ON RECENT ADVANCES IN SPECIAL FUNCTIONS AND FRACTIONAL CALCULUS WITH APPLICATIONS

THESIS

SUBMITTED IN PARTIAL FULFILLMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

By

PRIYANKA ASHOK HARJULE

(ID. No. 2012RMA9538)

Supervisor

Prof. RASHMI JAIN

Professor

Department of Mathematics

DEPARTMENT OF MATHEMATICS

MALAVIYA NATIONAL INSTITUTE OF TECHNOLOGY, JAIPUR

JAIPUR-302017 (INDIA)

January, 2017

Prof. (Mrs.) Rashmi Jain

Department of Mathematics
MNIT, Jaipur-302017
E-mail: rashmiramessh1@gmail.com

5, Satya Vihar
Opp. Jain ENT Hospital
Lal Kothi, Jaipur 302015
Ph: 0141-2741135(R)
9549654373(M)

CERTIFICATE

This is to certify that Ms **Priyanka Ashok Harjule** has worked under my supervision for the award of degree of Doctor of Philosophy in Mathematics on the topic entitled, **"ON RECENT ADVANCES IN SPECIAL FUNCTIONS AND FRACTIONAL CALCULUS WITH APPLICATIONS".** The findings contained in this thesis are original and have not been submitted to any University/Institute, in part or full, for award of any degree.

RASHMI JAIN

(SUPERVISOR)

Malaviya National Institute of Technology

Jaipur -302017 (INDIA)

CANDIDATE'S DECLARATION

I hereby declare that the Ph.D. thesis entitled **'ON RECENT ADVANCES IN SPECIAL FUNCTIONS AND FRACTIONAL CALCULUS WITH APPLICATIONS'** is my own work conducted under the supervision of Prof. Rashmi Jain, Malaviya National Institute of Technology, Jaipur (Rajasthan), India.

I firmly declare that the presented work does not contain any part of any work that has been submitted for the award of any degree either in this University or in any other University/ Deemed University without proper citation.

Priyanka Harjule

ACKNOWLEDGEMENTS

I feel great privilege and pleasure in expressing my sincere and deepest sense of gratitude to my supervisor and mentor Prof. (Mrs.) Rashmi Jain, Department of Mathematics for her scholarly guidance and affection. It is because of her helpful attitude, incessant direction, invaluable support, uninterrupted and expert supervision that I could accomplish the present manuscript.

I acknowledge my sincere gratitude to Prof. I.K. Bhatt, Former Director, MNIT for his kind help, inspiration and encouragement. I am equally thankful to, Prof. A.B. Gupta, Director, MNIT his encouragement, suggestions, support and facility provided. I express my deep and profound sense of gratitude to my DPGC convener Prof. K.C. Jain, DREC members Dr. Vatsala Mathur (HOD), Dr. Sanjay Bhatter, Dr. Ritu Agarwal, Department of Mathematics, MNIT, Jaipur, for their persistent encouragement and suggestions.

My special thanks are due to Dr. Awdhesh Bhardwaj, Department of Management, whose prompt help and positive attitude helped me a lot in completing this investigation.

My heartfelt gratitude is due to Dr. K.C. Gupta, former Professor and Head, Department of Mathematics, MNIT, Jaipur for his all time invaluable support, matchless suggestions, uninching and timely help that paved the way for completion of my thesis.

My special regards are due to Mr. R.K. Jain , IAS (husband of Prof. Rashmi Jain) and Mrs. Prakash Gupta (wife of Prof K.C. Gupta) for showering their affectionate blessings and best wishes for the accomplishment of my work .

I am much grateful to all the faculty members, my fellow research scholar Mr. Manish Kumar Bansal for his support and kind cooperation, and my senior and junior research workers for enthusiastic encouragement.

I have no words to express my indebtedness to my husband Sandesh Nayak, I.A.S, my one year old son Vihaan and my beloved parents for their continuous support, inspiring moral encouragement and patience in waiting for the completion of my doctoral thesis.

Finally, I express my sincere indebtedness to the Almighty for his kind and everlasting blessings.

Priyanka Harjule

ABSTRACT

The main intent of this thesis is to establish some new results in the area of special functions and fractional calculus. The study is presented having divided into six chapters

Chapter 1 is intended to provide an introduction to various special functions, polynomials and fractional integral operators studied by some of the earlier researchers. Further, we present the brief chapter by chapter summary of the thesis.

In **chapter 2**, first of all we give definition of a generalized Riemann-Liouville fractional derivative operator $D_{a+}^{\mu,\nu}$ of order μ and type ν. Then, we introduce and investigate an integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ which contains H-function in its kernel. Next we find solutions to two different fractional differential equations in theorem form using these operators. Since $\mathcal{H}_{0+;p,q;\beta}^{w;m,n;\alpha}$ is general in nature, by specializing the parameters we can obtain a number of special cases of these theorems involving special cases of the integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ and giving appropriate values to $f(x)$. Furthermore numerical examples are calculated and using these examples graphical illustrations are presented.

Chapter 3 deals with general fractional Kinetic differintegral equation involving the fractional operator $D_{0+}^{\mu,\nu}$ and an integral operator whose kernel involves the general class of polynomials S_N^M. We make use of Laplace transform method to solve the fractional kinetic dif-

ferintegral equation. On account of general nature of S_N^M occuring in the fractional kinetic equation, a number of results involving simpler polynomials also follow as special cases of our main result. We give here six special cases involving Laguerre polynomial, Bessel polynomial, Gould and Hopper polynomial, Brafman polynomial and Cesaro polynomial in the kernel of the integral operator occuring in the fractional kinetic differintegral equation respectively.

Chapter 4 deals with the study of fractional differential integral operator. First, we define the operator $(D_{p+}^{\gamma,\mu,\nu})$ of our study and then obtain image of a product of H-function and $\overline{H}-$ function under this operator. Fractional integral operator involving a number of simpler functions in its kernel follow as its special cases. we record here four such special cases. Next we derrive two new and interesting composition formulae for the fractional integral operator $I_{a+}^{\gamma,\mu}$ and the integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$. Then we give the composition formulae for the fractional integral operators $I_{a+}^{\mu}, D_{a+}^{\mu}, D_{p+}^{\gamma,\mu,\nu}$ and integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$.

The object of **chapter 5** is to find solutions of two volterra-type integral equations associated with integral operators whose kernels involve $\overline{H}$-function and a product of general class of polynomials S_N^M and multivarible $H-$function respectively. We make use of convolution technique to solve these equations. We have obtained a large number of integral equations involving products of several useful polynomials and special functions as its special cases.

In **chapter 6** we evaluate a unified and general finite integral whose integrand involves the product of generalized modified Bessel function $\lambda_{\mu,\nu}^{(\eta)}$,general class of polynomials S_N^M and the multivariable $H-$function.

The arguments of the functions occurring in the integrand involve the product of factors of the form $x^{\rho-1}(a-x)^{\sigma}(1+(bx)^{\ell})^{-\lambda}$. Main integral is believed to be new and is capable of giving a large number of simpler integrals (new and known) involving several special functions and polynomials as its special cases.For the sake of illustration we record here six new integrals as its special cases.

Four research papers have already been published, two accepted for publication and two have been communicated for publication, in reputed journals having a bearing on the subject matter of the thesis.

Contents

1

INTRODUCTION TO THE TOPIC OF STUDY AND BRIEF CHAPTER BY CHAPTER SUMMARY OF THE THESIS

The present chapter deals with an introduction to the topic of the study as well as a brief review of the contributions made by some of the earlier workers on the subject matter presented in this thesis. Next a brief chapter by chapter summary of the thesis has been given. At the end of this chapter, list of research paper having a bearing on subject matter has been given.

1.1 SPECIAL FUNCTIONS

1.1.1 ${}_2F_1, {}_1F_1, {}_pF_q$ AND THE $G-$FUNCTION

The core of special functions is the Gaussian hypergeometric function ${}_2F_1$ and its confluent forms, the confluent hypergeometric functions ${}_1F_1$ and ψ. The confluent hypergeometric functions slightly modified are also known as Whittaker functions. The ${}_2F_1$ includes as special cases Legendre functions, the incomplete beta function, the complete elliptic functions of the first and second kinds, and most of classical orthogonal polynomials. The confluent hypergeometric functions include as special cases Bessel functions, parabolic cylinder functions, Coulomb wave functions, and incomplete gamma functions. Numerous properties of confluent hypergeometric functions flow directly from a knowledge of the ${}_2F_1$, and a basic understanding of the ${}_2F_1$ and ${}_1F_1$ is sufficient for the derivation of many characteristics of all the other above-named functions. A natural generalization of the ${}_2F_1$ is the generalized hypergeometric function, the so-called ${}_pF_q$, which in turn is generalized by Meijer $G-$function. The theory of the ${}_pF_q$ and the $G-$function is fundamental in the applications, since they contain as special cases all the commonly used functions of analysis. Further, these functions are the building blocks for many functions which are not members of the hypergeometric family.

1.1.2 THE $H-$FUNCTION

The $H-$function is defined by the following Mellin-Barnes type integral[49, p.10] with the integrand containing products and quotients of the Euler gamma functions. Such a function generalizes most of the known special functions.

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ \\ (b_j,\beta_j)_{1,q} \end{matrix} \right. \right] = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q) \end{matrix} \right. \right]$$

$$:= \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \Theta(\mathfrak{s}) z^{\mathfrak{s}}\, d\mathfrak{s}, \tag{1.1.1}$$

where $\omega = \sqrt{-1}$, $z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers, and

$$\Theta(\mathfrak{s}) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \mathfrak{s}) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \mathfrak{s})}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j \mathfrak{s}) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \mathfrak{s})}, \tag{1.1.2}$$

m, n, p and q are non-negative integers satisfying

$$1 \leqq m \leqq q \quad \text{and} \quad 0 \leqq n \leqq p \quad ; \alpha_j (j = 1, ..., p) \quad \text{and} \quad \beta_j (j = 1, ..., q)$$

are assumed to be positive quantities for standardization purposes. The definition of the H-function given by (1.1.1) will, however, have meaning even if some of these quantities are zero. Also, $a_j (j = 1, ..., p)$ and $b_j (j = 1, ..., q)$ are complex numbers such that none of the points

$$\mathfrak{s} = \frac{b_h + \nu}{\beta_h} \quad h = 1, ..., m; \nu = 0, 1, 2, ... \tag{1.1.3}$$

which are the poles of $\Gamma(b_h - \beta_h s), h = 1, ..., m$ and the points

$$\mathfrak{s} = \frac{a_i - 1 - \eta}{\alpha_i} \quad i = 1, ..., n; \eta = 0, 1, 2, ... \tag{1.1.4}$$

which are the poles of $\Gamma(1 - a_i + \alpha_i s)$ coincide with one another, i.e

$$\alpha_i(b_h + \nu) \neq b_h(a_i - \eta - 1) \tag{1.1.5}$$

for $\nu, \eta = 0, 1, 2, ...; h; 1, ..., m; i = 1, ..., n.$

Further, the contour $\mathfrak{L}$ runs from $-\omega\infty$ to $+\omega\infty$ such that the poles $\Gamma(b_h - \beta_h s), h = 1, ..., m,$ lie to the right of $\mathfrak{L}$ and and the poles of $\Gamma(1 - a_i + \alpha_i s) \quad i = 1, ...n,$ lie to the left of $\mathfrak{L}$. Such a contour is possible on account of (1.1.5). These assumptions will be adhered to throughout the present work.

SPECIAL CASES

We list here a few interesting cases of the $H-$function which may be useful for the workers on integral transforms and special functions.

1. **Confluent hypergeometric function of Kummer**

$$H_{1,2}^{1,1}\left[z \left| \begin{matrix} (1-a, 1) & \\ (0, 1), & (1-c, 1) \end{matrix} \right.\right] = \frac{\Gamma(a)}{\Gamma(c)} {}_1F_1(a; c; -z) \tag{1.1.6}$$

which is the so-called confluent hypergeometric function of Kummer[6, 6.1]

2. **Hypergeometric function**[49, p.18]

$$H_{2,2}^{1,2}\left[z \left| \begin{matrix} (1-a, 1), & (1-b, 1) \\ (0, 1), & (1-c, 1) \end{matrix} \right.\right] = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b; c; -z); \tag{1.1.7}$$

3. **Generalized Hypergeometric function**[49, p.18]

$$H^{1,p}_{p,q+1}\left[z\left|\begin{array}{cc}(1-a_i,1)_{1,p} & \\ (0,1), & (1-b_j,1)_{1,q}\end{array}\right.\right]=\frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q(a_1,...,a_p;b_1,...,b_q;-z). \tag{1.1.8}$$

4. **MacRobert $E-$function**

$$H^{p,1}_{q+1,p}\left[z\left|\begin{array}{cc}(1,1) & (b_j,1)_{1,q}\\ (a_i,1)_{1,p} & \end{array}\right.\right]=E(a_1,...,a_p:b_1,...,b_q;z), \tag{1.1.9}$$

where $E(a_1,...,a_p:b_1,...,b_q;z)$ is the MacRobert $E-$function[6, section 5.2].

Now we give the $H-$functions which are reduced to Bessel type functions:

5. **Bessel function of the first kind**

$$H^{1,0}_{0,2}\left[\frac{z^2}{4}\left|\begin{array}{cc}---- & \\ (\frac{a+\eta}{2},1), & (\frac{a-\eta}{2},1)\end{array}\right.\right]=\left(\frac{z}{2}\right)^a J_\eta(z), \tag{1.1.10}$$

where $J_\eta(z)$ is the Bessel function of the first kind[49, p.19]

6. **Modified Bessel function of the third kind or Macdonald function**

$$H^{2,0}_{0,2}\left[\frac{z^2}{4}\left|\begin{array}{cc}---- & \\ (\frac{a-\eta}{2},1), & (\frac{a+\eta}{2},1)\end{array}\right.\right]=2\left(\frac{z}{2}\right)^a K_\eta(z), \tag{1.1.11}$$

where $K_\eta(z)$ is the modified Bessel function of the third kind or Macdonald function[7, section 7.2.2 and (8.9.2)]

7. **Generalized Mittag-Leffler function**

$$H^{1,1}_{1,2}\left[-z\left|\begin{array}{cc}(1-\gamma,\kappa) & \\ (0,1), & (1-\beta,\alpha)\end{array}\right.\right]=\Gamma(\gamma)E^{\gamma,\kappa}_{\alpha,\beta}(z) \tag{1.1.12}$$

where $E^{\gamma,\kappa}_{\alpha,\beta}$ is the generalized Mittag-Leffler function given by [56]

8. **Reduced Green function**

$$H^{2,1}_{3,3}\left[z\left|\begin{array}{ccc}(1,1/\alpha), & (1,\beta/\alpha), & (1,\rho)\\ (1,1/\alpha), & (1,1), & (1,\rho)\end{array}\right.\right]=(\alpha z)K^{\theta}_{\alpha,\beta}(z) \tag{1.1.13}$$

Where $\rho = (\alpha - \theta)/2\alpha$ and

$K^{\theta}_{\alpha,\beta}$ stands for reduced Green function[19, p.11, eq. (10)]

9. **Lorenzo-Hartley R-function**

$$H^{1,1}_{1,2}\left[-az^{q}\left|\begin{array}{ll}(0,1) & \\ (0,1), & (1+\nu-q,q)\end{array}\right.\right] = \frac{1}{z^{q-\nu-1}}R_{q,\nu}[a,z] \tag{1.1.14}$$

Here $R_{q,\nu}$ is the Lorenzo-Hartley R-function[14, p.64, eq.(2.4)]see also[35]

10. **Lorenzo-Hartley G-function**

$$H^{1,1}_{1,2}\left[-az^{q}\left|\begin{array}{ll}(1-r,1) & \\ (0,1), & (1+\nu-rq,q)\end{array}\right.\right] = \frac{\Gamma(r)}{z^{rq-\nu-1}}G_{q,\nu,r}[a,z] \tag{1.1.15}$$

Here $G_{q,\nu,r}$ is the Lorenzo-Hartley G-function[14, p.64, eq.(2.3)] see also [35]

11. **Miller-Ross functions**

$$H^{1,1}_{1,2}\left[-az\left|\begin{array}{ll}(0,1) & \\ (0,1), & (-\nu,1)\end{array}\right.\right] = \frac{1}{z^{\nu}}E_{z}[\nu,a] \tag{1.1.16}$$

$$H^{1,1}_{1,2}\left[a^{2}z^{2}\left|\begin{array}{ll}(0,1) & \\ (0,1), & (-\nu,2)\end{array}\right.\right] = \frac{1}{z^{\nu}}C_{z}[\nu,a] \tag{1.1.17}$$

Here E_z and C_z are the Miller-Ross functions[19, p.14,eq. (21,22)]

Finally we present the special cases of the $H-$function which cannot be obtained from the $G-$function:

11. **Wright's generalized hypergeometric function**

$$H^{1,p}_{p,q+1}\left[-z\left|\begin{array}{ll}(1-a_i,\alpha_i)_{1,p} & \\ (0,1), & (1-b_j,\beta_j)_{1,q}\end{array}\right.\right] = {}_p\Psi_q\left[\begin{array}{l}(a_i,\alpha_i)_{1,p};\\ (b_j,\beta_j)_{1,q};\end{array} z\right], \tag{1.1.18}$$

where

$${}_p\Psi_q\left[\begin{array}{l}(a_i,\alpha_i)_{1,p};\\ (b_j,\beta_j)_{1,q};\end{array} z\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+k\alpha_i)}{\prod_{j=1}^{q}\Gamma(b_j+k\beta_j)}\frac{z^k}{k!} \tag{1.1.19}$$

is Wright's generalized hypergeometric function[6, section 4.1]

12. **Generalized Modified Bessel function**[10, p.152, eq.(1.2); p.155, eq.(2.6)]

The following special case has been used in the thesis in chapter 6:

$$H_{1,2}^{2,0}\left[z\left|\begin{array}{ll}(1-\frac{\sigma+1}{\beta},\frac{1}{\beta}) & \\ (0,1), & (-\gamma-\frac{\sigma}{\beta},\frac{1}{\beta})\end{array}\right.\right]=\lambda_{\gamma,\sigma}^{(\beta)}(z) \tag{1.1.20}$$

with

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)=\frac{\beta}{\Gamma(\gamma+1-\frac{1}{\beta})}\int_1^\infty (t^\beta-1)^{\gamma-\frac{1}{\beta}}e^{-zt}dt \tag{1.1.21}$$

$$\left(\beta>0;\Re(\gamma)>\frac{1}{\beta}-1;\sigma\in\mathbb{C},\Re(z)>0\right).$$

13. **Modified Bessel Function**[30, p.66 eq.(2.9.33)-(2.9.34)]

$$H_{1,2}^{2,0}\left[z\left|\begin{array}{ll}(\gamma+1-\frac{1}{n},\frac{1}{n}) & \\ (\gamma n,1), & (0,\frac{1}{n})\end{array}\right.\right]=(2\pi)^{\frac{(1-n)}{2}}n^{\gamma n+\frac{1}{2}}\lambda_\gamma^{(n)}(z) \tag{1.1.22}$$

with

$$\lambda_\gamma^{(n)}(z)=\frac{(2\pi)^{(n-1)/2}\sqrt{n}}{\Gamma(\gamma+1-\frac{1}{n})}\left(\frac{z}{n}\right)^{\gamma n}\int_1^\infty (t^n-1)^{\gamma-\frac{1}{n}}e^{-zt}dt \tag{1.1.23}$$

$$\left(n\in\mathbb{N};\Re(\gamma)>\frac{1}{n}-1,\Re(z)>0\right)$$

According to the following integral representations in Erdélyi, Magnus, Oberhettinger and Tricomi[7, 7.12(23) and 7.12(19)] for the modified Bessel function of the third kind or the Macdonald function $K_\nu(z)$:

$$K_\nu(z)=\frac{1}{2}\int_0^\infty e^{-z(t+1/t)/2}t^{-\nu-1}dt$$

$$=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-\nu)}\left(\frac{2}{z}\right)^\nu\int_1^\infty e^{-zt}(t^2-1)^{-\nu-\frac{1}{2}}dt \quad (\Re(z)>0), \tag{1.1.24}$$

the functions (1.1.23) and (1.1.21) are connected with the function $K_{-\gamma}(z)$ by the relations

$$\lambda_\gamma^{(2)}(z)=2\left(\frac{z}{2}\right)^\gamma K_{-\gamma}(z); \tag{1.1.25}$$

$$\lambda_{\gamma,0}^{(2)}(z) = \frac{2}{\sqrt{\pi}} \left(\frac{2}{z}\right)^{\gamma} K_{-\gamma}(z). \tag{1.1.26}$$

We also note that the function $\lambda_{\gamma}^{(n)}(z)$ in (1.1.23) is expressed via $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ in (1.1.21) when $\sigma = 0$ and $\beta = n \in \mathbb{N}$:

$$\lambda_{\gamma}^{(n)}(z) = (2\pi)^{(n-1)/2} n^{-(\gamma n+1/2)} z^{\gamma n} \lambda_{\gamma,0}^{(n)}(z). \tag{1.1.27}$$

1.1.3 THE $\overline{H}$−FUNCTION

Though the H-function is sufficiently general in nature, many useful functions notably generalized Riemann Zeta function [6], the polylogarithm of complex order [6], the exact partition of the Gaussian model in statistical mechanics [24],a certain class of Feynman integrals [6] and others do not form its special cases. Inayat Hussain [24] introduced a generalization of the H-function popularly known as $\overline{H}$ -function which includes all the above mentioned functions as its special cases. This function is developing fast and stands on a firm footing through the publications of Buschman and Srivastava [5], Rathie[40], Saxena [42, 43], Gupta and Soni[20], Jain and Sharma[28], Gupta, Jain and Sharma[18], Gupta, Jain and Agrawal[17] and several others.

The $\overline{H}$-function[24] is defined and represented in the following manner:

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, & (a_j, \alpha_j)_{N+1,P} \\ \\ (b_j, \beta_j)_{1,M}, & (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right]$$

$$:= \frac{1}{2\pi\omega} \int_{\mathfrak{L}} \overline{\Theta}(\xi) z^{\xi} d\xi \tag{1.1.28}$$

where, $\omega = \sqrt{-1}$,$z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi) = \frac{\prod\limits_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod\limits_{j=1}^{N} \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod\limits_{j=M+1}^{Q} \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod\limits_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)} \tag{1.1.29}$$

It may be noted that $\overline{\Theta}(\xi)$ contains fractional powers of some of the gamma functions. M, N, P, Q are integers such $1 \leq M \leq Q, 0 \leq N \leq P, \quad (\alpha_j)_{1,P}, (\beta_j)_{1,Q}$ and $(A_j)_{1,N}, (B_j)_{M+1,Q}$ are positive quantities for standardization purpose. $(a_j)_{1,P}$ and $(b_j)_{1,Q}$ are complex numbers such that the points $\xi = \dfrac{b_j + k}{\beta_j}; \quad j = 1, ..., M; \quad k = 0, 1, 2, ...$ which are the poles of $\Gamma(b_j - \beta_j \xi)$, and the points $\xi = \dfrac{a_j - 1 - k}{\alpha_j} \quad j = 1, ..., N; \quad k = 0, 1, 2, ...$ which are the singularities of $\{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}$, do not coincide. We retain these assumptions throughout the thesis.

The contour $\mathfrak{L}$ is the line from $c - i\infty$ to $c + i\infty$, suitably intended to keep the poles of $\Gamma(b_j - \beta_j \xi); \quad j = 1, ..., M$ to the right of the path, and the singularities of $\{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}; \quad j = 1, ..., N$ to the left of the path. If we take $A_i = B_j = 1 \quad (i = 1, ..., N; j = M + 1, ..., Q)$, then the $\overline{H}-$function reduces to the familiar $H-$function.

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}-$function given by (1.1.28) have been given by Gupta, Jain and Agarwal[17]

$$\begin{aligned} &(i)\, |\arg(z)| < \frac{1}{2}\Omega\pi \quad and \quad \Omega > 0 \\ &(ii)\, |\arg(z)| = \frac{1}{2}\Omega\pi \quad and \quad \Omega \geq 0 \end{aligned} \tag{1.1.30}$$

and

(a) $\mu \neq 0$ and the contour $\mathfrak{L}$ is so chosen that $(c\mu + \lambda + 1) < 0$

(b) $\mu = 0$ and $(\lambda + 1) < 0$

where

$$\Omega = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} \alpha_j A_j - \sum_{j=M+1}^{Q} \beta_j B_j - \sum_{j=N+1}^{P} \beta_j \tag{1.1.31}$$

$$\mu = \sum_{j=1}^{N} \alpha_j A_j + \sum_{j=N+1}^{P} \alpha_j - \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q} \beta_j B_j \tag{1.1.32}$$

$$\begin{aligned}\lambda =& Re\left(\sum_{j=1}^{M} b_j + \sum_{j=M+1}^{Q} b_j B_j - \sum_{j=1}^{N} a_j A_j - \sum_{j=N+1}^{P} a_j\right)\\ &+ \frac{1}{2}\left(-M - \sum_{j=M+1}^{Q} B_j + \sum_{j=1}^{N} A_j + p - N\right)\end{aligned} \tag{1.1.33}$$

The following series representation for the $\overline{H}-$Function given by Rathie [40] and Saxena[42]:

$$\overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P} \\ \\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] = \sum_{t=0}^{\infty}\sum_{h=1}^{M} \overline{\Theta}\mathfrak{s}_{t,h} z^{\mathfrak{s}_{t,h}} \tag{1.1.34}$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t,h}) = \frac{\prod\limits_{j=1,j\neq h}^{M} \Gamma(b_j - \beta_j \mathfrak{s}_{t,h}) \prod\limits_{j=1}^{N} \{\Gamma(1 - a_j + \alpha_j \mathfrak{s}_{t,h})\}^{A_j}}{\prod\limits_{j=M+1}^{Q} \{\Gamma(1 - b_j + \beta_j \mathfrak{s}_{t,h})\}^{B_j} \prod\limits_{j=N+1}^{p} \Gamma(a_j - \alpha_j \mathfrak{s}_{t,h})} \frac{(-1)^t}{t!\beta_h} \tag{1.1.35}$$

$$\mathfrak{s}_{t,h} = \frac{b_h + t}{\beta_h} \tag{1.1.36}$$

The following behaviour of the $\overline{H}_{P,Q}^{M,N}(z)$function for small and large values of z as recorded by Saxena et al[45, p.112, eqs.(2.3-2.4)],

$\overline{H}_{P,Q}^{M,N}(z) = O(|z|^g)$, for small z, where $g = \min\limits_{1\leq j\leq M} \Re\left(\frac{b_j}{\beta_j}\right)$

$\overline{H}_{P,Q}^{M,N}(z) = O(|z|^h)$, for large z, where $h = \max_{1\le j\le N} \Re\left(A_j \frac{a_j - 1}{\alpha_j}\right)$ provided that either of the following conditions are satisfied:

$$(i) \quad \mu < 0 \quad and \quad 0 < |z| < \infty \tag{1.1.37}$$

$$(ii) \quad \mu = 0 \quad and \quad 0 < |z| < \delta^{-1} \tag{1.1.38}$$

where

$$\mu = \sum_{1}^{N} \alpha_j A_j + \sum_{N+1}^{P} \alpha_j - \sum_{1}^{M} \beta_j - \sum_{M+1}^{Q} \beta_j B_j \tag{1.1.39}$$

$$\delta = \prod_{1}^{N} (\alpha_j)^{\alpha_j A_j} \prod_{N+1}^{P} (\alpha_j)^{\alpha_j} \prod_{1}^{M} (\beta_j)^{-\beta_j} \prod_{M+1}^{Q} (\beta_j)^{-\beta_j B_j} \tag{1.1.40}$$

SPECIAL CASES

The following special cases of the $\overline{H}-$function have been made use in this thesis:

1. **Polylogarithm function**

$$z\overline{H}_{1,2}^{1,1}\left[-z \left| \begin{array}{cc} (0,1;p+1) & \\ (0,1), & (-1,1;p) \end{array} \right.\right] = F(z,p) = \sum_{r=1}^{\infty} \frac{z^r}{r^p} \tag{1.1.41}$$

Here $F(z,p)$ is the polylogarithm function of order p [6, p.30]

2. **The $H-$function**

$$\overline{H}_{P,Q}^{M,N}\left[z \left| \begin{array}{cc} (a_j,\alpha_j;1)_{1,N}, & (a_j,\alpha_j)_{N+1,P} \\ \\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;1)_{M+1,Q} \end{array} \right.\right] = H_{P,Q}^{M,N}\left[z \left| \begin{array}{c} (a_i,\alpha_i)_{1,P} \\ \\ (b_j,\beta_j)_{1,Q} \end{array} \right.\right] \tag{1.1.42}$$

Naturally, all functions which are special cases of the $H-$function are also special cases of the $\overline{H}-$function.

