ 1-\rho, & 1-\rho+\eta-\nu, & & b_1,...,b_q; \end{array} \frac{z}{x}\right] \tag{2.71}$$

$$(x>0;\ \ \Re(\omega)\geqq 0;\ \Re(\rho)<1+\min\{\Re(-\nu-\eta),\Re(\omega+\eta)\}),$$

*where it is assumed that the right-sided hypergeometric fractional derivatives in (2.70) and (2.71) exist.*

Upon setting $\nu=-\omega$ and $\nu=0$ respectively, in Theorem 2.9.1, if we use the relationships in (1.110), we can deduce the following corollaries.

**Corollary 2.9.3.** *Under the conditions stated with (1.25) and (1.26), the following*

*Riemann-Liouville fractional derivative formulas hold true:*

$$\begin{aligned}
&{}_{0}^{RL}\mathbb{D}_{t}^{\omega}\left(t^{\rho-1}{}_{p}\gamma_{q}\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} zt\right]\right)(x) \\
&\quad = x^{\rho+\omega-1}\frac{\Gamma(\rho)}{\Gamma(\rho-\omega)}\cdot {}_{p+1}\gamma_{q+1}\left[\begin{array}{cc} \rho, & (a_1,x),\ a_2,...,a_p; \\ \rho-\omega, & b_1,...,b_q; \end{array} zx\right]
\end{aligned} \tag{2.72}$$

$$(x>0;\ \ \mathfrak{R}(\omega)\geqq 0;\ \ \mathfrak{R}(\rho)>0),$$

*and*

$$\begin{aligned}
&{}_{0}^{RL}\mathbb{D}_{t}^{\omega}\left(t^{\rho-1}{}_{p}\Gamma_{q}\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} zt\right]\right)(x) \\
&\quad = x^{\rho+\omega-1}\frac{\Gamma(\rho)}{\Gamma(\rho-\omega)}\cdot {}_{p+1}\Gamma_{q+1}\left[\begin{array}{cc} \rho, & [a_1,x],\ \ a_2,...,a_p; \\ \rho-\omega, & b_1,...,b_q; \end{array} zx\right]
\end{aligned} \tag{2.73}$$

$$(x>0;\ \ \mathfrak{R}(\omega)\geqq 0;\ \ \mathfrak{R}(\rho)>0),$$

*where it is assumed that the Riemann-Liouville fractional derivatives in (2.72) and (2.73) exist.*

**Corollary 2.9.4.** *Under the conditions stated already with (1.25) and (1.26), the following left-sided Erdélyi-Kober fractional derivative formulas hold true:*

$$\begin{aligned}
&{}^{\mathrm{EK}}\mathbb{D}_{0+}^{\omega,\eta}\left(t^{\rho-1}{}_{p}\gamma_{q}\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} zt\right]\right)(x) \\
&\quad = x^{\rho-1}\frac{\Gamma(\rho+\omega+\eta)}{\Gamma(\rho+\eta)}\cdot {}_{p+1}\gamma_{q+1}\left[\begin{array}{cc} \rho+\omega+\eta, & (a_1,x),a_2,...,a_p; \\ \rho+\eta, & b_1,...,b_q; \end{array} zx\right]
\end{aligned} \tag{2.74}$$

$$(x>0;\ \ \mathfrak{R}(\omega)\geqq 0;\ \ \mathfrak{R}(\rho)>-\min\{0,\mathfrak{R}(\eta)\}).$$

*and*

$$
\begin{aligned}
&{}^{\mathrm{EK}}\mathbb{D}_{0+}^{\omega,\eta}\left(t^{\rho-1}{}_p\Gamma_q\left[\begin{array}{cc}(a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q;\end{array} zt\right]\right)(x) \\
&\quad = x^{\rho-1}\frac{\Gamma(\rho+\omega+\eta)}{\Gamma(\rho+\eta)}\cdot{}_{p+1}\Gamma_{q+1}\left[\begin{array}{cc}\rho+\omega+\eta, & [a_1,x],\ a_2,...,a_p; \\ \rho+\eta, & b_1,...,b_q;\end{array} zx\right]
\end{aligned}
\tag{2.75}
$$

$$(x>0;\ \ \Re(\omega)\geqq 0;\ \ \Re(\rho)>-\min\{0,\Re(\eta)\}).$$

*where it is assumed that the left-sided Erdélyi-Kober fractional derivatives in (2.74) and (2.75) exist.*

Corollaries 2.9.5 and Corollary 2.9.6 below would follow from the Theorem 2.9.2, respectively by setting $\nu=-\omega$ and $\nu=0$ and making use of the relationships in (1.114).

**Corollary 2.9.5.** *Under the conditions stated already with (1.25) and (1.26), the following Weyl fractional derivative formula holds true:*

$$
\begin{aligned}
&\mathbb{W}_{\infty-}^{\omega}\left(t^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc}(a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q;\end{array} \frac{z}{t}\right]\right)(x) \\
&= x^{\rho-\omega-1}\frac{\Gamma(1-\rho+\omega)}{\Gamma(1-\rho)}{}_{p+1}\gamma_{q+1}\left[\begin{array}{cc}1-\rho+\omega, & (a_1,x),\ a_2,...,a_p; \\ 1-\rho, & b_1,...,b_q;\end{array} \frac{z}{x}\right]
\end{aligned}
\tag{2.76}
$$

$$(x>0;\ \ \Re(\omega)\geqq 0;\ \ \Re(\rho)<1+\Re(\omega)),$$

*and*

$$
\begin{aligned}
&\mathbb{W}_{\infty-}^{\omega}\left(t^{\rho-1}{}_p\Gamma_q\left[\begin{array}{cc}(a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q;\end{array} \frac{z}{t}\right]\right)(x) \\
&= x^{\rho-\omega-1}\frac{\Gamma(1-\rho+\omega)}{\Gamma(1-\rho)}{}_{p+1}\Gamma_{q+1}\left[\begin{array}{cc}1-\rho+\omega, & (a_1,x),a_2,...,a_p; \\ 1-\rho, & b_1,...,b_q;\end{array} \frac{z}{x}\right]
\end{aligned}
\tag{2.77}
$$

$$(x > 0;\ \ \Re(\omega) \geqq 0;\ \ \Re(\rho) < 1 + \Re(\omega)),$$

where it is assumed that the Weyl fractional derivative in (2.76) and (2.77) exists.

**Corollary 2.9.6.** *Under the conditions stated with (1.25) and (1.26), the following right-sided Erdélyi-Kober fractional derivative formulas hold true:*

$$\begin{aligned} &{}^{\mathrm{EK}}\mathbb{D}^{\omega,\eta}_{\infty-}\left(t^{\rho-1}{}_p\gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} \frac{z}{t}\right]\right)(x) \\ &= x^{\rho-1}\frac{\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho+\eta)}{}_{p+1}\gamma_{q+1}\left[\begin{array}{cc} 1-\rho+\omega+\eta, & (a_1,x),a_2,...,a_p; \\ 1-\rho+\eta, & b_1,...,b_q; \end{array} \frac{z}{x}\right] \end{aligned} \tag{2.78}$$

*and*

$$\begin{aligned} &{}^{\mathrm{EK}}\mathbb{D}^{\omega,\eta}_{\infty-}\left(t^{\rho-1}{}_p\Gamma_q\left[\begin{array}{cc} (a_1,x), & a_2,...,a_p; \\ & b_1,...,b_q; \end{array} \frac{z}{t}\right]\right)(x) \\ &= x^{\rho-1}\frac{\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho+\eta)}{}_{p+1}\Gamma_{q+1}\left[\begin{array}{cc} 1-\rho+\omega+\eta, & (a_1,x),\ a_2,...,a_p; \\ 1-\rho+\eta, & b_1,...,b_q; \end{array} \frac{z}{x}\right] \end{aligned} \tag{2.79}$$

$$(x > 0;\ \ \Re(\omega) \geqq 0;\ \ \Re(\rho) < 1 + \min\{\Re(-\eta), \Re(\omega+\eta)\}).$$

where it is assumed that the right-sided Erdélyi-Kober fractional derivatives in (2.78) and (2.79) exist.

## 2.10 Conclusion

The family of the extended generalized hypergeometric type functions and the incomplete hypergeometric functions and incomplete hypergeometric functions, defined by (1.18), (1.25) and (1.26), respectively has the distinct advantage that most of the known and widely-investigated special functions are expressible in terms of the extended generalized hypergeometric functions.
In conclusion, therefore, it may be remarked that the results deduced above are significant and can yield numerous other integral transforms and fractional derivative formulas involving various special functions by some suitable specializations of the essentially arbitrary parameters which are involved in these results. More significantly, they are expected to lead to some potential applications in several diverse fields of mathematics. Further, applications of these functions in communication theory, probability theory and groundwater pumping modeling are shown by Srivastava et al. [167], More significantly, they are expected to lead to some potential applications in several diverse fields of mathematics. Also, these results are expected to find some applications in finding the solutions of the integral equations.

# 3

# Certain Image Formulas of Generlized Lommel-Wright Function

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The main findings of this chapter have been published as detailed below:

1. **R. Agarwal and S. Jain**, Generalized Lommel-Wright function associated with Saigo-Maeda fractional derivative operator, Communicated.

In this chapter, we establish certain new image formulas of generalized Lommel-Wright function by applying the operators of fractional integration involving Appells function $F_3(\cdot)$ due to *Marichev-Saigo-Maeda*. Furthermore, by employing some integral transforms on the resulting formulas, we present some more image formulas. All the results derived here are of general character and can yield a number of results in the theory of special functions

## 3.1 Introduction

The fractional integral operators involving various special functions, have found significant importance and applications in various sub-fields of applicable mathematical analysis. During last four decades, a number of workers have studied, in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research work by a number of authors, (see, for example [106, 151]). Many earlier works on the subject of fractional calculus contain interesting accounts of the theories of fractional calculus operators and their applications in diverse research areas [155, 158]. In particular, *Srivastava* and *Saxena* [174] presented a survey-cum-expository paper which gives a remarkably, insightful, and systematic exposition of the investigations carried out by many authors in the field of fractional calculus and its applications and contains a fairly comprehensive bibliography of as many as 190 further references on the subject.

## 3.2 Image Formulas Associated with Fractional Integral Operator

In this section, we establish image formulas for the generalized Lommel-Wright function of first kind involving Saigo-Meada fractional integral operators (1.119) and (1.120), in term of the generalized Wright function $\Psi$ as defined in eq. (1.7). These formulas are given by the following theorems:

**Theorem 3.2.1.** *Let* $\alpha, \alpha^{'}, \beta, \beta', \gamma, \lambda \in \mathbb{C}, m \in \mathbb{N},\ \mu > 0$ *and* $x > 0$ *be such that*

$$\Re(\gamma) > 0,\ \ \Re(\nu) > -1,\ \Re(\rho+\nu) > \max\{0, \Re(\alpha+\alpha^{'}+\beta-\gamma), \Re(\alpha^{'}-\beta^{'})\} \quad (3.1)$$

*then there holds the formula*

$$\left[\mathbb{I}_{0+}^{\alpha,\alpha^{'},\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(tz)\right](x) = x^{A-\alpha-\alpha^{'}+\gamma-1}\left(\frac{z}{2}\right)^{\nu+2\lambda}$$
$${}_4\Psi_{4+m}\left[\begin{array}{c|c} (A,2), (S-\beta,2), (A+\beta^{'}-\alpha^{'},2), (1,1) & \\ (A+\beta^{'},2), (S,2), (A+\gamma-\alpha^{'}-\beta,2), (\nu+\lambda+1,\mu), \underbrace{(\lambda+1,1)}_{m-times} & -\dfrac{(zx)^2}{4}\end{array}\right] \quad (3.2)$$

*where* $A = \rho+\nu+2\lambda,\ S = A+\gamma-\alpha-\alpha^{'}$.

Proof: Taking the fractional integral of (1.44) using the equation (1.119) therein and changing the order of integration and summation, which is justified under the conditions stated with Theorem 3.2.1, we get

$$\begin{aligned}&\left[\mathbb{I}_{0+}^{\alpha,\alpha^{'},\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(tz)\right](x)\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{\nu+2\lambda+2k}\Gamma(k+1)}{(\Gamma(\lambda+k+1))^m\Gamma(\nu+k\mu+\lambda+1)k!}\left(I_{0+}^{\alpha,\alpha^{'},\beta,\beta',\gamma} t^{\nu+2\lambda+2k+\rho-1}\right)(x)\end{aligned} \quad (3.3)$$

Applying the known result (1.121) with $\rho$ replaced by $\rho+\nu+2\lambda+2k$, we obtain

$$\left[\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(tz)\right](x) = x^{A-\alpha-\alpha'+\gamma-1}\left(\frac{z}{2}\right)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(A+2k)\Gamma(k+1)}{\Gamma(A+\beta'+2k)(\Gamma(\lambda+1+k))^m}$$
$$\times\frac{\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)\Gamma(A+\beta'-\alpha'+2k)}{\Gamma(A+\gamma-\alpha'-\beta+2k)\Gamma(\nu+\lambda+1+\mu k)\Gamma(A+\gamma-\alpha-\alpha'+2k)}\frac{(zx)^{2k}}{4^k k!} \tag{3.4}$$

Here $A=\rho+\nu+2\lambda$.

Interpreting the right-hand side of the above equation, in view of the definition (1.7), we arrive at the result (3.2).

**Theorem 3.2.2.** *Let $\alpha,\alpha',\beta,\beta',\gamma,\lambda\in\mathbb{C}, m\in\mathbb{N},\ \mu>0$ and $x>0$ be such that*

$$\Re(\gamma)>0,\ \ \Re(\nu)>-1,\ \ \Re(\rho-\nu)>1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\} \tag{3.5}$$

*then there holds the formula*

$$\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(z/t)\right](x) = x^{\gamma-\alpha-\alpha'-A}\left(\frac{z}{2}\right)^{\nu+2\lambda}$$
$$\times\, {}_4\Psi_{4+m}\left[\left.\begin{array}{c}(S,2),(A+\alpha+\beta'-\gamma,2),(A-\beta,2),(1,1)\\ (A,2),(S+\beta',2),(A+\alpha-\beta,2),(\nu+\lambda+1,\mu),\underbrace{(\lambda+1,1)}_{m-times}\end{array}\right| -\frac{z^2}{4x^2}\right] \tag{3.6}$$

*where $A=1-\rho+\nu+2\lambda,\ S=A-\gamma+\alpha+\alpha'$.*

Proof: On making use of the definitions (1.120) and (1.44) and changing the order of integration and summation, which is justified under the conditions stated with Theorem 3.2.2, we have

$$\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(z/t)\right](x)$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{\nu+2\lambda+2k}\Gamma(k+1)}{(\Gamma(\lambda+k+1))^m\Gamma(\nu+k\mu+\lambda+1)k!}\left(\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-\nu-2\lambda-2k-1}\right)(x) \tag{3.7}$$

Here, on applying the formula (1.122) with $\rho$ replaced by $\rho-\nu-2\lambda-2k$, we obtain

$$\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(z/t)\right](x) = x^{\gamma-\alpha-\alpha'-A}\left(\frac{z}{2}\right)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(A-\beta+2k)}{\Gamma(A+2k)(\Gamma(\lambda+k+1))^m}$$
$$\times\frac{\Gamma(k+1)\Gamma(S+2k)\Gamma(A+\alpha+\beta'-\gamma+2k)}{\Gamma(A+\alpha-\beta+2k)\Gamma(\nu+k\mu+\lambda+1)\Gamma(S+\beta'+2k)}\frac{(z)^{2k}}{(4x^2)^k k!} \tag{3.8}$$

where $A=1-\rho+\nu+2\lambda$, $S=A+\alpha+\alpha'-\gamma$.

In view of the definition of the generalized Lommel-Wright function given by (1.44), the equation (3.8) leads to the result (3.6).

For $m=1$ and in the light of eq. (1.45), Theorem 3.2.1 leads to the following corollaries respectively.

**Corollary 3.2.3.** *Under the conditions stated already with (3.1), the following image formula*

$$\left[\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,1}(zt)\right](x) = x^{A-\alpha-\alpha'+\gamma-1}\left(\frac{z}{2}\right\rangle^{\nu+2\lambda}$$
$$\times\, {}_4\Psi_5\left[\begin{matrix}(A,2),(S-\beta,2),(A+\beta'-\alpha',2),(1,1)\\(A+\beta',2),(S,2),(A+\gamma-\alpha'-\beta,2),(\nu+\lambda+1,\mu),(\lambda+1,1)\end{matrix}\middle|-\frac{(zx)^2}{4}\right] \tag{3.9}$$

$A=\rho+\nu+2\lambda$, $S=A+\gamma-\alpha-\alpha'$ *for generalized Bessel function* $J_{\nu,\lambda}^{\mu,1}(zt)$ *given by eq.* (1.45) *holds true.*

**Corollary 3.2.4.** *Under the conditions stated already with (3.5), the image formula*

$$\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,1}(z/t)\right](x) = x^{\gamma-\alpha-\alpha'-A}\left(\frac{z}{2}\right)^{\nu+2\lambda}$$
$$\times\, {}_4\Psi_5\left[\begin{matrix}(S,2),(A+\alpha+\beta'-\gamma,2),(A-\beta,2),(1,1)\\(A,2),(S+\beta',2),(A+\alpha-\beta,2),(\nu+\lambda+1,\mu),(\lambda+1,1)\end{matrix}\middle|-\frac{z^2}{4x^2}\right] \tag{3.10}$$

$A = 1 - \rho + \nu + 2\lambda,\ S = A - \gamma + \alpha + \alpha^{'}$ *for generalized Bessel function* $J_{\nu,\lambda}^{\mu,1}(z/t)$ *holds true.*

If we take $m = 1,\ \mu = 1$ and $\lambda = \frac{1}{2}$ in (3.2), then we obtain the corresponding result for the Struve function $H_\nu(\cdot)$ given by eq. (1.48) mentioned in Chapter 1.

**Corollary 3.2.5.** *Under the conditions stated already with (3.1), the following image formula*

$$\begin{aligned}&\left[\mathbb{I}_{0+}^{\alpha,\alpha^{'},\beta,\beta',\gamma} t^{\rho-1} H_\nu(zt)\right](x) = x^{A-\alpha-\alpha^{'}+\gamma-1}\left(\frac{z}{2}\right)^{\nu+1}\\&\times {}_4\Psi_5\left[\begin{matrix}(A,2),(S-\beta,2),(A+\beta^{'}-\alpha^{'},2),(1,1)\\(A+\beta^{'},2),(S,2),(A+\gamma-\alpha^{'}-\beta,2),(\nu+\frac{3}{2},1),(\frac{3}{2},1)\end{matrix}\middle| -\frac{(zx)^2}{4}\right]\end{aligned}\tag{3.11}$$

$A = \rho + \nu + 1,\ S = A + \gamma - \alpha - \alpha^{'}$ *for Struve function* $H_\nu(zt)$ *holds true.*

**Corollary 3.2.6.** *Under the conditions stated already with (3.5), the following image formula*

$$\begin{aligned}&\left[\mathbb{I}_{0-}^{\alpha,\alpha^{'},\beta,\beta',\gamma} t^{\rho-1} H_\nu(z/t)\right](x) = x^{\rho-\nu-\alpha-\alpha^{'}+\gamma-2}\left(\frac{z}{2}\right)^{\nu+1}\\&\times {}_4\Psi_5\left[\begin{matrix}(S,2),(A+\alpha+\beta^{'}-\gamma,2),(A-\beta,2),(1,1)\\(A,2),(S+\beta',2),(A+\alpha-\beta,2),(\nu+\frac{3}{2},1),(\frac{3}{2},1)\end{matrix}\middle| -\frac{z^2}{4x^2}\right]\end{aligned}\tag{3.12}$$

*where* $A = 2 - \rho + \nu,\ A - \gamma + \alpha + \alpha^{'}$ *for Struve function* $H_\nu(z/t)$ *holds true.*

### 3.2.1 Special Cases

(1) On taking $\mu = 1,\ m = 1,\ \lambda = 0$ and $z = 1$ in Theorem 3.2.1, we obtain the image formula for the Bessel function considered by *Purohit et al.* [142, Theorem 1].

**Corollary 3.2.7.** *Under the conditions stated already with (3.1), the following image* *formula*

$$\left[\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\nu(t)\right](x) = \frac{x^{\rho+\nu-\alpha-\alpha'+\gamma-1}}{2^\nu} \times$$
$${}_3\Psi_4 \left[\begin{matrix} (\rho+\nu,2),(\rho+\nu+\gamma-\alpha-\alpha'-\beta,2),(\rho+\nu+\beta'-\alpha',2) \\ (\rho+\nu+\beta',2),(\rho+\nu+\gamma-\alpha-\alpha',2),(\rho+\nu+\gamma-\alpha'-\beta,2),(\nu+1,1) \end{matrix}\middle| -\frac{x^2}{4}\right] \tag{3.13}$$

*for Bessel function $J_\nu(t)$ holds true.*

(2) Further, on taking $\mu = 1, m = 1$ and $\lambda = 0$ in Theorem 3.2.2, we arrive the Right sided image formula for the Bessel function considered by *Purohit et al.* [142, Theorem 2].

**Corollary 3.2.8.** *Under the conditions stated already with (3.5), the image formula*

$$\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\nu(1/t)\right](x) = \frac{x^{\gamma-\alpha-\alpha'-1+\rho-\nu}}{2^\nu}$$
$$\times\, {}_3\Psi_4 \left[\begin{matrix} (A-\gamma+\alpha+\alpha',2),(A+\alpha+\beta'-\gamma,2),(A-\beta,2) \\ (A,2)(A+\alpha+\alpha'+\beta'-\gamma,2),(A+\alpha-\beta,2),(\nu+1,1) \end{matrix}\middle| -\frac{1}{4x^2}\right] \tag{3.14}$$

*where $A = 1-\rho+\nu$, for Bessel function $J_\nu(1/t)$ holds true.*

## 3.3 Image Formulas Associated with Integral Transforms

In this section, we obtain certain theorems involving the results obtained in previous section associated with the integral transforms like Beta transforms, pathway transforms, Laplace transforms and Whittaker transforms.