1.1.4 THE MULTIVARIABLE $H-$FUNCTION

The multivariable $H-$function occuring in the thesis was introduced and studied by Srivastava and Panda [52, p. 130, eq. (1.1)]. This function involves r complex variables and will be defined and represented in the following contracted form [49, p. 251-252, eqs. (C.1-C.3)]

$$H^{0,B:A_1,B_1;...;A_r,B_r}_{C,D:C_1,D_1;...;C_r,D_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,C}:(c_j^{(1)},\gamma_j^{(1)})_{1,C_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r} \\ . & \\ . & \\ . & \\ z_r & (b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,D}:(d_j^{(1)},\delta_j^{(1)})_{1,D_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,D_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathfrak{L}_1}...\int_{\mathfrak{L}_r}\psi(\xi_1,...,\xi_r)\prod_{i=1}^{r}(\phi_i(\xi_i)z_i^{\xi_i})d\xi_1...d\xi_r \qquad (i=1,...,r) \quad (1.1.43)$$

where $\omega=\sqrt{-1}$,

$$\phi_i(\xi_i)=\frac{\prod\limits_{i=1}^{A_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod\limits_{j=1}^{B_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod\limits_{j=A_i+1}^{D_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod\limits_{j=B_i+1}^{C_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)} \qquad (i=1,...,r) \quad (1.1.44)$$

$$\psi(\xi_1,...,\xi_r)=\frac{\prod\limits_{j=1}^{B}\Gamma(1-a_j+\sum\limits_{i=1}^{r}d_j^{(i)}\xi_i)}{\prod\limits_{j=1}^{D}\Gamma(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod\limits_{j=B+1}^{C}\Gamma(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i)} \quad (1.1.45)$$

All the greek letters occuring on the left-hand side of (1.1.43) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable $H-$function will, however, be meaningful even if some of these quantities are zero such that

$$\Lambda_i \equiv \sum_{j=1}^{C} \alpha_j^{(i)} + \sum_{j=B_i+1}^{C_i} \gamma_j^{(i)} - \sum_{j=1}^{D} \beta_j^{(i)} - \sum_{j=1}^{D_i} \delta_j^{(i)} > 0 \quad (i = 1, 2, ..., r) \tag{1.1.46}$$

$$\Omega_i \equiv - \sum_{j=B+1}^{C} \alpha_j^{(i)} + \sum_{j=1}^{B_i} \gamma_j^{(i)} - \sum_{j=B_i+1}^{C_i} \gamma_j^{(i)} - \sum_{j=1}^{D} \beta_j^{(i)} + \sum_{j=1}^{A_i} \delta_j^{(i)} - \sum_{j=A_i+1}^{D_i} \delta_j^{(i)} > 0 \quad (i = 1, 2, ..., r) \tag{1.1.47}$$

where $B, C, D, A_i, B_i, C_i, D_i$ are non negative integers such that $0 \leq B \leq C$, $D \geq 0, 0 \leq B_i \leq C_i$ and $1 \leq A_i \leq D_i, \quad (i = 1, ..., r)$.

The sequences of the parameters in (1.1.43) are such that none of the poles of the integrand coincide i.e. the poles of the integrand in (1.1.43) are simple. The contour $\mathfrak{L}_i$ in the complex ξ_i- plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i) \quad (j = 1, ..., A_i)$ are separated from those of $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)}\xi_i)$ $(j = 1, ..., B_i)$ and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)}\xi_i) \quad (i = 1, ..., r; j = 1, ..., B)$

It is known that multiple Mellin-Barnes contour integral representing the multivariable $H-$ function (1.1.43) converges absolutely[53, p.130, eq.(1.4)] under the condition (1.1.47) when

$$|\arg(z_i)| < \frac{1}{2}\Omega_i\pi, \quad (i = 1, ..., r) \tag{1.1.48}$$

The point $z_i = 0 (i = 1, ..., r)$ and various exceptional parameter values are excluded.

SPECIAL CASES

By suitably specializing the various parameters occuring in the multivariable $H-$ function defined by (1.1.43), it reduces to the simpler special functions of one and more variables.

Some of them which have been used in this thesis are given below:

(i) If we take $\alpha_j^{(1)} = \alpha_j^{(2)} = ... = \alpha_j^{(r)} \quad (j = 1, ..., D)$ and $\beta_j^{(1)} = \beta_j^{(2)} = ... = \beta_j^{(r)} \quad (j = 1, ..., D)$ in (1.1.43), it reduces to a special multivariable $H-$ function studied by Saxena[42].

(ii) If we take all the Greek letters $\alpha's, \beta's, \gamma's$, and $\delta's$ equal to unity in (1.1.43) , it reduces to the corresponding $G-$function of several variables studied by Khadia and Goyal[29].

(iii) If we take $r = 2$, in (1.1.43), we get $H-$function of two variables defined in [49, p.82, eq.(6.1.1)].

(iv) A relation between $H-$function of two variable and the Appell function[49, p.89,eq.(6.4.6)] is given as below:

$$H_{0,1:2,1;2,1}^{0,0:1,2;1,2}\left[\begin{array}{c} -x \\ -y \end{array}\middle|\begin{array}{cccc} - : & (1-c,1),(1-c',1); & (1-e,1),(1-e',1) \\ (1-b;1,1) : & (0,1); & (0,1) \end{array}\right]$$

$$= \frac{\Gamma(c)\Gamma(c')\Gamma(e)\Gamma(e')}{\Gamma(b)} F_3(c,e,c',e';b;x,y), \quad |x| < 1, |y| < 1 \qquad (1.1.49)$$

1.1.5 GENERAL CLASS OF POLYNOMIALS

Srivastava [46, p.1 eq. (1)] has introduced the general class of polynomials

$$S_V^U[x] = \sum_{R=0}^{[V/U]} \frac{(-V)_{UR} \quad A_{V,R} \quad x^R}{R!} \qquad (V = 0, 1, 2, ...), \qquad (1.1.50)$$

where U is an arbitrarty positive integer, and the coefficients $A_{V,R}(V, R \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{V,R}, S_V^U[x]$ yields a number of known polynomials as its special cases. These include, among others, Jacobi polynomial, Laguerre polynomial and several others [55, p.158-161].If $x = 0, A_{0,0} = 1$, then $S_V^U[x]$ reduces to unity.

SPECIAL CASES

The following special cases of the general class of polynomials S_V^U will be required in the thesis:

(i) Laguerre Polynomials

On taking $U = 1, A_V, R = \begin{pmatrix} V+\alpha \\ V \end{pmatrix} \frac{1}{(\alpha+1)_R}$ in (1.1.50),

$$S_V^1 \to L_V^{(\alpha)}(x), \tag{1.1.51}$$

where,

$$L_N^{(\alpha)}(x) = \frac{(1+\alpha)_V}{V!} {}_1F_1(-V; 1+\alpha; x) \tag{1.1.52}$$

is the Laguerre Polynomial[57, p.101,eq.(5.1.6)]

(ii) Jacobi Polynomials

On taking $U = 1, A_V, R = \begin{pmatrix} V+\alpha \\ V \end{pmatrix} \frac{(\alpha+\beta+V+1)_R}{(\alpha+1)_R}$ in (1.1.50),

$$S_V^1 \to P_V^{(\alpha,\beta)}(1-2x), \tag{1.1.53}$$

where

$$\begin{aligned} P_V^{(\alpha,\beta)}(x) &= \sum_{R=0}^{V} \begin{pmatrix} V+\alpha \\ V-R \end{pmatrix} \begin{pmatrix} V+\beta \\ R \end{pmatrix} \left(\frac{x-1}{2}\right)^R \left(\frac{x+1}{2}\right)^{V-R} \\ &= \frac{(1+\alpha)_V}{V!} \sum_{R=0}^{V} \frac{(-V)_R(1+\alpha+\beta+V)_R}{(1+\alpha)_R R!} \left(\frac{1-x}{2}\right)^R \end{aligned} \tag{1.1.54}$$

is the Jacobi Polynomial.[57, p.68, eq.(4.3.2)]

(iii) Gould & Hopper Polynomial

Taking $A_{V,R} = 1$ in (1.1.50),

$$S_V^U(x) \to \left(\frac{-x}{h}\right)^{V/U} g_V^U\left[\left(\frac{-h}{x}\right)^{1/U}, h\right], \tag{1.1.55}$$

where

$$g_V^U(x,h) = \sum_{R=0}^{[V/U]} \frac{V!}{R!(V-UR)!} h^R x^{V-UR},$$

$$= x^V {}_MF_0 \left[\Delta(U;-V); -; h\left(\frac{-U}{x}\right)^U \right] \tag{1.1.56}$$

is the Gould & Hopper polynomial.[12, p.58, eq.(6.2)]

Here $\Delta(U;V)$ denotes the array of U parameters $\frac{V}{U}, \frac{V+1}{U}, ..., \frac{V+U-1}{U}, \quad U \geq 1$, the set $\Delta(0,V)$ being empty.

(iv) Cesaro Polynomial

Taking $U = 1, A_{V,R} = \dfrac{(s+1)_V R!}{V!(-s-V)_R}$

$$S_V^1(x) \rightarrow g_V^{(s)}(x) \tag{1.1.57}$$

where

$$g_V^{(s)}(x) = \begin{pmatrix} s+V \\ V \end{pmatrix} {}_2F_1 \left(\begin{matrix} -V, 1; \\ -s-V; \end{matrix} \; x \right) \tag{1.1.58}$$

is the Cesaro polynomial[51, p.449, eq.(20)].

(v) Bessel Polynomial

Taking $V = 1, A_{V,R} = (\alpha + V - 1)_R$ in (1.1.50),

$$S_V^1(x) \rightarrow y_V(-\beta x, \alpha, \beta), \tag{1.1.59}$$

where

$$y_V(-\beta x, \alpha, \beta) = \sum_{R=0}^{V} \frac{(-V)_R(\alpha+V-1)_R}{R!} \left(\frac{-x}{\beta}\right)^R$$

$$= {}_2F_0[-V, \alpha + V - 1; -; -x/\beta] \tag{1.1.60}$$

is the Bessel Polynomial consideres by Krall and Frink[34, p.108, eq.(34)]

(vi) Brafman Polynomial

Taking $A_{V,R} = \dfrac{(\alpha_1)_R ... (\alpha_p)_R}{(\beta_1)_R ... (\beta_q)_R}$ in (1.1.50),

$$S_V^U(x) \to B_V^U[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q : xU^U], \tag{1.1.61}$$

where

$$B_V^U[\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q : x] = {}_pF_q[\Delta(U; -V), \alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q : x] \tag{1.1.62}$$

is the Brafman polynomial.[3, p.186]

Here $\Delta(U; V)$ denotes the array of U parameters $\frac{V}{U}, \frac{V+1}{U}, ..., \frac{V+U-1}{U}, \quad U \geq 1$, the set $\Delta(0, V)$ being empty.

1.2 FRACTIONAL CALCULUS

The concept of differentiation operator $D = d/dx$ is familiar to all who have studied the elementary calculus. And for suitable functions f, namely $D^n f(x) = d^n f(x)/dx^n$ is well-defined provided that n is a positive integer. In 1695 L'Hospital inquired of Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as, Euler, Laplace, Fourier, Abel, Liouville, Riemann, Lacrocx, Pracoch, Heaviside, Weyl, Kober, Erdélyi and Laurent.

During the second half of the twentieth century, considerable amount of research in fractional calculus was published in engineering literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering.

1.2.1 FRACTIONAL DIFFERENTIAL INTEGRAL OPERATOR

A detailed account of various fractional integral operators studied from time to time has been given by Srivastava and Saxena[54]

The Riemann-Liouville fractional integral operator I^{μ}_{a+} and the Riemann-Liouville fractional derivative operator D^{μ}_{a+}, which are defined by (see, for details, [33], [37] and [41]):

$$(I^{\mu}_{a+}f)(x) = \frac{1}{\Gamma(\mu)}\int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\,dt \qquad \big(\Re(\mu) > 0\big) \tag{1.2.1}$$

and

$$(D^{\mu}_{a+}f)(x) = \left(\frac{d}{dx}\right)^n (I^{n-\mu}_{a+}f)(x) \qquad \big(\Re(\mu) > 0;\ n = [\Re(\mu)] + 1\big), \tag{1.2.2}$$

where $[x]$ denotes the greatest integer in the real number x. Hilfer [56] generalized the operator in (1.2.2) and defined a general fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 \leqq \nu \leqq 1$ with respect to x as follows:

$$(D^{\mu,\nu}_{a+}f)(x) = \left(I^{\nu(1-\mu)}_{a+}\frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+}f\right)\right)(x). \tag{1.2.3}$$

The generalization in (1.2.3) yields the classical Riemann-Liouville fractional derivative operator D^{μ}_{a+} when $\nu = 0$. When $\nu = 1$, (1.2.3) reduces to the fractional derivative operator introduced by Joseph Liouville, which is often attributed now-a-days to Caputo (see [33] and [68]; see also [11]).

The following generalized form of fractional integral operator studied earlier by Hilfer[22] will be used in chapter 4 of the thesis.

$$(D^{\gamma,\mu,\nu}_{p+}f)(x) = \left(I^{\gamma,\nu(1-\mu)}_{p+}\left(\frac{d}{dx}\right)^m (I^{\gamma,(1-\nu)(1-\mu)}_{p+}f)\right)(x) \tag{1.2.4}$$

with $0 < \mu < 1$ and $0 \leq \nu \leq 1$

where

$$(I^{\gamma,\mu}_{a+}f)(x) = \frac{(x-a)^{-\mu-\gamma}}{\Gamma(\mu)}\int_a^x \frac{t^{\gamma}f(t)}{(x-t)^{1-\mu}}dt \quad (Re(\gamma) > -1, \quad Re(\mu) > 0) \tag{1.2.5}$$

For $m = 1$ and $\gamma = 0$, (1.2.4) reduces to the Hilfer fractional derivative operator[22, p.43, see 11]

1.2.2 FRACTIONAL DIFFERENTIAL EQUATIONS

The first form of the Fractional Integral Equation is given by the following form:

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau = f(t), \quad 0 < \alpha < 1 \tag{1.2.6}$$

This may also be written as

$$J^{\alpha} u(t) = f(t) \tag{1.2.7}$$

The solution of this kind is straightforward, and written

$$u(t) = D^{\alpha} f(t) \tag{1.2.8}$$

In the present work we solve a set of following general fractional differential equations:

$$\left(D_{0+}^{\mu,\nu}\, y\right)(x) = \lambda \left(\mathcal{H}_{0+;p,q,\beta}^{w;m,n;\alpha}\, 1\right)(x) + f(x), \tag{1.2.9}$$

and

$$x\left(D_{0+}^{\mu,\nu}\, y\right)(x) = \lambda \left(\mathcal{H}_{0+;p,q,\beta}^{w;m,n;\alpha}\, 1\right)(x), \tag{1.2.10}$$

For definition of $\mathcal{H}_{0+;p,q,\beta}^{w;m,n;\alpha}$ see (2.2.1)

FRACTIONAL KINETIC EQUATION

Fractional Kinetic equations have gained popularity during the past decade mainly due to the discovery of their relation with the CTRW-theory in [21]. These equations are investigated in order to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on [22, 23].

In a recent investigation by Saxena and Kalla[44], the following fractional kinetic equation was considered:

$$N(t) - N_0 f(t) = -c^{\nu}(I_{0+}^{\nu}N)(t) \qquad (\Re(\nu) > 0), \tag{1.2.11}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and (for convenience) $f \in L(0, \infty)$, it being tacitly assumed that $f(0) = 1$ in order to satisfy the initial condition $N(0) = N_0$.

1.3 BRIEF CHAPTER BY CHAPTER SUMMARY

In **chapter 2**, first of all we give definition of a generalized Riemann-Liouville fractional derivative operator $D_{a+}^{\mu;\nu}$ of order μ and type ν. Then, we introduce and investigate an integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ which contains H-function in its kernel. Next we find solutions to two different fractional differential equations in theorem form using this operator. Since $\mathcal{H}_{0+;p,q;\beta}^{w;m,n;\alpha}$ is general in nature, by specializing the parameters we can obtain a number of special cases of these theorems involving special cases of the fractional integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ and giving appropriate values to $f(x)$. Furthermore numerical examples are calculated and using these graphical illustrations are presented.

chapter 3 deals with general fractional Kinetic differintegral equation involving the fractional operator $D_{0+}^{\mu;\nu}$ and an integral operator whose kernel involves the general class of polynomials S_N^M. We make use of Laplace transform method to solve the fractional kinetic differintegral equation. On account of general nature of S_N^M occuring in the fractional kinetic equation, a number of results involving simpler polynomials also follow as special cases of our main result. We give here six special cases involving Laguerre polynomial, Bessel polynomial, Gould and Hopper polynomial, Brafman polynomial and Cesaro polynomial in the kernel of the integral operator occuring in the fractional kinetic differintegral equation respectively.

chapter 4 deals with the study of a fractional differential integral operator. First, we define the operator of our study $D_{p+}^{\gamma,\mu,\nu}$ and then obtain image of a product of H-function and $\overline{H}-$ function under this operator. Fractional integrals involving a number of simpler functions follow as its special cases. we record here four special cases. Next we derive two new and interesting composition formulae for the fractional integral operator $I_{a+}^{\gamma,\mu}$ and the integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$. Then we give the composition formulae for the fractional integral operators $I_{a+}^{\gamma,\mu}, I_{a+}^{\mu}, D_{a+}^{\mu}, D_{a+}^{\mu,\nu}$ and integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$.

The object of **chapter 5** is to find solutions of two Volterra-type integral equations associated with integral operators whose kernels involve $\overline{H}$-function and a product of general class of polynomials S_N^M and multivarible $H-$function respectively. We make use of convolution technique to solve these equations. We have obtained a large number of integral equations involving products of several useful polynomials and special functions as its special cases.

In **chapter 6** we evaluate a unified and general finite integral whose integrand involves the product of generalized modified Bessel function $\lambda_{\mu,\nu}^{(\eta)}$,general class of polynomials S_N^M and the multivariable $H-$function. The arguments of the functions occurring in the integrand involve the product of factors of the form $x^{\rho-1}(a-x)^{\sigma}(1+(bx)^{\ell})^{-\lambda}$. Main integral is believed to be new and is capable of giving a large number of simpler integrals (new and known) involving several special functions and polynomials as its special cases.For the sake of illustration we record here six new integrals as its special cases involving modified Bessel function of third kind, Gould & Hopper polynomial and Mittag-leffler function;

generalized modified Bessel function of third kind, general class of polynomials, reduced Green function, Lorenzo-Hartley R-function, Miller-Ross functions,Cesaro polynomial and Lorenzo-Hartley G-function.

1.4 LIST OF RESEARCH PAPERS CONTRIBUTED BY THE AUTHOR

(i) A General Fractional Differential Equation Associated with an Integral operator with the H-function in the Kernel, Russian Journal of Mathematical Physics, **22(1)** (2015),112-126.

(ii) A study of Fractional Differential Equation with an Integral operator containing Fox's H-function in the Kernel, Int. Bull. of Mathematical research(IBMR)ISSN:2394-7802, **2(1)** (2015), 1-8.

(iii) A General Volterra type Fractional Equation Associated with an Integral operator with the H-function in the Kernel, Journal of Rajasthan Academy of Physical Sciences, **14(3)** (2015) 289-294.

(iv) A Study of Fractional Differential Integral operator,Proc. of the 12th annual conf. SSFA, **12** (2013), 73-77.

(v) A General Volterra type Fractional Equation Associated with an Integral operator with the $\overline{H}$-function in the Kernel, International Journal of Pure and Applied Mathematics(IJPAM) ISSN:1311-8080. (Accepted for Publication)

(vi) A General Volterra-type Integral Equation Associated with an Integral Operator involving the product of general class of polynomials and multivarible H-Function in the Kernel,Journal of Rajasthan Academy of Physical Sciences, Vol. 3 (2016) (In press)

(vii) A General Fractional kinetic Differintegral Equation Associated

with an Integral Operator with the general class of polynomial in the Kernel, Proceedings of the Jangjeon Mathematical Society.(communicated)

(viii) A Study of unified finite integral involving generalized modified Bessel function of third kind, general class of polynomials and the multivariable H-function, Journal of the Indian Academy of Mathematics. (Communicated)

2

ON GENERAL FRACTIONAL DIFFERENTIAL EQUATIONS ASSOCIATED WITH AN INTEGRAL OPERATOR HAVING THE H-FUNCTION IN THE KERNEL

The main findings of this chapter have been published as given below:

A General Fractional Differential Equation Associated with an Integral operator with the H-function in the Kernel, Russian J. of Math. Phy., **22(1)** (2015),112-126.

A study of Fractional Differential Equation with an Integral operator containing H-function in the Kernel, Int.Bull.of Math.Research,**2(1)** (2015),1-8.

In this chapter, First of all we give definition of a generalized Riemann-Liouville fractional derivative operator $D^{\mu,\nu}_{a+}$ of order μ and type ν, which was introduced and investigated in several earlier works of Hilfer[22].Then, we introduce and investigate an integral operator $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ which contains H-function in its kernel. Further we give its three special cases involving Fox-Wright function, hypergeometric function and Bessel function respectively . Then we prove our first theorem corresponding to the boundedness property of the integral operator $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$. Next we find solutions to two different fractional differential equations in theorem form using this operator.

In theorem 2 we find the solution to our first fractional differential equation involving the operators $\mathcal{H}^{w;m,n;\alpha}_{0+;p,q;\beta}$ and $D^{\mu,\nu}_{0+}$. Since $\mathcal{H}^{w;m,n;\alpha}_{0+;p,q;\beta}$ is general in nature, by specializing the parameters we can obtain a number of special cases of theorem 2. Here we give five corollaries of theorem 2 involving special cases of the fractional integral operator $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ and giving appropriate values to $f(x)$. In the third theorem we prove the companion of theorem 2 in which we find solution to our second fractional differential equation involving the operators $\mathcal{H}^{w;m,n;\alpha}_{0+;p,q;\beta}$ and $D^{\mu,\nu}_{0+}$. Then we give three corollaries of theorem 3 by specializing the oprator $\mathcal{H}^{w;m,n;\alpha}_{0+;p,q;\beta}$. Furthermore numerical examples of theorem 2 are calculated using these examples graphical illustrations are presented and it is found that the graphs given here are quite comparable to the physical phenomena involving ordinary calculus, especially when the parameters $\nu > 0$ and $\mu > 0$ get closer and closer to an integer.

The results derived in this chapter generalize the results obtained in earlier works by Kilbas *et al.* [32] and Srivastava and Tomovski [56].

2.1 INTRODUCTION AND DEFINITIONS

In this chapter we make use of the Riemann-Liouville fractional integral operator I^{μ}_{a+} and the Riemann-Liouville fractional derivative operator D^{μ}_{a+}, which are defined by (see, for details, [33], [37] and [41]):

$$(I^{\mu}_{a+}f)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\, dt \qquad \big(\Re(\mu) > 0\big) \tag{2.1.1}$$

and

$$(D^{\mu}_{a+}f)(x) = \left(\frac{d}{dx}\right)^n (I^{n-\mu}_{a+}f)(x) \qquad \big(\Re(\mu) > 0;\ n = [\Re(\mu)] + 1\big), \tag{2.1.2}$$

where $[x]$ denotes the greatest integer in the real number x. Hilfer [56] generalized the operator in (2.1.2) and defined a general fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 \leqq \nu \leqq 1$ with respect to x as follows:

$$(D^{\mu,\nu}_{a+}f)(x) = \left(I^{\nu(1-\mu)}_{a+} \frac{d}{dx} \left(I^{(1-\nu)(1-\mu)}_{a+} f\right)\right)(x). \tag{2.1.3}$$

The generalization in (2.1.3) yields the classical Riemann-Liouville fractional derivative operator D^{μ}_{a+} when $\nu = 0$. When $\nu = 1$, (2.1.3) reduces to the fractional derivative operator introduced by Joseph Liouville, which is often attributed nowadays to Caputo (see [33] and [68]; see also [11]).
Now, assuming that the Laplace transform $\mathcal{L}[f(x)](s)$ of the function $f(x)$ exists, that is, the integral in

$$\mathcal{L}[f(x)](s) = \int_0^{\infty} e^{-sx}\, f(x)dx \qquad \big(\Re(s) > 0\big), \tag{2.1.4}$$

is convergent [68].

The well-known convolution theorem for Laplace transform

$$L\left\{\int_0^x f(x-u)g(u)du; s\right\} = L\{f(x);s\}L\{g(x);s\} \tag{2.1.5}$$

holds provided that the various Laplace transforms occuring in (2.1.5) exist.

$$\mathcal{L}[(D^{\mu,\nu}_{0+}f)(x)](s) = s^{\mu}\mathcal{L}[f(x)](s) - s^{-\nu(1-\mu)}\left(I^{(1-\nu)(1-\mu)}_{0+}f\right)(0+) \qquad \left(\Re(s)>0;\ 0<\mu<1\right), \tag{2.1.6}$$

where the initial-value term:

$$\left(I^{(1-\nu)(1-\mu)}_{0+}f\right)(0+)$$

involves the Riemann-Liouville fractional integral (2.1.1) $(with\ a = 0)$ of the function $f(t)$ of order

$$\mu \mapsto (1-\nu)(1-\mu)$$

evaluated in the limit as $x \to 0+$.