### 3.3.1 Beta Transform

The *beta transform* (see, e.g.[164]) of complex valued function $f(z)$ of real variable $z$ is defined as:

$$B\{f(z):a,b\}=\int\limits_0^1 z^{a-1}(1-z)^{b-1}f(z)\,dz,\quad \Re(z)>0,\quad \Re(a),\Re(b)>0 \tag{3.15}$$

Beta transform of power function $z^{\rho-1}$ is given by:

$$B\left(z^{\rho-1};a,b\right)=\int_0^1 z^{a+\rho-2}(1-z)^{b-1}dz=\frac{\Gamma(a+\rho-1)\Gamma(b)}{\Gamma(a+\rho+b-1)}\quad \Re(z)>0,\quad \Re(a),\Re(b)>0 \tag{3.16}$$

**Theorem 3.3.1.** *Let $\alpha,\alpha',\beta,\beta',\gamma,\lambda\in\mathbb{C}, m\in\mathbb{N},\ \mu>0$ and $x>0$ be such that*

$$\Re(l)>0,\ \ \Re(n)>0\ \ \Re(\gamma)>0,\ \ \Re(\nu)>-1,\ \ \Re(\rho+\nu)>\max\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\} \tag{3.17}$$

*then the following beta transform formula holds:*

$$B\left(\left(\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\nu,\lambda}^{\mu,m}(tz)\right)(x):l,n\right)=\frac{x^{A-\alpha-\alpha'+\gamma-1}\Gamma(n)}{2^{\nu+2\lambda}}$$
$$\times\,{}_5\Psi_{5+m}\left[\begin{matrix}(A,2),(S-\beta,2),(A+\beta'-\alpha',2),(C-n,2)(1,1)\\(A+\beta',2),(S,2),(A+\gamma-\alpha'-\beta,2),(\nu+\lambda+1,\mu),(C,2),\underbrace{(\lambda+1,1)}_{m-times}\end{matrix}\middle|-\frac{x^2}{4}\right] \tag{3.18}$$

*Here $A=\rho+\nu+2\lambda$, $S=A+\gamma-\alpha-\alpha'$ and $C=l+\nu+2\lambda+n$.*

Proof: For convenience, let the left-hand side of the formula (3.18) be denoted by $\varsigma$.

Applying (3.15) to eq. (3.18) we get,

$$\varsigma = \int_0^1 z^{l-1}(1-z)^{n-1}\left(\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(tz)\right)(x)\right) dz.$$

Here, applying eq. (3.2) to the integral, we obtain the following expression

$$\varsigma = \int_0^1 z^{l-1}(1-z)^{n-1} z^{\nu+2\lambda} \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(A+2k)\Gamma(k+1)}{\Gamma(A+\beta'+2k)\Gamma(A+\gamma-\alpha-\alpha'+2k)}$$
$$\times \frac{\Gamma(A+\beta'-\alpha'+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(A+\gamma-\alpha'-\beta+2k)\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m} \frac{(zx^2)^k}{4^k k!} dz$$

here $A = \rho + \nu + 2\lambda$.

Interchanging the order of integration and summation, we have

$$\begin{aligned}
\varsigma = {} & \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}} \sum_{k=0}^{\infty} \frac{\Gamma(A+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(A+\gamma-\alpha-\alpha'+2k)\Gamma(A+\gamma-\alpha'-\beta+2k)} \\
& \times \frac{\Gamma(A+\beta'-\alpha'+2k)\Gamma(k+1)(-1)^k}{\Gamma(A+\beta'+2k)\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m} \frac{(x^2)^k}{4^k k!} \\
& \times \int_0^1 z^{l+\nu+2\lambda+2k-1}(1-z)^{n-1} dz \\
= {} & \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+2\lambda+2k)\Gamma(n)\Gamma(A+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(l+\nu+2\lambda+2k+n)\Gamma(A+\beta'+2k)\Gamma(A+\gamma-\alpha-\alpha'+2k)} \\
& \times \frac{\Gamma(A+\beta'-\alpha'+2k)\Gamma(k+1)}{\Gamma(A+\gamma-\alpha'-\beta+2k)\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m} \frac{(-x^2)^k}{4^k k!}
\end{aligned} \tag{3.19}$$

Interpreting the right-hand side of the above equation, in the view of the definition (1.7) we arrive at the required result (3.18).

**Theorem 3.3.2.** *Let* $\alpha, \alpha^{'}, \beta, \beta', \gamma, \lambda, \nu \in \mathbb{C}, m \in \mathbb{N},\ \mu > 0$ *and* $x > 0$ *be such that*

$$\Re(\gamma) > 0,\ \ \Re(\nu) > -1,\ \ \Re(l) > 0,\ \ \Re(n) > 0,\ \ \Re(\rho - \nu) > 1 + \min\{\Re(-\beta), \Re(\alpha + \alpha^{'} - \gamma), \Re(\alpha + \beta^{'} - \gamma)\} \tag{3.20}$$

*then the following beta transform formula holds:*

$$B\left(\left(\mathbb{I}_{0-}^{\alpha,\alpha^{'},\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(z/t)\right)(x) : l, n\right) = \frac{x^{\gamma-\alpha-\alpha^{'}-A}\Gamma(n)}{2^{\nu+2\lambda}}$$
$$\times\, {}_5\Psi_{5+m}\left[\begin{array}{c} (S,2), (A+\alpha+\beta^{'}-\gamma,2), (A-\beta,2), (C-n,2), (1,1) \\ (A,2), (S+\beta',2), (A+\alpha-\beta,2), (\nu+\lambda+1,\mu), (C,2), \underbrace{(\lambda+1,1)}_{m-times} \end{array}\middle| -\frac{1}{4x^2}\right] \tag{3.21}$$

*where* $A = 1 - \rho + \nu + 2\lambda,\ S = A - \gamma + \alpha + \alpha^{'}$ *and* $C = l + \nu + 2\lambda + n.$

Proof: The proof of the fractional integral formula (3.21) would run parallel to the formula (3.18) given in Theorem 3.3.1. There for we choose to skip the details involved.

**Remark 3.3.3.** *(1) For* $m = 1$ *Theorem 3.3.1 and Theorem 3.3.2 leads to the corresponding results for fractional integral of generalized Bessel function defined by* (1.45).

*(2) If we take* $m = 1,\ \mu = 1$ *and* $\lambda = \frac{1}{2}$ *in (3.18) and* (3.21), *we get the corresponding results for fractional integral of Struve function defined in* (1.48).

*(3) On taking* $m = 1,\ \mu = 1$ *and* $\lambda = 0$ *in (3.18) and* (3.21), *we get the results for fractional integral of Bessel function defined in* (1.49).

### 3.3.2 $P_\delta$-Transform

**Theorem 3.3.4.** *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, \lambda \in \mathbb{C}, m \in \mathbb{N},\ \mu > 0,\ \Re(\rho) > 0,\ \Re(s) > 0,\ \delta > 1$ *and* $x > 0$ *be such that*

$$\Re(\gamma) > 0,\ \ \Re(\nu) > -1,\ \ \Re(s) > 0,\ \Re(\rho+\nu) > \max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\} \tag{3.22}$$

*then the following $P_\delta$-transform formula holds:*

$$P_\delta\left(z^{l-1}\left(\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(tz)\right)(x) : s\right) = (\Lambda(\delta; s))^{l+\nu+2\lambda}\ \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}}\times$$
$$_5\Psi_{4+m}\left[\begin{array}{c} (A,2), (S-\beta,2), (A+\beta'-\alpha',2), (l+\nu+2\lambda,2), (1,1) \\ (A+\beta',2), (S,2)(A+\gamma-\alpha'-\beta,2), (\nu+\lambda+1,\mu), \underbrace{(\lambda+1,1)}_{m-times} \end{array}\middle| -\frac{(\Lambda(\delta;s)x)^2}{4}\right] \tag{3.23}$$

*where* $A = \rho+\nu+2\lambda,\ S = A+\gamma-\alpha-\alpha'$ *and* $\Lambda(\delta; s) = \left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}$.

Proof: For convenience, let the left-hand side of the formula (3.23) be denoted by $\varphi$. Applying (1.81) to eq. (3.18) we get,

$$\varphi = \int_0^\infty [1+(\delta-1)s]^{-\frac{z}{\delta-1}} z^{l-1}\left(\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(tz)\right)(x)\right) dz$$

Here, applying eq. (3.4) to the integral, we obtain the following expression

$$\varphi = \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(A+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(A+\beta'+2k)\Gamma(A+\gamma-\alpha-\alpha'+2k)\Gamma(A+\gamma-\alpha'-\beta+2k)}$$
$$\frac{\Gamma(A+\beta'-\alpha'+2k)\Gamma(k+1)}{\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m} \frac{(x)^{2k}}{4^k k!} \times \int_0^\infty [1+(\delta-1)s]^{-\frac{z}{\delta-1}} z^{\nu+2\lambda+2k+l-1} dz$$

Here making use of the result (1.82) and interchanging the order of integration and summation, we obtain,

$$\varphi = \{\Lambda(\delta;s)\}^{l+\nu+2\lambda}\frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}}\sum_{k=0}^{\infty}\frac{\Gamma(A+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(A+\beta'+2k)\Gamma(A+\gamma-\alpha-\alpha'+2k)}$$
$$\times\frac{\Gamma(\nu+2\lambda+2k+l)\Gamma(A+\beta'-\alpha'+2k)\Gamma(k+1)(-1)^k}{\Gamma(A+\gamma-\alpha'-\beta+2k)\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m}\frac{\{\Lambda(\delta;s)x\}^{2k}}{4^k k!} \tag{3.24}$$

where $A=\rho+\nu+2\lambda$ and $\Lambda(\delta;s)=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}$,
in the view of the definition (1.7) we arrive at the required result (3.23).

**Theorem 3.3.5.** *Let $\alpha,\alpha',\beta,\beta',\gamma,\lambda\in\mathbb{C}, m\in\mathbb{N},\ \mu>0\ \Re(\rho)>0,\ \Re(s)>0,\ \delta>1$ and $x>0$ be such that*

$$\Re(\gamma)>0,\ \ \Re(\nu)>-1,\ \ \Re(s)>0,\ \ \Re(\rho-\nu)>1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),$$
$$\Re(\alpha+\beta'-\gamma)\} \tag{3.25}$$

*then the following $P_\delta$- transform formula holds:*

$$P_\delta\left(z^{l-1}\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\nu,\lambda}^{\mu,m}(z/t)\right](x):s\right)=\{\Lambda(\delta;s)\}^{l+\nu+2\lambda}\frac{x^{\rho-\nu-2\lambda-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}}$$
$$\times\,{}_5\Psi_{4+m}\left[\begin{matrix}(S,2),(A+\alpha+\beta'-\gamma,2),(A-\beta,2),(l+\nu+2\lambda,2),(1,1)\\(A,2),(S+\beta',2),(A+\alpha-\beta,2),(\nu+\lambda+1,\mu),\underbrace{(\lambda+1,1)}_{m-times}\end{matrix}\;\middle|\;-\frac{\{\Lambda(\delta;s)\}^2}{4x^2}\right] \tag{3.26}$$

*where $A=1-\rho+\nu+2\lambda,\ S=A-\gamma+\alpha+\alpha'$ and $\Lambda(\delta;s)=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}$.*

Proof: Our demonstration of the $P_\delta$- transform of generalized Lommel Wright function (3.26) is based upon the known result (3.6). The details involved are being left as and exercise for the interested reader.

It is interesting to observe that, for taking $\delta\to1$ in (1.81), the $P_\delta$-transform defined by (1.81)) reduces to the well-known Laplace transform (1.84).

A limit case of theorem 3.3.4 and 3.3.5 when $\delta \to 1$ yields the following corollaries.

**Corollary 3.3.6.** *Under the conditions stated already with (3.22), the following Laplace transform formula holds true:*

$$P_\delta\left(z^{l-1}\left(\mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\nu,\lambda}^{\mu,m}(tz)\right)(x):s\right)=\frac{x^{A-\alpha-\alpha'+\gamma-1}}{s^l\,2^{\nu+2\lambda}}$$
$$\times\,{}_5\Psi_{4+m}\left[\begin{array}{c}(A,2),(S-\beta,2),(A+\beta'-\alpha',2),(l+\nu+2\lambda,2),(1,1)\\(A+\beta',2),(S,2),(A+\gamma-\alpha'-\beta,2),(\nu+\lambda+1,\mu),\underbrace{(\lambda+1,1)}_{m-times}\end{array}\middle|-\frac{(x)^2}{s^{2l}4}\right] \tag{3.27}$$

*where* $A=\rho+\nu+2\lambda$, $S=A+\gamma-\alpha-\alpha'$.

**Corollary 3.3.7.** *Under the conditions stated already with (3.25), the following Laplace transform formula holds true:*

$$P_\delta\left(z^{l-1}\left[\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\nu,\lambda}^{\mu,m}(z/t)\right](x):s\right)=\frac{x^{\rho-\nu-2\lambda-\alpha-\alpha'+\gamma-1}}{s^l\,2^{\nu+2\lambda}}$$
$$\times\,{}_5\Psi_{4+m}\left[\begin{array}{c}(S,2),(A+\alpha+\beta'-\gamma,2),(A-\beta,2),(l+\nu+2\lambda,2),(1,1)\\(A,2),(S+\beta',2),(A+\alpha-\beta,2),(\nu+\lambda+1,\mu),\underbrace{(\lambda+1,1)}_{m-times}\end{array}\middle|-\frac{1}{s^{2l}\,4x^2}\right] \tag{3.28}$$

*where* $A=1-\rho+\nu+2\lambda$, $S=A-\gamma+\alpha+\alpha'$.

**Remark 3.3.8.** *(1) On taking $m=1$ Theorem 3.3.4 and 3.3.5 leads to the $P_\delta$-transform formulas for fractional integral of generalized Bessel function.*

*(2) A limit case of Theorem 3.3.4 and 3.3.5 when $\delta \to 1$ and $m=1$ yield the Laplace transform formulas for fractional integral of generalized Bessel function.*

*(3) On taking* $m = 1, \ \mu = 1$ *and* $\lambda = \frac{1}{2}$, *Theorems 3.3.4 and 3.3.5, yield to* $P_\delta$*-transform formulas for fractional integral of Struve function.*

*(4) A limit case of Theorem 3.3.4 and 3.3.5, when* $\delta \to 1$ *and* $m = 1, \ \mu = 1$ *and* $\lambda = \frac{1}{2}$ *yield the Laplace transform formulas for fractional integral of Struve function.*

*(5) On taking* $m = 1, \ \mu = 1$ *and* $\lambda = 0$ *in Theorem 3.3.4 and 3.3.5, yield the corresponding results for fractional integral of Bessel function.*

*(6) A limit case of Theorem 3.3.4 when* $\delta \to 1$ *and* $m = 1, \ \mu = 1$ *and* $\lambda = 0$ *yield the corresponding Laplace transform formulas for fractional integral of Bessel function.*

### 3.3.3 Whittaker Transform

**Theorem 3.3.9.** *Let* $\alpha, \alpha', \beta, \beta', \gamma, \lambda, \eta, \sigma \in \mathbb{C}, \ m \in \mathbb{N}, \ \mu > 0$ *and* $x > 0$ *be such that*

$$\Re(\gamma) > 0, \Re(\nu) > -1, \ \Re(\xi \pm \eta) > -1/2, \ \Re(\rho + \nu) > \max\{0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\} \tag{3.29}$$

*then the following Whittaker transform formula holds:*

$$\int_0^\infty z^{\sigma-1} e^{-z/2} W_{\sigma,\eta}(zt) \left( \left( I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(zt) \right) \right) dz = \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}}$$
$$\times \, {}_6\Psi_{5+m} \left[ \begin{matrix} (A,2), (S-\beta,2), (A+\beta'-\alpha',2), (E+\eta,2), (E-\eta,2), (1,1) \\ (A+\beta',2), (S,2), (A+\gamma-\alpha'-\beta,2), (\nu+\lambda+1,\mu), (E-\sigma,2), \underbrace{(\lambda+1,1)}_{m-times} \end{matrix} \middle| -\frac{x^2}{4} \right] \tag{3.30}$$

*where* $A = \rho + \nu + 2\lambda$, $S = A + \gamma - \alpha - \alpha'$ *and* $E = \xi + \nu + 2\lambda + 1/2$.

Proof: For simplicity and convenience, let $\omega$ be the left-hand side of the formula (3.30). Applying (1.55) to eq. (3.30) we have,

$$\omega = \int_0^\infty z^{\sigma-1} e^{-z/2} W_{\sigma,\eta}(zt) \left( \left( \mathbb{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(zt) \right) \right) dz. \tag{3.31}$$

Here, applying eq. (3.2) to the integral, we obtain the following expression

$$\omega = \int_0^\infty z^{\sigma+\nu+2\lambda-1} e^{-z/2} W_{\sigma,\eta}(zt) \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}} \times \sum_{k=0}^\infty \frac{\Gamma(A+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(A+\beta'+2k)\Gamma(A+\gamma-\alpha-\alpha'+2k)}$$
$$\times \frac{\Gamma(A+\beta'-\alpha'+2k)\Gamma(k+1)(-1)^k}{\Gamma(A+\gamma-\alpha'-\beta+2k)\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m} \frac{(zx)^{2k}}{4^k k!}$$

where $A = \rho + \nu + 2\lambda$. Interchanging the order of integration and summation we have

$$\omega = \frac{x^{A-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}} \sum_{k=0}^\infty \frac{\Gamma(E+\eta+2k)\Gamma(E-\eta+2k)\Gamma(A+\gamma-\alpha-\alpha'-\beta+2k)}{\Gamma(E-\sigma+2k)\Gamma(A+\gamma-\alpha'-\beta+2k)}$$
$$\times \frac{(-1)^k \Gamma(A+2k)\Gamma(A+\beta'-\alpha'+2k)\Gamma(k+1)}{\Gamma(A+\beta'+2k)\Gamma(A+\gamma-\alpha-\alpha'+2k)\Gamma(\nu+\lambda+1+\mu k)(\Gamma(\lambda+1+k))^m} \frac{((x)^{2k})}{4^k k!} \tag{3.32}$$

where $A = \rho + \nu + 2\lambda$ and $E = \xi + \nu + 2\lambda + 1/2$.

Interpreting the right-hand side of the above equation, in the view of the definition (1.7) we arrive at the required result (3.30).

**Theorem 3.3.10.** *Let* $\alpha, \alpha', \beta, \beta', \gamma, \lambda, \eta, \sigma \in \mathbb{C}, m \in \mathbb{N},\ \mu > 0$ *and* $x > 0$ *be such that*

$$\Re(\gamma) > 0,\ \ \Re(\nu) > -1,\ \ \Re(\xi \pm n) > -1/2,\ \ \Re(\rho-\nu) > 1 + \min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma),$$
$$\Re(\alpha+\beta'-\gamma)\} \tag{3.33}$$

*then there holds the formula*

$$\int_0^\infty z^{\sigma-1}e^{-z/2}W_{\sigma,\eta}(zt)\left(\left(\mathbb{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\nu,\lambda}^{\mu,m}(zt)\right)\right)dz=\frac{x^{\rho-\nu-2\lambda-\alpha-\alpha'+\gamma-1}}{2^{\nu+2\lambda}}$$

$$\times\,{}_6\Psi_{5+m}\left[\left.\begin{matrix}(S,2),(A+\alpha+\beta'-\gamma,2),(A-\beta,2),(E+\eta,2),(E-\eta,2),(1,1)\\(A,2),(S+\beta',2),(A+\alpha-\beta,2),(\nu+\lambda+1,\mu),(E-\sigma,2),\underbrace{(\lambda+1)}_{m-times}\end{matrix}\right|-\frac{1}{4x^2}\right] \tag{3.34}$$

*where* $A=1-\rho+\nu+2\lambda$, $S=A-\gamma+\alpha+\alpha'$ *and* $E=\xi+\nu+2\lambda+1/2$.

Proof: it is easy to see that a similar lines as in the proof of Theorem 3.3.9 will establish the Theorem 3.3.10.

**Remark 3.3.11.** *(1) For* $m=1$*, Theorem 3.3.9 and 3.3.10 leads to the corresponding results for fractional integral of generalized Bessel function defined in* (1.45).

*(2) If we take* $m=1$*,* $\mu=1$ *and* $\lambda=\frac{1}{2}$ *in Theorem 3.3.9 and 3.3.10, yield the corresponding results for fractional integral of Struve function defined in* (1.48).

*(3) On taking* $m=1$*,* $\mu=1$ *and* $\lambda=0$ *in Theorem 3.3.9 and 3.3.10, yield the corresponding results for fractional integral of Bessel function defined in* (1.49).

## 3.4 Conclusion

In this section, we try to briefly consider some special cases of the our main results involving in Theorems 3.2.1-3.3.10 which can easily be derived by setting (for example) $\alpha'=0$. Such interesting consequences of our results would involve the Saigo fractional integral operator $\mathbb{I}_{0,x}^{\alpha,\beta,\eta}$ and $\mathbb{I}_{x,\infty}^{\alpha,\beta,\eta}$ can be deduced from Theorems 3.2.1-3.3.10 by appropriately applying the relationships given in the definitions (1.105) and (1.106). If we set $\beta=-\alpha$, in the Theorems 3.2.1-3.3.10, then from the eq. (1.87) and (1.91) we obtain the corresponding results for Riemann-Liouville and

Weyl fractional integral operators respectively. Again, if we put $\beta = 0$ in the Theorems 3.2.1-3.3.10 then from the eq. (1.93) and (1.94) we obtain the corresponding results for Erdélyi-Kober type fractional integral operators.

We conclude our present investigation by remarking further that the results obtained here are useful in deriving various image formulas involving such relatively more familiar fractional integral operators given by above relations. We may also emphasize that results derived in this paper are of general character and can specialize to give further interesting and potentially useful formulas involving fractional integral operators.

# 4

# Analytic Solution of some Non-Linear Fractional Partial Differential Equations

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The main findings of this chapter have been published as detailed below:

1. **Agarwal R., Jain S., and Agarwal R.P.**, Analytic solution of generalized space time advection-dispersion equation with fractional Laplace operator, *J. Nonlinear Sci. Appl.*, **9**(6), (2016), 3545-3554.