2.2 AN INTEGRAL OPERATOR $\mathcal{H}^{w;m,n;\alpha}_{\mathfrak{a}+;p,q,\beta}$ INVOLVING THE H-FUNCTION

Various general families of operators of fractional integration involving the H-function and its extensions including those in two and more variables were considered extensively by Srivastava and Saxena (see, for details, [66, Sections 6 to 9]). Here, in our present investigation, we find it to be convenient to use the following special case of one of these general families of fractional integral operators with the H-function in their kernels (see [66, p. 15, eq. (6.3)] and the references cited therein):

$$\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\, H^{m,n}_{p,q}[w(x-t)^{\alpha}]\varphi(t)dt \tag{2.2.1}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C}\backslash\{0\};\ 1 \leqq m \leqq q;\ 0 \leqq n \leqq p;\ \Re(\beta) + \min_{1\leqq j \leqq m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0\right).$$

For $a = 0$, by using the Convolution Theorem for the Laplace Transform in (2.1.4), we find from the definition (2.2.1) that

$$\begin{aligned}\mathcal{L}\left[\left(\mathcal{H}^{w;m,n;\alpha}_{0+;p,q;\beta}\,\varphi\right)(x)\right](s) &= \mathcal{L}\left[x^{\beta-1}\,H^{m,n}_{p,q}[wx^{\alpha}]\right](s)\cdot\mathcal{L}[\varphi(x)](s)\\ &= s^{-\beta}H^{m,n+1}_{p+1,q}\left[ws^{-\alpha}\,\middle|\,\begin{array}{c}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{array}\right]\Phi(s)\end{aligned} \tag{2.2.2}$$

$$\left(\Re(s) > 0;\ \alpha > 0;\ \Re(\beta) + \min_{1\leqq j \leqq m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0\right),$$

where, for convenience,

$$\Phi(s) := \mathcal{L}[\varphi(x)](s) \qquad \left(\Re(s) > 0\right).$$

In its special case when $\varphi(x) \equiv 1$, (2.2.2) immediately yields

$$\mathcal{L}\left[\left(\mathcal{H}^{w;m,n;\alpha}_{0+;p,q;\beta}\,1\right)(x)\right](s) = s^{-\beta-1}H^{m,n+1}_{p+1,q}\left[ws^{-\alpha}\,\middle|\,\begin{array}{c}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{array}\right] \tag{2.2.3}$$

$$\left(\Re(s) > 0;\ \alpha > 0;\ \Re(\beta) + \min_{1\leqq j \leqq m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0\right).$$

2.2.1 SPECIAL CASES OF $\mathcal{H}^{w;m,n;\alpha}_{\mathfrak{a}+;p,q,\beta}$

1. For the Fox-Wright function ${}_p\Psi_q$, it is known that (see, for example, [33, p. 67, eq. (1.12.68)])

$$
{}_p\Psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}; & \\ & z\\ (b_j,\beta_j)_{1,q}; & \end{matrix}\right] = {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p); & \\ & z\\ (b_1,\beta_1),\cdots,(b_q,\beta_q); & \end{matrix}\right]
$$

$$
:= \sum_{n=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n)\cdots\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n)\cdots\Gamma(b_q+\beta_q n)}\;\frac{z^n}{n!}
$$

$$
= H^{1,p}_{p,q+1}\left[-z\,\middle|\,\begin{matrix}(1-a_j,\alpha_j)_{1,p}\\ \\ (0,1),(1-b_j,\beta_j)_{1,q}\end{matrix}\right]. \tag{2.2.4}
$$

Thus, the following fractional integral operator becomes the special case of (2.2.1):

$$
\left(\Psi^{w;p;\alpha}_{a+;q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\;{}_p\Psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}; & \\ & w(x-t)^{\alpha}\\ (b_j,\beta_j)_{1,q}; & \end{matrix}\right]\varphi(t)dt \tag{2.2.5}
$$

$$
\big(\Re(\beta)>0;\; w\in\mathbb{C}\setminus\{0\};\; p\leqq q+1\big),
$$

2. Since (see, for example, [33, p. 65, eq. (1.12.54)])

$$
{}_p\Psi_q\left[\begin{matrix}(a_1,1),\cdots,(a_p,1); & \\ & z\\ (b_1,1),\cdots,(b_q,1); & \end{matrix}\right] = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}\;{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p; & \\ & z\\ b_1,\cdots,b_q; & \end{matrix}\right], \tag{2.2.6}
$$

we get the following special case of fractional integral operator (2.2.1):

$$
\left(\mathcal{F}^{w;p;\alpha}_{a+;q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\;{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p; & \\ & w(x-t)^{\alpha}\\ b_1,\cdots,b_q; & \end{matrix}\right]\varphi(t)dt \tag{2.2.7}
$$

$$\big(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\big),$$

3. For Bessel function it is known that[49, p.19, eq.(2.6.10)]

$$\mathcal{J}_{\lambda}^{\sigma}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!\Gamma(1+\lambda+\sigma r)}$$

$$= H_{0,2}^{1,0}\left[x \Big| \begin{matrix} - \\ (0,1),(-\lambda,\sigma) \end{matrix}\right] \tag{2.2.8}$$

we get the following special case of fractional integral operator (2.2.1):

$$\left(\mathcal{J}_{a+;0,2;\beta}^{w;1,0;\alpha}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\mathcal{J}_{\lambda}^{\sigma}(w(x-t)^{\alpha})\varphi(t)dt \tag{2.2.9}$$

$$\big(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\}$$

2.2.2 BOUNDEDNESS PROPERTY OF $\mathcal{H}_{\mathfrak{a}+;p,q,\beta}^{w;m,n;\alpha}$

By assuming, in general, that $(\mathfrak{a},\mathfrak{b})$ $(-\infty \leqq \mathfrak{a} < b \leqq \infty)$ is a finite or infinite interval on the real axis $\mathbb{R} = (-\infty,\infty)$, we denote by $L(\mathfrak{a},\mathfrak{b})$ the space of Lebesgue measurable functions on a finite interval $[\mathfrak{a},\mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ on the real line $\mathbb{R}$ given by (see, for details, [11],[33])

$$L(\mathfrak{a},\mathfrak{b}) := \left\{ f : ||f||_1 := \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|dx < \infty \right\}. \tag{2.2.10}$$

Theorem 2.2.1. *Under the various parametric constraints stated already with the definition (2.2.1) let the function φ be in the space $L(\mathfrak{a},\mathfrak{b})$ of Lebesgue measurable functions on a finite interval $[\mathfrak{a},\mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ as defined by (2.2.10) Then the integral operator $\mathcal{H}_{\mathfrak{a}+;p,q,\beta}^{w;m,n;\alpha}$ is bounded on $L(\mathfrak{a},\mathfrak{b})$ and*

$$\left|\left|\left(\mathcal{H}_{\mathfrak{a}+;p,q,\beta}^{w;m,n;\alpha}\,\varphi\right)(x)\right|\right|_1 \leqq M \cdot ||\varphi||_1 \qquad (0 < M < \infty), \tag{2.2.11}$$

where the constant M is given by

$$M = \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \Theta(\mathfrak{s}) w^{\mathfrak{s}} \left(\frac{(\mathfrak{b}-\mathfrak{a})^{\beta+\alpha\mathfrak{s}}}{\beta+\alpha\mathfrak{s}} \right) d\mathfrak{s} \qquad (0 < M < \infty). \tag{2.2.12}$$

Proof. We apply the definitions (2.2.1) and (2.2.10) in conjunction with the definition (6.1.7) of the H-function. Upon interchanging the order of integration by means of the Dirichlet formula [37, p. 56], we thus find that

$$\begin{aligned}
\left|\left|\left(\mathcal{H}^{w;m,n;\alpha}_{\mathfrak{a}+;p,q,\beta}\,\varphi\right)(x)\right|\right|_1 &= \int_{\mathfrak{a}}^{\mathfrak{b}} \left| \int_{\mathfrak{a}}^{x} (x-t)^{\beta-1}\, H^{m,n}_{p,q}\left[w(x-t)^{\alpha}\right]\varphi(t)\,dt \right| dx \\
&\leqq \int_{\mathfrak{a}}^{\mathfrak{b}} |\varphi(t)| \left(\int_{t}^{\mathfrak{b}} (x-t)^{\beta-1} \left|H^{m,n}_{p,q}\left[w(x-t)^{\alpha}\right]\right| dx \right) dt \\
&= \int_{\mathfrak{a}}^{\mathfrak{b}} |\varphi(t)| \left(\int_{0}^{\mathfrak{b}-t} \tau^{\beta-1} \left|H^{m,n}_{p,q}\left[w\tau^{\alpha}\right]\right| d\tau \right) dt \\
&\leqq \int_{\mathfrak{a}}^{\mathfrak{b}} |\varphi(t)| \left(\int_{0}^{\mathfrak{b}-\mathfrak{a}} \tau^{\beta-1} \left|H^{m,n}_{p,q}\left[w\tau^{\alpha}\right]\right| d\tau \right) dt \\
&\leqq ||\varphi||_1 \cdot \left[\frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \Theta(\mathfrak{s}) w^{\mathfrak{s}} \left(\frac{(\mathfrak{b}-\mathfrak{a})^{\beta+\alpha\mathfrak{s}}}{\beta+\alpha\mathfrak{s}} \right) d\mathfrak{s} \right] \\
&= M \cdot ||\varphi||_1 \qquad \big(\Re(\beta) > 0\big),
\end{aligned}$$

where the constant $M \quad (0 < M < \infty)$ is given by (2.2.12).This completes our proof of the boundedness property of the integral operator $\mathcal{H}^{w;m,n;\alpha}_{\mathfrak{a}+;p,q,\beta}$ as asserted by (2.2.11). □

Remark 1. Throughout the present investigation, it is tacitly assumed that, in such situations as those occurring in the definitions (2.1.1), (2.1.2) and (2.2.1), the number $\mathfrak{a}$ in the function space $L(\mathfrak{a}, \mathfrak{b})$ coincides precisely with the lower terminal a in the integrals involved in the definitions (2.1.1), (2.1.2) and (2.2.1).

Remark 2. The results obtained by Kilbas *et al.* [32] and Srivastava and Tomovski [56] can be deduced as special cases of (2.2.11).

2.3 FRACTIONAL DIFFERENTIAL EQUATIONS BASED UPON HILFER DIFFERENTIAL OPERATOR

2.3.1 FIRST FRACTIONAL DIFFERENTIAL EQUATION

Theorem 2.3.1. *The following fractional differential equation*:

$$\left(D_{0+}^{\mu,\nu}\, y\right)(x) = \lambda \left(\mathcal{H}_{0+;p,q,\beta}^{w;m,n;\alpha}\, 1\right)(x) + f(x), \tag{2.3.1}$$

$$\Bigg(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ 1 \leqq m \leqq q;$$

$$0 \leqq n \leqq p;\ \Re(\beta) + \min_{1 \leqq j \leqq m} \left\{ \Re\left(\frac{\alpha b_j}{\beta_j}\right) \right\} > 0 \Bigg).$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\, y\right)(0+) = C, \tag{2.3.2}$$

has its solution in the space $L(0,\infty)$ *given by*

$$y(x) = C\, \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + \lambda x^{\beta+\mu}\, H_{p+1,q+1}^{m,n+1} \left[w x^{\alpha} \,\middle|\, \begin{matrix} (1-\beta,\alpha), (a_j,\alpha_j)_{1,p} \\ \\ (b_j,\beta_j)_{1,q}, (-\beta-\mu,\alpha) \end{matrix} \right]$$

$$+ \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1}\, f(t)dt, \tag{2.3.3}$$

where C *and* λ *are arbitrary constants and the function* f *is suitably prescribed.*

Proof. We denote by $Y(s)$ the Laplace transform of the function $y(x)$, which is given as in (2.1.4). Then, by applying the Laplace transform operator $\mathcal{L}$ to each side of (2.3.1), and using the formulas (2.1.6) and (2.2.3), the initial condition (2.3.2), and the Laplace convolution theorem, we find that

$$s^{\mu}Y(s) - Cs^{-\nu(1-\mu)} = \lambda s^{-\beta-1}\ H^{m,n+1}_{p+1,q}\left[ws^{-\alpha}\ \middle|\ \begin{matrix}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right] + F(s),$$

which readily yields

$$Y(s) = Cs^{-\mu-\nu(1-\mu)} + \lambda s^{-\beta-\mu-1}\ H^{m,n+1}_{p+1,q}\left[ws^{-\alpha}\ \middle|\ \begin{matrix}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right] + s^{-\mu}\ F(s). \tag{2.3.4}$$

Now, by taking the inverse Laplace transformation of each side of (2.3.4), we get

$$y(x) = C\ \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + \lambda\left(\frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{L}}\Theta(\mathfrak{s})w^{\mathfrak{s}}\ \Gamma(\beta+\alpha\mathfrak{s})\mathcal{L}^{-1}\left[s^{-\beta-\mu-\alpha\mathfrak{s}-1}\right](x)\ d\mathfrak{s}\right)$$

$$+\left(\frac{x^{\mu-1}}{\Gamma(\mu)} * f(x)\right)$$

$$= C\ \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma(\mu+\nu(1-\mu))} + \lambda x^{\beta+\mu}\ H^{m,n+1}_{p+1,q+1}\left[wx^{\alpha}\ \middle|\ \begin{matrix}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q},(-\beta-\mu,\alpha)\end{matrix}\right]$$

$$+\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\ f(t)dt, \tag{2.3.5}$$

which completes our proof of (2.3.1) under the various already-stated parametric constraints. □

2.3.2 SPECIAL CASES

We give below some corollaries and consequences of (2.3.1). First of all, if we reduce the H-function to the Mittag-Leffler function (see [56] and [56]) in the integral operator on the right-hand side of (2.3.1), we get the result obtained by Srivastava and Tomovski [56, p. 207, Theorem 8] (with ν replaced by $-\nu$).

Corollary 2.1. with the help of (2.2.4) if we reduce the H-function to Wright function in the integral operator on the right hand side of (2.3.1) we get the following fractional differential equation:

$$\left(D_{0+}^{\mu,\nu}\,y\right)(x)=\lambda\left(\Psi_{0+;q,\beta}^{w;p;\alpha}\,1\right)(x)+f(x), \tag{2.3.6}$$

$$\big(0<\mu<1;\ 0\leqq\nu\leqq 1;\ \Re(\beta)>0;\ w\in\mathbb{C}\setminus\{0\};\ p\leqq q+1\big)$$

under the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x)=C\,\frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)}+\lambda x^{\beta+\mu}\ {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(\beta,\alpha),(a_j,\alpha_j)_{1,p};\\ \\ (b_j,\beta_j)_{1,q},(\beta+\mu+1,\alpha);\end{matrix}\ wx^{\alpha}\right]$$

$$+\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\,f(t)dt, \tag{2.3.7}$$

where C and λ are arbitrary constants and the function f is suitably prescribed. By setting

$$\alpha_j=\beta_k=1\qquad(j=1,\cdots,p;\ k=1,\cdots,q)\qquad\text{and}\qquad\lambda\mapsto\lambda\,\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}$$

in Corollary 2.1, we immediately deduce Corollary 2.2 below.

Corollary 2.2. The following fractional differential equation:

$$\left(D_{0+}^{\mu,\nu}\, y\right)(x) = \lambda\left(\mathcal{F}_{0+;q,\beta}^{w;p;\alpha}\, 1\right)(x) + f(x), \tag{2.3.8}$$

$$\left(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\right)$$

under the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\,\frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\left(\mu+\nu(1-\mu)\right)} + \frac{\lambda\Gamma(\beta)}{\Gamma(\beta+\mu+1)}\, x^{\beta+\mu}\ {}_{p+1}F_{q+1}\left[\begin{matrix} \beta, a_1, \cdots, a_p; \\ \\ b_1, \cdots, b_q, \beta+\mu+1; \end{matrix}\ wx^{\alpha}\right]$$

$$+ \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\, f(t)dt, \tag{2.3.9}$$

where C and λ are arbitrary constants and the function f is suitably prescribed.

Corollary 2.3. With the help of (2.2.9) we get the following special case of (2.3.1):

$$\left(D_{0+}^{\mu,\nu}\, y\right)(x) = \lambda\left(\mathcal{J}_{0+;0,2;\beta}^{w;1,0;\alpha}\, 1\right)(x) + f(x), \tag{2.3.10}$$

$$\left(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\right.$$

under the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\,\frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\left(\mu+\nu(1-\mu)\right)} + x^{\beta+\mu} H_{1,3}^{1,1}\left[wx^{\alpha}\ \middle|\ \begin{matrix} (1-\beta,\alpha) \\ (0,1),(-\lambda,\sigma),(-\beta-\mu,\alpha) \end{matrix}\right]$$

$$+ \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\, f(t)dt, \tag{2.3.11}$$

where C and λ are arbitrary constants and the function f is suitably prescribed.

Remark 3. In every situation in which the function f is prescribed appropriately, each of the above results (Theorem 2.3.1, Corollary 2.1, Corollary 2.2 and corollary 2.3) would provide us with an explicit solution of the initial-value problem for the corresponding fractional differential equation. Thus, in terms of the

$\overline{H}$-function of Inayat-Hussain [24] (see, for details, [5, p. 4707, eq. (1)]; see also [65]), if we set

$$f(x) = x^{\rho}\, \overline{H}_{P,Q}^{M,N}[\mathfrak{w}x^{\kappa}] = x^{\rho}\, \overline{H}_{P,Q}^{M,N}\left[\mathfrak{w}x^{\kappa}\ \middle|\ \begin{array}{c} (c_j, \gamma_j; A_j)_{1,N}, (c_j, \gamma_j)_{N+1,P} \\ \\ (d_j, \delta_j)_{1,M}, (d_j, \delta_j; B_j)_{M+1,Q} \end{array}\right]$$

$$= x^{\rho}\, \overline{H}_{P,Q}^{M,N}\left[\mathfrak{w}x^{\kappa}\ \middle|\ \begin{array}{c} (c_1, \gamma_1; A_1), \cdots, (c_N, \gamma_N; A_N), (c_{N+1}, \gamma_{N+1}), \cdots, (c_P, \gamma_P) \\ \\ (d_1, \delta_1), \cdots, (d_M, \delta_M), (d_{M+1}, \delta_{M+1}; B_{M+1}), \cdots, (d_Q, \delta_Q; B_Q) \end{array}\right], \tag{2.3.12}$$

we are led eventually to the following consequence of Theorem 2.3.1.

Corollary 2.4. The following fractional differential equation:

$$\left(D_{0+}^{\mu,\nu}\, y\right)(x) = \lambda \left(\mathcal{H}_{0+;p,q,\beta}^{w;m,n;\alpha}\, 1\right)(x) + x^{\rho}\, \overline{H}_{P,Q}^{M,N}\left[\mathfrak{w}x^{\kappa}\ \middle|\ \begin{array}{c} (c_j, \gamma_j; A_j)_{1,N}, (c_j, \gamma_j)_{N+1,P} \\ \\ (d_j, \delta_j)_{1,M}, (d_j, \delta_j; B_j)_{M+1,Q} \end{array}\right] \tag{2.3.13}$$

$$\Big(\kappa > 0;\ 0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w, \mathfrak{w} \in \mathbb{C} \setminus \{0\};$$

$$1 \leqq m \leqq q;\ 0 \leqq n \leqq p;\ 1 \leqq M \leqq Q;\ 0 \leqq N \leqq P;$$

$$\Re(\beta) + \min_{1 \leqq j \leqq m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0;\ \Re(\rho) + \min_{1 \leqq j \leqq M}\left\{\Re\left(\frac{\kappa d_j}{\delta_j}\right)\right\} > -1\Big).$$

with the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\,\frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + \lambda x^{\beta+\mu}\, H^{m,n+1}_{p+1,q+1}\left[wx^{\alpha} \;\middle|\; \begin{matrix} (1-\beta,\alpha),(a_j,\alpha_j)_{1,p} \\ \\ (b_j,\beta_j)_{1,q},(-\beta-\mu,\alpha) \end{matrix} \right]$$

$$+\, x^{\mu+\rho}\, \overline{H}^{M,N+1}_{P+1,Q+1}\left[\mathfrak{w}x^{\kappa} \;\middle|\; \begin{matrix} (-\rho,\kappa;1),(c_j,\gamma_j;A_j)_{1,N},(c_j,\gamma_j)_{N+1,P} \\ \\ (d_j,\delta_j)_{1,M},(d_j,\delta_j;B_j)_{M+1,Q},(-\mu-\rho,\kappa;1) \end{matrix} \right], \tag{2.3.14}$$

where C and λ are arbitrary constants.

We now turn to one of the fundamentally important higher transcendental functions of Analytic Number Theory, that is, the general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (see, for example, [7, p. 27. eq. 1.11 (1)]; see also [58], [62, p. 121 *et seq.*] and [63, p. 194 *et seq.*])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{2.3.15}$$

$$\big(a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\; s\in\mathbb{C} \quad \text{when} \quad |z|<1;\; \Re(s)>1 \quad \text{when} \quad |z|=1\big),$$

where $\mathbb{Z}_0^-$ denotes the set of nonpostive integers. It contains, as its *special* cases, not only the Riemann Zeta function $\zeta(s)$, the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ (see [1] and [38]) and the Lerch Zeta function $\ell_s(\xi)$, but also such other important functions of Analytic Number Theory as (for example) the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$ defined by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1) \tag{2.3.16}$$

$$\big(s\in\mathbb{C} \quad \text{when} \quad |z|<1;\; \Re(s)>1 \quad \text{when} \quad |z|=1\big)$$

and the Lipschitz-Lerch Zeta function $\phi(\xi, a, s)$ (see [62, p. 122, eq 2.5 (11)]) given by

$$\phi(\xi, s, a) := \sum_{n=0}^{\infty} \frac{e^{2n\pi \mathrm{i}\xi}}{(n+a)^s} = \Phi\Big(e^{2\pi \mathrm{i}\xi}, s, a\Big) \tag{2.3.17}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\big),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see also [59, Section 5]). Recently, Srivastava *et al.* [67] introduced and systematically studied various properties and results involving a natural multiparameter extension and generalization of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined by (2.3.15) (see also [61] for a further generalization). In order to recall their definition (which was motivated essentially by several earlier works), each of the following notations will be employed:

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right) \tag{2.3.18}$$

and

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \qquad \text{and} \qquad \omega := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2} \tag{2.3.19}$$

The extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}(z, s, a)$$

is then defined by [67, p. 503, eq. (6.2)] (see also [59], [60] and [64])

$$\Phi^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}(z, s, a) := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod\limits_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{2.3.20}$$

$$\Big(p, q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j = 1, \cdots, p);\ a, \mu_j \in \mathbb{C} \setminus Z_0^-\ (j = 1, \cdots, q);$$

$$\rho_j, \sigma_k \in \mathbb{R}^+ \ (j = 1, \cdots, p; \ k = 1, \cdots, q);$$

$$\Delta > -1 \ \text{ when } \ s, z \in \mathbb{C};$$

$$\Delta = -1 \ \text{ and } \ s \in \mathbb{C} \ \text{ when } \ |z| < \nabla^*;$$

$$\Delta = -1 \ \text{ and } \ \Re(\omega) > \frac{1}{2} \ \text{ when } \ |z| = \nabla^* \Bigg),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the Pochhammer symbol (or the *shifted* factorial) which is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \ \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the above Γ-quotient exists. In terms of the extended Hurwitz-Lerch zeta function defined by (2.3.20), if we set

$$f(x) = x^\rho \, \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\big(\mathfrak{w}x^\kappa, s, a\big), \tag{2.3.21}$$

Theorem 2.3.1 would readily yield the following corollary.

Corollary 2.5. The following fractional differential equation:

$$\big(D^{\mu,\nu}_{0+}\, y\big)(x) = \lambda\big(\mathcal{H}^{w;m,n;\alpha}_{0+;p,q,\beta}\, 1\big)(x) + x^\rho \, \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\big(\mathfrak{w}x^\kappa, s, a\big), \tag{2.3.22}$$

$$\Bigg(\kappa > 0; \ 0 < \mu < 1; \ 0 \leqq \nu \leqq 1; \ \Re(\beta) > 0; \ w \in \mathbb{C} \setminus \{0\}; \ 1 \leqq m \leqq q; \ 0 \leqq n \leqq p;$$

$$\Re(\beta) + \min_{1 \leqq j \leqq m} \left\{ \Re\left(\frac{\alpha b_j}{\beta_j}\right) \right\} > 0; \ \Re(\rho) > -1 \Bigg).$$

with the initial condition (2.3.2) has its solution in the space $L(0, \infty)$ given by

$$y(x) = C \, \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + \lambda x^{\beta+\mu} \, H^{m,n+1}_{p+1,q+1}\left[wx^{\alpha} \,\middle|\, \begin{array}{c} (1-\beta,\alpha),(a_j,\alpha_j)_{1,p} \\ \\ (b_j,\beta_j)_{1,q},(-\beta-\mu,\alpha) \end{array} \right]$$

$$+ \frac{\Gamma(\rho+1)}{\Gamma(\mu+\rho+1)} \, x^{\mu+\rho} \, \Phi^{(\rho_1,\cdots,\rho_p,\kappa,\sigma_1,\cdots,\sigma_q,\kappa)}_{\lambda_1,\cdots,\lambda_p,\rho+1;\mu_1,\cdots,\mu_q,\mu+\rho+1}\big(\mathfrak{w}x^{\kappa}, s, a\big), \qquad (2.3.23)$$

where C and λ are arbitrary constants.

By letting $w \to 0$ in Corollary 2.2, the hypergeometric functions occurring in (2.2.7) and (2.3.9) would obviously reduce to their first term 1. We are thus led immediately to the following result.

Corollary 2.6. The following fractional differential equation:

$$\left(D^{\mu,\nu}_{0+} \, y\right)(x) = \lambda \left(\mathcal{F}^{0;p}_{0+;q,\beta} \, 1\right)(x) + f(x), \qquad (2.3.24)$$

$$\big(0 < \mu < 1; \; 0 \leqq \nu \leqq 1; \; \Re(\beta) > 0\big)$$

under the initial condition (2.3.2) has its solution in the space $L(0, \infty)$ given by

$$y(x) = C \, \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + \frac{\lambda\Gamma(\beta)}{\Gamma(\beta+\mu+1)} \, x^{\beta+\mu} + \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \, f(t)dt, \qquad (2.3.25)$$

where C and λ are arbitrary constants and the function f is suitably prescribed.

Remark 5. In its further special case when we set

$$f(x) = x^{\rho} \qquad \big(\Re(\rho) > -1\big),$$

Corollary 2.6 reduces to Corollary 2.7 below.

Corollary 2.7. The following fractional differential equation:

$$\left(D^{\mu,\nu}_{0+} \, y\right)(x) = \lambda \left(\mathcal{F}^{0;p}_{0+;q,\beta} \, 1\right)(x) + x^{\rho}, \qquad (2.3.26)$$

$$\big(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ \Re(\rho) > -1\big)$$

under the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\,\frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + \frac{\lambda\Gamma(\beta)}{\Gamma(\beta+\mu+1)}\,x^{\beta+\mu} + \frac{\Gamma(\rho+1)}{\Gamma(\mu+\rho+1)}\,x^{\mu+\rho}, \tag{2.3.27}$$

where C and λ are arbitrary constants and the function f is suitably prescribed.

2.3.3 SECOND FRACTIONAL DIFFERENTIAL EQUATION

Theorem 2.3.2. *The following fractional differential equation:*

$$x\left(D_{0+}^{\mu,\nu}\,y\right)(x) = \lambda\left(\mathcal{H}_{0+;p,q,\beta}^{w;m,n;\alpha}\,1\right)(x), \tag{2.3.28}$$

$$\Bigg(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\};\ 1 \leqq m \leqq q;\ 0 \leqq n \leqq p;$$

$$\Re(\beta) + \min_{1\leqq j\leqq m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0\Bigg).$$

with the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\,\frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + C^*\,\frac{x^{\mu-1}}{\Gamma(\mu)}$$

$$+\,\lambda x^{\beta+\mu-1}\,H_{p+2,q+2}^{m,n+2}\left[wx^{\alpha}\,\middle|\,\begin{matrix}(1-\beta,\alpha),(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q},(-\beta,\alpha),(1-\beta-\mu,\alpha)\end{matrix}\right], \tag{2.3.29}$$

where C, C^* and λ are arbitrary constants.

Proof. Since

$$\frac{d^n}{ds^n}\{\mathcal{L}[f(x)](s)\} = (-1)^n\,\mathcal{L}[x^n\,f(x)](s), \tag{2.3.30}$$

if we denote by $Y(s)$ the Laplace transform of the function $y(x)$ and apply the Laplace transform operator $\mathcal{L}$ to each side of (2.3.28) and use the formulas (2.1.6) and (2.2.3), the initial condition (2.3.2), and the Laplace convolution theorem, we get

$$\frac{d}{ds}\left\{s^{\mu}Y(s) - Cs^{-\nu(1-\mu)}\right\} = -\lambda s^{-\beta-1}\,H^{m,n+1}_{p+1,q}\left[ws^{-\alpha}\,\middle|\,\begin{array}{c}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{array}\right]. \tag{2.3.31}$$

Upon integrating both sides, this last equation (2.3.31) yields

$$Y(s) = Cs^{-\mu-\nu(1-\mu)} + C^{*}s^{-\mu} + \lambda s^{-\beta-\mu}\,H^{m,n+2}_{p+2,q+1}\left[ws^{-\alpha}\,\middle|\,\begin{array}{c}(1-\beta,\alpha),(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q},(-\beta,\alpha)\end{array}\right], \tag{2.3.32}$$

where C^{*} is a constant of integration. The solution (2.3.29) asserted by (2.3.28) would now follow when we take the inverse Laplace transform of each term in (2.3.32). □

2.3.4 SPECIAL CASES

Corollary 3.1.

with the help of (2.2.5) we get the following special case of theorem 2.3.2:

$$x\left(D^{\mu,\nu}_{0+}\,y\right)(x) = \lambda\left(\Psi^{w;p;\alpha}_{0+;q,\beta}\,1\right)(x), \tag{2.3.33}$$

$$\big(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\big)$$

with the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\ \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + C^{*}\ \frac{x^{\mu-1}}{\Gamma(\mu)}$$

$$+ \lambda x^{\beta+\mu-1}\ {}_{p+2}\Psi_{q+2}\left[\begin{array}{c} (1-\beta,\alpha),(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}; \\ \\ (b_j,\beta_j)_{1,q},(-\beta,\alpha)(1-\beta-\mu,\alpha); \end{array} wx^{\alpha}\right] \tag{2.3.34}$$

where C, C^{*} and λ are arbitrary constants.