2. **Agarwal R., Jain S., and Agarwal R.P.**, Analytic Solution of Generalized Space Time fractional Reaction Diffusion Equation, *Fractional Differential Calculus*, **7(2)**, 169-184.

we divide this chapter into two parts. In part 'A' we investigate the analytic solution of the solutions of time-space fractional advection-dispersion equation involving fractional Laplace operator, following with some illustrations and application. In part 'B' we investigate the analytic solution of the generalized space-time fractional reaction-diffusion equation involving fractional Laplace operator, following with some illustrations and concrete applications.

**Part A**

## 4.1 Advection Dispersion Equation

The description of transport is closely related to the terms convection, diffusion, dispersion, and retardation as well as decomposition. First, it is assumed that there are no interactions between the species dissolved in water and the surrounding solid phase. The primary mechanism for the transport of improperly discarded hazardous waste through the environment is by the movement of water through the subsurface and surface waterways. Studying this movement requires that one must be able to measure the quantity of waste present at a particular point in space time. The universal measure for chemical pollution is the concentration. Analytical methods that handle solute transport in porous media are relatively easy to use *Javandel et.al* [69]. However, because of complexity of the equations involved, the analytical solutions are generally available restricted to either radial flow problems or to cases where velocity is uniform over the area of interest. Numerous analytical solutions are available for time-dependent solute transport within media having steady state and uniform flow.

An equation commonly used to describe solute transport in aquifers is the advection-dispersion equation (ADE) (*Liu et al.* [94–96])

$$\frac{\partial u}{\partial t} = -\eta \frac{\partial u}{\partial x} + \zeta \frac{\partial^2 u}{\partial x^2} \tag{4.1}$$

where $u$ is solute concentration, the positive constants $\eta$, $\zeta$ represent the average fluid velocity and the dispersion coefficient, $x$ is the spatial domain, $t$ is time. The ADE is a deterministic homogeneous equation describing a probability function for the location of particles in a continuum. The fundamental solutions of the ADE over time will be Gaussian densities with means and variances based on the values of the macroscopic transport coefficients $\nu$ and $\zeta$. The classical ADE with a local (or asymptotically constant) dispersion tensor is a very handy predictive equation, since solutions are easily gained. The fractional order forms of the ADE are similarly useful. Some partial differential equations of space-time fractional order were successfully used for modeling relevant physical processes (*Basu* and *Acharya* [9], *El-Sayed* and *Aly* [161] and *Benson* et al.[11]. Numerous authors have shown the equivalence between the transport equations that used fractional-order derivatives and some heavy-tailed motions which extended the predictive capability of models built on the stochastic process of Brownian motion, which is the basis for the classical ADE. The motions can be heavy-tailed, implying extremely long-term correlation and fractional derivatives in time and/or space. For example, Benson and his collaborator have derived the application of a fractional ADE (see *Benson* et al. [11], *Meerschaert* et al. [112]). There are some other authors who considerd the fractional ADE. A space fractional ADE with Eulerian derivation was derived by *Schumer* et al.[162] which is used to describe solute plume evolution with a large probability of particles moving significantly ahead of and behind the mean solute velocity.

The physical interpretation of space-time fractional advection-dispersion equation (FADE) is given by *Schumer et.al* [162]. (FADE) is a generalization of the classical ADE in which the first-order space derivative is replaced with Hilfer composite fractional derivatives (see *Hilfer* [63]) of order $0 < \mu < 1$ and $0 \leq \nu \leq 1$ and the second-order space derivative is replaced with the space fractional Laplacian operator of order $0 < \alpha \leq 2$.

## 4.2 Analytic Solution of Unified Space Time Fractional Advection Dispersion Equation

In this section, we investigate the analytic solution of the Space time fractional advection dispersion equation involving fractional Laplace operator contained in the following theorem:

**Theorem 4.2.1.** *Consider the generalized Cauchy type problem for fractional advection dispersion equation*

$$ {}_0\mathbb{D}_{t+}^{\mu,\nu}(u(x,t)) = -\eta\mathbb{D}_x u(x,t) + \zeta\Delta^{\frac{\lambda}{2}}u(x,t),\ 0<\lambda\leq 2,\ x\in\mathbb{R},\ t\in\mathbb{R}^+, \tag{4.2}$$

*subject to the initial condition*

$$\mathbb{I}_{0+}^{(1-\nu)(1-\mu)}u(x,0+) = g(x),\ x\in\mathbb{R},\ 0<\mu<1,\ 0\leq\nu\leq 1 \tag{4.3}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.4}$$

*where* ${}_0\mathbb{D}_t^{\mu,\nu}$ *is the generalized Riemann-Liouville fractional derivative operator defined by Hilfer as* (1.99). $\mathbb{I}_{0+}^{(1-\nu)(1-\mu)}u(x,0+)$ *involves the Riemann Liouville fractional integral operator of order* $(1-\nu)(1-\mu)$ *evaluted in the limit as* $t\to 0+$. $\Delta^{\frac{\lambda}{2}}$ *is the fractional generalized Laplace operator of order* $\lambda$, *where* $0<\lambda\leq 2$.
$u(x,t)$ *and* $g(x)$ *are both the (real) field variable, and sufficiently well behaved functions.*
*Then the solution of equation* (4.2), *subject to the above constraints, is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{4.5}$$

*where*

$$G(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu+\nu(1-\mu)}[(i\eta k - \zeta|k|^{\lambda})t^{\mu}]exp(-ikx)dk \tag{4.6}$$

*is the Green's function,* $E_{\mu,\nu}$ *is the two parameter Mittag-Leffler function given in the eq.* (1.31)*.*

Proof: In order to prove the theorem, we take the Fourier transform of equation (4.2) with respect to the space variable $x$ and use equation (1.126) to obtain

$$_0\mathbb{D}_{t+}^{\mu,\nu}(u^*(k,t)) = \eta iku^*(k,t) - \zeta|k|^{\lambda}u^*(k,t) \tag{4.7}$$

where $u^*(k,t)$ is the Fourier transform of function $u(x,t)$.
Now we apply Laplace transform on (4.7) with respect to variable $t$, and use equation (1.100), we get

$$s^{\mu}\ \bar{u^*}(k,s) - s^{\nu(\mu-1)}\mathbb{I}_{0+}^{(1-\nu)(1-\mu)}u(k,0+) = i\eta k\bar{u^*}(k,s) - \zeta|k|^{\lambda}\bar{u^*}(k,s) \tag{4.8}$$

where $L[u(k,t);s] = \bar{u}(k,s)$.
Now using the initial condition (4.3) and boundary condition (4.4) and solving the equation (4.8) we get

$$(s^{\mu} - i\eta k + \zeta|k|^{\lambda})\bar{u^*}(k,s) = s^{\nu(\mu-1)}g(k) \tag{4.9}$$

$$\implies \bar{u^*}(k,s) = \frac{s^{\nu(\mu-1)}}{s^{\mu} + (\zeta|k|^{\lambda} - i\eta k)}g(k) \tag{4.10}$$

On taking inverse Laplace transform of equation (4.10), by means of the following result by *Haubold* et. al [110, Eq.18]

$$L^{-1}\left\{\frac{s^{\beta-1}}{s^{\alpha}+a}\right\} = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}) \tag{4.11}$$

where $\Re(s) > 0, \Re(\alpha) > 0, \Re(\alpha - \beta) > -1$, we obtain

$$u^*(k,t) = t^{\mu+\nu(1-\mu)-1} E_{\mu,\mu+\nu(1-\mu)}((i\eta k - \zeta|k|^\lambda)t^\mu)g(k) \tag{4.12}$$

Further, taking the inverse Fourier transform, we get

$$u(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} g(k) E_{\mu,\mu+\nu(1-\mu)}((i\eta k - \zeta|k|^\lambda)t^\mu) \exp(-ikx)dk \tag{4.13}$$

If we apply the convolution theorem of the Fourier transform to equation (4.13), it gives the solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{4.14}$$

where the Green's function is given by

$$G(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu+\nu(1-\mu)}((i\eta k - \zeta|k|^\lambda)t^\mu) exp(-ikx)dk \tag{4.15}$$

It is interesting to observe that as an particular case of Theorem 4.2.1, we can obtain solution of homogeneous Schrödinger equation occurring in the quantum mechanics.

(1) On taking $\eta = 0$, $\zeta \equiv \frac{ih}{2m}$ in Theorem 4.2.1 and using (1.57), we arrive at the main result of the paper by *Saxena et al.* [156] given as below:

**Corollary 4.2.2.** *Consider the following one dimensional space-time fractional Schrödinger equation of a free particle of mass m, defined by*

$$_0\mathbb{D}^{\mu;\nu}_{t+}(u(x,t)) = \frac{ih}{2m} \Delta^{\frac{\lambda}{2}} u(x,t), \ x \in \mathbb{R}, \ t \in \mathbb{R}^+, \tag{4.16}$$

*subject to the initial condition*

$$\mathbb{I}^{(1-\nu)(1-\mu)}_{0+} u(x,0+) = g(x), \ x \in \mathbb{R} \tag{4.17}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0, \ t > 0 \tag{4.18}$$

*where* ${}_0\mathbb{D}^{\mu,\nu}_{t+}$ *is the fractional derivative operator defined by Hilfer as* (1.99) *with* $0 < \mu < 1$, $0 \leq \nu \leq 1$, $\Delta^{\frac{\lambda}{2}}$ *is the fractional Laplace operator of order* $\lambda$, $0 < \lambda \leq 2$ *and* $h = 6.625 \times 10^{-27}$ *erg sec* $= 4.21 \times 10^{-21}$ *Mev sec is the Planck constant.*
*Then the solution of equation* (4.16) *is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk \tag{4.19}$$

*where*

$$G(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\lambda|x|} H^{2,1}_{3,3}\left[\frac{|x|}{\left(\frac{ih}{2m}\right)^{\frac{1}{\lambda}} t^{\frac{\mu}{\lambda}}}\middle|\begin{matrix}(1,\frac{1}{\lambda}),(\mu+\nu(1-\mu),\frac{\mu}{\lambda}),(1,\frac{1}{2})\\(1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2})\end{matrix}\right] \tag{4.20}$$

(2) Further, on taking $\nu = 0$, $\eta = 0$, $\zeta = \frac{ih}{2m}$ in Theorem 4.2.1 and using (1.57), we obtain Corollary 1.1 of *Saxena et al.* [156]

(3) On taking $\nu = 1$, $\eta = \nu$, $\lambda = 2$ and $g(x) = C_o(x)$ in Theorem 4.2.1, we obtain following result which was considered by *Liu et al.* [96]

$${}^C_0\mathbb{D}^{\mu}_t(u(x,t)) = -\nu\mathbb{D}_x u(x,t) + \zeta\frac{\partial^2}{\partial x^2}(u(x,t)), \tag{4.21}$$

subject to the initial condition

$$u(x,0+) = C_o(x), \tag{4.22}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,t) = 0. \tag{4.23}$$

## 4.3 Illustrative Examples

**Example 4.3.1.** *Consider the generalized fractional advection dispersion equation to describe solute transport in aquifers*

$$ {}_0\mathbb{D}_{t+}^{\mu,\nu}(u(x,t)) = -\mathbb{D}_x(u(x,t)) + \mu'\Delta^{\frac{\lambda}{2}}(u(x,t)), \tag{4.24}$$

*subject to the initial condition*

$$ \mathbb{I}_{0+}^{(1-\nu)(1-\mu)}(u(x,0)) = e^{-x},\ 0<x<1, t>0, \tag{4.25}$$

*and boundary condition*

$$ \lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.26}$$

*where* $\mu' = \frac{d}{\nu L}$ *and the number* $Pe = \frac{1}{\mu'}$ *is called the Peclet number. The Peclet number describes the relative influence of the effects characterized by advection-dispersion problems which involve a non-dissipative component and a dissipative component,* $d$ *is the dispersion coefficient* $[L^2T^{-1}]$ *and* $\nu$ *is the Darcy velocity* $[LT^{-1}]$.

In view of Theorem 4.2.1, we conclude that the analytical expression of solute concentration $u(x,t)$ is given by

$$ u(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi}\int_{-\infty}^{\infty} g(k)e^{-ikx}E_{\mu,\mu-\nu(\mu-1)}((ik-\mu'|k|^{\lambda})t^{\mu})\ dk \tag{4.27}$$

where $g(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1+ik)}-1}{1+ik}\right]$

It is interesting to observe that for $\nu = 0,\ \mu = 1$ and $\lambda = 2$, equations (4.24)-(4.26) reduces to problem considered by *Pandey et. al* [128].

Next, we take an example where in the initial condition we put $g(x) = \delta(x)$, the Dirac-delta function.

**Example 4.3.2.** *Consider the generalized fractional advection dispersion equation*

$$ {}_0\mathbb{D}_{t+}^{\mu;\nu}(u(x,t)) = -\mathbb{D}_x(u(x,t)) + \mu' \Delta^{\frac{\lambda}{2}}(u(x,t)), \tag{4.28} $$

*subject to the initial condition*

$$ \mathbb{I}_{0+}^{(1-\nu)(1-\mu)}(u(x,0)) = \delta(x), \tag{4.29} $$

*where $\delta(x)$ is Dirac-delta function and boundary condition*

$$ \lim_{|x|\to\infty} u(x,t) = 0,\ t > 0 \tag{4.30} $$

In view of Theorem 4.2.1, the solution of 4.28 is given by

$$ u(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\mu,\mu-\nu(\mu-1)}((ik - \mu'|k|^{\lambda})t^{\mu})\, dk \tag{4.31} $$

## 4.4 Concrete Applications

If we set $\nu = 0$, then the Hilfer fractional derivative (1.99) reduces to a Riemann-Liouvile fractional derivative (1.87) and the Theorem 4.2.1 yields the following:

**Corollary 4.4.1.** *Consider the generalized Cauchy type problem for fractional advection dispersion equation*

$$ {}_0^{RL}\mathbb{D}_t^{\mu}(u(x,t)) = -\eta\, \mathbb{D}_x u(x,t) + \zeta \Delta^{\frac{\lambda}{2}} u(x,t),\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{4.32} $$

*subject to the initial conditions*

$$ {}_0\mathbb{D}_t^{\mu-1}(u(x,0+)) = g(x), \quad {}_0\mathbb{D}_t^{\mu-2}(u(x,0+)) = 0 \quad 1 < \mu \leq 2, x \in \mathbb{R} \tag{4.33} $$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0 \tag{4.34}$$

*Here $u$ is solute concentration, the positive constants, $\eta$ and $\delta$ represent the average fluid velocity and the dispersion coefficient respectively, $x$ is the spatial domain, $u(x,t)$ and $g(x)$ are both the (real) field variable. Then the solution of 4.32, subject to the above constraints, is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk \tag{4.35}$$

*where*

$$G(x,t) = \frac{t^{\mu-1}}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\mu,\mu}((i\eta k - \zeta|k|^{\lambda})t^{\mu})\,dk \tag{4.36}$$

When $\nu = 1$, then the Hilfer fractional space derivative (1.99) get reduced to Caputo fractional derivative operator (1.97) and it yields the following result:

**Corollary 4.4.2.** *Consider the generalized Cauchy type problem for fractional advection dispersion equation*

$${}_0^C\mathbb{D}_t^{\mu}(u(x,t)) = -\eta\mathbb{D}_x u(x,t) + \zeta\Delta^{\frac{\lambda}{2}}u(x,t), 0 < \alpha \le 2,\ x \in \mathbb{R},\ t \in \mathbb{R}^{+}, \tag{4.37}$$

*subject to the initial condition*

$$u(x,0+) = g(x),\ x \in \mathbb{R} \tag{4.38}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0, \tag{4.39}$$

*where ${}_0^C\mathbb{D}_t^{\mu}$ is the Caputo fractional derivative operator defined as (1.97) with $0 < \mu < 1$, $\Delta^{\frac{\lambda}{2}}$ is the fractional Laplace operator, defined by (1.125) of order $\lambda$, $0 < \lambda \le 2$.*

*Then the solution of equation 4.4.2, subject to the above constraints, is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{4.40}$$

*where*

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_{\mu,1}((i\eta k - \zeta|k|^{\lambda})t^{\mu})\, dk. \tag{4.41}$$

On taking $\nu = 0$, $\eta = 0$, $\lambda = 2$, $\zeta = 1$ in Theorem 4.2.1 and using (1.57), we obtain the following result [110, Eq.25]:

**Corollary 4.4.3.** *Consider the generalized Cauchy type problem for fractional heat equation*

$${}_0^{RL}\mathbb{D}_t^{\mu}(u(x,t)) = \frac{\partial^2}{\partial x^2}(u(x,t)),\; x \in \mathbb{R},\; t \in \mathbb{R}^+, \tag{4.42}$$

*subject to the initial condition*

$$\mathbb{I}_{0+}^{(1-\mu)} u(x,0+) = g(x),\; x \in \mathbb{R} \tag{4.43}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\; t > 0, \tag{4.44}$$

*where* ${}_0^{RL}\mathbb{D}_t^{\mu}$ *is the Riemann-Liouville fractional derivative operator defined by* (1.87) *with* $0 < \mu \leq 1$*, Then the solution of 4.42 is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk, \tag{4.45}$$

*where*

$$G(x,t) = \frac{t^{\mu-1}}{2|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t^{\frac{\mu}{2}}}\middle|\begin{matrix}(1,\frac{1}{2}),(\mu,\frac{\mu}{2}),(1,\frac{1}{2})\\ (1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right]. \tag{4.46}$$

On taking $\nu = 1$, $\eta = 0$, $\zeta = 1$ and $\lambda = 2$ in Theorem 4.2.1 and using (1.57), we obtain the following result:

**Corollary 4.4.4.** *Consider the generalized Cauchy type problem for fractional heat equation*

$$ {}_{0}^{C}\mathbb{D}_t^{\mu}(u(x,t)) = \mathbb{D}^2(u(x,t)),\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{4.47}$$

*subject to the initial condition*

$$u(x,0+) = g(x),\ x \in \mathbb{R} \tag{4.48}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0 \tag{4.49}$$

*where* ${}_{a}^{C}D_t{}^{\mu}$ *is the Caputo fractional derivative operator defined by* (1.97) *with* $0 < \mu \leq 1$, *Then the solution of* (4.47) *is given by*

$$u(x,t) = \int_{-\infty}^{\infty} G(x-k,t)g(k)dk \tag{4.50}$$

*where*

$$G(x,t) = \frac{1}{2|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t^{\frac{\mu}{2}}}\left|\begin{matrix}(1,\frac{1}{2}),(1,\frac{\mu}{2}),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right.\right] \tag{4.51}$$

**Part B**

## 4.5 Reaction Diffusion Equation

Reaction-diffusion equations have found many applications in applied science and engineering. In recent work, many authors have explained some significant physical issues of reaction-diffusion equations such as oscillations, stationary, spatio-temporal dissipative pattern formation, waves etc. A reaction-diffusion equation comprises a reaction term and a diffusion term, i.e. the typical form of this equation is as follows:

$$u(x,t) = k\Delta u + f(u)$$

$u(x,t)$ is a state variable and describe density or concentration of a substance or a population at position $x \in \Omega \subset \mathbb{R}$ at time t ($\Omega$ being an open set). $\Delta$ denotes the Laplace operator. The first term on the right hand side describes the diffusion, $k$ being diffusion coefficient. The second term, $f(u)$ is a smooth function $f : \mathbb{R} \to \mathbb{R}$ and describes processes which really change the present $u$, i.e. something happens to it (birth, death, chemical reaction , etc.), not just diffuse in the space. Analytical solution of generalized reaction-diffusion equation studied by *Saxena et al.* and [159], [160]. Linear fractional reaction-diffusion equation on a finite domain is solved by *Yildirim* and *Sezer* [188] using homotopy perturbation method and Yu *et al.* [189] using Adomian decomposition method. Recently, *Garg* and *Manohar* [45] obtained analytical solution of linear space-time fractional reaction-diffusion equation using generalized differential transform method.

Linear space-time fractional reaction diffusion equation on finite domain $0 < x < L$, $t > 0$ with $0 < \mu \leq 1$ and $0 < \nu \leq 2$ as discussed by *Yildirim* and *Sezer* [188] and *Yu* et al. [189]

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} = b(x)\frac{\partial^\nu u(x,t)}{\partial x^\nu} - c(x)u(x,t) + f(x,t) \tag{4.52}$$

where $\frac{\partial^\mu u(x,t)}{\partial t^\mu}$ is the Caputo time fractional derivative of order $0 < \mu \leq 1$, $\frac{\partial^\nu u(x,t)}{\partial x^\nu}$ is the Caputo Space fractional derivative of order $1 < \nu \leq 2$ and $0 < b(x) \leq b_{\max}$ and $0 < c(x) \leq c_{\max}$ are continuous for $0 < x < L$ and the function $u(x,t)$ represent source or sink and $f(x,t)$ is a sufficiently well behaved function.

## 4.6 Analytic Solution of Unified Space Time Fractional Reaction Diffusion Equation

In this section, we investigate the analytic solution of the generalized space-time fractional reaction-diffusion equation involving fractional Laplace operator contained in the following theorem:

**Theorem 4.6.1.** *Consider the generalized Cauchy type problem for unified generalized linear space-time reaction-diffusion equation*

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(x,t) = k\,\Delta^{\frac{\lambda}{2}}(u(x,t)) + c\,u(x,t) + b\,\varphi(x,t),\ t>0,\ x\in\mathbb{R}, \tag{4.53}$$

*with initial condition*

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(x,0^+)\right] = g(x) \tag{4.54}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.55}$$

*with* $\mu \in (0,1), \nu \in [0,1],\ \ \omega \in \mathbb{R},\ \ \rho > 0,\ \ \gamma \geq 0$ *and* $k > 0$ *is diffusion coefficient. Here,* $\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}$ *is the Hilfer-Prabhakar fractional derivative operator as defined in (1.103).* $\Delta^{\frac{\lambda}{2}}$ *is the fractional generalized Laplace operator of order* $\lambda$*, where* $0 < \lambda \leq 2$*,* $u(x,t)$ *represent source or sink.* $\varphi(x,t)$ *and* $g(x)$ *are both sufficiently well behaved functions and* $b$*,* $c$ *are arbitrary constants. Then the solution of Eq. (4.53),* *subject to the above constraints, is given by*

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1} E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho}) \int_{-\infty}^{\infty} (c-k|\eta|^{\lambda})^{n} e^{-i\eta x} g^{*}(\eta) d\eta$$
$$+ \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} (c-k|\eta|^{\lambda})^{n} \tau^{\mu(n+1)-1} E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\, \varphi^{*}(\eta, t-\tau) e^{-i\eta x} d\eta\, d\tau, \tag{4.56}$$

*where $g^{*}(\eta)$ and $\varphi^{*}(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$ respectively and $E^{\gamma}_{\rho,\mu}(\cdot)$ is the three parameter Mittag- Leffler function as in eq.* (1.30).