Corollary 3.2.

with the help of (2.2.7) we get the following special case of theorem 2.3.2:

$$x\left(D_{0+}^{\mu,\nu}\ y\right)(x) = \lambda\left(\mathcal{F}_{0+;q,\beta}^{w;p;\alpha}\ 1\right)(x), \tag{2.3.35}$$

$$\big(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\big)$$

with the initial condition (2.3.2) has its solution in the space $L(0,\infty)$ given by

$$y(x) = C\ \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + C^{*}\ \frac{x^{\mu-1}}{\Gamma(\mu)}$$

$$+ \lambda x^{\beta+\mu-1}\ {}_{p+2}F_{q+2}\left[\begin{array}{c} 1-\beta,1-\beta,a_1,\cdots,a_p; \\ \\ b_1,\cdots,b_q,-\beta,1-\beta-\mu; \end{array} wx^{\alpha}\right] \tag{2.3.36}$$

where C, C^{*} and λ are arbitrary constants.

Corollary 3.3.

with the help of (2.2.9) we get the following special case of (2.3.28):

$$x\left(D_{0+}^{\mu,\nu}\ y\right)(x) = \lambda\left(\mathcal{J}_{0+;0,2;\beta}^{w;1,0;\alpha}\ 1\right)(x), \tag{2.3.37}$$

$$\big(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};$$

with the initial condition (2.3.2) has its solution in the space $L(0, \infty)$ given by

$$y(x) = C\ \frac{x^{\mu+\nu(1-\mu)-1}}{\Gamma\big(\mu+\nu(1-\mu)\big)} + C^*\ \frac{x^{\mu-1}}{\Gamma(\mu)}$$

$$+ \lambda x^{\beta+\mu-1} H^{1,2}_{2,4}\left[w x^{\alpha} \,\middle|\, \begin{matrix} (1-\beta, \alpha), (1-\beta, \alpha) \\ (0,1), (-\beta, \alpha), (-\lambda, \sigma), (1-\beta-\mu, \alpha) \end{matrix} \right] \tag{2.3.38}$$

where C, C^* and λ are arbitrary constants.

Remark 4. By suitably specializing the H-function occurring in the definition (2.2.1), we can derive a number of simpler results by appealing similarly to Theorem 2.3.1 and 2.3.2.

2.4 NUMERICAL EXAMPLES AND GRAPHICAL REPRESENTATIONS

Example 1. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = \frac{1}{2}, \quad \nu = 0 \qquad \text{and} \qquad \rho = 0.$$

Then the following fractional differential equation:

$$\left(D^{\frac{1}{2},0}_{0+}\, y\right)(x) = \lambda \left(\mathfrak{F}^{0;p}_{0+;q,\frac{1}{4}}\, 1\right)(x) + 1, \tag{2.4.1}$$

together with the initial condition:

$$\left(I^{\frac{1}{2}}_{0+}\, y\right)(0+) = C,$$

has its solution in the space $L(0, \infty)$ given by

$$y_{0.5}(x) = \frac{C}{\sqrt{\pi x}} + \lambda\, \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}\, x^{\frac{3}{4}} + 2\, \sqrt{\frac{x}{\pi}}\ , \tag{2.4.2}$$

where C and λ are arbitrary constants.

Example 2. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = 0.7, \quad \nu = 0 \qquad \text{and} \qquad \rho = 0.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{0.7,0}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + 1, \tag{2.4.3}$$

together with the initial condition:

$$\left(I_{0+}^{0.3}\, y\right)(0+) = C,$$

has its solution in the space $L(0, \infty)$ given by

$$y_{0.7}(x) = C\, \frac{x^{-0.3}}{\Gamma(0.7)} + \lambda\, \frac{\Gamma(\frac{1}{4})}{\Gamma(1.95)}\, x^{0.95} + \frac{x^{0.7}}{\Gamma(1.7)}, \tag{2.4.4}$$

where C and λ are arbitrary constants.

Example 3. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = 0.9, \quad \nu = 0 \qquad \text{and} \qquad \rho = 0.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{0.9,0}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + 1, \tag{2.4.5}$$

together with the initial condition:

$$\left(I_{0+}^{0.1}\, y\right)(0+) = C,$$

has its solution in the space $L(0, \infty)$ given by

$$y_{0.9}(x) = C\, \frac{x^{-0.1}}{\Gamma(0.9)} + \lambda\, \frac{\Gamma(\frac{1}{4})}{\Gamma(2.15)}\, x^{1.15} + \frac{x^{0.9}}{\Gamma(1.9)}, \tag{2.4.6}$$

where C and λ are arbitrary constants.

Example 4. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = 0.95, \quad \nu = 0 \qquad \text{and} \qquad \rho = 0.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{0.95,0}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + 1, \tag{2.4.7}$$

together with the initial condition:

$$\left(I_{0+}^{0.05}\, y\right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$y_{0.95}(x) = C\, \frac{x^{-0.05}}{\Gamma(0.95)} + \lambda\, \frac{\Gamma(\frac{1}{4})}{\Gamma(2.2)}\, x^{1.2} + \frac{x^{0.95}}{\Gamma(1.95)}, \tag{2.4.8}$$

where C and λ are arbitrary constants.

Example 5. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu \to 1-, \quad \nu = 0 \qquad \text{and} \qquad \rho = 0.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{1,0}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + 1, \tag{2.4.9}$$

together with the initial condition:

$$\left(I_{0+}^{0}\, y\right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$y_1(x) = C + \frac{16}{5}\,\lambda\, x^{\frac{5}{4}} + x, \tag{2.4.10}$$

where C and λ are arbitrary constants.

Example 6. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = \frac{1}{2}, \quad \nu = \frac{1}{2} \qquad \text{and} \qquad \rho = 2.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{\frac{1}{2},\frac{1}{2}}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + x^2, \tag{2.4.11}$$

together with the initial condition:

$$\left(I_{0+}^{\frac{1}{4}}\, y\right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$\mathfrak{y}_{0.5}(x) = C\, \frac{x^{-0.25}}{\Gamma(0.75)} + \lambda\, \frac{\Gamma(\frac{1}{4})}{\Gamma(1.75)}\, x^{0.75} + \frac{2x^{2.5}}{\Gamma(3.5)}, \tag{2.4.12}$$

where C and λ are arbitrary constants.

Example 7. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = \frac{1}{2}, \quad \nu = 0.7 \qquad \text{and} \qquad \rho = 2.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{\frac{1}{2},0.7}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + x^2, \tag{2.4.13}$$

together with the initial condition:

$$\left(I_{0+}^{0.15}\, y\right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$\mathfrak{y}_{0.7}(x) = C\, \frac{x^{-0.15}}{\Gamma(0.85)} + \lambda\, \frac{\Gamma(\frac{1}{4})}{\Gamma(1.75)}\, x^{0.75} + \frac{2x^{2.5}}{\Gamma(3.5)}, \tag{2.4.14}$$

where C and λ are arbitrary constants.

Example 8. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = \frac{1}{2}, \quad \nu = 0.9 \qquad \text{and} \qquad \rho = 2.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{\frac{1}{2},0.9}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + x^2, \tag{2.4.15}$$

together with the initial condition:

$$\left(I_{0+}^{0.05}\, y\right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$\mathfrak{y}_{0.9}(x) = C\,\frac{x^{-0.05}}{\Gamma(0.95)} + \lambda\,\frac{\Gamma(\frac{1}{4})}{\Gamma(1.75)}\,x^{0.75} + \frac{2x^{2.5}}{\Gamma(3.5)}, \tag{2.4.16}$$

where C and λ are arbitrary constants.

Example 9. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = \frac{1}{2}, \quad \nu = 0.95 \qquad \text{and} \qquad \rho = 2.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{\frac{1}{2},0.95}\, y\right)(x) = \lambda \left(\mathcal{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1\right)(x) + x^2, \tag{2.4.17}$$

together with the initial condition:

$$\left(I_{0+}^{0.025}\, y\right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$\mathfrak{y}_{0.95}(x) = C\,\frac{x^{-0.025}}{\Gamma(0.975)} + \lambda\,\frac{\Gamma(\frac{1}{4})}{\Gamma(1.75)}\,x^{0.75} + \frac{2x^{2.5}}{\Gamma(3.5)}, \tag{2.4.18}$$

where C and λ are arbitrary constants.

Example 10. In Corollary 2.7, we set

$$\beta = \frac{1}{4}, \quad \mu = \frac{1}{2}, \quad \nu = 1 \qquad \text{and} \qquad \rho = 2.$$

Then the following fractional differential equation:

$$\left(D_{0+}^{\frac{1}{2},1}\, y \right)(x) = \lambda \left(\mathfrak{F}_{0+;q,\frac{1}{4}}^{0;p}\, 1 \right)(x) + x^2, \tag{2.4.19}$$

together with the initial condition:

$$\left(I_{0+}^{0}\, y \right)(0+) = C,$$

has its solution in the space $L(0,\infty)$ given by

$$\mathfrak{y}_1(x) = C + \lambda\, \frac{\Gamma(\frac{1}{4})}{\Gamma(1.75)}\, x^{0.75} + \frac{2x^{2.5}}{\Gamma(3.5)}, \tag{2.4.20}$$

where C and λ are arbitrary constants.

The following graphs (see Figure 1 and Figure 2) are obtained by using MATLAB. Figure 1 exhibits a comparison between the behaviors of the solutions $y_\mu(x)$ given by (2.4.2), (2.4.4), (2.4.6)and (2.4.10) for different values of the parameter μ. On the other hand, Figure 2 illustrates a comparison between the behaviors of the solutions $\mathfrak{y}_\nu(x)$ given by (2.4.12), (2.4.14), (2.4.16) and (2.4.20) for different values of the parameter ν.

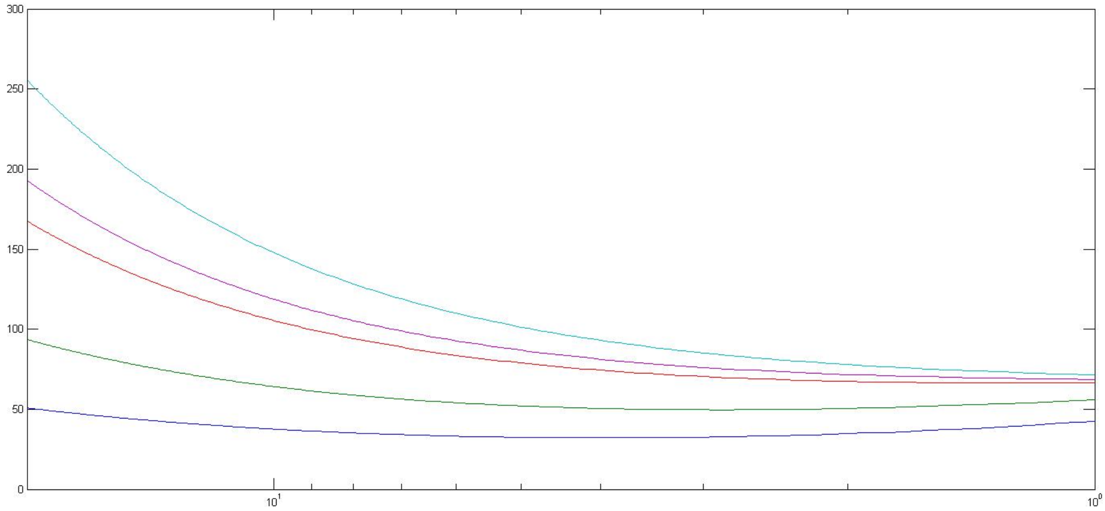

Figure 1. Solutions $y_\mu(x)$ for different values of μ when $C = 66.4$ and $\lambda = 1$

[Here $y_1(x)$ is the uppermost graph and $y_\mu(x)$ is approaching $y_1(x)$ as $\mu \to 1$]

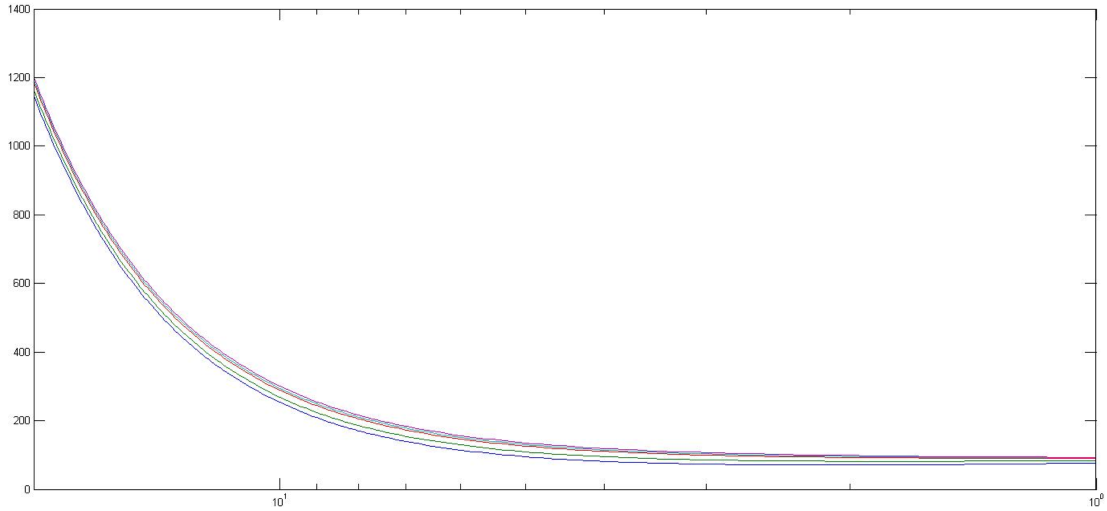

Figure 2. Solutions $\mathfrak{y}_\nu(x)$ for different values of ν when $C = 88.4$ and $\lambda = 1$

[Here $\mathfrak{y}_1(x)$ is the lowermost graph and $\mathfrak{y}_\nu(x)$ is approaching $\mathfrak{y}_1(x)$ as $\nu \to 1$]

Remark 6: It is found that the graphs (see Figure 1 and Figure 2) given here are quite comparable to the corresponding physical phenomena involving ordinary calculus, especially when the parameters $\nu > 0$ and $\mu > 0$ get closer and closer to an integer.

2. ON GENERAL FRACTIONAL DIFFERENTIAL EQUATIONS ASSOCIATED WITH AN INTEGRAL OPERATOR HAVING THE H-FUNCTION IN THE KERNEL

3

A GENERAL FRACTIONAL KINETIC DIFFERINTEGRAL EQUATION ASSOCIATED WITH AN INTEGRAL OPERATOR $\mathcal{S}_{N;\beta}^{M;\alpha}$

The main findings of this chapter have been accepted for publication as given below:

A General Fractional kinetic Differintegral Equation Associated with an Integral Operator with the general class of polynomial in the Kernel, Proceedings of the Jangjeon Mathematical Society.

In this chapter, we solve a general fractional Kinetic differintegral equation involving the fractional operator $D_{0+}^{\mu,\nu}$ and an integral operator whose kernel involves the general class of polynomials S_N^M. We make use of Laplace transform method to solve the fractional kinetic differintegral equation.

On account of general nature of S_N^M occuring in the fractional kinetic equation, a number of results involving simpler polynomials also follow as special cases of our main result. We give here six special cases of the main equation. In the first special case we give appropriate value to the function $f(t)$ occuring in the fractional kinetic differintegral equation. Second, third, fourth, fifth and sixth special cases involve Laguerre polynomial, Bessel polynomial, Gould and Hopper polynomial, Brafman polynomial and Cesaro polynomial in the kernel of the integral operator occuring in the fractional kinetic differintegral equation respectively.Our main findings generalizes the result obtained by Tomovski et al.[56].

3.1 INTRODUCTION AND DEFINITIONS

Fractional Kinetic equations have gained popularity during the past decade mainly due to the discovery of their relation with the CTRW-theory in [21]. These equations are investigated in order to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on [22, 23].

AN INTEGRAL OPERATOR INVOLVING S_N^M POLYNOMIAL IN ITS KERNEL

In our present investigation we use the following integral operator with S_N^M polynomial[16] in its kernel

$$\left(\mathcal{S}_{N;\beta}^{M;\alpha}\ \varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\ S_N^M[(x-t)^\alpha]\varphi(t)dt \tag{3.1.1}$$

By using the *Convolution Theorem* for the Laplace Transform, we find from the definition (3.1.1) that

$$\mathcal{L}\left[\left(\mathcal{S}_{N;\beta}^{M;\alpha}\ \varphi\right)(x)\right](s) = s^{-\beta}\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}\Gamma(\beta+\alpha r)s^{-\alpha r}\Phi(s) \tag{3.1.2}$$

where, for convenience,

$$\Phi(s) := \mathcal{L}[\varphi(x)](s) \qquad \big(\Re(s)>0\big).$$

3.2 A FRACTIONAL KINETIC DIFFERINTEGRAL EQUATION INVOLVING $\mathcal{S}_{N;\beta}^{M;\alpha}$

In a recent investigation by Saxena and Kalla[44], the following fractional kinetic equation was considered:

$$N(t) - N_0 f(t) = -c^{\nu}(I_{0+}^{\nu}N)(t) \qquad (\Re(\nu) > 0), \tag{3.2.1}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and (for convenience) $f \epsilon L(0, \infty)$, it being tacitly assumed that $f(0) = 1$ in order to satisfy the initial condition $N(0) = N_0$.

We consider the following general fractional kinetic defferintegral equation associated with S_N^M polynomial:

$$a\left(D_{0+}^{\mu,\nu}\, N\right)(t) - N_0 f(t) = b\left(\mathcal{S}_{N,\beta}^{M;\alpha}\, N\right)(t) \tag{3.2.2}$$

$$\left(0 < \mu < 1;\ 0 \leqq \nu \leqq 1;\ \Re(\beta) > 0\right)$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\, N\right)(0+) = c, \tag{3.2.3}$$

where a, b and c are constants and $f \in L(0, \infty)$.

Solution of (3.2.2) is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha-\beta)k_p-\mu-l-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha-\beta)k_p-\mu-l\right)} \frac{d^l f(t)}{dt^l} + ac \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)\right)} \tag{3.2.4}$$

provided $f^{(i)}(0) = 0$ for $0 \leq i \leq l-1$, l being a positive integer

where

$$\Delta_r = \frac{1}{a} \sum_{k_1+k_2+....+k_{[N/M]}=r} \binom{r}{k_1 k_2 ... k_{[N/M]}} \prod_{1 \leq p \leq [N/M]} (\lambda_p s^{-[N/M]\alpha-\beta})^{k_p} \tag{3.2.5}$$

and

$$\lambda_p = b\frac{(-N)_{Mp}}{p!} A_{N,p} \Gamma(\beta+\alpha p) \tag{3.2.6}$$

Proof. Taking Laplace Transform on both sides of (3.2.2), we get

$$a\left[s^{\mu}\overline{N}(s) - Cs^{-\nu(1-\mu)}\right] - N_0 F(s) = s^{-\beta} \sum_{n=0}^{[N/M]} \frac{(-N)_{Mn} A_{N,n}}{n!} \Gamma(\beta+\alpha n) s^{-\alpha n} \overline{N}(s) \tag{3.2.7}$$

where $\overline{N}(s)$ is the Laplace transform of $N(t)$

Rearranging the terms in (3.2.7) we have

$$\overline{N}(s) = N_0 F(s) + acs^{-\nu(1-\mu)} \left(as^{\mu} - \sum_{n=0}^{[N/M]} \lambda_n s^{-\alpha n-\beta}\right)^{-1} \tag{3.2.8}$$

where λ_n is given by (3.2.6).

consider,

$$\left(as^{\mu}-\sum_{n=0}^{[N/M]}\lambda_n s^{-\alpha n-\beta}\right)^{-1}=as^{-\mu}\left(1-\sum_{n=0}^{[N/M]}\lambda_n s^{-\alpha n-\beta}\right)^{-1} \tag{3.2.9}$$

Using the following multinomial formula for any positive integer m and any non-negative integer n, the multinomial formula is as follows:

$$(x_1+x_2+...+x_m)^n=\sum_{k_1+k_2+...+k_m=n}\binom{n}{k_1,k_2,...,k_m}\prod_{1\le t\le m}x_t^{k_t} \tag{3.2.10}$$

where

$$\binom{n}{k_1,k_2,...,k_m}=\frac{n!}{k_1!k_2!...k_m!}$$

is a multinomial coefficient.

The RHS of (3.2.9) takes the following form

$$as^{-\mu}\left(1-\sum_{n=0}^{[N/M]}\lambda_n s^{-\alpha n-\beta}\right)^{-1}=\frac{s^{-\mu}}{a}\sum_{r=0}^{\infty}\left(\sum_{n=0}^{[N/M]}\lambda_n s^{-\alpha n-\beta-\mu}\right)^r$$

$$=\sum_{r=0}^{\infty}\Delta_r s^{\left(\left(\sum_{p=1}^{[N/M]}-[N/M]\alpha-\beta\right)k_p-\mu\right)} \tag{3.2.11}$$

provided $\left|\sum_{n=0}^{[N/M]}\lambda_n s^{-\alpha n-\beta}\right|<1$

where Δ_r is given by (3.2.5)

Substituting (3.2.11) in (3.2.8) we get

$$\overline{N}(s)=N_0\sum_{r=0}^{\infty}\Delta_r s^{\left(\left(\sum_{p=1}^{[N/M]}-[N/M]\alpha-\beta\right)k_p-\mu-l\right)}s^lF(s)+ac\sum_{r=0}^{\infty}\Delta_r s^{\left(\left(\sum_{p=1}^{[N/M]}-[N/M]\alpha-\beta\right)k_p-\mu-\nu(1-\mu)\right)} \tag{3.2.12}$$

Taking Laplace inverse of (3.2.12) we get the required result (3.2.4). □

3.2.1 SPECIAL CASES

SPECIAL CASE 1

If we put $f(t) = t^{\rho-1}$ in (3.2.2) we get the following equation:

$$a\left(D_{0+}^{\mu,\nu}\,N\right)(t) - N_0 t^{\rho-1} = b\left(\mathcal{S}_{N,\beta}^{M;\alpha}\,N\right)(t) \tag{3.2.13}$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\,N\right)(0+) = c, \tag{3.2.14}$$

where a, b and c are constants.

whose solution is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l\right)} \frac{\Gamma(\rho)}{\Gamma(\rho-l)} t^{\rho-l-1}$$

$$+ ac\sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\nu(1-\mu)\right)} \tag{3.2.15}$$

where Δ_r is given by (3.2.5) and λ_p is given by (3.2.6)

SPECIAL CASE 2

If we reduce S_N^M polynomial to Laguerre polynomial[4] we get the following equation:

$$a\left(D_{0+}^{\mu,\nu}\,N\right)(t) - N_0 f(t) = b\left(\mathcal{L}_{N,\beta}^{1;\alpha}\,N\right)(t) \tag{3.2.16}$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\,N\right)(0+) = c, \tag{3.2.17}$$

where a, b and c are constants.

whose solution is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha - \beta)k_p - \mu - l - 1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha - \beta)k_p - \mu - l\right)} \frac{d^l}{dt^l)} f(t)$$

$$+ ac \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha - \beta)k_p - \mu - \nu(1-\mu) - 1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]} (-[N/M]\alpha - \beta)k_p - \nu(1-\mu)\right)} \tag{3.2.18}$$

where Δ_r is given by (3.2.5)

and

$$\lambda_n = b \frac{(-1)^n}{n!} \binom{N}{k} \Gamma(\beta + \alpha n) \tag{3.2.19}$$

SPECIAL CASE 3

If we reduce S_N^M polynomial to Bessel polynomial[4] we get the following equation:

$$a \left(D_{0+}^{\mu,\nu} N\right)(t) - N_0 f(t) = b \left(\mathcal{J}_{N,\beta}^{1;\alpha} N\right)(t) \tag{3.2.20}$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)} N\right)(0+) = c, \tag{3.2.21}$$

where a, b and c are constants.

whose solution is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l\right)} \frac{d^l}{dt^l)} f(t)$$

$$+ ac \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\nu(1-\mu)\right)} \tag{3.2.22}$$

where Δ_r is given by (3.2.5)

and

$$\lambda_n = b \sum_{n=0}^{N} \frac{(N+n)!}{(N-n)!n!2^n} \Gamma(\beta+\alpha n) \tag{3.2.23}$$

SPECIAL CASE 4

If we reduce S_N^M polynomial to Gould and Hopper polynomial[4] we get the following equation:

$$a\left(D_{0+}^{\mu,\nu}\, N\right)(t) - N_0 f(t) = b\left(\mathcal{G}_{N,\beta}^{M;\alpha}\, N\right)(t) \tag{3.2.24}$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\, N\right)(0+) = c, \tag{3.2.25}$$

where a, b and c are constants.

whose solution is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum\limits_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l-1\right)}}{\Gamma\left(\sum\limits_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l\right)} \frac{d^l f(t)}{dt^l}$$

$$+ ac \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum\limits_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)-1\right)}}{\Gamma\left(\sum\limits_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)\right)} \tag{3.2.26}$$

provided $f^{(i)}(0) = 0$ for $0 \leq i \leq l-1$, l being a positive integer

where Δ_r is given by (3.2.5)

and

$$\lambda_n = b\frac{(-N)_{Mn}}{n!}\Gamma(\beta+\alpha n) \tag{3.2.27}$$

SPECIAL CASE 5

If we reduce S_N^M polynomial to Brafman polynomial[4] we get the following equation:

$$a\left(D_{0+}^{\mu,\nu}\, N\right)(t) - N_0 f(t) = b\left(\mathcal{B}_{N,\beta}^{M;\alpha}\, N\right)(t) \tag{3.2.28}$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\, N\right)(0+) = c, \tag{3.2.29}$$

where a, b and c are constants.

whose solution is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-l\right)} \frac{d^l f(t)}{dt^l}$$

$$+ ac \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)-1\right)}}{\Gamma\left(\sum_{p=1}^{[N/M]}(-[N/M]\alpha-\beta)k_p-\mu-\nu(1-\mu)\right)} \tag{3.2.30}$$

provided $f^{(i)}(0) = 0$ for $0 \le i \le l-1$, l being a positive integer

where Δ_r is given by (3.2.5)

and

$$\lambda_n = b\frac{(-N)_{Mn}}{n!}\frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n}\Gamma(\beta+\alpha n) \tag{3.2.31}$$

SPECIAL CASE 6

If we reduce S_N^M polynomial to Cesaro polynomial[4] we get the following equation:

$$a\left(D_{0+}^{\mu,\nu}\, N\right)(t) - N_0 f(t) = b\left(\mathcal{C}_{N,\beta}^{M;\alpha}\, N\right)(t) \tag{3.2.32}$$

with the initial condition:

$$\left(I_{0+}^{(1-\nu)(1-\mu)}\, N\right)(0+) = c, \tag{3.2.33}$$

where a, b and c are constants.

whose solution is given by:

$$N(t) = N_0 \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum\limits_{p=1}^{N}(-N\alpha-\beta)k_p-\mu-l-1\right)}}{\Gamma\left(\sum\limits_{p=1}^{N}(-N\alpha-\beta)k_p-\mu-l\right)} \frac{d^l f(t)}{dt^l}$$

$$+ ac \sum_{r=0}^{\infty} \Delta_r \frac{t^{\left(\sum\limits_{p=1}^{N}(-N\alpha-\beta)k_p-\mu-\nu(1-\mu)-1\right)}}{\Gamma\left(\sum\limits_{p=1}^{N}(-N\alpha-\beta)k_p-\mu-\nu(1-\mu)\right)} \tag{3.2.34}$$

provided $f^{(i)}(0) = 0$ for $0 \leq i \leq l-1$, l being a positive integer

where Δ_r is given by (3.2.5)

and

$$\lambda_n = b\frac{(-N)_n}{n!} \frac{(S+1)_N n!}{N!(-S-N)_n} \Gamma(\beta + \alpha n) \tag{3.2.35}$$

Finally, it may be noted that if we reduce S_N^M polynomial involved in the R.H.S of (3.2.2) to unity, we get the results obtained by Tomovski, Hilfer and Srivastava[56, p.813].