Proof: In order to prove the theorem, we take the Fourier transform of Eq. (4.53) with respect to the space variable $x$ and using boundary condition (4.55) and Eq. (1.126) therein, to obtain

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^{+}}(u^{*}(\eta,t)) = -k|\eta|^{\lambda}\,(u^{*}(\eta,t)) + c\,u^{*}(\eta,t) + b\,\varphi^{*}(\eta,t),\ t>0 \tag{4.57}$$

where $u^{*}(\eta,t)$ is the Fourier transform of the function $u(x,t)$.
Now, taking Laplace transform of (4.57) with respect to variable $t$ and making use of the Eq. (1.125), we get

$$\begin{aligned} &s^{\mu}[1-\omega s^{-\rho}]^{\gamma}\bar{u^{*}}(\eta,s) - s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^{+}} u^{*}(\eta,0^{+})\right] \\ &= -k|\eta|^{\lambda}\bar{u^{*}}(\eta,s) + c\,\bar{u^{*}}(\eta,s) + b\,\bar{\varphi^{*}}(\eta,s) \end{aligned} \tag{4.58}$$

where $L[u(\eta,t);s] = \bar{u}(\eta,s)$.
Next, taking the Fourier transform of the initial condition 4.54) and putting in (4.58), we get

$$s^{\mu}[1-\omega s^{-\rho}]^{\gamma}\bar{u^{*}}(\eta,s) - s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu} g^{*}(\eta) = -k|\eta|^{\lambda}\bar{u^{*}}(\eta,s) + c\,\bar{u^{*}}(\eta,s) + b\,\bar{\varphi^{*}}(\eta,s).$$

Simplifying,

$$\left[s^{\mu}(1-\omega s^{-\rho})^{\gamma}+k|\eta|^{\lambda}-c\right]\bar{u^{*}}(\eta,s)=s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}g^{*}(\eta)+b\,\bar{\varphi^{*}}(\eta,s),$$

which gives

$$\bar{u^{*}}(\eta,s)=\frac{s^{\nu(\mu-1)}[1-\omega s^{-\rho}]^{\gamma\nu}g^{*}(\eta)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+k|\eta|^{\lambda}-c}+\frac{b\,\bar{\varphi^{*}}(\eta,s)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+k|\eta|^{\lambda}-c}. \tag{4.59}$$

Hence,

$$\begin{aligned}\bar{u^{*}}(\eta,s)=&s^{-\mu+\nu(\mu-1)}(1-\omega s^{-\rho})^{-\gamma(1-\nu)}g^{*}(\eta)\left[1+\frac{k|\eta|^{\lambda}-c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\right]^{-1}\\&+bs^{-\mu}(1-\omega s^{-\rho})^{-\gamma}\bar{\varphi^{*}}(\eta,s)\left[1+\frac{k|\eta|^{\lambda}-c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\right]^{-1}.\end{aligned} \tag{4.60}$$

Finally,

$$\begin{aligned}\bar{u^{*}}(\eta,s)=&\sum_{n=0}^{\infty}(c-k|\eta|^{\lambda})^{n}s^{-\mu(n+1)+\nu(\mu-1)}(1-\omega s^{-\rho})^{-\gamma[(n+1)-\nu]}g^{*}(\eta)\\&+b\sum_{n=0}^{\infty}(c-k|\eta|^{\lambda})^{n}s^{-\mu n}(1-\omega s^{-\rho})^{-\gamma n}\bar{\varphi^{*}}(\eta,s),\quad\left(\left|\frac{k|\eta|^{\lambda}-c}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}\right|<1\right).\end{aligned} \tag{4.61}$$

On taking inverse Laplace transform of Eq.(4.61) and using convolution theorem, we get

$$\begin{aligned}u^{*}(\eta,t)=&\sum_{n=0}^{\infty}(c-k|\eta|^{\lambda})^{n}t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})g^{*}(\eta)\\&+b\int_{0}^{t}(c-k|\eta|^{\lambda})^{n}\tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\varphi^{*}(\eta,t-\tau)d\tau.\end{aligned} \tag{4.62}$$

Further, taking the inverse Fourier transform of (4.62), we get

$$
\begin{aligned}
u(x,t) =& \frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1} E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho}) \int_{-\infty}^{\infty} (c-k|\eta|^{\lambda})^{n} e^{-i\eta x} g^{*}(\eta) d\eta \\
&+ \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} (c-k|\eta|^{\lambda})^{n} \tau^{\mu(n+1)-1} E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\, \varphi^{*}(\eta,t-\tau) e^{-i\eta x} d\eta\, d\tau
\end{aligned}
$$

where $g^*(\eta)$ and $\varphi^*(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$, respectively.

It is interesting to observe that as an particular case of Theorem 4.6.1, we can obtain solution of homogeneous Schrödinger equation occurring in the quantum mechanics, solution of non homogeneous fractional generalized diffussion wave equation and the solution of fractional partial differential equation that arises in the study of heat transfer through diathermanous materials.

(1) If we set $\gamma = 0$ then the Hilfer-Prabhakar fractional derivative (1.103) reduces to a Hilfer fractional derivative (1.99) and we get the following result:

**Theorem 4.6.2.** *Consider the generalized Cauchy type problem for fractional linear space-time reaction-diffusion equation*

$$
\mathbb{D}_t^{\mu,\nu} u(x,t) = k\,\Delta^{\frac{\lambda}{2}}(u(x,t)) + c\,u(x,t) + b\,\varphi(x,t),\ t>0,\ x\in\mathbb{R}, \tag{4.63}
$$

*with initial condition*

$$
\mathbb{I}_{0+}^{(1-\nu)(1-\mu)} u(x,0^+) = \left[\mathbb{P}^{0}_{\rho,(1-\nu)(1-\mu),\omega,0^+} u(x,0^+)\right] = g(x),\ \ x\in\mathbb{R} \tag{4.64}
$$

*and boundary condition*

$$
\lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.65}
$$

*with* $\mu\in(0,1),\ \ \nu\in[0,1],\ \ \omega\in\mathbb{R},\ \ \rho>0,\ \ 0<\lambda\le 2$. *Then, the solution of (4.63)* *is given by*

$$
\begin{aligned}
u(x,t) = & \frac{t^{(\mu-1)(1-\nu)}}{2\pi} \int_{-\infty}^{\infty} g^*(\eta) E_{\mu,\mu+\nu(1-\nu)}(c-k|\eta|^{\lambda}) t^{\mu} e^{-i\eta x} d\eta \\
& + \frac{b}{2\pi} \int_{-\infty}^{\infty} \int_0^t \xi^{\mu-1} E_{\mu,\mu}(c-k|\eta|^{\lambda}) t^{\mu} \varphi^*(\eta, t-\xi) d\xi d\eta
\end{aligned}
\tag{4.66}
$$

*where $g^*(\eta)$ and $\varphi^*(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$, respectively and $E_{\rho,\mu}(\cdot)$ is the two parameter Mittag-Leffler function.*

Proof: In order to prove the theorem, we take the Fourier transform of Eq. (4.53) with respect to the space variable $x$ and using boundary condition (4.65) and Eq. (1.126) therein, to obtain

$$
\mathbb{D}_t^{\mu,\nu}(u^*(\eta,t)) = -k|\eta|^{\lambda}\ (u^*(\eta,t)) + c\, u^*(\eta,t) + b\, \varphi^*(\eta,t),\ t > 0 \tag{4.67}
$$

where $u^*(\eta,t)$ is the Fourier transform of the function $u(x,t)$.

Now, taking Laplace transform of (4.57) with respect to variable $t$ and making use of the Eq. (1.100), we get

$$
s^{\mu}\ \bar{u^*}(\eta,s) - s^{\nu(\mu-1)} I_{0+}^{(1-\nu)(1-\mu)} u(\eta,0+) = -k|\eta|^{\lambda} \bar{u^*}(\eta,s) + c\, \bar{u^*}(\eta,s) + b\, \bar{\varphi^*}(\eta,s) \tag{4.68}
$$

where $L[u(\eta,t);s] = \bar{u}(\eta,s)$.

Next, taking the Fourier transform of the initial condition (4.64) and putting in (4.68), we get

$$
s^{\mu}\ \bar{u^*}(\eta,s) - s^{\nu(\mu-1)} u(\eta,0+) g^*(\eta) = -k|\eta|^{\lambda} \bar{u^*}(\eta,s) + c\, \bar{u^*}(\eta,s) + b\, \bar{\varphi^*}(\eta,s)
$$

Simplifying,

$$
\left[s^{\mu} + k|\eta|^{\lambda} - c\right] \bar{u^*}(\eta,s) = s^{\nu(\mu-1)} g^*(\eta) + b\, \bar{\varphi^*}(\eta,s),
$$

which gives

$$\bar{u^*}(\eta, s) = \frac{s^{\nu(\mu-1)} g^*(\eta)}{s^\mu + k|\eta|^\lambda - c} + \frac{b\, \bar{\varphi^*}(\eta, s)}{s^\mu + k|\eta|^\lambda - c}. \tag{4.69}$$

On taking inverse Laplace transform of equation (4.69), by means of the following result by Haubold et al. ([110, Eq.18]) and using convolution theorem,

$$L^{-1}\left\{\frac{s^{\beta-1}}{s^\alpha + a}\right\} = t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}(-at^\alpha) \tag{4.70}$$

where $\Re(s) > 0, \Re(\alpha) > 0, \Re(\alpha - \beta) > -1$, we obtain

$$\begin{aligned} u^*(\eta, s) =& t^{(\mu-1)(1-\nu)} E_{\mu,\mu+\nu(1-\nu)}(c - k|\eta|^\lambda) t^\mu g^*(\eta) \\ &+ \int_0^t \xi^{\mu-1} E_{\mu,\mu}(c - k|\eta|^\lambda) t^\mu \varphi^*(\eta, t-\xi) d\xi \end{aligned} \tag{4.71}$$

Further, taking the inverse Fourier transform of (4.62), we get

$$\begin{aligned} u(x,t) =& \frac{t^{(\mu-1)(1-\nu)}}{2\pi} \int_{-\infty}^{\infty} g^*(\eta) E_{\mu,\mu+\nu(1-\nu)}(c - k|\eta|^\lambda) t^\mu e^{-i\eta x} d\eta \\ &+ \frac{b}{2\pi} \int_{-\infty}^{\infty} \int_0^t \xi^{\mu-1} E_{\mu,\mu}(c - k|\eta|^\lambda) t^\mu \varphi^*(\eta, t-\xi) d\xi d\eta. \end{aligned}$$

(2) Further, on taking, $c = 0$ and $k = \frac{ih}{2m}$, the above result yields the solution of the non-homogeneous fractional generalized Schrödinger equation considered in Corollary 3.1 by *Purohit* [141].

**Corollary 4.6.3.** *Consider the following one dimensional non-homogeneous generalized fractional Schrödinger equation of a particle of mass m, defined by*

$$\mathbb{D}_t^{\mu,\nu} u(x,t) = \left(\frac{ih}{2m}\right) \Delta^{\frac{\lambda}{2}} u(x,t) + b\, \varphi(x,t), \quad t > 0,\ 0 < \lambda \leq 2\ \ x \in \mathbb{R}, \tag{4.72}$$

*with initial condition*

$$\mathbb{I}_{0^+}^{(1-\nu)(1-\mu)} u(x, 0+) = g(x), \quad -\infty < x < \infty, \quad 0 < \mu < 1, \quad 0 \leq \nu \leq 1 \tag{4.73}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0, \tag{4.74}$$

*where $b$ is arbitrary, $h = 2\pi\hbar$ is the Plank constant and $g(x)$ and $\varphi(x,t)$ are given functions.*

Then, the solution of (4.72), under the given conditions, is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)g(\xi)\,d\xi + b\int_0^t (t-\tau)\left[\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)\varphi(\xi,\tau)d\xi\right]d\tau, \tag{4.75}$$

where the Green's function $G_1(x,t)$ is given by

$$G_1(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\lambda|x|} H_{3,3}^{2,1}\left[\frac{|x|}{a^{\frac{1}{\lambda}}t^{\frac{\mu}{\lambda}}}\left|\begin{matrix}(1,\frac{1}{\lambda}),(\mu+\nu(1-\mu),\frac{\mu}{\lambda}),(1,\frac{1}{2})\\(1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2})\end{matrix}\right.\right] \tag{4.76}$$

and the function $G_2(x,t)$ is given by

$$G_2(x,t) = \frac{1}{\lambda|x|} H_{3,3}^{2,1}\left[\frac{|x|}{a^{\frac{1}{\lambda}}t^{\frac{\mu}{\lambda}}}\left|\begin{matrix}(1,\frac{1}{\lambda}),(\mu,\frac{\mu}{\lambda}),(1,\frac{1}{2})\\(1,\frac{1}{\lambda}),(1,1),(1,\frac{1}{2})\end{matrix}\right.\right] \tag{4.77}$$

where $a = \frac{ih}{2m}$ and $H_{p,q}^{m,n}$ is well known H-function defined by (see, e.g. *Mathai* et al. [110, Eq. Chapter 1]).

(3) On taking $c = 0$ and $k = \psi^2$, in Eq. (4.63) we get the solution of non-homogeneous fractional generalized diffusion wave equation considered in Corollary 3.2 by *Purohit* [141].

**Corollary 4.6.4.** *Consider the following one dimensional non-homogeneous generalized fractional diffusion wave equation, defined by*

$$\mathbb{D}_t^{\mu,\nu} u(x,t) = \psi^2 \Delta^{\frac{\lambda}{2}} u(x,t) + b\, \varphi(x,t),\ t > 0,\ 0 < \lambda \leq 2,\ \ x \in \mathbb{R}, \tag{4.78}$$

*with initial condition*

$$\mathbb{I}_{0+}^{(1-\nu)(1-\mu)} u(x,0^+) = g(x), \quad -\infty < x < \infty, \quad 0 < \mu < 1, \quad 0 \leq \nu \leq 1 \tag{4.79}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0 \tag{4.80}$$

*where $b$ is arbitrary constant and $g(x)$ and $\varphi(x,t)$ are given functions.*

Then, the solution of (4.78) under the given conditions, is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t) g(\xi)\ d\xi + b \int_0^t (t-\tau) \left[ \int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)\varphi(\xi,\tau) d\xi \right] d\tau, \tag{4.81}$$

where the Green's function $G_1(x,t)$ and $G_2(x,t)$ are, respectively, given by (4.76) and (4.77) with $a = \psi^2$.

(4) On taking $b = 0$, $c = 0$ and $\lambda = 2$ in Theorem 4.6.1, we arrive at the following result by *Garra et al.* [46, Theorem 5.1]:

**Corollary 4.6.5.** *Consider the Cauchy problem*

$$\mathbb{D}_{\rho,\omega,0+}^{\gamma,\mu,\nu} u(x,t) = k\, \frac{\partial^2}{\partial x^2} u(x,t), \ \ t > 0,\ x \in \mathbb{R} \tag{4.82}$$

*with initial condition*

$$\left[ \mathbb{P}_{\rho,(1-\nu),(1-\mu),\omega,0+}^{-\gamma(1-\nu)} u(x,0^+) \right] = g(x) \tag{4.83}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0, \qquad t > 0 \tag{4.84}$$

*with* $\mu \in (0,1), \nu \in [0,1], \omega \in \mathbb{R}, \quad \rho > 0, \quad \gamma \geq 0.$

Then, the solution of equation (4.82) is given by

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-k)^n t^{\mu(n+1)-\nu(\mu-1)-1} E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho}) \int_{-\infty}^{\infty} \eta^{2n}\cos\eta x \; g^*(\eta)d\eta, \tag{4.85}$$

where $g^*(\eta)$ is the Fourier transform of the function $g(x)$.

(5) Further, if we take $\gamma = 0$, $c = 0$, $k = \alpha$, $b = \beta$ and $\varphi(x,t) = e^{-\tau x}$, Theorem 4.6.2 yields the solution of fractional partial differential equation arising in the study of heat transfer through diathermanous materials considered by *Kachhia* and *Prajapati* [74].

**Corollary 4.6.6.** *Consider the fractional partial differential equation that arise in the study of heat transfer through diathermanous materials as*

$$\mathbb{D}_t^{\mu,\nu} u(x,t) = \alpha \Delta^{\frac{\lambda}{2}} u(x,t) + \beta e^{-\tau x}, \quad 0 < \lambda \leq 2 \tag{4.86}$$

*with initial condition*

$$\mathbb{I}_{0+}^{(1-\nu)(1-\mu)} u(x, 0^+) = 0, \tag{4.87}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0, \; t > 0. \tag{4.88}$$

with $\mu \in (0,1), \nu \in [0,1], \alpha > 0.$

Then, the solution of (4.86) under the given conditions, is given by

$$u(x,t) = \frac{\beta t^{\mu} e^{-\tau x}}{\lambda} \int_{-\infty}^{\infty} \frac{e^{\tau\mu}}{|\xi|} H^{2,1}_{3,3}\left[ \frac{|\xi|}{\alpha^{\frac{1}{\tau}} t^{\frac{\mu}{\lambda}}} \middle| \begin{matrix} (1,\frac{1}{\lambda}), (\mu+1, \frac{\mu}{\lambda}), (1, \frac{1}{2}) \\ (1,\frac{1}{\lambda}), (1,1), (1,\frac{1}{2}) \end{matrix} \right] d\xi \tag{4.89}$$

## 4.7 Illustrative Examples

**Example 4.7.1.** *Consider the generalized Cauchy type problem for unified generalized linear space-time reaction-diffusion equation*

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(x,t) = k\,\Delta^{\frac{\lambda}{2}}(u(x,t)) + c\,u(x,t) + b\,\varphi(x,t),\ t>0,\ x\in\mathbb{R}, \tag{4.90}$$

*with initial condition*

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(x,0^+)\right] = e^{-x} \tag{4.91}$$

*and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.92}$$

*with* $\mu\in(0,1), \nu\in[0,1], \omega, k\in\mathbb{R}, k, \rho>0, \gamma\geq 0,\ 0<\lambda\leq 2.$

*In view of Theorem 4.6.1, the solution of equation (4.90) is given by*

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})\int_{-\infty}^{\infty}(c-k|\eta|^{\lambda})^n e^{-i\eta x}G(\eta)d\eta \\ &+\frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t (c-k|\eta|^{\lambda})^n\tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\ \varphi^*(\eta,t-\tau)e^{-i\eta x}d\eta\ d\tau \end{aligned} \tag{4.93}$$

*where* $\varphi^*(\eta,t)$ *is Fourier transform of the functions* $\varphi(x,t)$ *and*
$G(\eta) = F\left\{e^{-x};\eta\right\} = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1+i\eta)}-1}{1+i\eta}\right].$

Next, we take an example where, in the initial condition, we put $g(x)=\delta(x)$, the Dirac delta function.