4

A STUDY OF FRACTIONAL DIFFERENTIAL INTEGRAL OPERATOR

The main findings of this chapter have been published as detailed below:
A Study of Fractional Differential Integral operator,Proc.of the 12th annual conf. SSFA, **12** (2013), 73-77.

In this chapter we introduce and study a fractional differential integral operator. First, we define the operator of our study and then obtain image of a product of H-function and $\overline{H}-$ function under this operator. Fractional integrals involving a number of simpler functions follow as its special cases. we record here four special cases.
Next we derive two new and interesting composition formulae for the fractional integral operator $I_{a+}^{\gamma;\mu}$ and the integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$. Then we give the composition formulae for the fractional integral operators $I_{a+}^{\mu}, D_{a+}^{\mu}$ and integral operator $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$. Further, we find that the results obtained by Srivastava et al.[56] follow as particular cases of our composition formulae.

4.1 INTRODUCTION AND DEFINITIONS

The main aim of the present chapter is to introduce and study the following generalized form of fractional integral operator studied earlier by Hilfer[22].

$$(D_{p+}^{\gamma,\mu,\nu}f)(x) = \left(I_{p+}^{\gamma,\nu(1-\mu)}\left(\frac{d}{dx}\right)^m (I_{p+}^{\gamma,(1-\nu)(1-\mu)}f)\right)(x) \tag{4.1.1}$$

with $0 < \mu < 1$ and $0 \leq \nu \leq 1$
where

$$(I_{a+}^{\gamma;\mu}f)(x) = \frac{(x-a)^{-\mu-\gamma}}{\Gamma(\mu)}\int_a^x \frac{t^\gamma f(t)}{(x-t)^{1-\mu}}dt \quad (\Re(\gamma) > -1, \quad \Re(\mu) > 0) \tag{4.1.2}$$

We shall also assume throughout this chapter that $\mathbf{A}$ denotes the class of functions $f(t)$ for which

$$\int_\omega |f(t)|dt < \infty \tag{4.1.3}$$

for every bounded region ω excluding the origin

and

$$f(t) = \begin{cases} 0\{|t|^{\xi}\} & , \max\{|t|\} \to 0 \\ \\ 0\{|t|^{w_1} e^{-w_2|t|}\} & , \min\{|t|\} \to \infty \end{cases} \tag{4.1.4}$$

Such a class of functions will be represented symbollically as $f(t) \in \mathbf{A}$.

For $m = 1$ and $\gamma = 0$, (4.1.1) reduces to the Hilfer fractional derivative operator[22, p.43, see 11]

4.2 IMAGE OF A PRODUCT OF H-FUNCTION AND $\overline{H}$-FUNCTION

$$D_{p+}^{\gamma,\mu,\nu}\left[x^r(x-p)^{\delta-1}H_{P_1,Q_1}^{M_1,N_1}[x^{-\lambda}(x-p)^{\eta}]\overline{H}_{P_2,Q_2}^{M_2,N_2}[(x-p)^{\sigma}]\right] = \sum_{h=1}^{M_2}\sum_{t=0}^{\infty}\frac{p^{2\gamma+r}}{\Gamma(-\gamma)}\overline{\theta}(s_{t,h})$$

$$(x-p)^{\delta+\sigma s_{t,h}-2\gamma-m-1}$$

$$H_{4,3:P_1,Q_1+1;0,1;1,1}^{0,4:M_1,N_1;1,0;1,1}\left[\begin{array}{c|ccccc} \frac{(x-p)^{\eta}}{p^{\lambda}} & A: & (a_j,\alpha_j)_{1,P_1}, & -; & -; & (1+\gamma,1) \\ \frac{(x-p)}{p} & & & & & \\ \frac{(x-p)}{p} & B: & (bj,\beta j)_{1,Q_1}, & (1+r+\gamma;\lambda); & (0,1); & (0,1) \end{array}\right] \tag{4.2.1}$$

where,

$$A = (1-\delta-\sigma s_{t,h}+\gamma+m;\eta,1,1),(1-\delta-\sigma s_{t,h}+\gamma;\eta,1,0),$$

$$(1-\delta-\sigma s_{t,h};\eta,1,0),(1+\gamma+r;\lambda,1,0)$$

$$B = (1 - \delta - \sigma s_{t,h} + \gamma + m - \nu(1-\mu); \eta, 1, 1),$$

$$(1 - \delta - \sigma s_{t,h} + \gamma + m; \eta, 1, 0), (1 - \delta - \sigma s_{t,h} - (1-\nu)(1-\mu); \eta, 1, 0)$$

provided the following conditions are satisfied
$\Re\Big((1-\nu)(1-\mu) - 1\Big) > 0$,
$\Re\Big(\delta + \eta \min\limits_{1\le j\le M_1}[(b_j^{(i)}/\beta_j^{(i)})] + \sigma \min\limits_{1\le j\le M_2}[(f_j/F_j)] - 1\Big) > 0$ and
$\arg(\frac{x-p}{p}) < \pi$

Proof. To prove (4.2.1), first of all we express $I_{a+}^{\gamma,(1-\nu)(1-\mu)}$ involved in the right hand side of (4.1.1) in the integral form with help of (4.1.2). Then, we express H-function and $\overline{H}$ function in contour form and series form respectively with help of (1.1.1) and (1.1.34) respectively. Next, we change the order of integration and summation therein (which is permissible under the conditions stated), we arrive at the following expression (say Δ_1)

$$\Delta_1 = \frac{(x-p)^{-(1-\nu)(1-\mu)-\gamma}}{\Gamma[(1-\nu)(1-\mu)]} \sum_{t=0}^{\infty} \sum_{h=1}^{M_1} \overline{\theta}(S_{t,h}) \frac{1}{2\pi\omega_1} \int_{L1} \theta(\xi_1)$$

$$\int_p^x (x-t)^{(1-\nu)(1-\mu)-1} t^{\gamma+r-\lambda\xi_1} (t-p)^{\delta-1+\eta\xi_1+\sigma s_{t,h}} dt d\xi_1 \tag{4.2.2}$$

Now, substituting $(t-p) = U$ in the above equation and then evaluate the integral using [13, p.299]. Next, expressing ${}_2F_1$ thus obtained in its contour form we arrive at the following equation (say Δ_2)

$$\Delta_2 = \sum_{t=0}^{\infty} \sum_{h=1}^{M_1} \overline{\theta}(S_{t,h}) \frac{1}{2\pi\omega_1} \int_{L1} \theta(\xi_1) p^{r-\lambda\xi_1+\gamma-\xi_2}$$

$$\frac{1}{2\pi\omega_2} \int_{L2} \frac{\Gamma(-\xi_2)\Gamma(-r+\lambda\xi_1-\gamma+\xi_2)}{\Gamma[(1-\nu)(1-\mu)+\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2]}$$

$$\frac{\Gamma(\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2)}{\Gamma[-r+\lambda\xi_1-\gamma]} (x-p)^{\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2-\gamma-1} d\xi_2 d\xi_1 \tag{4.2.3}$$

Next, we Take the mth derivative with respect to x of the above equation (4.2.3), we get the following:

$$\Delta_3 = \sum_{t=0}^{\infty} \sum_{h=1}^{M_1} \overline{\theta}(S_{t,h}) \frac{1}{2\pi\omega_1} \int_{L1} \theta(\xi_1) p^{r-\lambda\xi_1+\gamma-\xi_2}$$

$$\frac{1}{2\pi\omega_2} \int_{L2} \frac{\Gamma(-\xi_2)\Gamma(-r+\lambda\xi_1-\gamma+\xi_2)}{\Gamma[(1-\nu)(1-\mu)+\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2-m]}$$

$$\frac{\Gamma(\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2)}{\Gamma[-r+\lambda\xi_1-\gamma]} (x-p)^{(1-\nu)(1-\mu)+\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2-m-1} d\xi_2 d\xi_1 \tag{4.2.4}$$

Applying $I_{p+}^{\gamma,\nu(1-\mu)}$ operator to the above expression, with the help of (4.1.2) and interchange the order of contour integration and t-integral using[13, p.299]. Substitute $(t-p)=U$ and evaluating the integral we get the following.

$$\Delta_4 = \sum_{t=0}^{\infty}\sum_{h=1}^{M_1}\overline{\theta}(S_{t,h})\frac{1}{2\pi\omega_1}\int_{L1}\theta(\xi_1)p^{r-\lambda\xi_1+\gamma-\xi_2}$$

$$\frac{1}{2\pi\omega_2}\int_{L2}\frac{\Gamma(-\xi_2)\Gamma(-r+\lambda\xi_1-\gamma+\xi_2)}{\Gamma[(1-\mu)+\delta+\eta\xi_1+\sigma s_{s_{t,h}}+\xi_2-m]}$$

$$\frac{\Gamma(\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2)}{\Gamma[-r+\lambda\xi_1-\gamma]}(x-p)^{(1-\mu)+\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2-m-1}$$

$${}_2F_1\left[-\gamma,(1-\nu)(1-\mu)+\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2-m;\right.$$

$$\left.1-\mu+\delta+\eta\xi_1+\sigma s_{t,h}+\xi_2-m;\frac{-(x-p)}{p}\right]d\xi_2 d\xi_1 \tag{4.2.5}$$

Thereafter, express ${}_2F_1$ in its contour form. Finally, reinterpreting the result thus obtained in terms of multivariable H-function [49, P. 251-252], we easily arrive at the required result. □

4.2.1 SPECIAL CASES

FIRST IMAGE

If we let r=0, $\lambda=0$, and reduce $\overline{H}$ function to unity[24], in the main result we get the following:

$$D_{p+}^{\gamma,\mu,\nu}\left[(x-p)^{\delta-1}H_{P,Q}^{M,N}[(x-p)^{\eta}]\right]=\frac{p^{2\gamma}}{\Gamma(-\gamma)\Gamma(-\gamma)}(x-p)^{\delta-2\gamma-m-1}H_{3,3:P,Q+1;1,1;1,1}^{0,3:M,N;1,1;1,1}$$

$$\left[\begin{array}{c|ccccc} (x-p)^{\eta} & C: & (a_j,\alpha_j)_{1,P}, & -; & (1+\gamma,1); & (1+\gamma,1) \\ \frac{(x-p)}{p} & & & & & \\ \frac{(x-p)}{p} & D: & (b_j,\beta_j)_{1,Q}, & (1-\delta,\eta); & (0,1); & (0,1) \end{array}\right] \tag{4.2.6}$$

where,

$$C=(1-\delta+\gamma+m;\eta,1,1),(1-\delta+\gamma;\eta,1,0),(1-\delta;\eta,1,0)$$

$$D=(1-\delta+\gamma+m-\nu(1-\mu);\eta,1,1),(1-\delta+\gamma+m;\eta,1,0),(1-\delta-(1-\nu)(1-\mu);\eta,1,0)$$

SECOND IMAGE

if we let r =0, and reduce H function to unity [49], in the main result we get the following:

$$D_{p+}^{\lambda,\mu,\nu}\left[(x-p)^{\delta-1}\overline{H}_{P,Q}^{M,N}[(x-p)^{\eta}]\right]=\sum_{h=1}^{M}\sum_{t=0}^{\infty}\overline{\theta}(s_{t,h})\frac{p^{2\gamma}}{\Gamma(-\gamma)\Gamma(-\gamma)}(x-p)^{\delta+\eta s_{t,h}-2\gamma-m-1}$$

$$H_{1,1:3,3;1,1}^{0,1:1,3;1,1}\left[\begin{array}{c|cccc} \frac{(x-p)}{p} & A^*: & C^*, & (1+\gamma,1); & (1+\gamma,1) \\ \frac{(x-p)}{p} & B^*: & D^*, & (0,1); & (0,1) \end{array}\right] \tag{4.2.7}$$

where

$$A^*=(1-\delta-\eta s_{t,h}+\gamma+m;1,1)$$

$$B^* = (1 - \nu(1-\mu) - \delta - \eta s_{t,h} + \gamma + m; 1, 1)$$

$$C^* = (1 - \delta - \eta s_{t,h} + \gamma, 1), (1 - \delta - \eta s_{t,h}, 1)$$

$$D^* = (1 - \delta - \eta s_{t,h} + \gamma + m, 1), (1 - \delta - \eta s_{t,h} - (1-\nu)(1-\mu), 1)$$

Following two interesting results of Polylogarithm function[4, p.195-196, eq.C.23] were found

THIRD IMAGE

Further in second image (4.2.7), if we reduce $\overline{H}-$ function to polylogarithm function $F(x-p,\alpha)$[4, p.195-196, eq.C.23] we get the following:

$$D_{p+}^{\gamma,\mu,\nu}[(x-p)^{\delta-1}F(x-p,\alpha)] = \sum_{r=0}^{\infty}(x-p)^{\delta+r-2\gamma-m-1}\frac{p^{2\gamma}}{(r+1)^{\alpha}}\frac{1}{\Gamma(-\gamma)(-\gamma)}$$

$$H_{1,1:3,3;1,1}^{0,1:1,3;1,1}\left[\begin{array}{c} \frac{(x-p)}{p} \\ \frac{(x-p)}{p} \end{array}\middle| \begin{array}{ccc} (1-\delta-r+\gamma+m;1,1): & C^{*}*; & (1+\gamma,1) \\ (1-\delta-r+\gamma+m-\nu(1-\mu);1,1): & D^{*}*; & (0,1) \end{array}\right] \tag{4.2.8}$$

where

$$C^{*}* = (1-\delta-r+\gamma,1), (1-\delta-r,1), (1+\gamma,1)$$

$$D^{*}* = (1-\delta-r+\gamma+m,1), (1-(1-\nu)(1-\mu)-\delta-r,1), (0,1)$$

FOURTH IMAGE

when $\gamma = 0$ in the above result (4.2.8) we get the following:

$$D_{p+}^{\mu,\nu}[(x-p)^{\delta-1}F(x-p,\alpha)] = (x-p)^{\delta-\mu}\overline{H}_{1,3}^{1,2}\left[-(x-p)\ \middle|\ \begin{matrix} (0,1;\alpha+1), & (-\lambda,1;1) & \\ (0,1), & (-1,1;\alpha), & (\mu-\lambda,1;1) \end{matrix}\right] \tag{4.2.9}$$

4.3 COMPOSITION FORMULAE

COMPOSITION FORMULA FOR THE OPERATORS $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ AND $I_{a+}^{\gamma,\mu}$

$$\left(\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha} I_{a+}^{\gamma,\mu}\phi\right)(x) = \int_a^x u^\gamma (x-u)^{\mu+\beta-1}\frac{(x-a)^{-\mu-\gamma}}{\Gamma(\mu+\gamma)}$$

$$H_{1,1:p,q;1,1}^{0,1:m,n;1,1}\left[\begin{matrix} w(x-u)^\alpha \\ -\frac{x-u}{x-a} \end{matrix}\ \middle|\ \begin{matrix} (1-\beta;\alpha,1): & (a_j,\alpha_j)_{1,p};(1-\mu-\gamma,1) \\ (1-\mu-\beta;\alpha,1): & (b_j,\beta_j)_{1,q};(0,1) \end{matrix}\right]$$

$$\phi(u)du \tag{4.3.1}$$

where $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ is given by (2.2.1) and $I_{a+}^{\gamma,\mu}$ is given by (4.1.2)
provided $\left|\frac{x-u}{x-a}\right| < 1, \Re(\gamma) > -1$

Proof. To prove (4.3.1), we first express both $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ and $I_{a+}^{\gamma,\mu}$ involved in its left hand side, in the integral form with the help of (2.2.1) and (4.1.2) respectively, we have

$$\left(\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha} I_{a+}^{\gamma,\mu}\phi\right)(x) = \int_a^x (x-t)^{\beta-1}\ H_{p,q}^{m,n}[w(x-t)^\alpha]\frac{(t-a)^{-\mu-\gamma}}{\Gamma(\mu)}\int_a^t \frac{u^\gamma}{(t-u)^{1-\mu}}\phi(u)dudt \tag{4.3.2}$$

Now, we interchange the order of $u-$integral and $t-$integral, which is permissible under the conditions stated, we easily arrive at the following after a little simplification:

$$\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta} I^{\gamma,\mu}_{a+} \phi\right)(x) = \int_a^x \frac{u^\gamma}{\Gamma(\mu)} \Delta\phi(u)du \tag{4.3.3}$$

where

$$\Delta = \int_u^x (t-u)^{\mu-1}(x-t)^{\beta-1}(t-a)^{-\mu-\gamma} H^{m,n}_{p,q}[w(x-t)^\alpha]dt \tag{4.3.4}$$

To evaluate Δ, we first replace the $H-$ function occuring in it in terms of its Mellin-Barnes contour integral with the help of (1.1.1) and interchange the order of contour integral and $t-$integral, which is permissible under the given conditions.

The above equation (4.3.4) now takes the following form after a little simplification:

$$\Delta = \frac{1}{2\pi i} \int_L \varphi(\xi) w^\xi \int_u^x (t-u)^{\mu-1}(x-t)^{\beta+\alpha\xi-1}(t-a)^{-\mu-\gamma} dt d\xi \tag{4.3.5}$$

On setting $z = \frac{x-t}{x-u}$ in the $t-$integral involved in (4.3.5) and evaluating the resulting $z-$integral with the help of the known result [13, p.286, eq.(3.197(3))], we arrive at the following result after a little simplication:

$$\Delta = \frac{1}{2\pi i} \int_L \varphi(\xi) w^\xi (x-u)^{\mu+\beta+\alpha\xi-1}(x-a)^{-\mu-\gamma} B(\mu, \beta+\alpha\xi)$$
$$ {}_2F_1\left[\mu+\gamma, \beta+\alpha\xi; \mu+\beta+\alpha\xi; -\frac{x-u}{x-a}\right] d\xi \tag{4.3.6}$$

Now, writing ${}_2F_1$ in terms of its contour and reinterpreting the above equation(4.3.6) in terms of the $H-$function of two variables and on substituting the value of Δ thus obtained, in (4.3.3), we easily arrive at the desired result (4.3.1) after a little simplification. $\square$

COMPOSITION FORMULA FOR THE OPERATORS $I_{a+}^{\gamma,\mu}$ AND $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$

$$\left(I_{a+}^{\gamma,\mu}\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}\phi\right)(x)=\frac{x^{\gamma}(x-a)^{-\mu-\gamma}}{\Gamma(\mu)\Gamma(-\gamma)}\int_a^x x^{\gamma}(x-u)^{\mu+\beta-1}$$

$$H_{0,1:p+1,q;2,1}^{0,0:m,n+1;1,2}\left[\begin{array}{c|c} w(x-u)^{\alpha} & -:(1-\beta,\alpha),(a_j,\alpha_j)_{1,p};(1+\gamma,1),(1-\mu,1) \\ -\frac{x-u}{x} & (1-\mu-\beta;\alpha,1):(b_j,\beta_j)_{1,q};(0,1) \end{array}\right]$$

$$\phi(u)du \tag{4.3.7}$$

where $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ is given by (2.2.1) and $I_{a+}^{\gamma,\mu}$ is given by (4.1.2)
provided $\left|\frac{x-u}{x}\right|<1$

Proof. To prove (4.3.7), we first express both $\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}$ and $I_{a+}^{\gamma,\mu}$ involved in its left hand side, in the integral form with the help of (2.2.1) and (4.1.2) respectively, we have

$$\left(I_{a+}^{\gamma,\mu}\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}\phi\right)(x)=\frac{(x-a)^{-\mu-\gamma}}{\Gamma(\mu)}\int_a^x t^{\gamma}(x-t)^{\mu-1}\int_a^t(t-u)^{\beta-1}H_{p,q}^{m,n}[w(x-t)^{\alpha}]\phi(u)dudt \tag{4.3.8}$$

Now, we interchange the order of $u-$integral and $t-$integral, which is permissible under the conditions stated, we easily arrive at the following after a little simplification:

$$\left(I_{a+}^{\gamma,\mu}\mathcal{H}_{a+;p,q;\beta}^{w;m,n;\alpha}\phi\right)(x)=\frac{(x-a)^{-\mu-\gamma}}{\Gamma(\mu)}\int_a^x \Delta\phi(u)du \tag{4.3.9}$$

where

$$\Delta=\int_u^x t^{\gamma}(t-u)^{\beta-1}(x-t)^{\mu-1}H_{p,q}^{m,n}[w(x-t)^{\alpha}]dt \tag{4.3.10}$$

To evaluate Δ, we first replace the $H-$ function occuring in it in terms of its Mellin-Barnes contour integral with the help of (1.1.1) and interchange the order of contour integral and $t-$integral, which is permissible under the given conditions.

The above equation (4.3.10) now takes the following form after a little simplification:

$$\Delta = \frac{1}{2\pi i}\int_L \varphi(\xi)w^\xi \int_u^x t^\gamma (t-u)^{\beta+\alpha\xi-1}(x-t)^{\mu-1}dtd\xi \tag{4.3.11}$$

On setting $z = \frac{x-t}{x-u}$ in the $t-$integral involved in (4.3.11) and evaluating the resulting $z-$integral with the help of the known result [13, p.286, eq.3.197(3)], we arrive at the following result after a little simplication:

$$\Delta = \frac{1}{2\pi i}\int_L \varphi(\xi)w^\xi x^\gamma (x-u)^{\mu+\beta+\alpha\xi-1}B(\mu,\beta+\alpha\xi)$$
$$ {}_2F_1\left[-\gamma,\mu;\mu+\beta+\alpha\xi;-\frac{x-u}{x}\right]d\xi \tag{4.3.12}$$

Now, expressing ${}_2F_1$ in its contour form and reinterpreting the above equation(4.3.12) in terms of the $H-$function of two variables and on substituting the value of Δ thus obtained, in (4.3.9), we easily arrive at the desired result (4.3.7) after a little simplification. □

COMPOSITION FORMULA FOR THE OPERATORS $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ AND I^μ_{a+}

For $m = 1$ and $\gamma = 0$ in (4.1.2),under the various parameteric constraints listed already with the definition (2.2.1), the following composition relationship is obtained as special cases of (4.3.1) and (4.3.7):

$$\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta} I^\mu_{a+}\phi\right)(x) = \left(\mathcal{H}^{w;m,n+1;\alpha}_{a+;p+1,q+1;\beta+\mu}\phi\right)(x) \tag{4.3.13}$$

Proof. To prove (4.3.13), we first express both $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ and I^μ_{a+} involved in its left hand side, in the integral form with the help of (2.2.1) and (2.1.1) respectively, we have

$$\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta} I^{\mu}_{a+}\phi\right)(x) = \int_a^x (x-t)^{\beta-1} H^{m,n}_{p,q}[w(x-t)^{\alpha}] \frac{1}{\Gamma(\mu)} \int_a^t (t-u)^{\mu-1}\phi(u)dudt \tag{4.3.14}$$

Next, we change the order of $u-$integral and $t-$integral, which is permissible under the conditions stated, we easily arrive at the following after a little simplification:

$$\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta} I^{\mu}_{a+}\phi\right)(x) = \frac{1}{\Gamma(\mu)} \int_a^x \Delta\phi(u)du \tag{4.3.15}$$

where

$$\Delta = \int_u^x (t-u)^{\mu-1}(x-t)^{\beta-1} H^{m,n}_{p,q}[w(x-t)^{\alpha}]dt \tag{4.3.16}$$

To evaluate Δ, we first replace the $H-$ function occuring in it in terms of its Mellin-Barnes contour integral with the help of (1.1.1) and interchange the order of contour integral and $t-$integral, which is permissible under the given conditions.

The above equation (4.3.16) now takes the following form after a little simplification:

$$\Delta = \frac{1}{2\pi i} \int_L \varphi(\xi) w^{\xi} \int_u^x (t-u)^{\mu-1}(x-t)^{\beta+\alpha\xi-1} dtd\xi \tag{4.3.17}$$

On setting $z = \frac{x-t}{x-u}$ in the $t-$integral involved in (4.3.17) and evaluating the resulting $z-$integral, we arrive at the following result after a little simplication:

$$\Delta = \frac{1}{2\pi i} \int_L \varphi(\xi) w^{\xi} (x-u)^{\mu+\beta+\alpha\xi-1} \frac{\Gamma(\beta+\alpha\xi)}{\Gamma(\mu+\beta+\alpha\xi)} d\xi \tag{4.3.18}$$

Now, reinterpreting the above equation(4.3.18) in terms of the $H-$function and on substituting the value of Δ thus obtained, in (4.3.15), we easily arrive at the desired result (4.3.13) after a little simplification. □

on similar lines we can also prove the following:

$$\left(I^{\mu}_{a+} \mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x) = \left(\mathcal{H}^{w;m,n+1;\alpha}_{a+;p+1,q+1;\beta+\mu}\phi\right)(x) \tag{4.3.19}$$

COMPOSITION FORMULA FOR THE OPERATORS $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ AND D^{μ}_{a+}

$$D^{\mu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x)=\left(\mathcal{H}^{w;m,n+2;\alpha}_{a+;p+2,q+2;\beta-\mu}\phi\right)(x) \tag{4.3.20}$$

where D^{μ}_{a+} is given by (2.1.2) and $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ is given by (2.2.1).

Proof. To prove (4.3.20) we make use of definition (2.1.2) of D^{μ}_{a+} involved in the left hand side of (4.3.20), we get

$$D^{\mu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x)=\left(\frac{d}{dx}\right)^{n} I^{n-\lambda}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right) \tag{4.3.21}$$

With the help of result (4.3.13) the above equation (4.3.21) takes the following form:

$$D^{\mu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(\mathcal{H}^{w;m,n+1;\alpha}_{a+;p+1,q+1;\beta+n-\lambda}\phi\right)$$

$$=\left(\frac{d}{dx}\right)^{n}\int_a^x (x-t)^{\beta-1+n-\lambda}H^{m,n+1}_{p+1,q+1}[w(x-t)^{\alpha}]\phi(t)dt$$

$$=\int_a^x (x-t)^{\beta-1-\lambda}H^{m,n+2}_{p+2,q+2}[w(x-t)^{\alpha}]\phi(t)dt$$

$$=\left(\mathcal{H}^{w;m,n+2;\alpha}_{a+;p+2,q+2;\beta-\mu}\phi\right)(x) \tag{4.3.22}$$

□

On similar lines we can prove the following:

$$\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}D^{\mu}_{a+}\phi\right)(x)=\left(\mathcal{H}^{w;m,n+2;\alpha}_{a+;p+2,q+2;\beta-\mu}\phi\right)(x) \tag{4.3.23}$$

COMPOSITION FORMULA FOR THE OPERATORS $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ AND $D^{\mu,\nu}_{a+}$

$$D^{\mu,\nu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x) = \left(\mathcal{H}^{w;m,n+3;\alpha}_{a+;p+3,q+3;\beta-\mu}\phi\right)(x) \tag{4.3.24}$$

where $D^{\mu,\nu}_{a+}$ is given by (2.1.3) and $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ is given by (2.2.1).