**Example 4.7.2.** *Consider the generalized Cauchy type problem for unified generalized linear space-time reaction-diffusion equation*

$$\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u(x,t) = k\,\Delta^{\frac{\lambda}{2}}(u(x,t)) + c\,u(x,t) + b\,\varphi(x,t), \quad t > 0,\ x \in \mathbb{R}, \tag{4.94}$$

*with initial condition*

$$\left[\mathbb{P}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+}u(x,0^+)\right] = \delta(x), \tag{4.95}$$

*where $\delta(x)$ is the Dirac delta function and boundary condition*

$$\lim_{|x|\to\infty} u(x,t) = 0,\ t > 0 \tag{4.96}$$

*with $\mu \in (0,1), \nu \in [0,1],\ \ \omega \in \mathbb{R},\ \ k,\rho > 0,\ \gamma \geq 0,\ 0 < \lambda \leq 2.$*
*In view of Theorem 4.6.1, the solution of equation (4.94), is given by*

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\sum_{n=0}^{\infty} t^{\mu(n+1)-\nu(\mu-1)-1}E^{\gamma(n+1-\nu)}_{\rho,\mu(n+1)-\nu(\mu-1)}(\omega t^{\rho})\int_{-\infty}^{\infty}(c-k|\eta|^{\lambda})^n e^{-i\eta x}d\eta \\ &+\frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t (c-k|\eta|^{\lambda})^n \tau^{\mu(n+1)-1}E^{\gamma(n+1)}_{\rho,\mu(n+1)}(\omega\tau^{\rho})\ \varphi^*(\eta,t-\tau)e^{-i\eta x}d\eta\, d\tau \end{aligned} \tag{4.97}$$

*where $\varphi^*(\eta,t)$ is Fourier transform of the function $\varphi(x,t)$ and $F\{\delta(x);\eta\} = 1$.*

## 4.8 Concrete Applications

When $\gamma = 0,\ \nu = 1$, the Hilfer-Prabhakar fractional space derivative (1.103) get reduced to Caputo fractional derivative ${}^{C}_{0}\mathbb{D}^{\mu}_{t}$ defined in (1.97) and it yields the following result:

**Corollary 4.8.1.** *Consider the generalized Cauchy type problem for fractional linear* *space-time reaction-diffusion equation*

$$ {}_0^C\mathbb{D}_t^{\mu} u(x,t) = k\, \Delta^{\frac{\lambda}{2}}(u(x,t)) + c\, u(x,t) + b\, \varphi(x,t),\ t>0,\ x\in\mathbb{R}, \tag{4.98} $$

*with initial condition*

$$ \mathbb{I}_t^{(1-\mu)} u(x,0^+) = g(x), \tag{4.99} $$

*and boundary condition*

$$ \lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.100} $$

*with* $\mu \in (0,1)$, $0<\lambda\leq 2$. *Then the solution of equation (4.98), is given by*

$$ \begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(\eta) E_{\mu,0}(c-k|\eta|^{\lambda}) t^{\mu} e^{-i\eta x} d\eta \\ &+ \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t \xi^{\mu-1} E_{\mu,\mu}(c-k|\eta|^{\lambda}) t^{\mu} \varphi^*(\eta, t-\xi) d\xi d\eta. \end{aligned} $$

*where* $g^*(\eta)$ *and* $\varphi^*(\eta,t)$ *are Fourier transform of the functions* $g(x)$ *and* $\varphi(x,t)$ *respectively and* $E_{\rho,\mu}(\cdot)$ *is the two parameter Mittag- Leffler function.*

On taking $\gamma = 0$, $\nu = 0$, the Hilfer-Prabhakar fractional derivative (1.103) reduces to a Riemann-Liouville fractional derivative ${}_0^{RL}\mathbb{D}_t^{\mu}$ as defined by (1.87) and the Theorem 4.6.2 yields the following corollary:

**Corollary 4.8.2.** *Consider the generalized Cauchy type problem for fractional linear space-time reaction-diffusion equation*

$$ {}_0^{RL}\mathbb{D}_t^{\mu} u(x,t) = k\, \Delta^{\frac{\lambda}{2}}(u(x,t)) + c\, u(x,t) + b\, \varphi(x,t),\ t>0,\ x\in\mathbb{R}, \tag{4.101} $$

*with initial condition*

$$ \mathbb{I}_t^{(1-\mu)} u(x,0^+) = g(x), \tag{4.102} $$

*and boundary condition*

$$ \lim_{|x|\to\infty} u(x,t) = 0,\ t>0 \tag{4.103} $$

*with* $\mu \in (0,1), 0 < \lambda \leq 2.$

Then, the solution of equation (4.101) is given by

$$\begin{aligned} u(x,t) = & \frac{t^{(\mu-1)}}{2\pi}\int_{-\infty}^{\infty} g^*(\eta)E_{\mu,\mu}(c-k|\eta|^{\lambda})t^{\mu}e^{-i\eta x}d\eta \\ & + \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_0^t \xi^{\mu-1}E_{\mu,\mu}(c-k|\eta|^{\lambda})t^{\mu}\varphi^*(\eta,t-\xi)d\xi d\eta. \end{aligned} \tag{4.104}$$

where $g^*(\eta)$ and $\varphi^*(\eta,t)$ are Fourier transforms of the functions $g(x)$ and $\varphi(x,t)$, respectively and $E_{\rho,\mu}(\cdot)$ is the two parameter Mittag- Leffler function.

## 4.9 Conclusion

The solution of a unified generalized linear space-time fractional reaction-diffusion equation involving Hilfer-Prabhakar time fractional derivative and the space fractional generalized Laplace operators is obtained in terms of Mittag-Leffler function by using Laplace transform and Fourier transform and the solution of time-space fractional advection-dispersion equation is obtained in terms of Mittag-Leffler function and H-function by using Laplace transform and Fourier transform. Usually, this method is very useful to study various problems arising in fluid dynamics, control theory, aerodynamics and applied sciences The analytic solutions are the exact solutions. Efficient numerical techniques can be developed to find solution of fractional PDE by considering these analytic solutions as base.

# 5

# Solution of Fractional Partial Differential Equation using Integral Transform of Pathway Type

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The main findings of this chapter have been published as detailed below:

1. **Agarwal R., Jain S. and Agarwal R.P.**, Solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type, *Prog. Fract. Differ. Appl.*, **1**(5), (2015), 145-155.

Motivated by the work of *Kumar* [87], in the present chapter we find the $P_\alpha$-transform of Caputo fractional derivatives and derive $P_\alpha$-transform for Volterra and Abel integral equation. Further, in Section 3 we find the solution of fractional Volterra integral equation. We discuss its application for solving singular integral equation having Bessel function in its kernel. The solution of non homogeneous time fractional heat equation in a spherical domain has been discussed.

## 5.1 Introduction

The subject of fractional calculus deals with the investigation of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notations of fractional order derivative and n-fold integral. Fractional calculus is now considered as a partial technique in many branches of science including physics (*Oldham* and *Spanier* [124]). Recently *Srivastava* et al. [175] gave the model of under-actuated mechanical system with fractional order derivative and *Sharma* et al. [40] studied advanced generalized fractional kinetic equation in Astrophysics.

In an integral equation, an unknown function to be determined, appears under one and more integral signs. The integral equation has been a subject of interest of mathematicians as well as physicists and engineers also. The development of integral equation has led to the formation of many real world engineering and physical problems and also in mathematical physics models, such as scattering in quantum mechanism, diffraction problem, conformal mapping and water waves. A large number of initial and boundary value problems can be converted into Volterra integral equation. The Volterra's population growth model, biological species living together, the heat transformation and heat radiation are many areas which are described by integral equations. Many scientific problems give rise to integral equations often arises in low frequency electromagnetic problems, electrostatic, electromagnetic scattering problems and elastic waves and many more (see, e.g. [151, 182]). The fractional order integral equations has numerous applications in porous media, rheology, control, electro chemistry, viscoelasticity, electromagnetism fluid structure, coupling and particle mechanics (see, e.g. [51], [115], [124], [177]).

The general form of integral equation (*Wazwaz* [182]) is given by

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{5.1}$$

where $g(x)$ and $h(x)$ are the limits of integration, $\lambda$ is a constant parameter, and $K(x,t)$ is called the *kernel* or the *nucleus* of the integral equation. The function $u(x)$ to be determined appears under the integral sign. The kernel $K(x,t)$ and the function $f(x)$ in equation (5.1) are given and the limits of integration $g(x)$and $h(x)$ may be both variables, constant or mixed.

The general form of Volterra integral equations (*Rahman* [145]) is

$$u(x) = f(x) + \lambda \int_{a}^{x} K(x,t)u(t)dt, \tag{5.2}$$

where the limits of integration are functions of $x$ and the unknown function $u(x)$ appears linearly under the integral sign.

Abel's integral equation (see, e.g. *Gorenflo* and *Vessela* [52], *Kilbas* and *Saigo* [78]) is given by

$$f(t) = \frac{\lambda}{\Gamma(\mu)} \int_0^t \frac{u(\tau)}{(t-\tau)^{\mu}} d\tau, \ 0 < \mu < 1 \tag{5.3}$$

## 5.2 Main Results

**Theorem 5.2.1.** *If Caputo fractional derivatives of function $f(t)$ of order $\nu$ exist and are $P_\alpha$- transformable and if $P_\alpha[f(t);s] = F(s)$, then for $\alpha > 1$, we have*

$$P_\alpha \left[ {}_0^C\mathbb{D}_t^\nu f(t); s \right] = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\nu} F(s) - \sum_{k=o}^{n-1} \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\nu-k-1} f^{(k)}(0) \tag{5.4}$$

*where* $n-1<\nu\leq n$.

*Proof*: Using the fact that ${}_0^C\mathbb{D}_t^{\nu}f(t)={}_0\mathbb{D}_t^{\nu-n}(f^{(n)}(t))={}_0\mathbb{D}_t^{-(n-\nu)}(f^{(n)}(t)),\quad n\geq\nu>n-1$, Lemma (1.2.3) gives,

$$P_\alpha\left[{}_0^C\mathbb{D}_t^{\nu}f(t);s\right]=\left\{\frac{\alpha-1}{\ln[1+(\alpha-1)s]}\right\}^{n-\nu}P_\alpha(f^{(n)}(t)).\tag{5.5}$$

So

$$P_\alpha\left[{}_0^C\mathbb{D}_t^{\nu}f(t);s\right]=\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu-n}P_\alpha(f^{(n)}(t))\tag{5.6}$$

Applying Theorem 1.2.4, we get

$$\begin{aligned}P_\alpha\left[{}_0^C\mathbb{D}_t^{\nu}f(t);s\right]=\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu-n}&\left[\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{n}F(s)\right.\\&\left.-\sum_{k=0}^{n-1}\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{n-k-1}f^{(k)}(0)\right]\end{aligned}$$

Finally ,

$$P_\alpha\left[{}_0^C\mathbb{D}_t^{\nu}f(t);s\right]=\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}F(s)-\sum_{k=o}^{n-1}\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu-k-1}f^{(k)}(0).\tag{5.7}$$

**Theorem 5.2.2.** *The solution of Volterra integral equation (5.2) using $P_\alpha$-transform is given by* $P_\alpha^{-1}\left\{\frac{1}{1-\lambda P_\alpha K(x)}\right\}=\psi(x)$, *where* $P_\alpha K(x)\neq\frac{1}{\lambda}$, $\alpha>1$

*Proof*: Apply $P_\alpha$- transform on both side of (5.2) and using Theorem 1.2.2, we obtain

$$P_\alpha\left\{u(x)\right\}=P_\alpha\left\{f(x)\right\}+\lambda P_\alpha\left\{K(x)\right\}P_\alpha\left\{u(x)\right\}\tag{5.8}$$

Let the $P_\alpha$- transform of $u(x)$ and $K(x-t)$ be $U(s)$ and $K(s)$, respectively, then by Theorem (1.2.2),

$$U(s)=F(s)+\lambda K(s)U(s)\tag{5.9}$$

Hence

$$U(s) = \frac{F(s)}{1-\lambda(K(s))}; K(s) \neq \frac{1}{\lambda} \tag{5.10}$$

and inverse transform gives

$$u(x) = \int_0^x \psi(x-t) f(t) dt \tag{5.11}$$

where it is assumed that $P_\alpha^{-1}\left\{\frac{1}{1-\lambda P_\alpha K(x)}\right\} = \psi(x)$.

The expression (5.11) is the solution of second kind Volterra integral equation of convolution type.

**Theorem 5.2.3.** *For $\alpha > 1$ and $0 < \mu < 1$, then the solution of the Abel integral equation (5.3) is given by*

$$u(t) = \frac{sin\pi\mu}{\pi} \int_0^t (t-\tau)^{\mu-1} G(\tau)\, d\tau, \tag{5.12}$$

where $G(t) = P_\alpha^{-1}\left\{F(s)\left(\frac{\alpha-1}{\ln[1+(\alpha-1)s]}\right)\right\}$.

Proof: The Abel integral equation is given by

$$f(t) = \int_0^t \frac{u(\tau)}{(t-\tau)^\mu} d\tau, t > 0 \tag{5.13}$$

Applying the $P_\alpha$-transform on both side of equation (5.13) and using Theorem 5.2.1, we get

$$P_\alpha\{f(t)\} = P_\alpha\{u(t)\} P_\alpha\{t^{-\mu}\} \tag{5.14}$$

If we take $P_\alpha\{f(t)\} = F(s)$, $P_\alpha\{u(t)\} = U(s)$ and using formula of $P_\alpha$-transform for power function given in (1.82), we get

$$F(s) = U(s)\Gamma(1-\mu)\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\mu-1} \tag{5.15}$$

which leads to

$$U(s) = \frac{F(s)}{\Gamma(1-\mu)\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\mu-1}} = \frac{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}F(s)}{\Gamma(1-\mu)\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\mu}} \tag{5.16}$$

Using duplication formula for Gamma function (*Rainville* [146, p.24, Eq.2]), we get

$$P_\alpha^{-1}\left\{\frac{1}{\Gamma(1-\mu)\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\mu}};t\right\} = \frac{t^{(\mu-1)}}{\Gamma(1-\mu)\Gamma(\mu)} = \frac{sin\pi\mu}{\pi}t^{\mu-1} \tag{5.17}$$

Finally, application of Theorem 1.2.2 gives

$$u(t) = \frac{sin\pi\mu}{\pi}\int_0^t (t-\tau)^{\mu-1}G(\tau)\,d\tau, \tag{5.18}$$

where $G(t) = P_\alpha^{-1}\left\{F(s)\left(\frac{\alpha-1}{\ln[1+(\alpha-1)s]}\right)\right\}$.

## 5.3 Solution of Fractional Volterra Integral Equation by Using $P_\alpha$ - Transform

**Theorem 5.3.1.** *Consider fractional Volterra singular integral equation of the form*

$${}_0^C\mathbb{D}_t^\nu f(x) = g(x) + \lambda\int_x^{+\infty} K(x-t)f(t)dt,\ f(0)=0, \tag{5.19}$$

*in which $K(x,t) = K(x-t)$ is the kernel, $g(x)$ satisfies all conditions of Lemma (1.2.5) and $0 < \nu \leq 1$, then (5.19) has solution of the form*

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{\frac{G(s)}{\lambda K(-s) - \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}}[1+(\alpha-1)s]^{\frac{t}{\alpha-1}}\right\}ds. \tag{5.20}$$

*Proof*: Apply $P_\alpha$ -transform on both sides of Eq. (5.19) denote $P_\alpha[f(x)] = F(s)$, $P_\alpha[g(x)] = G(s)$. Let $K(-s)$ be the $P_\alpha$-transform of $K(x)$. Then by using Theorem

1.2.2, we obtain

$$\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu} F(s) = G(s) + \lambda K(-s)F(s) \tag{5.21}$$

which gives,

$$F(s) = \frac{-G(s)}{\lambda K(-s) - \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}}, \tag{5.22}$$

and consequently by taking the inverse Bromwich's integral we get the following relation

$$f(x) = \frac{1}{2\pi\imath}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{\frac{G(s)}{\lambda K(-s) - \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}}[1+(\alpha-1)s]^{\frac{t}{\alpha-1}}\right\} ds \tag{5.23}$$

which can be solved further by the use of Residue theorem (see Brown and *Churchill* [13]).

Here, we illustrate the application of the above theorem in finding solutions of some singular integral equations:

(i) Consider singular integral equation having Bessel function $J_0(2\sqrt{(x-t)}$ as its kernel

$${}_0^C\mathbb{D}_t^{\nu} f(x) = e^{-ax} + \lambda\int_x^{+\infty} J_0(2\sqrt{(x-t)})f(t)dt,\ f(0)=0,\ 0<\nu\le 1 \tag{5.24}$$

In view of (5.23), one can obtain solution of (5.24) as

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\} e^{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}x}}{\left\{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}+a)(\lambda e^{\frac{1}{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}}} + \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu+1}\right\}} ds \tag{5.25}$$

(a) By setting $\alpha\to 1$ in Eq. (5.25), we obtain the corresponding results for the classical Laplace transform as follows:

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{se^{sx}}{(s+a)(\lambda e^{\frac{1}{s}}+s^{\nu+1})}ds \tag{5.26}$$

(ii) Taking $\nu = 0.5$ in Eq. (5.25), we obtain an interesting result:

Solution of integral equation

$${}_0^C\mathbb{D}_t^{0.5}f(x) = e^{-ax} + \lambda\int_x^{+\infty} J_0(2\sqrt{(x-t)}f(t)dt,\ f(0)=0,\ 0<\nu\leq 1 \tag{5.27}$$

is given by

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-\infty}^{\gamma+i\infty}\frac{e^{Ax}}{\sqrt{A}(A+a)(\lambda e^{\frac{1}{A}}A^{-\frac{3}{2}}+1)}ds \tag{5.28}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$

Proof: We apply the $P_\alpha$-transform of convolution of function and using the fact that

$$P_\alpha^{-1}\left\{\frac{1}{\sqrt{A}(A+a)}\right\} = \int_0^x \frac{e^{a(\eta-x)}}{\sqrt{\pi x}}dx, \tag{5.29}$$

and also the following relationship

$$\begin{aligned}
P_\alpha^{-1}\left\{\frac{1}{1+\lambda e^{\frac{1}{A}}A^{-\frac{3}{2}}}\right\} &= P_\alpha^{-1}\left\{1-\left(\lambda e^{\frac{1}{A}}A^{-\frac{3}{2}}\right)+\left(\lambda e^{\frac{1}{A}}A^{-\frac{3}{2}}\right)^2-...\right\}\\
&= P_\alpha^{-1}\left\{1+\sum_{k=1}^{\infty}(-1)^k\lambda^k e^{\frac{k}{A}}A^{-}\frac{3k}{2}\right\}\\
&= \delta(x)+\sum_{k=1}^{\infty}(-1)^k\lambda^k\left(\frac{x}{k}\right)^{\frac{3k-2}{4}} I_{\frac{3k-2}{2}}(2\sqrt{kx})
\end{aligned} \tag{5.30}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$. From equations (5.29) and (5.30), one gets the formal solution of equation (5.27) as follows:

$$f(x) = \left\{ \int_0^x \frac{e^{a(\eta - x)}}{\sqrt{\pi x}} d\eta \right\} * \left\{ \delta(x) + \sum_{k=1}^{\infty} (-1)^k \lambda^k \left(\frac{x}{k}\right)^{\frac{3k-2}{4}} I_{\frac{3k-2}{2}}(2\sqrt{kx}) \right\}.$$

(iii) The solution of the following system of fractional singular integral equations of the form,

$$\begin{aligned} {}_0^C\mathbb{D}_t^\nu \phi(x) &= g(x) - \lambda \int_x^{+\infty} k(x-t)\psi(t)dt \\ {}_0^C\mathbb{D}_t^\nu (x)\psi(x) &= h(x) + \lambda \int_x^{+\infty} k(x-t)\phi(t)dt, \end{aligned} \tag{5.31}$$

with conditions $\phi(0) = 0$, $\psi(0) = 0$ and $0 \le \nu \le 1$, is given by

$$\begin{aligned} \Phi(x) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{A^\nu G(s) + \lambda K(-s) H(s)}{\lambda^2 (K(-s))^2 + A^{2\nu}} e^{Ax} ds, \\ \psi(x) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{A^\nu H(s) + \lambda K(-s) G(s)}{\lambda^2 (K(-s))^2 + A^{2\nu}} e^{Ax} ds. \end{aligned} \tag{5.32}$$

where $A = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha - 1} \right\}$

*Proof*: Multiplying second equation of (5.31) by $i$ and adding to the first equation leads to

$${}_0^C\mathbb{D}_t^\nu (\phi + i\psi) = (g + ih)(x) + i\lambda \int_x^{\infty} k(x-t)(\phi + i\psi)(t)dt. \tag{5.33}$$

Now let $(\phi + i\psi)(x) = \zeta(x)$, $(g + ih)(x) = f(x)$, $i\lambda = \xi$, then we can rewrite the above equation in the form

$${}_0^C\mathbb{D}_t^\nu \zeta(x) = f(x) + \xi \int_x^{\infty} k(x-t)\zeta(t)dt. \tag{5.34}$$

In view of (5.23), one can obtain solution of (5.34) as below:

Taking $p_\alpha$ transform of equation (5.34) leads to

$$\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}\Phi(s) = F(s) + \xi K(-s)\Phi(s) \tag{5.35}$$

where $\Phi(s), F(s), K(s)$ are $P_\alpha$-transform of the functions $\zeta(x), f(x), k(x)$, respectively.
Hence we get the following relationship

$$\Phi(s) = \frac{A^{\nu}G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} + i\frac{A^{\nu}H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} \tag{5.36}$$

$G(s), H(s)$ being $P_\alpha$ - transform, of $g(x), h(x)$, respectively. So we get

$$\tilde{\phi}(s) = \frac{A^{\nu}G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}}, \tilde{\psi}(s) = \frac{A^{\nu}H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}}$$

Finally, applying the complex inversion formula, the solution of (5.31) is obtained as

$$\begin{aligned}\Phi(x) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{A^{\nu}G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}}e^{Ax}ds,\\ \psi(x) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{A^{\nu}H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}}e^{Ax}ds\end{aligned} \tag{5.37}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$.
(iv) Solution of the fractional Volterra singular integral equation of the form,

$${}_0^C\mathbb{D}_t^{\nu}\ln(x-t)\phi(t)dt, \;\; \phi(0) = 0, \;\; 0 \le \nu \le 1, \tag{5.38}$$

is given by

$$\phi(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{AF(s)e^{Ax}}{A^{\nu+1} + \lambda(\xi + \ln A)}ds \tag{5.39}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$.

Proof: After taking $P_\alpha$ - transform of above integral equation (5.38) and simplifying, one gets

$$P_\alpha[\Phi(x); s] = \frac{sF(s)}{s^{A+1} + \lambda(\xi + \ln A)} \tag{5.40}$$

in which $\xi \approx 0.577$ is Euler constant. Applying complex inversion formula to the above relation leads to

$$\phi(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{AF(s)e^{Ax}}{A^{\nu+1} + \lambda(\xi + \ln A)} ds \tag{5.41}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$.

## 5.4 Non-Homogeneous Time Fractional Heat Equation in a Spherical Domain

**Theorem 5.4.1.** *Let $f(t)$ be $P_\alpha$- transformable function. For $0 \le r < 1, t > 0, 0 < \nu \le 1$, the solution of the non-homogeneous time fractional heat equation*

$$ {}_0^C\mathbb{D}_t^\nu u(r,t) = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r,t)}{\partial r} - \lambda u(r,t) - f(t), \quad t > 0 \tag{5.42}$$

*satisfying the boundary conditions $\lim_{r\to 0} |u(r,t)| < \infty$, $u_r(1,t) = 1$ and the initial conditions $u(r,0) = 0$, f(0)=0,*

*is given by*

$$u(r,t) = \frac{1}{r}\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \left( \frac{\left(\sinh r\sqrt{\lambda + A^\nu}\right)}{A\left(\lambda + A\cosh\left(\sqrt{\lambda + A^\nu}\right) - \sinh\left(\sqrt{\lambda + A^\nu}\right)\right)} - \frac{F(s)}{\lambda + A^\nu}\right) e^{st} ds. \tag{5.43}$$

Proof. Let us define $\nu(r,t) = ru(r,t)$. Then equation (5.42) becomes

$$ {}_0^C\mathbb{D}_t^\nu v(r,t) = \frac{\partial^2 \nu(r,t)}{\partial r^2} - \lambda\nu(r,t) - rf(t) \tag{5.44}$$

By taking the $P_\alpha$- transform of equation (5.44) with respect to variable $t$ and applying boundary conditions, we get

$$\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu} V(r,s) = \frac{d^2V(r,s)}{dr^2} - \lambda V(r,s) - rF(s), f(0)=0 \tag{5.45}$$

where $V(r,s) = P_\alpha[\nu(r,t)]$.
or

$$\frac{d^2V(r,s)}{dr^2} - \left(\lambda + \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}\right) V = rF(s) \tag{5.46}$$

with the boundary conditions

$$\lim_{r\to 0} |V(r,s)| = 0, \; and \; V_r(1,s) - V(1,s) = \frac{1}{s}.$$

Equation (5.46) is second order ordinary differential equation. Its solution is given by

$$V(r,s) = \frac{\left(\sinh r\sqrt{\lambda + A^\nu}\right)}{A\left(\lambda + A^\nu \cosh\left(\sqrt{\lambda+A^\nu}\right) - \sinh\left(\sqrt{\lambda+A^\nu}\right)\right)} - \frac{F(s)}{\lambda + A^\nu} \tag{5.47}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$. By using Bromwich's integral and taking inverse $P_\alpha$-transform we get

$$v(r,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{\left(\sinh r\sqrt{\lambda + A^\nu}\right)}{s\left(\lambda + A \cosh\left(\sqrt{\lambda+A^\nu}\right) - \sinh\left(\sqrt{\lambda+A^\nu}\right)\right)} - \frac{F(s)}{\lambda + A^\nu}\right) e^{st} ds, \tag{5.48}$$

and hence we obtain

$$u(r,t) = \frac{1}{r}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{\left(\sinh r\sqrt{\lambda + A^\nu}\right)}{s\left(\lambda + A \cosh\left(\sqrt{\lambda+A^\nu}\right) - \sinh\left(\sqrt{\lambda+A^\nu}\right)\right)} - \frac{F(s)}{\lambda + A^\nu}\right) e^{st} ds. \tag{5.49}$$

The $P_\alpha$- transforms are useful when the boundary conditions are time dependent. Now consider the case when one of the boundary is moving. This type of problem arises in combustion problems where the boundary moves due to the burning of the fuel (see, e.g. Duffy[33]).

**Example 5.4.1.** *Consider the following time dependent heat equation*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{5.50}$$

*where* $\beta t < x < \infty$, $t > 0$, $\beta \in \mathbb{R}$ *and subject to the initial condition* $u(x,0) = 0,\ 0 < x < \infty$ *and boundary conditions* $u(x,t)|_{x=\beta t} = f(t)$, $\lim\limits_{x\to\infty} |u(x,t)| < \infty$, $t > 0$.

*Then the solution of (5.50) is given by*

$$u(x,t) = e^{\frac{-\beta(x-\beta t)}{2a^2}} \int_0^t f(t-\tau)\Phi(x-\beta\tau,\tau)d\tau \tag{5.51}$$

*where*

$$\Phi(x-\beta t,t) = \frac{1}{2}\left[e^{\frac{-\beta(x-\beta t)}{2a^2}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} - \frac{\beta\sqrt{t}}{2a}\right) + e^{\frac{\beta(x-\beta t)}{2a^2}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} + \frac{\beta\sqrt{t}}{2a}\right)\right] \tag{5.52}$$

Proof: By introducing the new coordinate $\eta = x - \beta t$, the problem can be reformulated as

$$\frac{\partial u}{\partial t} - \beta\frac{\partial u}{\partial \eta} = a^2 \frac{\partial^2 u}{\partial \eta^2} \tag{5.53}$$

where $0 < \eta < \infty$, $t > 0$ and subject to the boundary conditions

$$u(0,t) = f(t),\ \lim_{\eta\to\infty} |u(\eta,t)| < \infty, t > 0$$

and the initial condition $u(\eta,0) = 0,\ 0 < \eta < \infty$.

Taking the $P_\alpha$- transform of the equation (5.53) with respect to $t$ and denoting $P_\alpha[u(\eta,t)] = U(\eta,s)$ we obtain

$$\frac{d^2U(\eta,s)}{d\eta^2} + \frac{\beta}{a^2}\frac{dU(\eta,s)}{d\eta} - \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}\frac{1}{a^2}U(\eta,s) = 0 \tag{5.54}$$

with

$$U(0,s) = F(s), \quad \lim_{\eta\to\infty} |U(\eta,s)| < \infty$$

The solution to the differential equation (5.54) is

$$U(\eta,s) = F(s)exp\left(\frac{-\beta\eta}{2a^2} - \frac{\eta}{a}\sqrt{A + \frac{\beta^2}{4a^2}}\right) \tag{5.55}$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$.

Referring the result by *Duffy* [33, p.89, Eq. (2.274)], correspondingly for $P_\alpha$-transform, we have

$$P_\alpha[\Phi(\eta,t)] = exp\left(-\frac{\eta}{a}\sqrt{A + \frac{\beta^2}{4a^2}}\right) \tag{5.56}$$

where

$$\Phi(\eta,t) = \frac{1}{2}\left[e^{\frac{-\beta\eta}{2a^2}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} - \frac{\beta\sqrt{t}}{2a}\right) + e^{\frac{\beta\eta}{2a^2}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} + \frac{\beta\sqrt{t}}{2a}\right)\right] \tag{5.57}$$

By taking inverse $P_\alpha$- transform of (5.55) and applying the convolution theorem, we get

$$u(\eta,t) = e^{\frac{-\beta\eta}{2a^2}} \int_0^t f(t-\tau)\Phi(\eta,\tau)d\tau \tag{5.58}$$

and hence

$$u(x,t) = e^{\frac{-\beta(x-\beta t)}{2a^2}} \int_0^t f(t-\tau)\Phi(x-\beta\tau,\tau)d\tau. \tag{5.59}$$

## 5.5 Conclusion

This chapter provides some new results in the areas of singular integral equations and fractional calculus. Furthermore, the implementation of the new integral transform ($P_\alpha$-transform) for solving certain integral equation have been discussed. The importance of using $P_\alpha$-transform method is that we get a wider class of integrals varying from binomial to exponential function and it is very efficient technique for finding exact solution for certain singular integral equations. The method could lead to a promising approach for many applications in applied sciences.

# 6

# Study of Certain Polynomials and Matrix Function with Lie Algebraic Approach

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The main findings of this chapter have been published as detailed below:

1. **M.A. Pathan, R. Agarwal, S. Jain** (2017). A Unified Study of orthogonal Polynomials via Lie Algebra, *Reports in Mathematical Physics (Elesvier)*, **79(1)**, 1-14.

2. **R. Agarwal and S. Jain** (2015). Certain Properties of Some Special Matrix Functions via Lie Algebra, *International Bulletin of Mathematical Research*, **2**(1), 9-15.

Our objective in this chapter is to obtain a theorem using some operators defined on a Lie algebra of endomorphisms of a vector space which generalizes some results of researchers on the families of special functions and orthogonal polynomials. In particular, we present examples, how the Lie algebraic approach can be used to derive the differential recurrence relations, differential equation for extended Jacobi polynomials and Gegenbauer polynomials.

The method developed in this chapter can also be used to study some other special functions of mathematical physics. Certain properties of some special matrix functions via Lie algebra are studied in the section. We have established a general theorem concerning eigenvector for the product of two operators defined on a Lie algebra of endomorphisms of a vector space derived in section 6.2.
In the section 6.5 and 6.6, we apply this theorem to obtain differential recurrence relations and differential equations for 2-variable Hermite generalized Hermite matrix polynomials and 2-variables Laguerre matrix polynomials.

# 6.1 Introduction

## 6.1.1 Orthogonal Polynomials

Consider a simple set of real polynomials $\psi_n$ [146, Chapter 9, p. 148]. If there exists an interval $a < x < b$ and a function $f(x) > 0$ on that interval, and if

$$\int_a^b f(x)\psi_m(x)\psi_n(x) = 0, \quad m \neq n, \tag{6.1}$$

we say that the polynomials $\psi_n$ are orthogonal with respect to the weight function $f(x)$ over the interval $a < x < b$.
The orthogonality of Jacobi polynomials, Laguerre polynomials and Bessal functions was first observed by *Krall* [83]. In fact, these polynomials or functions satisfy their own family of differential equations. In 1929, *Bochner* classified all orthogonal polynomials on real line satisfying second-order differential equations. In the late thirties

*Krall* proved that all families of orthogonal polynomials that satisfy a fourth order differential equation [83]. Some years earlier, *Littlejohn* had discovered all families of orthogonal polynomials satisfying sixth-order differential equations. *J. Koekoek* and *R. Koekoek* showed in 1991 that orthogonal polynomials with respect to the weights satisfy infinite-order differential equations, except for the nonnegative integers.
The systems of orthogonal polynomials associated with the names of Hermite, Laguerre, and Jacobi (including the special cases named after Chebyshev, Legendre and Gegenbauer) are unquestionably the most extensively studied and widely applied systems. These three systems are called collectively the classical orthogonal polynomials. The derivatives of any classical orthogonal polynomials are also orthogonal. Conversely, Hahn showed that the only orthogonal polynomials, whose derivatives are also orthogonal, are the classical orthogonal polynomials. In fact, Hahn considered only orthogonal polynomials relative to positive definite moment functionals and *Krall* extended the result to the general orthogonal polynomials. Later, Hahn extended his result by showing that the only orthogonal polynomials, whose derivatives of any fixed order are also orthogonal, must be classical orthogonal polynomials [18, 28, 41, 47, 48].

The Jacobi polynomials, also known as hypergeometric polynomials, occur in the study of rotation groups and in the solution to the equations of motion of the symmetric top. They are solutions to the Jacobi differential equation, and give some other special named polynomials as special cases. The Jacobi polynomials appears naturally as extension of Legendre polynomials and Gegenbauer polynomials in the context of potential theory and harmonic analysis [20]. The Jacobi polynomial has been used extensively in mathematical analysis and many practical applications.

### 6.1.2 Extended Jacobi Polynomial

*Fujiwara* [43] studied the polynomial $F_n^{(\alpha,\beta)}(x;a,b,c)$ which is called the extended Jacobi polynomial defined by Rodrigue formula

$$F_n^{(\alpha,\beta)}(x;a,b) = \frac{(-1)^n}{n!}\left(\frac{\lambda}{b-a}\right)^n (x-a)^{-\alpha}(b-x)^{-\beta}\ D^n[(x-a)^{n+\alpha}(b-x)^{n+\beta}], \tag{6.2}$$

where $D = \frac{d}{dx}, \alpha, \beta > -1$.

*Thakare* [135] showed that,

$$\begin{aligned} F_n(\alpha,\beta;x) =& \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} {}_2F_1\left[\begin{matrix} -n, -n-a \\ 1+\beta \end{matrix} ; \frac{x-b}{x-a}\right] \\ =& \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} \sum_{n=0}^{\infty} \frac{(-n)_k(-n-a)_k}{k!(1+\beta)} \left(\frac{x-b}{x-a}\right)^k. \end{aligned} \tag{6.3}$$

He also expressed the extended Jacobi polynomials as

$$\begin{aligned} F_n(\alpha,\beta;x) =& \lambda^n \binom{n+\beta}{n} {}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+n \\ 1+\beta \end{matrix} ; \frac{b-x}{b-a}\right] \\ =& \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} \sum_{n=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{k!(1+\beta)} \left(\frac{x-b}{x-a}\right)^k. \end{aligned} \tag{6.4}$$

This polynomial satisfy the following differential equation:

$$[(x-a)(b-x)D^2 + \{(\alpha+1)(b-x)-(\beta+1)(x-a)\}\,D + n(1+\alpha+\beta+n)]y = 0 \tag{6.5}$$

The various types of generating functions for extended Jacobi functions are studied by *Pittaluga et. al* [136].

For $a = -1$, $b = 1$ and $\lambda = 1$, the extended Jacobi polynomial (6.3) reduces to

Jacobi polynomial (*Rainville* [146, p. 255])

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\alpha)_k(1+\alpha+\beta)_n} \left(\frac{x-1}{2}\right)^k . \tag{6.6}$$

The generating function for Jacobi polynomial is given by:

$$\sum_{n=0}^{\infty} p^{(\alpha,\beta)}\ (x)\ t^n = 2^{\alpha+\beta} R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta} \tag{6.7}$$

where $R = (1-2zt+t^2)^{\frac{1}{2}}$.

The Jacobi polynomial $p^{(\alpha,\beta)}\ (x)$ satisfies the differential equation

$$(1-x^2)\frac{d^2P_n^{(\alpha,\beta)}(x)}{dx^2} + [\beta-\alpha-(2+\alpha+\beta)x]\frac{dP_n^{(\alpha,\beta)}(x)}{dx} + n(1+\alpha+\beta+n)P_n^{(\alpha,\beta)}(x) = 0 \tag{6.8}$$

If we take $a = -1$, $b = 1$, $\lambda = 1$ and $\alpha = \beta = 2\nu - 1$ for $\nu \in \mathbb{R}$ then the extended Jacobi polynomial (6.2) reduces to Gegenbauer polynomial (*Rainville* [146, p. 255])

$$C_n^{\nu}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}}{k!(n-2k)!} \tag{6.9}$$

The generating function for $C_n^{\nu}(x)$ is given by

$$(1-2xt+t^2)^{-\nu} = \sum_{k=0}^{\infty} C_n^{\nu}(x)t^n \tag{6.10}$$

The Gegenbauer polynomials $C_n^{\nu}(x)$ satisfies the differential equation

$$(1-x^2)\frac{d^2}{dx^2}(C_n^{\nu}(x)) - (2\nu+1)x\frac{d}{dx}(C_n^{\nu}(x)) + n(2\nu+n)C_n^{\nu}(x) = 0. \tag{6.11}$$

For $a = -1$, $b = 1$, $\lambda = 1$ and $\alpha = \beta = 0$, the extended Jacobi polynomial (6.2) reduces to the Legendre polynomial $P_n(x)$ (see, e.g. *Rainville* [146])

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\frac{1}{2})_{n-k} (2x)^{n-2k}}{k!(n-2k)!}. \tag{6.12}$$

For the basic properties and addition theorem of Gegenbauer and Legendre polynomials one may refer book by *Rainville* [146].
Generating function for Legendre polynomial is given by

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{6.13}$$

The Legendre polynomials $P_n(x)$ satisfies the differential equation

$$(1 - x^2)\frac{d^2 P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(1+n)P_n = 0. \tag{6.14}$$

### 6.1.3 Lie Algebra and Special Function

The classical orthogonal functions of mathematical physics are closely related to Lie groups. Specifically, they are matrix elements of, or basis vectors for, unitary irreducible representations of low dimensional Lie groups.
The theory of generalized special functions has witnessed a rather significant evolution during the last years. The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases. One important method for studying special functions via their recurrence relations, differential equations lies in closely with the standard Lie algebraic techniques. Many important classical differential equations has connection with Lie theory. The interplay between differential equations, special functions and Lie theory plays an

important role in mathematical physics. When the Lie algebraic aspects of the special functions are considered in the literature, they are limited to the Lie algebra generated by the raising, lowering and maintaining operators.

*Radulescu* [143, 144] discussed some important properties of Hermite and Laguerre polynomials using some operators defined on a Lie algebra. *Mandal* [104] studied some properties of simple Bessel polynomials considered by *Krall* and *Frink* [83].*Pathan* and *Khan* [133, 134] extended the Lie algebraic approach discussed by *Radulescu* [144] and *Mandal* [104] to derive some properties of generalized Hermite polynomials of two variables (see, e.g. *Dattoli* et al. [22]), generalized Bessel functions of two variables (*Dattoli*, et al. [23, 24]). Recently, *Humi* [66] has shown that in addition to these operators, the dilation and the translation operators can be added to these Lie algebras for some families of factorisable equations using factorization method used in theoretical physics.

### 6.1.4 Matrix Polynomial

A matrix polynomial is a polynomial with matrices as variables. Given an ordinary, scalar-valued polynomial

$$p(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \tag{6.15}$$

this polynomial evaluated at a square matrix $A$ of order $N$ is

$$p(A) = \sum_{i=0}^{n} a_i A^i = a_0 I + a_1 A + a_2 x^2 + \ldots + a_n A^n, \tag{6.16}$$

where $I$ is the identity matrix.

A matrix polynomial equation is an equality between two matrix polynomials, which holds for the specific matrices in question. A matrix polynomial identity is a matrix polynomial equation which holds for all matrices A in a specified matrix ring $M_n(R)$.

The Hermite matrix polynomials (Jodar and Company [71]) $H_n(x;A)$ are defined as:

$$H_n^{\lambda}(x;A) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{(n-2k)!k!}(x\sqrt{2A})^{n-2k} \quad (n \geq 0). \tag{6.17}$$

Following Rodrigues formula holds for the Hermite matrix polynomials

$$H_n(x,A) = exp\left(\frac{Ax^2}{2}\right)(-1)^n\left(\frac{A}{2}\right)\left[\frac{d^n}{dx^n}exp\left(\frac{Ax^2}{2}\right)\right], \quad n \geq 0 \tag{6.18}$$

and it satisfies the three terms recurrence relationship

$$\begin{aligned} H_n(x,A) &= xI\sqrt{2A}H_{n-1} - 2(n-1)zH_{n-2}(x,A), n \geq 1 \\ H_{-1}(x,A) &= 0, H_0(x,A) = 1 \end{aligned} \tag{6.19}$$

where, $I$ is the identity matrix in $\mathbb{C}^{N\times N}$ is the space of all matrices whose entries are complex numbers with real part and imaginary part are natural numbers and its all eigenvalue has positive real part.

The generating function for the Hermite matrix polynomial is (Jodar and Company [71])

$$\exp\left(xt\sqrt{2A} - t^2I\right) = \sum_{n=0}^{\infty} H_n(x,A)\frac{t^n}{n!}. \tag{6.20}$$

The 2-variable generalized Hermite matrix polynomial $H_n^{\lambda}(x,y;A)$ (*Batahan* [10]) defined as:

$$H_n^{\lambda}(x,y;A) = n!\lambda^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x\sqrt{(\frac{A}{2})^{n-2k}}y^k}{\lambda^k(n-2k)!k!} \quad (n \geq 0) \tag{6.21}$$

and is specified by the generating function

$$\exp\left(\lambda(xt)\sqrt{\frac{A}{2}} + yt^2I\right) = \sum_{n=0}^{\infty} H_n^{\lambda}(x,y;A)\frac{t^n}{n!} \tag{6.22}$$

and satisfy the recurrence relations.

$$\begin{aligned} & H_n(x,y,A) = y^{\frac{n}{2}} H_n(x/\sqrt{y}, A), \\ & H_0(x.y,A) = I, H_1(x,y,A) = x\sqrt{2A}, \\ \text{and} \qquad & H_n(x,1,a) = H_n(x,A), \end{aligned} \tag{6.23}$$

where $H_n(x,a)$ is defined in (6.17).
The Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ (*Jodar et al.* [72]) is defined by

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}x^k, \quad n \geq 0 \tag{6.24}$$

where $(A)_n$ is the matrix Pochhammer symbol defined by

$$(A)_n = A(A+I)\cdots(A+(n-1)I), \; n \geq 1; \; (A)_0 = I \tag{6.25}$$

and is specified by the generating function

$$(1-t)^{-(A+I)} \exp\left(\frac{-\lambda x t}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n, \; x,t \in \mathbb{C}, \; |t| < 1. \tag{6.26}$$

The 2- variable Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ (Khan and Hasan [76]) is defined by

$$L_n^{(A,\lambda)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k x^k y^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}, \quad n \geq 0 \tag{6.27}$$

where $(A)_n$ is the matrix Pochhammer symbol defined by

$$(A)_n = A(A+I)\cdots(A+(n-1)I), \; n \geq 1; \; (A)_0 = I \tag{6.28}$$

Laguerre matrix polynomial can also be expressed in terms of the confluent hypergeometric function (*Andrews* [3]) as

$$L_n^{(A,\lambda)}(x,y) = \frac{\Gamma(A+(n+1)I)(\Gamma(A+I))^{-1}y^n}{\Gamma(n+1)}\ {}_1F_1\left[-n; A+I; \frac{\lambda x}{y}\right] \tag{6.29}$$

and is specified by the generating function

$$(1-yt)^{-A}\exp(-\lambda xt) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x,y)t^n,\ x,y,t \in \mathbb{C},\ |yt| < 1. \tag{6.30}$$

### 6.1.5 Certain Properties of Some Special Matrix Functions via Lie Algebra

Special matrix functions seen on statistics, Lie group theory and number theory are well known (see, e.g. *Constantine and Muirhead [21], Terras [176] and James [68]*). These types of functions are also useful in many subject viz. physics, chemistry and mechanics (see, e.g. *Keller* and Wolfe [75] and *Morse* and *Fesbach* [119]). Recently, the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials by *Defez* and *Jodar* [30, 31] and *Jodar et. al* [72, 73]. Motivated by their work, in this paper, we establish results for their polynomials using Lie algebra approach.

Throughout in Theorems 6.5.1 and 6.6.1, we assume that $A$ is a positive stable matrix in $\mathbb{C}^{N\times N}$, that is, $A$ satisfies the following condition:

$$\Re(\mu) > 0, \quad \text{for all } \mu \in \sigma(A), \tag{6.31}$$

where $\sigma(A)$ denotes the set of all the eigenvalues of $A$. If $D_0$ is the complex plane cut along the negative real axis and $\log(z)$ denotes the principle logarithm of $z$, then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log\ z)$. If the matrix $A \in \mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A}$ denote the image of the matrix functional calculus acting on the matrix $A$.