Proof. Making use of the composition relationships asserted by (4.3.20) and (4.3.13), we find that

$$D^{\mu+\nu-\mu\nu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x) = \left(\mathcal{H}^{w;m,n+1;\alpha}_{a+;p+1,q+3;\beta-\mu-\nu+\mu\nu}\phi\right)(x) \tag{4.3.25}$$

and

$$\begin{aligned} D^{\mu,\nu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x) &= I^{\nu(1-\mu)}_{a+} D^{\mu+\nu-\mu\nu}_{a+}\left(\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}\phi\right)(x) \\ &= I^{\nu(1-\mu)}_{a+}\left(\mathcal{H}^{w;m,n+1;\alpha}_{a+;p+1,q+3;\beta-\mu-\nu+\mu\nu}\phi\right)(x) \\ &= \left(\mathcal{H}^{w;m,n+3;\alpha}_{a+;p+3,q+3;\beta-\mu}\phi\right)(x) \end{aligned} \tag{4.3.26}$$

which would complete the proof of (4.3.24). □

If we reduce the $H-$ function involved in the integral operator $\mathcal{H}^{w;m,n;\alpha}_{a+;p,q;\beta}$ to Mittag-Leffler function in (4.3.13), (4.3.20) and (4.3.24) we get the results obtained by Srivastava et al.[56, p. 7, eq.(2.23), (2.24) and (2.25)] respectively.

5

SOLUTION OF GENERAL VOLTERRA-TYPE INTEGRAL EQUATIONS

The main findings of this chapter have been published/Accepted as below:

1.**A General Volterra type Fractional Equation Associated with an Integral operator with the H-function in the Kernel**, J. of Raj. Acad. of Phy. Sci., **14(3)** (2015) 289-294.

2. **A General Volterra type Fractional Equation Associated with an Integral operator with the $\overline{H}$-function in the Kernel**, Int. J. of Pure and Appl. Math. (Accepted)

3. **A General Volterra-type Integral Equation Associated with an Integral Operator involving the product of S_N^M and multivarible H-Function in the Kernel**,J. of Raj. Acad. of Phy. Sci. Vol. 3 (2016)(In press)

The object of this chapter is to find solutions of the Volterra-type integral equations associated with integral operators whose kernels involve various special functions and polynomials.We make use of convolution technique to solve the equations.

We first give solution to general Volterra-type integral equation associated with an integral operator involving $\overline{H}$-function in its kernel. Since $\overline{H}$-function is general in nature we can obtain a number of special cases of the Volterra-type integral equation, by specializing the parameters involved.First special case is a Volterra-type integral equation associated with an integral operator with H-function in its kernal. On account of general nature of H-function occuring in the operator herein we can obtain a number of special cases by specializing the parameters of the H-function. We record here five such special cases which involve Fox-Wright function ${}_p\Psi_q$, Mittag-Leffler function$E_{\alpha,\beta}^{\gamma,\kappa}$, hypergeometric function ${}_pF_q$, Bessel function$\mathcal{J}_\lambda^\sigma$ and giving appropriate value to $g(x)$. Thereafter we give two special cases involving Riemann Zeta function $\phi(t, \mu, \xi)$ and Polylogarithm function$F(t, \mu)$.

Further, we solve a general Volterra-type integral equation involving a product of general class of polynomials S_N^M and multivariable $H-$function occuring as kernels in the integral equation. We can obtain a large number of integral equations involving products of several useful polynomials and special functions as its special cases. We record here only two such special cases which involve the product of general class of polynomialsS_N^M & Appell function F_3 and a general class of polynomials.

The importance of the findings of this chapter lies in the fact that both the Volterra-type integral equations associated with integral operators involving several special functions and polynomials in their kernel are quite general in nature and generalize the results obtained by Srivastava et al.[49] and Rashmi

Jain[26][27] respectively. A large number of special cases of these equations, involving useful and simpler special functions and polynomials can be obtained by specializing the parameters, find practical importance in the fields of Physics and Engineering science.

5.1 INTRODUCTION AND DEFINITIONS

The $\overline{H}$-function occurring in the present work will be defined and represented here in the following manner [24]

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{ll} (a_j,\alpha_j;A_j)_{1,n}, & (a_j,\alpha_j)_{n+1,p} \\ \\ (b_j,\beta_j)_{1,m}, & (b_j,\beta_j;B_j)_{m+1,q} \end{array} \right]$$

$$:= \frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty} \overline{\Theta}(\xi) z^{\xi} d\xi \tag{5.1.1}$$

where, $\omega = \sqrt{-1}$,$z \in \mathbb{C} \setminus \{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\overline{\Theta}(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} \tag{5.1.2}$$

The sufficient condition for the absolute convergence of the integral have been established by Bushman and Srivastava[5, p.4708] The series representation for the $\overline{H}-$Function is as follows:

$$\overline{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{ll} (a_j,\alpha_j;A_j)_{1,n}, & (a_j,\alpha_j)_{n+1,p} \\ \\ (b_j,\beta_j)_{1,m}, & (b_j,\beta_j;B_j)_{m+1,q} \end{array} \right] = \sum_{t=0}^{\infty}\sum_{h=1}^{m}\overline{\Theta}\mathfrak{s}_{t,h} z^{\mathfrak{s}_{t,h}} \tag{5.1.3}$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t,h}) = \frac{\prod_{j=1,j\neq h}^{m} \Gamma(b_j - \beta_j \mathfrak{s}_{t,h}) \prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j \mathfrak{s}_{t,h})\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j \mathfrak{s}_{t,h})\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \mathfrak{s}_{t,h})} \tag{5.1.4}$$

The multivariable H-function occuring in this chapter is a special case of H-function of r-variables and is defined as follows:[52, p.271, eq.(4.1)][50, p.64, eq.(1.3)]

$$H \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{bmatrix} = H^{0,0:1,n_1;...;1,n_r}_{p,q:p_1,q_1+1;...;p_r,q_r+1}$$

$$\left[\begin{array}{c|c} z_1 & (a_j; \alpha_j^{(1)}, ..., \alpha_j^{(r)})_{1,p} : (c_j^{(1)}, \gamma_j^{(1)})_{1,p_1}; ...; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (b_j; \beta_j^{(1)}, ..., \beta_j^{(r)})_{1,q} : (0,1), (d_j^{(1)}, \delta_j^{(1)})_{1,q_1}; ...; (0,1), (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1, ..., \xi_r) \prod_{i=1}^{r} (\phi_i(\xi_i) z_i^{\xi_i}) \Gamma(-\xi_1) ... \Gamma(-\xi_r) z_1^{\xi_1} ... z_r^{\xi_r} d\xi_1 ... d\xi_r$$

(5.1.5)

(5.1.6)

$$= \sum_{k_1,...,k_r=0}^{\infty} \phi_1(k_1) ... \phi_r(k_r) \psi(k_1, ..., k_r) \frac{(-z_1)^{k_1}}{k_1!} ... \frac{(-z_r)^{k_r}}{k_r!} \tag{5.1.7}$$

where $\omega = \sqrt{-1}$ and

$$\phi_i(k_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} k_i)}{\prod_{j=1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} k_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} k_i)} \qquad (i = 1, ..., r) \qquad (5.1.8)$$

$$\psi(k_1, ..., k_r) = \frac{1}{\prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} k_i) \prod_{j=1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} k_i)} \qquad (5.1.9)$$

AN INTEGRAL OPERATOR INVOLVING $\overline{H}$-FUNCTION IN ITS KERNEL

In this chapter, we make use of the following integral operator with $\overline{H}$-function in its kernel

$$\left(\overline{\mathcal{H}}^{1,n;\sigma}_{0+;p,q;\rho}\,\varphi\right)(x) := \int_0^x (x-t)^{\rho-1}\,\overline{H}^{1,n}_{p,q}[(x-t)^{\sigma}]\varphi(t)dt \qquad (5.1.10)$$

$\Re(\rho) > 0$ By using the *Convolution Theorem* for the Laplace Transform, we find from the definition (5.1.10) that

$$\mathcal{L}\left[\left(\overline{\mathcal{H}}^{1,n;\sigma}_{0+;p,q;\rho}\,\varphi\right)(x)\right](s) = \mathcal{L}\left[x^{\rho-1}\,\overline{H}^{1,n}_{p,q}[x^{\sigma}]\right](s)\cdot\mathcal{L}[\varphi(x)](s)$$

$$= s^{-\rho}\overline{H}^{1,n+1}_{p+1,q}\left[s^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ \\(0,1),(b_j,\beta_j;B_j)_{2,q}\end{matrix}\right.\right]\Phi(s) \qquad (5.1.11)$$

where $\Re(s, \rho, \sigma) > 0$

AN INTEGRAL OPERATOR WITH THE H-FUNCTION IN ITS KERNEL

In our present investigation, we make use of the following special case of the operator given by (5.1.10) i.e. the integral operator with the H-function in its kernel[66]:

$$\left(\mathcal{H}^{w;1,n;\alpha}_{0+;p,q;\beta}\,\varphi\right)(x) := \int_0^x (x-t)^{\beta-1}\, H^{1,n}_{p,q}[w(x-t)^{\alpha}]\varphi(t)dt \tag{5.1.12}$$

$$\Bigg(\Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\};\ 0 \leqq n \leqq p;\Bigg).$$

By using the *Convolution Theorem* for the Laplace Transform given by (2.1.5) we find from the definition (5.1.12) that

$$\mathcal{L}\left[\left(\mathcal{H}^{w;1,n;\alpha}_{0+;p,q;\beta}\,\varphi\right)(x)\right](s) = \mathcal{L}\left[x^{\beta-1}\, H^{1,n}_{p,q}[wx^{\alpha}]\right](s)\cdot\mathcal{L}[\varphi(x)](s)$$

$$= s^{-\beta}H^{1,n+1}_{p+1,q}\left[ws^{-\alpha}\,\middle|\,\begin{matrix}(1-\beta,\alpha),(a_j,\alpha_j)_{1,p}\\ \\ (0,1),(b_j,\beta_j)_{2,q}\end{matrix}\right]\Phi(s) \tag{5.1.13}$$

$$\Bigg(\Re(s) > 0;\ \alpha > 0;\ \Re(\beta) > 0\Bigg),$$

where, for convenience,

$$\Phi(s) := \mathcal{L}[\varphi(x)](s) \qquad \big(\Re(s) > 0\big).$$

AN INTEGRAL OPERATOR INVOLVING THE PRODUCT OF GENERAL CLASS OF POLYNOMIALS AND MULTIVARIABLE H-FUNCTION IN ITS KERNEL

The following integral operator involving a product of general class of polynomials and multivariable H-function in its kernel will be used in this chapter[16]:

$$\int_0^x (x-t)^{\beta-1} S_N^M[-z_{r+1}(x-t)] H \begin{bmatrix} z_1(x-t) \\ . \\ . \\ . \\ z_r(x-t) \end{bmatrix} \phi(t)dt \tag{5.1.14}$$

$\Re(\beta) > 0$

where $H \begin{bmatrix} z_1(x-t) \\ . \\ . \\ . \\ z_r(x-t) \end{bmatrix}$ is given by (5.1.5) and (5.1.7)

5.2 GENERAL VOLTERRA-TYPE INTEGRAL EQUATION ASSOCIATED WITH THE OPERATOR $\overline{\mathcal{H}}_{0+;p,q;\rho}^{1,n;\sigma}$

A general Volterra-type integral equation associated with an integral operator with the $\overline{H}$-function in its kernel (5.1.10) is given by

$$\left(\overline{\mathcal{H}}_{0+;p,q;\rho}^{1,n;\sigma}\, y\right)(x) + \frac{a}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} y(t)dt := g(x) \tag{5.2.1}$$

$\Re(\rho,\sigma,\eta) > 0; 0 \leq n \leq p$

has the solution

$$y(x) = \int_0^x (x-t)^{l-\sigma k-\rho-1} \sum_{\lambda=0}^{\infty} \frac{C_\lambda (x-t)^{\sigma\lambda}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)} D_t^l\{g(t)\}dt, \tag{5.2.2}$$

where l is a positive integer such that $\Re(l-\sigma k-\rho) > 0$, where k denotes the least ν for which $C_\nu^{'} \neq 0$ where

$$C_\nu^{'} = \frac{\Gamma(\rho+\sigma\nu)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\nu)\}^{A_j}}{\prod_{j=2}^{q}\{\Gamma(1-b_j+\beta_j\nu)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\nu)\nu!}(-1)^\nu \tag{5.2.3}$$

g is prescribed such that $g^{(u)}(0) = 0$ for $0 \leq u \leq l-1$

C_λ are given by

$$C_\lambda = (-1)^\lambda (C_k^{'})^{-\lambda-1} \det \begin{bmatrix} C_{k+1}^{'} & C_k^{'} & ... & 0 & ... & 0 \\ C_{k+2}^{'} & C_{k+1}^{'} & ... & ... & ... & 0 \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ \left(C_{k+\frac{\eta-\rho}{\sigma}}+a\right) & . & & & & \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ C_{k+\lambda}^{'} & C_{k+\lambda-1}^{'} & ... & \left(C_{k+\frac{\eta-\rho}{\sigma}}+a\right) & ... & C_{k+1}^{'} \end{bmatrix} \tag{5.2.4}$$

Proof. To solve (5.2.1) we first take the Laplace transform of its both sides. Using (5.1.11) we get

$$s^{-\rho}\overline{H}^{1,n+1}_{p+1,q}\left[s^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\ \\ (0,1),(b_j,\beta_j;B_j)_{2,q}\end{matrix}\right.\right]Y(s)+aS^{-\eta}Y(s)=G(s) \tag{5.2.5}$$

Now, expressing $\overline{H}$- function involved in the left hand side of the above equation in terms of series with the help of (5.1.3) we have

$$s^{-\rho}\left[\sum_{\nu=0}^{\infty}C'_{\nu}s^{-\sigma\nu}+as^{-\eta+\rho}\right]Y(s)=G(s) \tag{5.2.6}$$

where C'_ν is given by (5.2.3).

Again, (5.2.6) is equivalent to

$$Y(s)=s^{\rho}\left[\sum_{\nu=0}^{\infty}C'_{\nu}s^{-\sigma\nu}+as^{-\eta+\rho}\right]^{-1}G(s) \tag{5.2.7}$$

If k denotes the least ν for which $C'_\nu\neq 0$ the series given by (5.2.7) can be reciprocated.

Writing

$$\left[\sum_{\nu=0}^{\infty}C'_{\nu+k}s^{-\sigma\nu}+as^{-\eta+\rho}\right]^{-1}=\sum_{\lambda=0}^{\infty}C_{\lambda}s^{-\sigma\lambda} \tag{5.2.8}$$

(5.2.7) takes the following form:

$$Y(s)=s^{\rho-l+\sigma k}\sum_{\lambda=0}^{\infty}C_{\lambda}s^{-\sigma\lambda}[s^{l}G(s)] \tag{5.2.9}$$

(5.2.9) can be written as

$$L\{y(x);s\}=L\left\{\sum_{\lambda=0}^{\infty}C_{\lambda}\frac{x^{l-\rho-\sigma k+\sigma\lambda-1}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)};s\right\}L\{g^{(l)}(x);s\} \tag{5.2.10}$$

Now using the convolution theorem in the RHS of (5.2.10) we get (using (2.3.30))

$$L\{y(x);s\}=L\left\{\int_0^x\sum_{\lambda=0}^{\infty}C_{\lambda}\frac{(x-t)^{l-\rho-\sigma k+\sigma\lambda-1}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)}g^{(l)}(x);s\right\} \tag{5.2.11}$$

□

Finally, on taking the inverse of the Laplace transform of both sides of (5.2.11) we arrive at the desired result (5.2.2).
If we take $\sigma = 1$, $A_j(j = 1, 2, ..., n) = B_j(j = 2, ...q) = 1$ and $a = 0$ in (5.2.1) we arrive at the result derived by Srivastava and Bushman[47][48] and If we take $a = 0$ in the (5.2.1) we arrive at the result obtained by Jain [26, theorem 2].

5.2.1 SPECIAL CASES

1. General Volterra-type integral equation associated with an integral operator with the H-function in its kernel (5.1.12) is given by

$$\left(\mathcal{H}^{w;1,n;\alpha}_{0+;p,q;\beta}\, y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt := g(x) \tag{5.2.12}$$

$$\left(\Re(\beta, \nu) > 0;\ w \in \mathbb{C} \setminus \{0\};\ 0 \leqq n \leqq p;\right)$$

has the solution

$$y(x) = \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r + l - \beta - \alpha\mu - 1}}{\Gamma(\alpha r + l - \beta - \alpha\mu)} g^{(l)}(t)dt \tag{5.2.13}$$

where $\Re(l - \beta - \alpha\mu) > 0$
provided that
$g^{(i)}(0) = 0$ for $0 \leqq i \leqq l-1$, l being a positive integer and $\nu - \beta(< r)$ is an

integer. Also

$$E_r = (-1)^r (\lambda_\mu)^{-r-1} w^r \det \begin{bmatrix} \lambda_{\mu+1} & \lambda_\mu & \cdots & \cdots & 0 & 0 \\ \lambda_{\mu+2} & \lambda_{\mu+1} & \cdots & .. & 0 & 0 \\ . & . & & & & \\ . & & & & & \\ . & . & & & & \\ w^{-\alpha}\left(\lambda_{\mu+\frac{\nu-\beta}{\alpha}} + \frac{a}{w^{\frac{\nu-\beta}{\alpha}}}\right). & . & & & & \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ \lambda_{\mu+r} & \lambda_{\mu+r-1} & \cdots\cdot & w^{-\alpha}\left(\lambda_{\mu+\frac{\nu-\beta}{\alpha}} + \frac{a}{w^{\frac{\nu-\beta}{\alpha}}}\right) & \cdots & \lambda_{\mu+1} \end{bmatrix} \tag{5.2.14}$$

and μ is the least k for which

$$\lambda_k = \frac{\Gamma(\beta+\alpha k)\prod\limits_{j=1}^{n}\Gamma(1-a_j+\alpha_j k)}{\prod\limits_{j=2}^{q}\Gamma(1-b_j+\beta_j k)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j k)k!}(-1)^k \neq 0 \tag{5.2.15}$$

SPECIAL CASES OF (5.2.12)

2. Reducing H-function in the R.H.S of (5.1.12) to the Fox-Wright function[33] and defining the integral operator as

$$\left(\Psi^{w;p;\alpha}_{a+;q;\beta}\,\varphi\right)(x) := \int_a^x (x-t)^{\beta-1}\,{}_p\Psi_q \left[\begin{matrix} (a_j,\alpha_j)_{1,p}; & \\ & w(x-t)^\alpha \\ (b_j,\beta_j)_{1,q}; & \end{matrix}\right] \varphi(t)dt \tag{5.2.16}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C}\setminus\{0\};\ p \leqq q+1\right),$$

We find that Volterra-type equation (5.2.12) takes the form as

$$\left(\Psi^{w;p;\alpha}_{0+;q;\beta}\, y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt = g(x) \tag{5.2.17}$$

$$(\Re(\beta) > 0; w \in \mathbb{C} \setminus \{0\}; p \leqq q+1; \Re(\nu) > 0)$$

whose solution is given by

$$y(x) = \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r + l - \beta - \alpha\mu - 1}}{\Gamma(\alpha r + l - \beta - \alpha\mu)} g^{(l)}(t)dt \tag{5.2.18}$$

where $\Re(l - \beta - \alpha\mu) > 0$

provided that

$g^{(i)} = 0$ for $0 \leqq i \leqq l-1, l$ being a positive integer and $\nu - \beta (< r)$ is an integer, E_r is given by (5.2.14) and μ is the least n for which

$$\lambda_n = \frac{\Gamma(a_1 + \alpha_1 n)...\Gamma(a_p + \alpha_p n)\Gamma(\beta - \alpha n)}{\Gamma(b_1 + \beta_1 n)...\Gamma(b_q + \beta_q n)n!} \neq 0 \tag{5.2.19}$$

3. Again reducing H-function in R.H.S of (5.1.12) to Mittag-Leffler function [56] and defining the integral operator as follows:

$$\left(\xi^{w;\gamma,\kappa}_{0+;\alpha,\beta} y\right)(x) = \int_0^x (x-t)^{\beta-1} E^{\gamma,\kappa}_{\alpha,\beta}[w(x-t)^{\alpha}] y(t)dt, \tag{5.2.20}$$

$$(\gamma, w \in \mathbb{C}; \Re(\alpha) > max\{0, \Re(\kappa) - 1\}; min\{\Re(\beta), \Re(\kappa)\} > 0)$$

then (5.2.12) can be written as

$$\left(\xi^{w;\gamma,\kappa}_{0+;\alpha,\beta} y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt = g(x) \tag{5.2.21}$$

$$(\gamma, w \in \mathbb{C}; \Re(\alpha) > max\{0, \Re(\kappa) - 1\}; \quad min\{\Re(\beta), \Re(\kappa)\} > 0; \Re(\nu) > 0)$$

whose solution is given by

$$y(x) = \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r + l - \beta - \alpha\mu - 1}}{\Gamma(\alpha r + l - \beta - \alpha\mu)} g^{(l)}(t)dt \tag{5.2.22}$$

where $\Re(l - \beta - \alpha\mu) > 0$

provided that

$g^{(i)} = 0$ for $0 \leqq i \leqq l-1, l$ being a positive integer and $\nu - \beta(< r)$ is an integer, E_r is given by (5.2.14) and μ is the least n for which

$$\lambda_n = \frac{\Gamma(\gamma + \kappa n)}{\Gamma(\gamma) n!} \neq 0 \tag{5.2.23}$$

4. Reducing H-function in R.H.S of (5.1.12) to hypergeometric function [49, p.18, eq.(2.6.3)] and defining the fractional operator as follows:

$$\left(\mathcal{F}_{a+;q;\beta}^{w;p;\alpha} \varphi\right)(x) := \int_a^x (x-t)^{\beta-1} {}_pF_q \left[\begin{array}{l} a_1, \cdots, a_p; \\ \\ \qquad\qquad\qquad w(x-t)^{\alpha} \\ \\ b_1, \cdots, b_q; \end{array} \right] \varphi(t) dt \tag{5.2.24}$$

$$\big(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ p \leqq q+1\big),$$

then (5.2.12) can be written as

$$\left(\mathcal{F}_{a+;q;\beta}^{w;p;\alpha} y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t) dt = g(x) \tag{5.2.25}$$

$$(\Re(\beta) > 0; w \in \mathbb{C} \setminus \{0\}; p \leqq q+1; \Re(\nu) > 0)$$

whose solution is given by

$$y(x) = \int_0^x (x-t)^{l-\beta-\alpha\mu-1} \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r}}{\Gamma(\alpha r + l - \beta - \alpha\mu)} g^{(l)}(t) dt \tag{5.2.26}$$

where $\Re(l - \beta - \alpha\mu) > 0$

provided that

$g^{(i)} = 0$ for $0 \leqq i \leqq l-1, l$ being a positive integer and $\nu - \beta(< r)$ is an integer, E_r is given by (5.2.14) and μ is the least n for which

$$\lambda_n = \frac{\Gamma(a_1+n)...\Gamma(a_p+n)\Gamma(\beta-\alpha n)}{\Gamma(b_1+n)...\Gamma(b_q+n)n!} \neq 0 \tag{5.2.27}$$

5. Reducing H-function in R.H.S of (5.1.12) to Bessel function [49, p.19, eq.(2.6.10)] and defining the integral operator as follows:

$$\left(\mathcal{J}_{a+;0,2;\beta}^{w;1,0;\alpha} \varphi\right)(x) := \int_a^x (x-t)^{\beta-1} \mathcal{J}_{\lambda}^{\sigma}(w(x-t)^{\alpha}) \varphi(t) dt \tag{5.2.28}$$

$$\big(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\}\Re(\nu) > 0\big)$$

then (5.2.12) can be written as

$$\left(\mathcal{J}_{a+;0,2;\beta}^{w;1,0;\alpha}\, y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt = g(x) \tag{5.2.29}$$

$$(\Re(\beta) > 0; w \in \mathbb{C} \setminus \{0\}; \Re(\nu) > 0)$$

whose solution is given by

$$y(x) = \int_0^x (x-t)^{l-\beta-\alpha\mu-1} \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r}}{\Gamma(\alpha r + l - \beta - \alpha\mu)} g^{(l)}(t)dt \tag{5.2.30}$$

where $\Re(l - \beta - \alpha\mu) > 0$

provided that

$g^{(i)} = 0$ for $0 \leqq i \leqq l-1, l$ being a positive integer and $\nu - \beta$ is an integer,

E_r is given by (5.2.14) and μ is the least n for which

$$\lambda_n = \frac{\Gamma(\beta + \alpha n)}{\Gamma(1 + \lambda + \sigma n)n!} \neq 0 \tag{5.2.31}$$

6. Substituting $g(x) = x^2$ in (5.2.12) we get

$$\left(\mathcal{H}_{0+;p,q;\beta}^{w;1,n;\alpha}\, y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt := x^2 \tag{5.2.32}$$

$$\left(\Re(\beta) > 0;\ w \in \mathbb{C} \setminus \{0\};\ 0 \leqq n \leqq p; \Re(\nu) > 0\right)$$

has the solution

$$y(x) = 2 \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r - \beta - \alpha\mu + 1}}{\Gamma(\alpha r - \beta - \alpha\mu + 2)} dt, \tag{5.2.33}$$

$\Re(l - \beta - \alpha\mu) > 0$

where E_r is given by (5.2.14) and μ is the least k for which $\lambda_k (\neq 0)$ is given by (5.2.15)

$\nu - \beta (< r)$ is an integer.