## 6.2 Main results

In this section, we shall prove results concerning eigenvector for the product of two operators defined on a Lie algebra of endomorphisms of a vector space.
Let End $V$ be the Lie algebra of endomorphisms of a vector space $V$, endowed with the Lie bracket $[\cdot,\cdot]$ defined by $[A,B] = AB - BA$, for every $A, B \in \text{End } V$. The main results of this chapter are contained in the following theorems:

**Theorem 6.2.1.** *Let $A, B \in \text{End } V$ be such that $[A,B]y_n = (a(2n+1)+b+c)y_n$ and the sequence $(y_n)_n \subset V$ is defined as follows: $Ay_1 = (c+1)y_0$ and $By_n = \left(\frac{a(n^2+2n)+bn+c(n+1)+1}{an+bn+c+1}\right) y_{n+1}$, $n \in N$. Then $Ay_n = (a(n-1)+b(n-1)+c+1)y_{n-1}$ and $y_n$ is an eigenvector of the eigenvalue $a(n^2-1)+b(n-1)+cn+1$, $\forall\, n \in N$ for $BA$.*

**Proof**: First, we shall show that

$$Ay_{n+1} = (an+bn+c+1)y_n \qquad \forall\, n \in N.$$

For $n = 1$, we have

$$[A,B]y_1 = (3a+b+c)y_1$$

$$A(By_1) - B(Ay_1) = (3a+b+c)y_1$$

Also, $Ay_1 = y_0$ and $By_0 = y_1$, $By_1 = \frac{3a+b+2c+1}{a+b+c+1} y_2$. Thus, we find that

$$Ay_2 = (a+b+c+1)y_1.$$

Now, suppose that $Ay_{n+1} = (an+bn+c+1)y_n$. Since we can write

$$[A,B]y_{n+1} = (a(2n+3)+b+c)y_{n+1}$$

$$\implies A(By_{n+1}) - B(Ay_{n+1}) = (a(2n+3)+b+c)y_{n+1}$$
$$\implies \frac{a\left((n+1)^2+2(n+1)\right)+b(n+1)+c(n+1)+1}{a(n+1)+b(n+1)+c+1}Ay_{n+2} - (an+bn+c+1)By_n$$
$$=(a(2n+3)+b+c)y_{n+1}$$
$$\implies \frac{a\left((n+1)^2+2(n+1)\right)+b(n+1)+c(n+2)+1}{a(n+1)+b(n+1)+c+1}Ay_{n+2}$$
$$-(a(n^2+2n)+bn+c(n+1)+1)y_{n+1} = (a(2n+3)+b+c)y_{n+1}$$
$$\implies \frac{a\left((n+1)^2+2(n+1)\right)+b(n+1)+c(n+2)+1}{a(n+1)+b(n+1)+c+1}Ay_{n+2}$$
$$=[a(n^2+4n+3)+b(n+1)+c(n+2)+1]y_{n+1}$$
$$\implies Ay_{n+2} = [a(n+1)+b(n+1)+c+1]y_{n+1}$$

Hence, by mathematical induction $Ay_{n+1} = (an+bn+c+1)y_n, \quad n \in \mathbb{N}$ and it follows that

$$BAy_n = [a(n-1)+b(n-1)+c+1]By_{n-1},$$

i.e. $\quad BAy_n = [a(n^2-1)+b(n-1)+cn+1]y_n$.
Hence, $y_n$ is an eigenvector of the eigenvalue $[a(n^2-1)+b(n-1)+cn+1]$ for $BA$, $\forall\, n \in \mathbb{N}$.

Many of the known results follow as special cases of the Theorem 6.2.1.
Firstly, if we take $a=0$, $b=1$ and $c=0$ in the Theorem 6.2.1, we arrive at the main result by *Radulescu* [143, Theorem 1] contained in the following corollary:

**Corollary 6.2.2.** *Let $G, H \in \mathrm{End}\ V$ be such that $[G,H]y_n = Iy_n$, where the sequence $(T_n)_n \subset V$ is defined as follows: $GH_0 = 0$ and $T_n = HT_{n-1}$, for every $n \geq 1$. Then $GT_n = (n)T_{n-1}$ and $T_n$ is an eigenvector of eigenvalue $n^2$, $\forall\, n \in N$ for $HG$.*

Next, if we take, $a=0$, $b=0$ and $c=0$ in the Theorem 6.2.1, we obtain the following result by *Mandal* [104]:

**Corollary 6.2.3.** *Let $I, J \in \mathrm{End}\ V$ be such that $[I,J]y_n = 0$, where the sequence*

*$(U_n)_n \subset V$ is defined as follows: $IU_1 = U_0$ and $U_{n+1} = JU_n$, for every $n \geq 1$. Then $IU_{n+1} = U_n$ and $U_n$ is an eigenvector of eigenvalue 1, $\forall\, n \in N$ for $JI$.*

Further taking, $c = 0$ in the Theorem 6.2.1, we get the main result by Pathan and Khan [133]:

**Corollary 6.2.4.** *Let $K, L \in \mathrm{End}\, V$ be such that $[K, L]S_n = (a(1+2n)+b)S_n$, where the sequence $(S_n)_n \subset V$ is defined as follows: $KS_1 = S_0$ and $LS_n = \frac{a(n^2+2n)+bn+1}{(an+bn+1)} S_{n+1}$, for every $n \geq 1$. Then $KS_{n+1} = (an + bn + 1)S_n$ and $S_n$ is an eigenvector of eigenvalue $a(n^2 - 1) + b(n + 1) + 1$, $\forall\, n \in N$ for $LK$.*

If we take $a = 1$, $b = 0$ and $c = \alpha$ in the Theorem 6.2.1, denoting by $P$ and $Q$, the operators $A$ and $B$ respectively and we obtain the following theorem for extended Jacobi polynomials.

**Theorem 6.2.5.** *Let $P, Q \in \mathrm{End}\, V$ be such that $[P, Q]y_n = (1 + 2n + \alpha)y_n$, where $\alpha \in R$ and the sequence $(y_n)_n \subset V$ is defined as follows: $Py_0 = 0$ and $Qy_n = (n + 1)y_{n+1}$, for every $n \geq 1$. Then $Py_n = (n + \alpha)y_{n-1}$ and $y_n$ is an eigenvector of eigenvalue $n(n + \alpha)$,$\forall\, n \in N$ for $QP$.*

Next, if we take $a = 0$, $b = 0$ and $c = \alpha = 2\nu - 1$, in Theorem 6.2.1, we denote by $E$ and $F$, the operators $A$ and $B$ respectively and we obtain the following theorem for the Gegenbauer polynomials:

**Theorem 6.2.6.** *Let $E, F \in \mathrm{End}\, V$ be such that $[E, F]W_n = (2n + 2\nu)W_n$, where $\nu \in \mathbb{R}$ and the sequence $(W_n)_n \subset V$ is defined as follows: $EW_0 = 0$ and $FW_n = (n + 1)W_{n+1}$, $\forall\, n \in N$. Then $EW_n = (2\nu + n - 1)W_{n-1}$ and $W_n$ is an eigenvector of eigenvalue $n(2\nu + n - 1)$, $\forall\, n \in N$ for $FE$.*

If we take $\alpha = 0$, in the Theorem 6.2.5, we obtain the main result by *Pathan* and *Khan* [134] in the following corollary:

**Corollary 6.2.7.** *Let $S, T \in \text{End } V$ be such that $[S, T]y_n = (1 + 2n)y_n$, where the sequence $(Z_n)_n \subset V$ is defined as follows: $SZ_0 = 0$ and $nZ_n = TZ_{n-1}$, for every $n \geq 1$. Then $SZ_n = (n)Z_{n-1}$ and $Z_n$ is an eigenvector of eigenvalue $n^2$, $\forall\, n \in N$ for $TS$.*

Let $End\,V$ be the Lie algebra of endomorphisms of a vector space $V$, endowed with the Lie bracket $[\cdot,\cdot]$ defined by $[C, D] = CD - DC$, for every $C, D \in End\,V$. The following theorems are results of the matrix function of this chapter:

**Theorem 6.2.8.** *Let $C, D \in End\,V$ be such that $[C, D]y_n = -y_n$, the sequence $(y_n)_n \subset V$ is defined as follows: $Cy_0 = 0$ and $Dy_n = -(n+1)y_{n+1}$, for every $n \geq 1$. Then $Cy_n = y_{n-1}$ and $y_n$ is an eigenvector of eigenvalue $(-n)$ for $DC$, for every $n \geq 1$.*

Proof: First, we show

$$Cy_n = y_{n-1}, \text{ for every } n \geq 1.$$

For $n = 1$, this equality is evident, because

$$[C, D]y_0 = -y_0,$$

$$C(Dy_0) - D(Cy_0) = -y_0,$$

also $Cy_0 = 0$ and $Dy_0 = -y_1$ and therefore,

$$Cy_1 = y_0$$

Now, suppose that $Cy_n = y_{n-1}$, then we have

$$[C, D]y_n = -y_n,$$

$$\Rightarrow C(Dy_n) - D(Cy_n) = -y_n,$$

$$\Rightarrow C((n+1)y_{n+1}) - D(n+\alpha)(y_{n-1}) = -y_n.$$

Using linearity property and $Dy_{n-1} = ny_n$, we get

$$(n+1)C(y_{n+1}) - (n+\alpha)ny_n = -y_n,$$

which on solving for $C(y_{n+1})$, gives

$$C(y_{n+1}) = y_n.$$

Therefore, by mathematical induction, $Cy_n = y_{n-1}$, for every $n \geq 1$. It follows immediately that $DCy_n = -ny_n$.
Hence, $y_n$ is an eigenvector of eigenvalue $(-n)$ for $DC$, for every $n \geq 1$.

## 6.3 Recurrence Relations and Differential Equation for the Extended Jacobi Polynomials

In this Section, we apply Theorem 6.2.5 to obtain the differential recurrence relations and the differential equation for the extended Jacobi polynomial. Let $V = C^\infty(\mathbb{R}^3)$ be the set of infinitely many times differentiable functions $u : \mathbb{R} \to \mathbb{C}$. We define the operators $A, B \in \text{End } V$ as

$$Au(x,y,z) = \frac{(x-a)y^{-1}z}{\lambda}u_x - \frac{z}{\lambda}u_y \tag{6.32}$$

$$\begin{aligned} Bu(x,y,z) =& \frac{\lambda}{(b-a)}\left[(x-a)(x-b)yz^{-1}u_x + (x-b)y^2z^{-1}u_y\right. \\ &\left. + (x-a)yu_z + (1+\alpha)(x-b)yz^{-1}\right] \end{aligned} \tag{6.33}$$

for $(x,y,z) \in \mathbb{R}^3$ and $u_x$, $u_y$ and $u_z$ denote $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ respectively.

We claim that the operators (6.32) and (6.33) obey the commutation relation

$[A, B]y_n = (1 + 2n + \alpha)y_n$. Indeed,

$$
\begin{aligned}
[A, B]u(x, y, z) =& A(Bu(x, y, z)) - B(Au(x, y, z)) \\
=& \left( \frac{(x-a)y^{-1}z}{\lambda} \frac{\partial}{\partial x} - \frac{z}{\lambda}\frac{\partial}{\partial y} \right) \\
& \times \frac{\lambda}{(b-a)} \big[ (x-a)(x-b)yz^{-1}u_x + (x-b)y^2z^{-1}u_y \\
& +(x-a)yu_z + (1+\alpha)(x-b)yz{-}1 ] - \frac{\lambda}{(b-a)} \Bigg[ (x-a)(x-b)yz^{-1}\frac{\partial}{\partial x} \\
& +(x-b)y^2z^{-1}\frac{\partial}{\partial y} + (x-a)y\frac{\partial}{\partial z} + (1+\alpha)(x-b)yz^{-1} \Bigg] \\
& \times \left( \frac{(x-a)y^{-1}z}{\lambda} u_x - \frac{z}{\lambda} u_y \right) \\
=& 2y\frac{\partial u}{\partial y} + u + \alpha u
\end{aligned}
\tag{6.34}
$$

Hence, this equation simplifies to

$$
[A, B]u(x, y, z) = \left( 2y\frac{\partial}{\partial y} + 1 + \alpha \right) u(x, y, z). \tag{6.35}
$$

Now, if $u(x, y, z)$ assumes the form $u(x, y, z) = F_n(\alpha, \beta; x)y^n z^\beta = y_n \in C^\infty(\mathbb{R}^3)$, then we have

$$
\begin{aligned}
[A, B](F_n(\alpha, \beta; x)y^n z^\beta) =& \left( 2y\frac{\partial}{\partial y} + 1 + \alpha \right) F_n(\alpha, \beta; x)y^n z^\beta \\
=& (1 + 2n + \alpha)F_n(\alpha, \beta; x)y^n z^\beta
\end{aligned}
\tag{6.36}
$$

and our claim is justified.

Now, the relation $By_n = (n + 1)y_{n+1}$ gives following differential recurrence relation on operator $B$ for $y_n$

$$\begin{aligned}\frac{\lambda}{(b-a)}\Bigg[&(x-a)(x-b)yz^{-1}\frac{\partial}{\partial x}+(x-b)y^2z^{-1}\frac{\partial}{\partial y}\\&+(x-a)y\frac{\partial}{\partial z}+(1+\alpha)(x-b)yz^{-1}\Bigg]F_n(\alpha,\beta;x)y^nz^\beta\\&=(n+1)F_{n+1}(\alpha,\beta;x)y^{n+1}z^\beta\end{aligned}$$

This yields

$$\begin{aligned}\Bigg[(x-a)(x-b)\frac{\partial}{\partial x}+n(x-b)\ +\beta(x-a)+(1+\alpha)(x-b)\Bigg]F_n(\alpha,\beta;x)y^{n+1}z^\beta\\=\frac{(b-a)(n+1)}{\lambda}F_{n+1}(\alpha,\beta;x)y^{n+1}z^{\beta+1}\end{aligned}$$

Writing $F_n(\alpha,\beta;x)y^nz^\beta=F_n(\alpha,\beta-1;x)y^nz^{\beta-1}$, on right hand side of the equation (6.37), we get

$$\begin{aligned}(x-a)(x-b)F_n^{'}(\alpha,\beta;x)+&[(n+\alpha+1)(x-b)+\beta(x-a)]F_n(\alpha,\beta;x)\\&=\frac{(b-a)(n+1)}{\lambda}F_{n+1}(\alpha,\beta-1;x)\end{aligned}\tag{6.37}$$

Next, if we use the relation $Ay_n=(n+\alpha)y_{n-1}$, we obtain the following differential recurrence relation on operator $A$

$$\left(\frac{(x-a)y^{-1}z}{\lambda}\frac{\partial}{\partial x}-\frac{z}{\lambda}\frac{\partial}{\partial y}\right)F_n(\alpha,\beta;x)y^nz^\beta=(n+\alpha)F_{n-1}(\alpha,\beta;x)y^{n-1}z^\beta$$

$$\begin{aligned}\Rightarrow\left[\frac{(x-a)F_n(\alpha,\beta;x)y^{n-1}z^{\beta+1}}{\lambda}-\frac{nF_n(\alpha,\beta;x)y^{n-1}z^{\beta+1}}{\lambda}\right]\\=(n+\alpha)F_{n-1}(\alpha,\beta;x)y^{n-1}z^\beta\end{aligned}$$

or equivalently

$$(x-a)F_n'(\alpha,\beta;x)z^{\beta} - nF_n(\alpha,\beta;x)z^{\beta} = \lambda(n+\alpha)F_{n-1}(\alpha,\beta;x)z^{\beta-1}.$$

Hence,

$$(x-a)F_n'(\alpha,\beta;x) - nF_n(\alpha,\beta;x) = \lambda(n+\alpha)F_{n-1}(\alpha,\beta+1;x). \tag{6.38}$$

From (6.37) and (6.38), we obtain

$$\begin{aligned}
&\frac{1}{(x-a)(x-b)}\left(\frac{(b-a)(n+1)}{\lambda}F_{n+1}(\alpha,\beta-1;x) - ((n+\alpha+1)(x-b)\right)\\
&+\beta(x-a))F_n(\alpha,\beta;x))\\
&= \frac{1}{(x-a)}\left(\lambda(n+\alpha)F_{n-1}(\alpha,\beta+1;x) + nF_n(\alpha,\beta;x)\right).
\end{aligned}$$

So,

$$\begin{aligned}
\frac{(b-a)(n+1)}{\lambda}F_{n+1}(\alpha,\beta-1;x) - ((2n+\alpha+1)(x-b)+\beta(x-a))F_n(\alpha,\beta;x)&\\
= \lambda(x-b)(n+\alpha)F_{n-1}(\alpha,\beta+1;x)&
\end{aligned} \tag{6.39}$$

Finally, the relation $BAy_n = n(n+\alpha)y_n$ gives

$$\begin{aligned}
&\frac{\lambda}{(b-a)}\Big\{(x-a)(x-b)yz^{-1}\frac{\partial}{\partial x} + (x-b)y^2z^{-1}\frac{\partial}{\partial y} + (x-a)y\frac{\partial}{\partial z}\\
&+(1+\alpha)(x-b)yz^{-1}\Big\} \times \left[\left(\frac{(x-a)y^{-1}z}{\lambda}\frac{\partial}{\partial x} - \frac{z}{\lambda}\frac{\partial}{\partial y}\right)\right] F_n(\alpha,\beta;x)y^nz^{\beta}\\
&= n(n+\alpha)F_n(\alpha,\beta;x)y^nz^{\beta}
\end{aligned}$$

or equivalently,

$$\begin{aligned}&(x-a)(b-xF_n''(\alpha,\beta;x)+\{(\alpha+1)(b-x)-(\beta+1)(x-a)\}\\&\times F_n'(\alpha,\beta;x)+n(1+\alpha+\beta+n)F_n(\alpha,\beta;x)=0\end{aligned} \tag{6.40}$$

Thus, we observe that extended Jacobi polynomial $F_n^{(\alpha,\beta)}(x;a,b)$ is the solution of the differential equation (6.40). Further, we note that the relation (6.37) and (6.38) are differential recurrence relations and (6.39) is the recurrence relation satisfied by the extended Jacobi polynomial $F_n^{(\alpha,\beta)}(x;a,b)$.

Special cases of the extended Jacobi Polynomial can be obtained by giving particular values to the parameters in the operators $A$ and $B$ above.

(1) If we take $a=-1$, $b=1$ and $\lambda=1$ in (6.32) and (6.33), we obtain

$$Au(x,y,z)=(x-1)y^{-1}zu_x-zu_y \tag{6.41}$$

and

$$\begin{aligned}Bu(x,y,z)=&\frac{1}{2}\left[(x^2-1)yz^{-1}u_x+(x-1)y^2z^{-1}u_y\right.\\&\left.+(x+1)yu_z+(1+\alpha)(x-1)yz^{-1}\right]\end{aligned} \tag{6.42}$$

Then by using the relation $BAy_n=n(n+\alpha)y_n$ and writing $F_n(\alpha,\beta;x)=P_n^{(\alpha,\beta)}(x)$ one can obtain differential equation

$$(1-x^2)\frac{d^2P_n^{(\alpha,\beta)}(x)}{dx^2}+[\beta-\alpha-(2+\alpha+\beta)x]\frac{dP_n^{(\alpha,\beta)}(x)}{dx}+n(1+\alpha+\beta+n)P_n^{(\alpha,\beta)}(x)=0 \tag{6.43}$$

which is a differential equation whose solution is classical Jacobi polynomials.

Also, by using the relations $Ay_n=(n+\alpha)y_{n-1}$ and $By_n=(n+1)y_{n+1}$, we get following differential recurrence relations for classical Jacobi polynomial.

$$(x^2-1)P_n'^{(\alpha,\beta)}(x)+((n+\alpha+1)(x-1)+\beta(x+1))P_n^{(\alpha,\beta)}(x) = 2(n+1)P_{n+1}^{(\alpha,\beta+1)}(x) \tag{6.44}$$

and

$$(x-1)P_n'^{(\alpha,\beta)}(x)-nP_n^{(\alpha,\beta)}(x)=(n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x) \tag{6.45}$$

(2) Again if we take $a=-1$, $b=1$, $\lambda=1$ and $\alpha=\beta=0$, writing $F_n(\alpha,\beta;x)=P_n(x)$ in (6.32) and (6.33), we obtain

$$Au(x,y,z)=(x-1)y^{-1}zu_x-zu_y \tag{6.46}$$

and

$$Bu(x,y,z)=\frac{1}{2}\left[(x^2-1)yz^{-1}u_x+(x-1)y^2z^{-1}u_y+(x+1)yu_z+(x-1)yz^{-1}\right] \tag{6.47}$$

Then, by using relation $BAy_n=n(n+\alpha)y_n$, one can obtain

$$(1-x^2)\frac{d^2P_n(x)}{dx^2}-2x\frac{dP_n(x)}{dx}+n(1+n)P_n(x)=0 \tag{6.48}$$

which is the differential equation whose solution is Legendre polynomial $P_n(x)$.

## 6.4 Recurrence Relations and Differential Equation for Gegenbauer Polynomials

In this section, we obtain certain properties viz. differential recurrence relations, differential equations of Gegenbauer polynomials, by application of Theorem 6.2.6 Let $V=C^\infty(\mathbb{R}^2)$, In view of Theorem 6.2.6, we define the operators $E$ and $F$ on

End $\in V$as follows:

$$Eu(x,y) = (x^2-1)y^{-1}\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} \tag{6.49}$$

$$Fu(x,y) = (x^2-1)y\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} + 2\nu xyu \tag{6.50}$$

For every $(x,y) \in \mathbb{R}\times\mathbb{R}$, where $u_x$ and $u_y$ denote $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, respectively. It can be easily seen that these operators satisfy the commutation relation $[E,F]y_n = (2n+2\nu)y_n$.

Now, if $u(x,y)$ assumes the form $u(x,y) = f_n(x)y^n = w_n \in C^\infty(\mathbb{R}^2)$, then we have

$$[E,F](f_n(x)y^n) = \left(2y\frac{\partial}{\partial y} + 2\nu y\right)(f_n(x)y^n) = (2n+2\nu)(f_n(x)y^n). \tag{6.51}$$

Above Eq. (6.51) and the relation

$$Ey_n = (n+1)y_{n+1}$$

by virtue of Theorem 6.2.6 gives

$$(x^2-1)\frac{\partial f_n(x)}{\partial x} = (n+1)f_{n+1}(x) - x(2\nu+n)f_n(x). \tag{6.52}$$

Again the relation $Ew_n = (2\nu+n-1)w_{n-1}$ yields

$$(x^2-1)\frac{\partial f_n(x)}{\partial x} = (2\nu+n+1)f_{n-1}(x) + xnf_n(x) \tag{6.53}$$

which on using (6.52) can be written as

$$(n+1)f_{n+1}(x) - 2x(v+n)f_n(x) + (2\nu+n-1)f_{n-1}(x) = 0. \tag{6.54}$$

Again, by virtue of Theorem 6.2.6 we can write $BAy_n = n(1-2\nu-n)y_n$

$$(x^2-1)^2 f_n''(x) + x(x^2-1)(2\nu+1)f_n'(x) - n(2\nu+n)(x^2-1)f_n(x) = 0,$$

which yields

$$(1-x^2)f_n''(x) - x(2\nu+1)f_n'(x) + n(2\nu+n)f_n(x) = 0. \tag{6.55}$$

Now, we observe that the Gegenbauer polynomial $C_n^\nu(x)$ is a solution of the differential equation (6.55). It is interesting to note that (6.52) and (6.53) are differential recurrence relations and relation (6.54) is the three-term recurrence relation satisfied by Gegenbauer polynomial $C_n^\nu(x)$.

## 6.5 2-variable Generalized Hermite Matrix Polynomials

In this section, we apply Theorem 6.2.8 to obtain differential recurrence relations and differential equations for 2-variable generalized Hermite matrix polynomials.

**Theorem 6.5.1.** *The 2-variable generalized Hermite matrix polynomial $H_n^\lambda(x,y;A)$ in Eq. (6.21) satisfies the following differential equation*

$$\left[\frac{\lambda A - 2yI}{A^2\lambda}\right]\frac{\partial^2}{\partial x^2}f_n(x,y;A) - xI\frac{\partial}{\partial x}f_n(x,y;A) + nIf_n(x,y;A) = 0 \tag{6.56}$$

*and the recurrence relation*

$$\left(\lambda I - \frac{2y}{A}\right)\frac{\partial}{\partial x}f_n(x,y;A) = \lambda xAf_n(x,y;A) - (n+1)If_{n+1}(x,y;A). \tag{6.57}$$

*or*

$$\frac{\partial}{\partial x}f_n(x,y,A) = A\lambda f_{n-1}(x,y,A) \tag{6.58}$$

Proof: Let $V = \mathbb{C}^{N\times N}$, we define the operator $C, D \in End\ V$ as

$$C(x,y,t;A) = \frac{1}{\lambda tA}\frac{\partial u}{\partial x} \tag{6.59}$$

$$D(x,y,t;A) = \left(\lambda I - \frac{2y}{A}\right) t\frac{\partial u}{\partial x} - \lambda xtA \tag{6.60}$$

For $x, y, t \in \mathbb{C}$ and $A$ is the matrix in $\mathbb{C}^{N\times N}$. We claim that the operators (6.59) and (6.60) obey the commutation relation $[C, D]y_n = -y_n$.

Indeed,

$$\begin{aligned}
[C,D]u(x,y,t;A) =& C(Du(x,y,t;A)) - D(Cu(x,y,t;A)) \\
=& \frac{1}{\lambda tA}\frac{\partial}{\partial x}\left[\left(\lambda I - \frac{2y}{A}\right) t\frac{\partial u}{\partial x} - \lambda xtA\right] \\
& - \left[\left(\lambda I - \frac{2y}{A}\right) t\frac{\partial}{\partial x} - \lambda xtA\right]\frac{I}{\lambda tA}\frac{\partial u}{\partial x} \\
=& -u(x,y,t;A)
\end{aligned} \tag{6.60}$$

i.e.

$$[C,D]u(x,y,t;A) = -u(x,y,t;A).$$

Now, if $u(x,y,t;A)$ assumes the form $y_n(x,y,t;A) = f_n(x,y;A)t^n \in \mathbb{C}^{N\times N}$, then we have

$$[C,D](f_n(x,y;A)t^n) = -f_n(x,y;A)t^n$$

and our claim is justified.

Further, the relation $Dy_n = -(n+1)y_{n+1}$ gives following differential recurrence relation on operator $D$

$$\left(\lambda I - \frac{2y}{A}\right) t\frac{\partial}{\partial x} - \lambda xtAf_n(x,y;A)t^n = -(n+1)f_{n+1}(x,y;A)t^{n+1}$$

$$\left(\lambda I - \frac{2y}{A}\right)\frac{\partial}{\partial x}f_n(x,y;A) = \lambda xAf_n(x,y;A) - (n+1)If_{n+1}(x,y;A) \tag{6.61}$$

The relation $Cy_n = y_{n-1}$ gives the following differential recurrence relation on matrix

$A$

$$\left[\frac{1}{\lambda t A}\frac{\partial}{\partial x}\right] f_n(x,y;A)t^n = f_{n-1}(x,y;A)t^{n-1}$$

or

$$\frac{1}{A\lambda} f_n'(x,y;A)t^{n-1} = f_{n-1}(x,y;A)t^{n-1}$$

or

$$\frac{\partial}{\partial x} f_n(x,y,A) = A\lambda f_{n-1}(x,y,A) \tag{6.62}$$

Finally, the relation $DCy_n = -ny_n$ gives

$$\left[\left(\lambda I - \frac{2y}{A}\right) t\frac{\partial}{\partial x} - \lambda x t A\right]\left[\frac{1}{\lambda t A}\frac{\partial}{\partial x}\right] f_n(x,y;A)t^n = -nf_n(x,y;A)t^n$$

$$\Rightarrow \left(\lambda I - \frac{2y}{A}\right)\frac{1}{\lambda A}\frac{\partial^2}{\partial x^2} f_n(x,y;A)t^n - xI\frac{\partial}{\partial x} f_n(x,y;A)t^n = -nIf_n(x,y;A)t^n$$

$$\Rightarrow \left[\frac{\lambda A - 2yI}{A^2\lambda}\right]\frac{\partial^2}{\partial x^2} f_n(x,y;A) - xI\frac{\partial}{\partial x} f_n(x,y;A) + nIf_n(x,y;A) = 0 \tag{6.63}$$

Now, we observe that 2-variable generalized Hermite matrix polynomial $H_n^\lambda(x,y;A)$ is the solution of the differential equation (6.63). Further we note that the relation (6.61) and (6.62) are differential recurrence relation satisfied by 2-variable generalized Hermite matrix polynomial $H_n^\lambda(x,y;A)$.

## 6.6 2- variables Laguerre Matrix Polynomials

In this section we apply Theorem 6.2.8 to obtain differential recurrence relations and differential equations for 2-variables Laguerre matrix polynomials.

**Theorem 6.6.1.** *The 2-variable Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ in Eq. (6.27) satisfies the following differential equation*

$$-\frac{xy}{\lambda}\frac{\partial^2}{\partial x^2} - \left[\frac{Ay}{\lambda} + \left\{\frac{y}{\lambda}(n+1) - x\right\} I\frac{\partial}{\partial x} - A\right] f_n(x,y;A) = -nf_n(x,y;A) \tag{6.64}$$

*and the recurrence relation*

$$\left[\frac{x}{\lambda y}\frac{\partial}{\partial x}+\frac{n+AI}{\lambda y}\right] f_n(x,y;A) = -f_{n-1}(x,y;A) \tag{6.65}$$

*or*

$$\left[-y^2 I\frac{\partial}{\partial x}+I\lambda y\right] f_n(x,y;A) = -(n+1)f_{n+1}(x,y;A) \tag{6.66}$$

Proof: Let $V=\mathbb{C}^{N\times N}$, we define the operator $C, D \in End\,V$ as

$$C = -\left[\frac{x}{\lambda t y} I\frac{\partial u}{\partial x}+\frac{I}{\lambda y}\frac{\partial u}{\partial t}+\frac{1}{\lambda t y}A\right] \tag{6.67}$$

$$D = -Iy^2 t\frac{\partial u}{\partial x}+I\lambda t y \tag{6.68}$$

For $x,y,t\in\mathbb{C}$, $A$ is the matrix in $\mathbb{C}^{N\times N}$ and $I$ is the unit matrix in $\mathbb{C}^{N\times N}$.

We claim that the operators (6.67) and (6.68) obey the commutation relation $[C,D]y_n = -y_n$.

Indeed,

$$\begin{aligned}
[C,D]u(x,y,t;A) =& C(Du(x,y,t;A)) - D(Cu(x,y,t;A))\\
=& \left[\frac{Ix}{\lambda t y}\frac{\partial}{\partial x}-\frac{I}{\lambda y}\frac{\partial}{\partial t}-\frac{1}{\lambda t y}A\right]\left[-y^2 t I\frac{\partial u}{\partial x}+I\lambda t y\right]\\
&-\left[-y^2 t I\frac{\partial}{\partial x}+I\lambda t y\right]-\left[\frac{Ix}{\lambda t y}\frac{\partial u}{\partial x}+\frac{I}{\lambda y}\frac{\partial u}{\partial t}+\frac{I}{\lambda t y}A\right]\\
=& -u(x,y,t;A)
\end{aligned} \tag{6.68}$$

i.e.

$$[C,D]u(x,y,t;A) = -u(x,y,t;A)$$

Now, if $u(x,y,t;A)$ assumes the form $u(x,y,t;A) = f_n(x,y;A)t^n \in \mathbb{C}^{N\times N}$, then we have

$$[C,D](f_n(x,y;A)t^n) = -f_n(x,y;A)t^n$$

and our claim is justified.

Now, the relation $Dy_n = -(n+1)y_{n+1}$ gives following differential recurrence relation on operator $C$

$$\left[-y^2tI\frac{\partial}{\partial x} + I\lambda ty\right] f_n(x,y;A)t^n = -(n+1)f_{n+1}(x,y;A)t^{n+1}$$

or

$$\left[-y^2I\frac{\partial}{\partial x} + I\lambda y\right] f_n(x,y;A) = -(n+1)f_{n+1}(x,y;A) \tag{6.69}$$

Also from the relation $Cy_n = y_{n-1}$ we obtain the following differential recurrence relation on matrix $A$

$$\left[\frac{-Ix}{\lambda ty}\frac{\partial}{\partial x} - \frac{I}{\lambda y}\frac{\partial}{\partial t} - \frac{1}{\lambda ty}A\right] f_n(x,y;A)t^n = f_{n-1}(x,y;A)t^{n-1}$$

$$\left[\frac{x}{\lambda y}\frac{\partial}{\partial x} + \frac{nI+A}{\lambda y}\right] f_n(x,y;A) = -f_{n-1}(x,y;A) \tag{6.70}$$

Finally, the relation $DCy_n = -ny_n$ gives

$$\left[-y^2t\frac{\partial}{\partial x} + \lambda ty\right]\left[\frac{-x}{\lambda ty}\frac{\partial}{\partial x} - \frac{1}{\lambda y}\frac{\partial}{\partial t} - \frac{1}{\lambda ty}A\right] f_n(x,y;A)t^n = -nIf_n(x,y;A)t^n$$

equivalently

$$-\frac{xy}{\lambda}\frac{\partial^2}{\partial x^2} - \left[\frac{Ay}{\lambda} + \left\{\frac{y}{\lambda}(n+1) - x\right\} I\frac{\partial}{\partial x} - A\right] f_n(x,y;A) = -nf_n(x,y;A) \tag{6.71}$$

Now, We observe that 2- variables matrix Laguerre polynomial $L_n^{(A,\lambda)}(x,y)$ is the solution of the differential equation (6.71). Further we note that the relation (6.69) and (6.70) are differential recurrence relations satisfied by the Laguerre Matrix polynomial $L_n^{(A,\lambda)}(x,y)$.

## 6.7 Conclusion

The approach used in this chapter can be applied to differential realization of dilatation and translation operators

$$D = x\frac{\partial}{\partial x}, T = \frac{\partial}{\partial x}$$

in one dimension. The action of one parameter groups generated by these operators $D$ and $T$ on a smooth function $f(x)$ is

$$e^{\alpha D} f(x) = f(e^{\alpha} x), e^{\beta D} f(x) = f(x + \beta)$$

These operators can be used to enlarge the Lie algebra for some classes of factorisable equations. Thus, Lie algebraic aspects of special functions which are limited to Lie algebra generated by raising and lowering operators can be extended by adding dilatation and translation operators to these Lie algebras for some families of factorisable equations. A more complete picture of the relation between some operators and some special functions is presented by *Humi* [66]. Many other interesting results can be obtained by appropriately applying the operators and action of one parameter groups generated by these operators $D$ and $T$ to generate new relations involving some special functions.

Further a new approach has been introduced in this chapter for studying some important properties of certain matrix special functions viz. recurrence relation, differential recurrence relation and differential equation. The method developed in this chapter can also be used to study some other special matrix functions which play vital role in mathematical physics, chemistry and mechanics.

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# List of Symbols

**Sets**

| | |
|---|---|
| $\in$ | belongs to |
| $\mathbb{N}$ | natural numbers, $\mathbb{N} := \{1, 2, 3, \ldots\}$ |
| $\mathbb{N}_0$ | counting numbers, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ |
| $\mathbb{Q}$ | rational numbers |
| $\mathbb{R}$ | real numbers |
| $\mathbb{C}$ | complex numbers, $\mathbb{C}_0 := \{x + iy \ : \ x, y \in \mathbb{R}, \ i = \sqrt{(-1)}\}$ |
| $C[a, b]$ | set of continuous function |
| $C^n[a, b]$ | set of function with continuous $n^th$ derivative |

**Functions**

| | |
|---|---|
| $\binom{n}{i}$ | Binomial coefficient |
| $[\cdot]$ | Celling function; $[x] = \min\{z \in \mathbb{Z} : \ z \leq x\}$ |
| $\Gamma(z)$ | Euler's continuous gamma function |
| $E_\alpha(z)$ | Mittag-Leffler function in one parameter, $\alpha$ |

| | |
|---|---|
| $E_{\alpha,\beta}(z)$ | Mittag-Leffler function in two parameter, $\alpha$, $\beta$ |
| $E^{\gamma}_{\alpha,\beta}(z)$ | Mittag-Leffler function in three parameter, $\alpha$, $\beta$, $\gamma$ |
| ${}_2F_1(a,b;c;z)$ | Gauss hypergeomatric function |
| $H^{m,n}_{p,q}(z)$ | Fox's H-function |
| ${}_p\Psi_q(z)$ | Fox-Wright function |
| ${}_pF_q^{(\alpha,\beta;\kappa,\mu)}$ | Extended generalized hypergeometric function |
| $B_{\mathfrak{p}}^{(\alpha,\beta;\kappa,\mu)}$ | Generalized beta function |
| ${}_p\gamma_q[z]$, ${}_p\Gamma_q[z]$ | Incomplete hypergeometric functions |
| $\mathrm{erf}(z)$ | Error function |
| $\mathrm{erfc}(z)$ | Complimentary error function |
| $J^{\mu,m}_{\nu,\lambda}(z)$ | Generalized Lommel-Wright function |
| $J^{\mu}_{\nu}(z)(z)$ | Bessel-Maitland functions |
| $H_\nu(z)$ | Struve function |
| $J_\nu(z)$ | Bessel function |
| $W_{\sigma,\eta}(z)$ | Whittaker function |
| **Fractional Operators** | |
| ${}_0\mathbb{D}^{\alpha}_t$ | Riemann-Liouville fractional derivative operator |
| ${}^C_0\mathbb{D}^{\alpha}_t$ | Caputo fractional derivative operator |
| $\mathbb{W}^{\alpha}_{\infty+}$ | Weyl fractional derivative operator |
| $\mathbb{E}^{\alpha,\eta}_{x+}$ | Erdélyi-Kober type fractional integral operator |
| ${}^{\mathrm{EK}}\mathbb{D}^{\alpha,\eta}_{\infty+}$ | Erdélyi-Kober fractional derivative operator |
| ${}_0\mathbb{D}^{\mu,\nu}_{a+}$ | Hilfer derivative operator |

| | |
|---|---|
| $\mathbb{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}$ | Hilfer-Prabhakar derivative operator |
| $\mathbb{I}^{\alpha,\beta,\eta}_{0+}$ | Saigo's integral operator |
| $\mathbb{D}^{\alpha,\beta,\eta}_{0+}$ | Saigo's derivative operator |
| $\mathbb{I}^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}$ | Marichev-Saigo-Maeda fractional integral operator |
| $\Delta^{\frac{\lambda}{2}}$ | Fractional Laplace operator |

**Integral Transforms**

| | |
|---|---|
| $P_\alpha(f)$ | $P_\alpha$-transform of function $f$ |
| $\mathbb{L}(f)$ | Laplace's transform of function $f$ |
| $\mathbb{J}^{(\alpha',\beta')}(f)$ | Jacobi transform of function $f$ |
| $\mathfrak{H}_\nu(f)$ | Hankel Transform of function $f$ |

# Bio-Data

| | | |
|---|---|---|
| Full Name | : | Sonal Jain |
| E-mail | : | sonaljainmnit@gmail.com |
| Education | | |
| July, 2013 – Till Date | : | Research Scholar, Department of Mathematics<br>National Institute of Technology, Jaipur. |
| July, 2004 – May, 2007 | : | Master of Science in Mathematics<br>Mohanlal Sukhadia University, Jaipur. |
| July, 2009–May, 2010 | : | Bachelor of Education, |
| | : | Janardan Rai Nagar Rajasthan Vidhyapeeth University,<br>Udaipur. |
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| Teaching Experience | | |
| February, 2013–July, 2013 | : | Assistant Professor, Department of Mathematics.<br>Vivekananda Institute of Technology, Jaipur. |
| December, 2011–February, 2013 | : | Assistant Professor, Department of Mathematics.<br>Apex Institute of Engineering & Technology College, Jaipu |

Publications

(1) Solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type, *Progr. Fract. Differ. Appl.*, **1(3)**, 145-155, 2015.

(2) Certain properties of some special matrix functions via Lie algebra, *International Bulletin of Mathematical Research*, **2(1)**, 9-15, 2015.

(3) Analytic solution of generalized space time advection-dispersion equation with fractional Laplace operator, *Journal of Nonlinear Science and Applications, (SCI Extended)*, **9(6)**, 3545-3554, 2016.

(4) Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions, *Mathematical Methods in the Applied Sciences, (Wiley Online Library, SCI Extended)*, **40**, 255-273, 2017.

(5) Analytic solution of generalized space time fractional reaction diffusion equation, *Fractional Differential Calculus*, **7(2)**, 169-184, 2017.

(6) A unified study of orthogonal polynomials via Lie algebra, *Reports on Mathematical Physics*, Elsevier, SCI Extended, *79(1)*, 2017.

(7) A family of the incomplete hypergeometric functions and associated integral transform and fractional derivative formulas, Filomat, *31(1)*, (125-140).

(8) Generalized Lommel-Wright function associated with Saigo-Maeda fractional derivative operator, Communicated.

National/ International Conferences

(1) International Conference of 'Society for Special Functions and Applications' (ICSFA) and Symposium on' Applications in Engineering and Technology' at Malaviya National Institute of Technology, Jaipur, December 13-15, 2013.

(2) National Conference on ‘Science Engineering’ (NCSE) at JK Lakshimipat University, Jaipur, July 27-28, 2014.

(3) National Conference on ‘Mathematical Analysis and Computation’ (NCMAC-2015) at Malaviya National Institute of Technology, Jaipur, February 20-21, 2015.

(4) International Conference of ‘Society for Special Functions and Applications’ (ICSFA) and Symposium on ’Fractional Calculus and its Applications to Special Function’s, Amity University, Noida, Uttar Pradesh, India, September 10-12, 2015.

(5) International Conference on ‘Recent Advances in Mathematics and their Applications’ at University of Rajasthan, India, July 10-12, 2016.

(6) 24th International Conference on ‘Finite or Infinite Dimensional Complex Analysis and Applications’, Anand International College of Engineering, Jaipur, India, 22-26 August, 2016.

Seminaar/ Workshop/ Short term training Programme

(1) Short term course on ‘Application of Matlab in Engineering’, Malaviya National Institute of Technology, Jaipur, September 16-20, 2013.

(2) Short term course on ‘Analysis and Applications’ (STCAA 2014), Malaviya National Institute of Technology, Jaipur, November 10-14, 2014.

(3) Workshop on ‘Self Empowerment: Coping with Workplace Harassment Conducted by women cell’, Malaviya National Institute of Technology, Jaipur, April 11, 2014.

(4) Workshop on ‘Professional Communication’, Malaviya National Institute of Technology, Jaipur, July 10-11, 2014.

(5) Seminar on 'MATLAB and Simulink for Engineering Education' by Math Works India, November 21, 2014.

(6) Advanced Level Workshop on 'Computational Methods for Control Problems' (CMPC) Jointly organized by Indian Institute of Space Science and Technology Trivendrum and Mar ivanios College, Trivendrum, India, March 16-21, 2015.

(7) 'Bluenose Applied and Computational Math Days' Workshop organized by Saint Marys University, Halifax, Nova Scotia, Canada, July 11-12, 2015.

(8) Workshop on 'Pattern Formation' Organized by Dalhousie University, Halifax, Canada, July 18-19, 2015.

(9) Instructional School for 'Teachers of on Algebra(NCM)', Malaviya National Institute of Technology, Jaipur with joint Support of TIFR Bombay, December 7-19, 2015.

(10) Short Term Course on 'Matlab: A Tool in Research', Malaviya National Institute of Technology, Jaipur, December 24-28, 2015.

(11) Workshop on 'Latex for Research', Malaviya National Institute of Technology, Jaipur, July 23-24, 2016.

Summer School

(1) 11th SERB School on 'Matrix Methods and Fractional Calculus held at Centre for Mathematical and Statistical Sciences' (CMSS) Peechi Campus, KFRI, Peechi- 680 653, Trichur, Kerala, 28th April- 24th May, 2014.

(2) AARMS-PIMS Summer School in 'Differential Equations and Numerical Analysis', Dalhousie University, Halifax, Canada, 6th July- 31st July, 2015.

Poster Presented

1. Poster presentation on 'Analytic Solution of Generalized Fractional Space Time Reaction Diffusion Equation' in Bluenose Applied and Computational Math Days Workshop at Saint Marys University, Halifax, Nova Scotia, Canada, July 11-12, 2015.

Awards/Travel Grant Received

(1) International Travel Grant From National Board of Higher Mathematics (NBHM) For AARMS-PIMS Summer School in Differential Equations and Numerical Analysis, Dalhousie University, Halifax, Canada, 6th July- 31st July, 2015.