SPECIAL CASES OF (5.2.1)

7. If we reduce the $\overline{H}$-function involved in (5.2.1) to the generalized Riemann Zeta function, $\phi\left((x-t)^{\sigma},\mu,\xi\right)$, [9, p.27], we arrive at the following result:

$$\int_0^x (x-t)^{\rho-1}\phi\left((x-t)^{\sigma},\mu,\xi\right)y(t)dt+\frac{a}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}y(t)dt:=g(x) \quad (5.2.34)$$

has the solution given by

$$y(x)=\int_0^x (x-t)^{l-\sigma k-\rho-1}\sum_{\lambda=0}^{\infty}\frac{C_{\lambda}(x-t)^{\sigma\lambda}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)}D_t^l\{g(t)\}dt \quad (5.2.35)$$

provided that $\min\Re(\rho,\sigma,l-\rho-\sigma k)>0, l$ is a positive integer and C_{λ} is given by (5.2.4)

where

$$C'_{\nu}=\frac{\Gamma(\rho+\sigma\nu)}{(\xi+\nu)^{\mu}},\nu=0,1,2,.... \quad (5.2.36)$$

Also $g^{(u)}(0)=0$ for $0\le u\le l-1$

8. Again, if we reduce the $\overline{H}$-function involved in (5.2.1) to the Polylogarithm function $F(t,\mu)$ of order μ[9, p.30,p.315], we get the following result:

$$\int_0^x (x-t)^{\rho-1}F\left((x-t)^{\sigma},\mu\right)y(t)dt+\frac{a}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}y(t)dt:=g(x) \quad (5.2.37)$$

has the solution given by

$$y(x)=\int_0^x (x-t)^{l-\sigma k-\rho-1}\sum_{\lambda=0}^{\infty}\frac{C_{\lambda}(x-t)^{\sigma\lambda}}{\Gamma(l-\sigma k+\sigma\lambda-\rho)}D_t^l\{g(t)\}dt \quad (5.2.38)$$

provided that $\min\Re(\rho,\sigma,l-\rho-\sigma k)>0, l$ is a positive integer and C_{λ} is given by (5.2.4) where

$$C'_{\nu}=\frac{\Gamma(\rho+\sigma+\sigma\nu)}{(1+\nu)^{\mu}},\nu=0,1,2,... \quad (5.2.39)$$

Also $g^{(u)}(0)=0$ for $0\le u\le l-1$

5.3 GENERAL VOLTERRA-TYPE INTEGRAL EQUATION INVOLVING A PRODUCT OF GENERAL CLASS OF POLYNOMIAL AND MULTIVARIABLE H-FUNCTION

The second volterra-type integral equation involving the operator (5.1.14) is given by:

$$\int_0^x (x-t)^{\beta-1} S_N^M[-z_{r+1}(x-t)] H \begin{bmatrix} z_1(x-t) \\ . \\ . \\ . \\ z_r(x-t) \end{bmatrix} y(t)dt + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt := g(x) \tag{5.3.1}$$

$\Re(\beta, \nu) > 0$

has the solution

$$y(x) = \int_0^x (x-t)^{l-\beta-\mu-1} \sum_{j=0}^{\infty} \frac{E_j(x-t)^j}{\Gamma(j+l-\beta-\mu)} g^{(l)}(t)dt \tag{5.3.2}$$

where$\Re(l-\beta-\mu) > 0$

provided that

$g^{(i)}(0) = 0$ for $0 \leqq i \leqq l-1$, l being a positive integer and $\nu - \beta (< j)$ is an integer. Also

$$E_j = (-1)^j (\lambda)_\mu^{-j-1} \det \begin{bmatrix} \lambda_{\mu+1} & \lambda_\mu & ... & 0 & ... & 0 \\ \lambda_{\mu+2} & \lambda_{\mu+1} & ... & ... & ... & 0 \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ (\lambda_{\mu+\nu-\beta}+a) & . & & & & \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ \lambda_{\mu+j} & \lambda_{\mu+j-1} & .... & (\lambda_{\mu+\nu-\beta}+a) & ... & \lambda_{\mu+1} \end{bmatrix} \tag{5.3.3}$$

and μ is the least B for which

$$\lambda_B = (-1)^B \sum_{k_1+...+k_{r+1}=B} \Delta(k_1,...,k_{r+1}) \frac{z_1^{k_1}}{k_1!} ... \frac{z_{r+1}^{k_{r+1}}}{k_{r+1}!} \tag{5.3.4}$$

where

$$\Delta(k_1,...,k_{r+1}) = \phi_1(k_1)...\phi_{r+1}(k_r+1)\psi(k_1,...,k_{r+1}) \tag{5.3.5}$$

$$\psi(k_1,...,k_{r+1}) = \Gamma(\beta+k_1+...+k_{r+1}) \left\{ \prod_{j=1}^{p} \Gamma\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} k_i\right) \prod_{j=1}^{q} \Gamma\left(1-b_j+\sum_{i=1}^{r} \beta_j^{(i)} k_i\right) \right\}^{-1} \tag{5.3.6}$$

$$\phi_i(k_i) = \prod_{j=1}^{n_i} \Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)} k_i\right) \left\{ \prod_{j=n_i+1}^{p_i} \Gamma\left(c_j - \gamma_j^{(i)} k_i\right) \prod_{j=1}^{q} \Gamma\left(1-d_j+\delta_j^{(i)} k_i\right) \right\}^{-1}$$

$$(i=1,...,r) \tag{5.3.7}$$

and

$$\phi_{r+1}(k_{r+1}) = \begin{cases} (-N)_M k_{r+1} A_{N,k_{r+1}}, & 0 \leq k_{r+1} \leq \left[\frac{N}{M}\right] \\ \\ 0, & k_{r+1} > \left[\frac{N}{M}\right] \end{cases} \tag{5.3.8}$$

Proof. To solve (5.3.1) we first take Laplace transform of its both sides. We easily obtain by definition of Laplace transform and its convolution property stated in (2.1.5), the following result

$$\int_0^\infty e^{-sx}(x)^{\beta-1} S_N^M[-z_{r+1}x] H \begin{bmatrix} z_1 x \\ . \\ . \\ . \\ z_r x \end{bmatrix} dx Y(s) + a s^{-\nu} Y(s) = G(s) \tag{5.3.9}$$

Now expressing the $S_N^M[-z_{r+1}x]$ and $H \begin{bmatrix} z_1 x \\ . \\ . \\ . \\ z_r x \end{bmatrix}$ involved in (5.3.9) in series using (1.1.50) and (5.1.7), changing the order of series and integration and evaluating the $x-$integral, we obtain

$$\left[\sum_{k_1,...,k_{r+1}=0}^{\infty} \Delta(k_1, ..., k_{r+1}) \frac{-z_1^{k_1}}{k_1!} ... \frac{-z_{r+1}^{k_{r+1}}}{k_{r+1}!} s^{-\beta-(k_1+...+k_{r+1})} + a s^{-\nu} \right] Y(s) = G(s) \tag{5.3.10}$$

where $\Delta(k_1, ..., k_{r+1})$ is defined by (5.3.5).

Re-writing (5.3.10), we get

$$s^{-\beta} \left[\sum_{B=0}^{\infty} \lambda_B s^{-B} + a s^{-\nu+\beta} \right] Y(s) = G(s) \tag{5.3.11}$$

where λ_B is given by (5.3.4).

Again, (5.3.11) is equivalent to

$$Y(s) = s^{\beta}\left[\sum_{B=0}^{\infty}\lambda_B s^{-B} + as^{-\nu+\beta}\right]^{-1} G(s) \tag{5.3.12}$$

If μ denotes the least B for which $\lambda_B \neq 0$, the series given by (5.3.12) can be reciprocated.

Writing

$$\left[\sum_{B=0}^{\infty}\lambda_{B+\mu} s^{-B} + as^{-\nu+\beta}\right]^{-1} = \sum_{j=0}^{\infty} E_j s^{-j} \tag{5.3.13}$$

(5.3.12) takes the following form:

$$Y(s) = s^{\beta-l+\mu}\sum_{j=0}^{\infty} E_j s^{-j}[s^l G(s)] \tag{5.3.14}$$

(5.3.14) can be written as (using (2.3.30))

$$L\{y(x); s\} = L\left\{\sum_{j=0}^{\infty} E_j \frac{x^{j+l-\mu-\beta-1}}{\Gamma(j+l-\mu-\beta)}; s\right\} L\{g^{(l)}(x); s\} \tag{5.3.15}$$

Now using the convolution theorem in the RHS of (5.3.15) we get

$$L\{y(x); s\} = L\left\{\int_0^x \sum_{j=0}^{\infty} E_j \frac{(x-t)^{j+l-\mu-\beta-1}}{\Gamma(j+l-\mu-\beta)} g^{(l)}(t)dt; s\right\} \tag{5.3.16}$$

Finally, on taking the inverse of the Laplace transform of both sides of (5.3.16) we arrive at the desired result (5.3.2). □

It is interesting to note that, if we put $a = 0$ in (5.3.1) we get the result obtained by Gupta et al.[16].

5.3.1 SPECIAL CASES

1. If we put $r = 2$ in (5.3.1) and reduce the H-function of two variables thus obtained to Appell's function F_3 [49, p.89, eq.(6.4.6)] we find after a little simplification that the Volterra-type integral equation given by

$$\int_0^x (x-t)^{\beta-1} S_N^M[-z_{r+1}(x-t)] F_3[c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_2^{(2)}; b; -z_1(x-t), -z_2(x-t)] y(t)dt$$

$$+ \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt = g(x) \tag{5.3.17}$$

has the solution

$$y(x) = \frac{\Gamma(c_1^{(1)})\Gamma(c_1^{(2)})\Gamma(c_2^{(1)})\Gamma(c_2^{(2)})}{\Gamma(b)} \int_0^x (x-t)^{l-\beta-\mu-1} \sum_{j=0}^{\infty} \frac{E_j (x-t)^j}{\Gamma(j+l-\beta-\mu)} g^{(l)}(t)dt \tag{5.3.18}$$

where$\Re(l-\beta-\mu) > 0, \Re(\beta) > 0, | z_1(x-t) | < 1, | z_2(x-t) | < 1$
provided that
$g^{(i)}(0) = 0$ for $0 \leqq i \leqq l-1$, l being a positive integer and $\nu - \beta (< j)$ is an integer and E_j are given by the relation (5.3.3) and μ is least B for which $\lambda_B \neq 0$

$$\lambda_B = (-1)^B \sum_{k_1+k_2+k_3=B} \Delta(k_1, k_2, k_3) \frac{z_1^{k_1}}{k_1!} \frac{z_2^{k_2}}{k_2!} \frac{z_3^{k_3}}{k_3!} \tag{5.3.19}$$

where in (5.3.19)

$$\Delta(k_1, k_2, k_3) = \frac{\Gamma(c_1^{(1)}+k_1)\Gamma(c_1^{(2)}+k_2)\Gamma(c_2^{(1)}+k_1)\Gamma(c_2^{(2)}+k_2)\Gamma(\beta+k_1+k_2+k_3)}{\Gamma(b+k_1+k_2)} \phi_3(k_3) \tag{5.3.20}$$

and

$$\phi_3(k_3) = \begin{cases} (-N)_M k_3 A_{N,k_3}, & 0 \leq k_3 \leq \left[\frac{N}{M}\right] \\ \\ 0, & k_3 > \left[\frac{N}{M}\right] \end{cases} \tag{5.3.21}$$

2. If we put $r = 1, p = q = 0, z_2 = -1$ in the LHS of (5.3.1), and further reduce the H-function thus obtained to e^{-z_1}[49, p.18, eq. (2.6.2)] and let $z_1 \to 0$, the

H-function reduces to unity then we arrive at the following special case of (5.3.1):

$$\int_0^x (x-t)^{\beta-1} S_N^M[(x-t)]y(t)dt + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}y(t)dt = g(x) \qquad (5.3.22)$$

has the solution

$$y(x) = \int_0^x (x-t^{l-\beta-\mu-1} \sum_{j=0}^{\infty} \frac{E_j(x-t)^j}{\Gamma(j+l-\beta-\mu)} g^{(l)}(t)dt \qquad (5.3.23)$$

where$\Re(l-\beta-\mu) > 0, \Re(\beta) > 0$

provided that

$g^{(i)}(0) = 0$ for $0 \leqq i \leqq l-1$, l being a positive integer and $\nu-\beta(< j)$ is an integer and E_j are given by the relation (5.3.3) and μ is least k for which $\lambda_k \neq 0$

$$\lambda_k = \frac{(-N)_M k A_{N,k}\Gamma(\beta+k)}{k!} \qquad (5.3.24)$$

$k = 0, 1, ..., [N/M], N = 0, 1, 2, ...,$

If we put $a = 0$ in (5.3.22) we get the result obtained by Jain[27, p. 102-103, eq. (3.5),eq.(3.6)]

Similary, on account of general nature of S_N^M and multivariable $H-$function involved in the R.H.S of (5.3.1), we can obtain a number of special cases of (5.3.1) by specializing the parameters involved therein.

6

UNIFIED FINITE INTEGRAL INVOLVING THE FUNCTION $\lambda_{\mu,\nu}^{(\eta)}$, THE MULTIVARIABLE H-FUNCTION AND S_N^M POLYNOMIALS

The main findings of this chapter have been communicated as detailed below:

A Study of unified finite integral involving generalized modified Bessel function of third kind, general class of polynomials and the multivariable H-function, J. of the Ind. Acad. of Math. (Communicated)

In this chapter we first define the functions and polynomial required to establish our main integral. Next, we evaluate a unified and general finite integral whose integrand involves the product of generalized modified Bessel function $\lambda_{\mu,\nu}^{(\eta)}$,general class of polynomials S_N^M and the multivariable $H-$function. The arguments of the functions occurring in the integrand involve the product of factors of the form $x^{\rho-1}(a-x)^{\sigma}(1+(bx)^{\ell})^{-\lambda}$.

Main integral is believed to be new and is capable of giving a large number of simpler integrals (new and known) involving several special functions and polynomials as its special cases.For the sake of illustration we record here six new integrals as its special cases. The first, second, third, fourth fifth and sixth special cases of the main integral are integrals whose integrands involve the product of the modified Bessel function of the third kind, Laguerre polynomial,hypergeometric function; the product of Meijer G-function, Jacobi polynomial and Appell function; the product of generalized modified Bessel function of third kind, Gould & Hopper polynomial and Mittag-leffler function; the product of generalized modified Bessel function of third kind, general class of polynomials, reduced Green function and Lorenzo-Hartley R-function; the product of generalized modified Bessel function of third kind, general class of polynomials and Miller-Ross functions and the product of generalized modified Bessel function of third kind, Cesaro polynomial and Lorenzo-Hartley G-function respectively. Several basic integrals obtained earlier by several authors also follow as special cases of our main findings.

6.1 INTRODUCTION AND DEFINITIONS

For the sake of continuity, completeness and to avoid frequent reference to other works we shall first briefly describe the polynomial and functions occuring in this chapter.

GENERALIZATION OF THE MODIFIED BESSEL FUNCTION

Generalization of the modified Bessel function of the third kind or Macdonald function will be represented by the following form:[10, p.152, eq.(1.2); p.155, eq.(2.6)]

$$\lambda_{\mu,\nu}^{(\eta)}[z] = \frac{\eta}{\Gamma(\mu+1-1/\eta)} \int_1^\infty (t^\eta - 1)^{\mu-1/\eta} t^\nu e^{-zt} dt$$

$$= H_{1,2}^{2,0}\left[z \left| \begin{matrix} (1-(\nu+1)/\eta, 1/\eta) \\ (0,1), (-\mu-\nu/\eta, 1/\eta) \end{matrix} \right. \right] \tag{6.1.1}$$

$$(\eta > 0; \quad \Re(\mu) > 1/\eta - 1; \quad \nu \in \Re; \quad \Re(z) > 0)$$

The function in (6.1.1) was introduced by Kilbas et al.[31]. Such a function was used by Bonilla et al.[2] to solve some homogeneous differential equations of fractional order and Volterra integral equations.

SPECIAL CASES OF GENERALIZATION OF MODIFIED BESSEL FUNCTION

1. If we take $\eta = 2$ and $\nu = 0$, in (6.1.1), we get

$$\lambda_{\mu,0}^{(2)}[z] = \frac{2}{\sqrt{\pi}} \left(\frac{2}{z}\right)^\mu K_{-\mu}(z) \qquad \left(\Re(\mu) > -\frac{1}{2}\right), \tag{6.1.2}$$

where $K_{-\mu}(z)$ is the modified Bessel function of third kind or Macdonald function[7, Section 7.2.2]

2. If we take $\eta = 1$ in (6.1.1) we get

$$\lambda_{\mu,\nu}[z] = G_{1,2}^{2,0}\left[z \left| \begin{matrix} -\nu \\ 0, -\mu-\nu \end{matrix} \right. \right] \tag{6.1.3}$$

where $G_{1,2}^{2,0}$ is Meijer G-function[36, p.10,eq.(1.7.1)].

MULTIVARIABLE H-FUNCTION

The multivariable $H-$function occuring in the thesis was introduced and studied by Srivastava and Panda [52, p. 130, eq. (1.1)]. This function involves r complex variables and will be defined and represented in the following contracted form [49, p. 251-252, eqn. (C.1-C.3)]

$$H_{p,q:p_1,q_1;...;p_r,q_r}^{0,n:m_1,n_1;...;m_r,n_r}\left[\begin{matrix} z_1 \\ . \\ . \\ . \\ z_r \end{matrix} \left| \begin{matrix} (a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,p} : (c_j^{(1)},\gamma_j^{(1)})_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \\ \\ \\ (b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,q} : (d_j^{(1)},\delta_j^{(1)})_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix} \right. \right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{L_r}\psi(\xi_1,...,\xi_r)\prod_{i=1}^{r}(\phi_i(\xi_i)z_i^{\xi_i})d\xi_1...d\xi_r \qquad (i=1,...,r) \tag{6.1.4}$$

where $\omega = \sqrt{-1}$,

$$\phi_i(\xi_i) = \frac{\prod_{i=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)} \qquad (i=1,...,r) \tag{6.1.5}$$

$$\psi(\xi_1,...,\xi_r)=\frac{\prod\limits_{j=1}^{n}\Gamma(1-a_j+\sum\limits_{i=1}^{r}d_j^{(i)}\xi_i)}{\prod\limits_{j=1}^{q}\Gamma(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod\limits_{j=n+1}^{p}\Gamma(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i)} \tag{6.1.6}$$

All the greek letters occuring on the left-hand side of (6.1.4) are assumed to be positive real numbers for standardization purposes; the definition of the multivariable $H-$function will, however, be meaningful even if some of these quantities are zero. The details about the nature of the contours $L_1,...,L_r$, conditions of convergence of the integrals given by the equation (6.1.4), the special cases of the multivariable $H-$function and its properties can be referred to in the paper cited above. Throughout the thesis it is assumed that this function satisfies its appropriate conditions of existence and convergence[49, p.252-253, eq.(C.4-C.6)].

THE H-FUNCTION

By taking $r=1$, the multivariable $H-$function (6.1.4) reduces to the $H-$function. The function $H[x]$ occurring in the present work stands for H-function[8] or simply the H-function will be defined and represented in the following manner[49, p.10].

$$H_{p,q}^{m,n}[z]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$:=\frac{1}{2\pi\mathrm{i}}\int_L\Theta(\mathfrak{s})z^{\mathfrak{s}}\,d\mathfrak{s}, \tag{6.1.7}$$

where $\mathrm{i}=\sqrt{-1}$, $z\in\mathbb{C}\setminus\{0\}$, $\mathbb{C}$ being the set of complex numbers,

$$\Theta(\mathfrak{s})=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-\beta_j\mathfrak{s})\prod\limits_{j=1}^{n}\Gamma(1-a_j+\alpha_j\mathfrak{s})}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\mathfrak{s})\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j\mathfrak{s})}, \tag{6.1.8}$$

and

$$1 \leqq m \leqq q \qquad \text{and} \qquad 0 \leqq n \leqq p \qquad (m, q \in \mathbb{N} = \{1, 2, 3, \cdots\};\ n, p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

The definition of the H-function given by (6.1.7) will, however, have meaning even if some of these quantities are zero, giving us in turn simple transformation formulas.

The nature of contour L in (6.1.7), a set of sufficient conditions for the convergence of this integral, the asymptotic expansions of the H-function, some of its properties and special cases can be referred to in the book by Srivastava et al. [49].

GENERAL CLASS OF POLYNOMIALS

Srivastava [46, p.1 eq. (1)] has introduced the general class of polynomials

$$S_N^M[x] = \sum_{R=0}^{[N/M]} \frac{(-N)_{MR} \quad A_{N,R} \quad x^R}{R!} \qquad (N = 0, 1, 2, ...), \tag{6.1.9}$$

where M is an arbitrarty positive integer, and the coefficients $A_{N,R}(N, R \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{N,R}$, $S_N^M[x]$ yields a number of known polynomials as its special cases. These include, among others, Jacobi polynomial, Laguerre polynomial and several others [4, 55, 57].

6.2 MAIN INTEGRAL

$$\int_0^a x^{\rho-1}(a-x)^{\sigma}[1+(bx)^{\ell}]^{-\lambda} \lambda_{\mu,\nu}^{(\eta)}(z_{r+1} x^{\rho_{r+1}} (a-x)^{\sigma_{r+1}} [1+(bx)^{\ell}]^{-\lambda_{r+1}})$$

$$H_{p,q:p_1,q_1;...;p_r,q_r}^{0,n:m_1,n_1;...;m_r,n_r}\left[\begin{array}{c|c} z_1 x^{\rho_1}(a-x)^{\sigma_1}[1+(bx)^\ell]^{-\lambda_1} & (a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,p}:C \\ . & \\ . & \\ . & \\ z_r x^{\rho_r}(a-x)^{\sigma_r}[1+(bx)^\ell]^{-\lambda_r} & (b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,q}:D \end{array}\right]$$

$$S_N^M[Yx^{\rho'}(a-x)^{\sigma'}[1+(bx)^\ell]^{-\lambda'}]dx$$

$$= a^{\rho+\sigma}\sum_{R=0}^{[N/M]}\frac{(-N)_{MR}A_{N,R}(Ya^{(\rho'+\sigma')})^R}{R!}H_{p+3,q+2:p_1,q_1;...;p_r,q_r;1,2;0,1}^{0,n+3:m_1,n_1;...;m_r,n_r;2,0;1,0}\left[\begin{array}{c|c} z_1 a^{\rho_1+\sigma_1} & A:C^* \\ . & \\ . & \\ . & \\ z_r a^{\rho_r+\sigma_r} & \\ z_{r+1}a^{\rho_{r+1}+\sigma_{r+1}} & \\ (ab)^\ell & B:D^* \end{array}\right] \tag{6.2.1}$$

where

$$A = (1-\lambda-R\lambda';\lambda_1,...,\lambda_r,\lambda_{r+1},1),(1-\rho-R\rho';\rho_1,...,\rho_r,\rho_{r+1},\ell),$$
$$(-\sigma-R\sigma';\sigma_1,...,\sigma_r,\sigma_{r+1},0),(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)},0,0)_{1,p}$$

$$B = (-\rho-\sigma-R(\rho'+\sigma');(\rho_1+\sigma_1),...,(\rho_r+\sigma_r),(\rho_{r+1}+\sigma_{r+1}),\ell),$$
$$(1-\lambda-R\lambda';\lambda_1,...,\lambda_r,\lambda_{r+1},0),(b_j;\beta_j^{(1)},...,\beta_j^{(r)},0,0)_{1,q}$$

$$C^* = C;(1-(\nu+1)/\eta,1/\eta);- \qquad D^* = D;(0,1),(-\mu-\nu/\eta,1/\eta);(0,1)$$

$$C = (c_j^{(1)},\gamma_j^{(1)})_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \qquad D = (d_j^{(1)},\delta_j^{(1)})_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}$$

provided

(i) $\Re(\lambda) \geq 0, min(\rho',\sigma',\lambda',\rho_i,\sigma_i,\lambda_i) \geq 0 \qquad (i=1,...,r+1)$

(not all zero simultaneously)

(ii) min $\Re(\rho + \sum_{i=1}^{r} \rho_i(d_j^{(i)})/\delta_j^{(i)} - \rho_{r+1}(\mu\eta+\nu)) > 0,$

min $\Re(\sigma + 1 + \sum_{i=1}^{r} \sigma_i(d_j^{(i)})/\delta_j^{(i)} - \sigma_{r+1}(\mu\eta+\nu)) > 0 \quad (j=1,...,m_i)$

Proof. First we express the modified Bessel function $\lambda_{\mu,\nu}^{(\eta)}$ in terms of the $H-$function of one variable[10, p.155,eq.(2.6)] and the $S_N^M[x]$ polynomials in terms of the series with the help of (6.1.9) occuring in the left-hand side of (6.2.1). Now, we express the multivariable $H-$function and $H-$function of one variable in terms of their respective Mellin-Barnes type contour integrals. Then we change the order of the series and $\xi_1,...,\xi_{r+1}$ contour integrals with the $x-$integral which is permissible under the conditions stated. The left-hand side of (6.2.1) takes the following form (say Δ):

$$\Delta = \sum_{R=0}^{[N/M]} \frac{(-N)_{MR} A_{N,R} Y^R}{R!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1, ..., \xi_r) \prod_{i=1}^{r} (\phi_i(\xi_i) z_i^{\xi_i})$$
$$\frac{1}{2\pi\omega} \int_{L_{r+1}} \frac{\Gamma(-\xi_{r+1})\Gamma(-\mu-(\nu+\xi_{r+1})/\eta)}{\Gamma(1-(\nu+1+\xi_{r+1})/\eta)} z_{r+1}^{\xi_{r+1}}$$
$$\int_0^a x^{\sum_{i=1}^{r+1}\rho_i\xi_i+\rho+R\rho'-1} (a-x)^{\sigma+R\sigma'+\sum_{i=1}^{r+1}\sigma_i\xi_i} [1+(bx)^\ell]^{-\lambda-R\lambda'-\sum_{i=1}^{r+1}\lambda_i\xi_i} dx d\xi_1 ... d\xi_r d\xi_{r+1} \tag{6.2.2}$$

Finally, on evaluating the above $x-$integral with the help of the following result [25, p.47, eqn.(1.3.3)]:

$$\int_0^\tau x^{\rho-1}(\tau-x)^\sigma(1+(Dx)^\ell)^{-\lambda}dx = \frac{\Gamma(\sigma+1)}{\Gamma(\lambda)}\tau^{\rho+\sigma} H_{2,2}^{1,2}\left[(D\tau)\ell \,\middle|\, \begin{matrix} (1-\rho,\ell),(1-\lambda,1) \\ \\ (0,1),(-\rho-\sigma,\ell) \end{matrix}\right] \tag{6.2.3}$$

where

$$\Re(\rho) > 0, \quad \Re(\sigma+1) > 0$$

the right-hand side of (6.2.2) takes the following form:

$$\Delta = \sum_{R=0}^{[N/M]} \frac{(-N)_{MR} A_{N,R} Y^R}{R!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1, ..., \xi_r) \prod_{i=1}^{r} (\phi_i(\xi_i) z_i^{\xi_i})$$

$$\frac{1}{2\pi\omega} \int_{L_{r+1}} \frac{\Gamma(-\xi_{r+1})\Gamma(-\mu-(\nu+\xi_{r+1})/\eta)}{\Gamma(1-(\nu+1+\xi_{r+1})/\eta)} z_{r+1}^{\xi_{r+1}} a^{\rho+R\rho'+\sum_{i}^{r+1}(\rho_i+\sigma_i)\xi_i+\sigma+R\sigma'} \tag{6.2.4}$$

$$H_{2,2}^{1,2}\left[(bx)\ell \,\middle|\, \begin{matrix} (1-\rho-R\rho'-\sum_{i=1}^{r+1}\rho_i\xi_i,\ell),(1-\lambda-R\lambda'-\sum_{i=1}^{r+1}\lambda_i\xi_i,1) \\ \\ (0,1),(-\rho-R\rho'-\sum_{i}^{r+1}(\rho_i+\sigma_i)\xi_i-\sigma-R\sigma',\ell) \end{matrix}\right] d\xi_1 ... d\xi_r d\xi_{r+1} \tag{6.2.5}$$

Now we express the $H-$function thus obtained in terms of its Mellin-Barnes type contour integral and re-interpret the result in terms of $(r+2)-$variable $H-$function, we easily arrive at the desired result after a little simplification. □

6.2.1 SPECIAL CASES OF THE MAIN INTEGRAL

FIRST INTEGRAL

If we reduce $\lambda_{\mu,\nu}^{(\eta)}$ into modified Bessel function of third kind $K_{-\mu}$ [10, p.152, eq.(1.3)], S_N^M into Laguerre polynomial[4, p.164, eq.(A.8)], and the $H-$function into pF_q by taking $r=1$[49, p.18, eq.(2.6.3)] in the main integral, we arrive at the following integral after a little simplification:

$$\int_0^a x^{\rho-\rho_2\mu-1}(a-x)^{\sigma-\sigma_2\mu}[1+(bx)^\ell]^{-\lambda+\lambda_2\mu}K_{-\mu}(z_2x^{\rho_2}(a-x)^{\sigma_2}[1+(bx)^\ell]^{-\lambda_2})$$

$$L_N^\alpha[Yx^{\rho'}(a-x)^{\sigma'}[1+(bx)^\ell]^{-\lambda'}]pF_q[(c_j)_p;(d_j)_q:-z_1x^{\rho_1-1}(a-x)^{\sigma_1}[1+(bx)^\ell]^{-\lambda_1}]dx$$

$$=2^{-(\mu+1)}\sqrt{\pi}\quad z_2^\mu\quad\frac{\prod\limits_{j=1}^{q}\Gamma(d_j)}{\prod\limits_{j=1}^{p}\Gamma(c_j)}a^{\rho+\sigma}\sum_{R=0}^{N}\frac{(-N)_R(Ya^{(\rho'+\sigma')})^R}{(\alpha+1)_RR!}{}^{(N+\alpha)}C_N$$

$$H^{0,3:1,p;2,0;1,0}_{3,2:p,q+1;1,2;0,1}\left[\begin{array}{c|llll} z_1a^{\rho_1+\sigma_1} & A^*: & (1-c_j,1)_{1,p}; & (1/2,1/2); & - \\ z_2a^{\rho_2+\sigma_2} & & & & \\ (ab)^\ell & B^*: & (0,1),(1-d_j,1)_{1,q}; & (0,1),(-\mu,1/2); & (0,1) \end{array}\right] \tag{6.2.6}$$

where

$$A^*=(1-\lambda-R\lambda';\lambda_1,\lambda_2,1),(1-\rho-R\rho';\rho_1,\rho_2,\ell),(-\sigma-R\sigma';\sigma_1,\sigma_2,0)$$

$$B^* = (-\rho - R(\rho' + \sigma') - \sigma; (\rho_1 + \sigma_1), (\rho_2 + \sigma_2), \ell), (1 - \lambda - R\lambda'; \lambda_1, \lambda_2, 0)$$

provided that the conditions easily obtainable from (6.2.1) are satisfied.

SECOND INTEGRAL

If we take $r = 2, \ell = 1, \lambda' = \sigma' = \sigma_i = \lambda_i = 0, (i = 1, ..., r+1)$ in the main integral and further reduce $\lambda_{\mu,\nu}^{(\eta)}$ to Meijer $G_{1,2}^{2,0}$ by taking $\eta = 1$, S_N^M into Jacobi polynomial $P_n^{\alpha,\beta}$[55, p.159, eq.(1.6)]and the $H-$function of two variable into Appell function F_3 [49, p.89,eq.(6.4.6)], we arrive at the following integral after a little simplification:

$$\int_0^a x^{\rho-1}(a-x)^{\sigma}[1+(bx)]^{-\lambda} G_{1,2}^{2,0}\left[z_3 x^{\rho_3} \left| \begin{matrix} \nu \\ 0, -\mu-\nu \end{matrix} \right.\right] P_N^{\alpha,\beta}[1 - 2Yx^{\rho'}]$$

$$F_3[k_1, h_1, k_2, h_2; s; z_1 x_1^{\rho_1}, z_2 x_2^{\rho_2}]dx$$

$$= \frac{\Gamma(s)\Gamma(1+\sigma)}{\Gamma(k_1)\Gamma(k_2)\Gamma(h_1)\Gamma(h_2)\Gamma(\lambda)} \quad a^{\rho+\sigma} \sum_{R=0}^{N} \frac{(-N)_R(\alpha+\beta+N+1)_R(Ya^{\rho'})^R}{(\alpha+1)_R R!} {}^{\alpha+N}C_N$$

$$H_{1,2:2,1;2,1;1,2;1,1}^{0,1:1,2;1,2;2,0;1,1}\left[\begin{matrix} -z_1 a^{\rho_1} \\ -z_2 a^{\rho_2} \\ z_3 a^{\rho_3} \\ (ab) \end{matrix} \left| \begin{matrix} A^{**} : C^{**}; (1-\lambda, 1) \\ \\ \\ B^{**} : D^{**}; (0, 1) \end{matrix} \right.\right] \tag{6.2.7}$$

where

$$A^{**} = (1 - \rho - \rho' R; \rho_1, \rho_2, \rho_3, 1)$$

$$B^{**} = (-\rho - \sigma - \rho' R; \rho_1, \rho_2, \rho_3, 1), (1 - s; 1, 1, 0, 0)$$

$$C^{**} = (1-k_1,1),(1-k_2,1);(1-h_1,1),(1-h_2,1);(-\nu,1)$$

$$D^{**} = (0,1);(0,1);(0,1),(-\mu-\nu,1)$$

provided that the conditions easily obtainable from (6.2.1) are satisfied.

THIRD INTEGRAL

If we take $r=1, \ell=1, \lambda'=\sigma'=\sigma_i=\lambda_i=0, (i=1,...,r+1)$ in the main integral and further reduce the general polynomial S_N^M into Gould & Hopper polynomial g_N^M[4, p.164, eq.(A.10)] and the $H-$function of one variable into Mittag Leffler function[56, p.193, eq.(1.15); p.192, eq.(1.9)] we arrive at the following integral after a little simplification:

$$\int_0^a x^{\rho+\rho' N/M-1}(a-x)^{\sigma}[1+(bx)]^{-\lambda}\lambda_{\mu,\nu}^{(\eta)}(z_2x^{\rho_2})g_N^M[(-\frac{h}{Yx^{\rho'}})^{1/M},h]E_{\alpha,\beta}^{\gamma,\kappa}[z_1x^{\rho_1}]dx$$

$$=\frac{(-1)^N}{\Gamma(\gamma)}(\frac{h}{Y})^{\frac{N}{M}}a^{\rho+\sigma}\sum_{R=0}^{[N/M]}\frac{(-N)_{MR}(Ya^{\rho'})^R}{R!}\frac{\Gamma(1+\sigma)}{\Gamma(\lambda)}$$

$$H_{1,1:1,2;1,2;1,1}^{0,1:1,1;2,0;1,1}\left[\begin{array}{c|c} -z_1a^{\rho_1} & (1-\rho-\rho' R;\rho_1,\rho_2,1): \quad C^{***}; \quad (1-\lambda,1) \\ z_2a^{\rho_2} & \\ (ab) & (-\rho-\sigma-\rho' R;\rho_1,\rho_2,1): \quad D^{***}; \quad (0,1) \end{array}\right] \tag{6.2.8}$$

where

$$C^{***} = (1-\gamma,\kappa);(1-(\nu+1)/\eta,1/\eta)$$

$$D^{***} = (0,1),(1-\beta,\alpha);(0,1),(-\mu-\nu/\eta,1/\eta)$$

provided that the conditions are easily obtainable from (6.2.1) satisfied.

FOURTH INTEGRAL

If in the main integral, we reduce the multivariable H-function into a product of two H-function by taking $p = q = 0; r = 2$,further reduce $H_{p_1,q_1}^{m_1,n_1}$ to the reduced Green function $K_{\alpha,\beta}^{\theta}$[19, p.11, eq. (10)] and $H_{p_2,q_2}^{m_2,n_2}$ to the Lorenzo-Hartley R-function $R_{q,u}$[19, p.14, eq.(24)][35, p.3,eq.(13)], S_N^M to Konhauser biorthogonal polynomial Z_N^A[4, p.165,eq.(A.12)] and take $\ell = 1, \sigma_1 = \sigma_2 = \lambda_2 = \lambda_3 = \sigma' = \lambda' = 0$ we arrive at the following integral after a little simplification:

$$\int_0^a x^{\rho+\rho_1+\rho_2(u+1-q)-1}(a-x)^{\sigma}[1+(bx)]^{-\lambda-\lambda_1}\lambda_{\mu,\nu}^{(\eta)}(z_3x^{\rho_3}(a-x)^{\sigma_3})Z_N^A[(Yx^{\rho'})^{1/k};k] \tag{6.2.9}$$

$$K_{\alpha,\beta}^{\theta}[z_1x^{\rho_1}(1+bx)^{-\lambda_1}]R_{q,u}(z_2,x^{\rho_2})dx$$

$$= \frac{1}{\alpha z_1}a^{\rho+\sigma}\sum_{R=0}^{N}\frac{(-N)_R\Gamma(1+A+kN)(Ya^{\rho'})^R}{N!\Gamma(1+A+kR)R!}$$

$$H_{2,1:3,4;1,2;2,2;0,1}^{0,2:2,1;1,1;2,1;1,0}\left[\begin{array}{c|c} \begin{array}{c} z_1a^{\rho_1} \\ (-z_2)a^{\rho_2 q} \\ z_3a^{\rho_3+\sigma_3} \\ (ab) \end{array} & \begin{array}{c} A^{\#}:C^{\#};- \\ \\ \\ B^{\#}:D^{\#};(0,1) \end{array} \end{array}\right] \tag{6.2.10}$$

where

$$A^{\#} = (1-\lambda;\lambda_1,0,0,1),(1-\rho-R\rho';\rho_1,\rho_2 q,\rho_3,1)$$

$$B^{\#} = (-\rho-\sigma-R\rho';\rho_1,\rho_2 q,(\rho_3+\sigma_3),1)$$

$$C^{\#} = (1,1/\alpha),(1,\beta/\alpha),(1,\frac{\alpha-\theta}{2\alpha});(0,1);(-\sigma,\sigma_3),(1-(\nu+1)/\eta,1/\eta)$$

$$D^{\#} = (1,1/\alpha),(1,1),(1,\frac{\alpha-\theta}{2\alpha}),(1-\lambda,\lambda_1);(0,1),(1+u-q,q);(0,1),(-\mu-\nu/\eta,1/\eta)$$

provided that the conditions easily obtainable from (6.2.1) are satisfied.

FIFTH INTEGRAL

Again, if in the main integral, we reduce the multivariable H-function into a product of two H-functions by taking $p = q = 0; r = 2$, and further reduce $H^{m_1,n_1}_{p_1,q_1}, H^{m_2,n_2}_{p_2,q_2}$ into Miller-Ross E_t & C_t functions[19, p.14, eq. (21,22)] respectively, S^M_N to Brafman polynomial B^M_N[4, p.165,eq.(A.11)] and take $\ell = 1, \sigma_1 = \sigma_2 = \lambda_1 = \lambda_3 = \sigma' = \lambda' = 0$ we get the following new integral involving the product of $\lambda^{(\eta)}_{\mu,\nu}, B^M_N$, E_t and C_t after a little simplification:

$$\int_0^a x^{\rho-\gamma(\rho_1+\rho_2)-1}(a-x)^{\sigma}[1+(bx)]^{-\lambda+\lambda_2\gamma}\lambda^{(\eta)}_{\mu,\nu}(z_3x^{\rho_3}(a-x)^{\sigma_3})B^M_N[\alpha_1,...,\alpha_p;\beta_1,...,\beta_q : Yx^{\rho'}M^M]$$

$$E_{x^{\rho_1}}(\gamma, z_1)C_{x^{\rho_2}(1+bx)^{-\lambda_2}}(\gamma, z_2)dx$$

$$= a^{\rho+\sigma}\sum_{R=0}^{[N/M]}\frac{(-N)_{MR}(\alpha_1)_R...(\alpha_p)_R(Ya^{\rho'})^R}{(\beta_R...(\beta_q)_R)R!}$$

$$H^{0,2:1,1;1,1;2,1;1,0}_{2,1:1,2;1,3;2,2;0,1}\left[\begin{array}{c|c} -z_1a^{\rho_1} & A' : C'; - \\ (z_2a^{\rho_2})^2 & \\ z_3a^{\rho_3+\sigma_3} & \\ (ab) & B' : D'; (0,1) \end{array}\right] \tag{6.2.11}$$

where

$$A' = (1-\lambda; 0, \lambda^2, 0, 1), (1-\rho-R\rho'; \rho_1, \rho_2, \rho_3, 1)$$

$$B' = (-\rho-\sigma-R\rho'; \rho_1, \rho_2, (\rho_3+\sigma_3), 1)$$

$$C' = (0,1); (0,1); (-\sigma, \sigma_3), (1-(\nu+1)/\eta, 1/\eta)$$

$$D' = (0,1), (-\gamma, 1); (0,1), (-\gamma, 2), (1-\lambda, \lambda_2^2); (0,1), (-\mu-\nu/\eta, 1/\eta)$$

provided that the conditions easily obtainable from (6.2.1) are satisfied.

SIXTH INTEGRAL

Now, we take $r = 1, \ell = 1, p = q = 0$ in the left hand side of (6.2.1) and further reduce $H_{p_1,q_1}^{m_1,n_1}$ to Lorenzo and Hartley $G_{q,u,r}$ function[35][14, p.64, eq.(2.3)] and S_N^M to Cesaro polynomial $g_N^{(s)}$[4, P.167, eq. (A.18)], we arrive at the following result after a little simplification:

$$\int_0^a x^{\rho-\rho_1(rq-u-1)-1}(a-x)^{\sigma}[1+(bx)]^{-\lambda+\lambda_1(rq-u-1)}\lambda_{\mu,\nu}^{(\eta)}(z_2x^{\rho_2})g_N^{(s)}[Yx^{\rho'}(a-x)^{\sigma'}] \tag{6.2.12}$$

$$G_{q,u,r}[z_1, x^{\rho_1}(1+bx)^{-\lambda_1}]dx$$

$$= \frac{1}{\Gamma(r)}a^{\rho+\sigma}\sum_{R=0}^{N}\frac{(-N)_R(s+1)_N(Ya^{\rho'+\sigma'})^R}{N!(-s-N)_R}\Gamma(\sigma+\sigma'R+1)$$

$$H_{2,1:1,3;1,2;0,1}^{0,2:1,1;2,0;1,0}\left[\begin{array}{c|c} & A'':C'';- \\ -z_1a^{\rho_1 q} & \\ (z_2)a^{\rho_2} & \\ (ab) & \\ & B'':D'';(0,1)\end{array}\right] \tag{6.2.13}$$

$$A'' = (1-\lambda;\lambda_1 q,0,1),(1-\rho-R\rho';\rho_1 q,\rho_2,1)$$

$$B'' = (-\rho-\sigma-\sigma'R-\rho'R;\rho_1 q,\rho_2,1)$$

$$C'' = (1-r,1);(1-(\nu+1)/\eta,1/\eta)$$

$$D'' = (0,1),(1+u-rq,q),(1-\lambda,\lambda_1 q);(0,1),(-\mu-\nu/\eta,1/\eta)$$

provided that the conditions easily obtainable from (6.2.1) are satisfied.

If we take $\ell = 1, \lambda = \lambda' = \lambda_i = 0(i = 1,...,r)$ and reduce $\lambda_{\mu,\nu}^{(\eta)}$ to unity in the main integral (6.2.1), we get a known integral obtained by Gupta, Goyal and

Verma[15, p.69, eq.(3.1)].

Again if we put $a = Y = \ell = \rho' = 1, r = 2, \lambda = \lambda' = \sigma' = \lambda_i = 0(i = 1, ..., r)$, reduce $\lambda_{\mu,\nu}^{(\eta)}$ to unity and general class of polynomial into Jacobi polynomial in the main integral (6.2.1), we arrive at an integral by Prasad and Singh[39, p.126]

The importance of the findings of this chapter lies in the fact that the main integral as well as all of its six special cases given here are unified in nature and of interest in themselves. Moreover, the function $\lambda_{\mu,\nu}^{(\eta)}$ involved in the integrand of the main integral has been used by several authors to solve some homogeneous differential equations of fractional order and Volterra integral equations. Further, they may find applications in practical problems occurring in several branches of engineering.

References

[1] Abramowitz, M. and Stegun, I.A. (Editors) (1964) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series* **55**, National Bureau of Standards, Washington, D.C.; Reprinted by Dover Publications, New York, 1965.

[2] Bonilla, B., Kilbas, A.A., Rivero, M.,Rodriguez, L. and Trujillo J.J.(2000) Modified Bessel-type function and solution of differential and integral equations, *Indian. J. Pure Appl. Math.* **31** 93-109.

[3] Brafman, F.(1957) Some generating Functions for Lagurre and Hermite polynomials, *Canad. J.Math.* **9** 80-187.

[4] Bundela, B. (2011) A study of Mellin-Barnes type Integral, Generalized Fractional calculus, Integral operators and general polynomials with Applications, Ph.D thesis M.N.I.T Jaipur, Rajasthan.

[5] Buschman, R.G. and Srivastava, H.M. (1990) The $\overline{H}$ function associated with a certain class of Feynman integrals, *J. Phys. A*: *Math. Gen.* **23** 4707-4710.

[6] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953) *Higher Transcendental Functions,* **I**, MacGraw-Hill, New York.

[7] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953) *Higher Transcendental Functions,* **II**, MacGraw-Hill, New York.

[8] Fox, C. (1961) The G- and H- functions as symmetrical Fourier Kernels. *Trans. Amer. Math. Soc. 98*, 395-429.

[9] Garg, M. and Mishra, R. (2003) On product of hypergeometric functions, general class of multivariable polynomials and a generalized hypergeometric series associated with Feynman Integrals, *Bull. Calc. Math. Soc.*, **95**, 313-324.

[10] Glaeske, H.J., Kilbas A.A. and Saigo. M (2000) A modified Bessel-type intehral transform and its compositions with fractional calculus operators on spaces $F_{p,\mu}$ and $F'_{p,\mu}$, *J. of comp. and App. Math.* **118** 151-168.

[11] Gorenflo, R., Mainardi, F. and Srivastava, H.M. (1997) Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Proceedings of the Eighth International Colloquium on Differential Equations* (Plovdiv, Bulgaria) (D. Bainov, Editor), VSP Publishers, Utrecht and Tokyo, 1998, 195–202.

[12] Gould, H.W. and Hopper, A.T. (1962) Operational Formulas connected with Two Generalizations of Hermite polynomials. *Duke Math. J.* **29** 51-63.

[13] Gradshteyn, I.S. and Ryzhik, I.M. (1965) *Tables of Integrals, Series and Products*, Academic Press inc.

[14] Gupta, K.C. (2001) New relationships of the H-function with functions of practical utility in fractional calculus, *Ganita Sandesh.* **15** 63-66.

[15] Gupta, K.C., Goyal, S.P. and Verma, S. (1987) Unified integrals involving a general class of polynomials and multivariable H-function, *Indian J. Pure Appl. Math.* **18** 65-73.

[16] Gupta, K.C. Jain, R. and Agrawal, P. (1995) Convolution integral equations involving a general class of polynomials and the multivariable H-function, *Proc. Indian Acad. Sci. (Math. Sci.)* **105** 187-192.

[17] Gupta, K.C. Jain, R. and Agrawal, R (2007) On Existence Conditions for Generalized Mellin-Barnes Type Integral, *Nat. Acad. Sci. Lett.* **30(5&6)** 169-172.

[18] Gupta, K.C. Jain, R. and Sharma, A.(2003) A study of Unified Finite Integral Transforms with Applications, *J. Raj. Aca. Phy. Sci.* **2(4)** 269-282.

[19] Gupta, K.C. and Soni, R.C. (2002) A unified inverse Laplace transform formula, functions of practical importance and H-functions, *Raj. Acad. Phy. Sci.* **I** 7-16.

[20] Gupta, K.C. and Soni, R.C (2001) New Properties of Generalization of Hypergeometric Series Associated with Feynman Integrals, *Kyungpook Math J.* **41(1)** 97-104.

[21] Hilfer, R. and Anton, L. (1995) Fractional master equations and fractal time random walks, *Phys. Rev.* **E 51** R848-R851.

[22] Hilfer, R. (ed.)(2000) *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.

[23] Hilfer, R. (ed.)(2000) *Fractional time evolution, in Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 87-130.

[24] Inayat-Hussain, A.A. (1987) New properties of hypergeometric series deriv-

able from Feynman integrals. II: A generalisations of the H-function, *J. Phys. A: Math. Gen.* **20**, 119-128.

[25] Jain, R. (1992) A study of Special functions, A general polynomial, integral equations and fractional calculus with applications, Ph.D thesis Univ. of Rajasthan.

[26] Jain, R. (2007) Convolution Integral Equations Involving the $\overline{H}$-function,*Tamsui Oxford Journal of Mathematical Sciences* **23** 317-324.

[27] Jain, R. (1990) Double convolution integral equations involving the general polynomials, *Ganita Sandesh* **4** 99-103.

[28] Jain, R. and Sharma, A.(2002) On Unified Integral Formula Involving H - function and General Class of Polynomials, *Aligarh Bull. of Maths.* **21** 7-12.

[29] Khadia, S.S. and Goyal, A.N. (1970) On the Generalized Function of n Variables,*Vijnana Parishad Anusandhan Patrika* **13**191-201.

[30] Kilbas, A.A. and Saigo, M. (2004) *H-Transforms theory and Applications*, CRC press company, Boa Raton London New York Washington, D.C.

[31] Kilbas, A.A., Saigo, M. and Glaeske, H.J. (1999) A modified transform of Bessel type and its compositions with operators of fractional integration and differentiation, *Dokl. Nats. Akad. Nauk Belarusi* **43** 26-30.

[32] Kilbas, A.A., Saigo, M. and Saxena, R.K. (2004) Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integral Transforms Spec. Funct.* **15**, 31–49.

[33] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006) *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical

Studies Vol. **204** Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York.

[34] Krall, H.L and Frink, O.(1949) A new class of Orthogonal Polynomials:The Bessal Polynomials, *Trans. Amer. Math. Soc.* **65** 100-115

[35] Lorenzo, C.F. and Hartley, T.T. (1999) Generalized functions for the fractional calculus, *NAZA*/TP-209424 1-17.

[36] Mathai, A.M. and Saxena, R.K. *The H-function with applications in statistics and other disciplines* John wiley and sons, New york, london sydney Toronto.

[37] Miller, K.S. and Ross, B. (1993) *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore.

[38] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (Editors) (2010) *NIST Handbook of Mathematical Functions*National Institute of Standards and Technology, Washington, D. C., 2010; Cambridge University Press, Cambridge, London and New York.

[39] Prasad, Y.N. and Singh, S.N. (1977) Some expansion formulae for the H-function of two variables involving Jacobi polynomials, *Bull. Math. Soc. Sci. Math.* **21** 123-132.

[40] Rathie, A.K. (1997) A New Generalization of Generalized Hypergeometric Functions, *Le Matematiche Fasc.* **II**, 52 297-310.

[41] Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993) *Fractional Integrals and*

Derivatives:Theory and Applications,Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania).

[42] Saxena, R.K. (1998) Functional Relations Involving Generalized H-function, *Le Mathematiche* **53** 123-131.

[43] Saxena, R.K. and Gupta, N.(1994) Some Abelian Theorems for Distributional H -function Transformation, *Indian J. Pure Appl. Math.* **25** 869-879.

[44] Saxena, R.K. and S.L. Kalla, S.L. (2008) On the solutions of certain fractional kinetic equations,*Appl. Math. Comput.* **199** 504-511.

[45] Saxena, R.K. and Nishimoto, K.(2010) NFractional Calculus of generalized Mittag-Leffler functions, *J. Fract. Calc.* **37** 43-52.

[46] Srivastava, H.M. (1972) A contour integral involving Fox's H-function, *Indian J. Math.,* **14**, 1-6.

[47] Srivastava, H.M. and Bushman, R.G. (1974) Some convolution integral equations, *Nederl. Akad. Wetensch. Proc. Ser. A77, Indag. Math.* **36** 211-216.

[48] Srivastava, H.M. and Bushman, R.G. (1992) Theory and Applications of Convolution Integral Equations, Kluwer Academic Publishers, Dordrecht, Boston, London.

[49] Srivastava, H.M. Gupta, K.C. and Goyal, S.P. (1982) *The H-functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi.

[50] Srivastava, H.M., Koul, C.L.,and Raina, R.K. (1985) A class of convolution integral equations, *J. Math. Anal. Appl.* **108** 63-71.

[51] Srivastava, H.M. and Manocha, H.L.(1984) *A Treatise on Generating Functions*,Ellis Hortwood Ltd. Chichester, John Wiley and sons, New York .

[52] Srivastava, H.M. and Panda, R. (1976) Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* 265-274.

[53] Srivastava, H.M. and Panda, R.(1976) Expansion Theorems for the H-function of Several Complex Variables, *J. Reine Angew. Math.* **288** 129-145.

[54] Srivastava, H.M. and Saxena, R.K. (2001) Operators of Fractional Integration and Their Applications, *Elsevier, Appl. Math. Comput.* **118** 1-52.

[55] Srivastava, H.M., and Singh, N.P. (1983) The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* **32** 157-187.

[56] Srivastava, H.M. and Tomovski, Ž. (2009) Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the Kernel, *Appl. Math. Comput.* **211** 198-210.

[57] Szego G. (1975) Orthogonal Polynomials, *Amer Math . Soc. Colloq. Publ.* **23** Fourth Edition Rhode Iseland.

[58] Srivastava, H.M. (2000) Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* **129** 77–84.

[59] Srivastava, H.M. (2011) Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* **5** 390–444.

[60] Srivastava, H.M. (2013) Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, *Springer-Plus* **2** Article ID 2:67, 1-14.

[61] Srivastava, H.M. (2014) A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications, *Appl. Math. Inform. Sci.* **8** 1485–1500.

[62] Srivastava, H.M. and Choi, J. (2001) *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London.

[63] Srivastava, H.M. and Choi, J. (2012) *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York.

[64] Srivastava, H.M., Jankov, D., Pogány, T.K. and Saxena, R.K. (2011) Two-sided inequalities for the extended Hurwitz-Lerch Zeta function, *Comput. Math. Appl.* **62** 516-522.

[65] Srivastava, H.M., Lin, S.-D. and Wang, P.-Y. (2006) Some fractional-calculus results for the $\overline{H}$-function associated with a class of Feynman integrals, *Russian J. Math. Phys.* **13** 94-100.

[66] Srivastava, H.M. and Saxena, R.K. (2001) Operators of fractional integration and their applications, *Appl. Math. Comput.* **118** 1-52.

[67] Srivastava, H.M., Saxena, R.K.,Pogány, T.K. and Saxena, R.(2011) Integral and computational representations of the extended Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.* **22** 487–506.

[68] Tomovski, Ž., Hilfer, R. and Srivastava, H.M. (2010) Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Integral Transforms Spec. Funct.* **21** 797–814.

Priyanka Harjule

Phone no. - 09785069289

E-mail priyankaharjule5@gmail.com

Educational Qualifications

Qualifying Degree	College/University	Year of Passing	Percentage obtained
MSc (Mathematics)	Mount Carmel College, Bangalore Univ. Bangalore	2012	85
BSc (Physics, Chemistry Mathematics)	Mount Carmel College, Bangalore Univ. Bangalore	2010	86
12^{th} std	Army School,Devlali,Nasik.CBSE	2007	86

Other qualifications

- Completed a 30 hour credit course in *Astronomy and Astrophysics.*
- Completed a 60 hour credit course in *Women's health and hygiene*.
- Completed a 80 hour credit course in *Hobby Electronics.*
- Completed a 45 hour credit course in *basic french.*

Publications

- H. M. Srivastava, P. Harjule and R. Jain (2015). A General Fractional Differential Equation Associated with an Integral Operator with the *H*-Function in the Kernel. *Russian Journal of Mathematical Physics*, **22(1),** 112-126.
- A Study of Fractional Differential Equation with an Integral Operator Containing Fox's H-Function in the Kernel Priyanka Harjule and Rashmi Jain. *International Bulletin of Mathematical Research Volume 02, Issue 01, March 2015*
- Rashmi Jain and Priyanka Harjule; 'A study of fractional differential integral operator'; procedings of the XIIth Annual Conference of *Society for Special Functions and their Applications*12, 73-77 (2013).

- Raisa D'Souza, Priyanka Harjule, Saranya D and Ashwini G S; 'The travelling salesman problem applied to inter-departmental travel in Mount

Carmel College'; *Carmelight – The Multidisciplinary Journal* Volume 9(1) June 2012.

Activities

- Worked on MSc. project titled 'Travelling Salesman Problem applied to interdepartmental travel in Mount Carmel College' under the guidance of Dr. Pradeep Kumar, Central College, Bangalore and co-supervised by Dr. Nagesh Kumar, IISc Bangalore.

- Worked on a UGC funded BSR project on 'Synthesis and UV characterisation of gold and silver nano particles'.

- Worked as secretary of Math Association 'Fermat', Mount Carmel College, Bangalore.

- Contributed to Mathematics department newsletter - 'Mathemalite', for the years 2011 and 2012.

Workshops/Conferences/Lectures

- Presented a paper titled *A study of fractional differential integral operator in lnternational Conference of Society for Special Functions and Applications held at MNIT Jaipur.*

- Presented a paper titled *novel green methods of synthesis of gold and silver nano particles and their characterization* at an international conference - *convergence of science and engineering in education and research, a global perspective in the new millennium* at Dayanand Sagar college of engineering, Bangalore.

- Attended public lectures on *science without boundaries* at TIFR, IISc campus.

- Attended a workshop on *mathematical analysis and applications* at Maharanilaxmi Amani College, Bangalore.

Academic awards/Other achievements

- TEQIP, MHRD Scholarship for Ph.D in Mathematics since July, 2013-till date

- First place in poster presentation in UGC sponsored national seminar on *science in the 21st century*.

- Third place in lecture contest in UGC sponsored national seminar on science in 21^{st} century.

- Third place in lecture contest in the intra-collegiate science fest by science association of Mount Carmel College.

- Merit scholarship under education scholarship scheme for army personnel for academic year 2009-2010 and 2010-11.

- Won second prize in the Ace Teacher quest-2012.

References

Prof. Rashmi Jain, Dept. of Mathematics, MNIT Jaipur.

Prof. K.C Gupta, Emeritus professor, MNIT Jaipur.