CHAPTER 1

INTRODUCTION

1.1 Background

With the advancement of new technologies and increasing global competition, today's manufacturers are facing a strong pressure to produce the high-quality products which are expected to perform their intended functions for years or even decades without any failures or flaws. This implies that the increased need for up-front reliability tests on systems, subsystems, & components (which we generically refer to as "units"), during the design stage of the products. With the short product development times, reliability tests must be conducted within the severe time and cost constraints. Traditional life tests (where time to failure is the response) may give result in few or no failures within very long period of time, even when the covariates are accelerated (e.g., by testing at higher-than-usual stress levels). Thus it is highly difficult to assess the reliability or performance of the products using the traditional life tests that record the only failure time. For this reason, degradation tests can be highly beneficial in manufacturing industries to obtain the reliability information more quickly.

For some products, the degradation response is the natural response. For example, for a luminescent light, degradation response may be the output of the light. Depending on the application, the degradation data may be available continuously or at some specific point of time where measurements are taken. With the degradation data, it is possible to make useful reliability predictions, even with the few or no failures. Direct observation of the physical degradation process or some closely related surrogate may also empower direct modeling of the physics-of-failure mechanisms, providing more justification and the credibility for the reliability estimates and a firmer basis for the extrapolation modeling.

Engineers usually increase levels of stresses or some other covariates (for example, temperature, voltage, humidity, or pressure) to the higher value than usual levels to accelerate the process of degradation. They expect that at the higher levels of applied stresses, the products will degrade more quickly and that they can estimate lifetime or degradation rates at lower, at normal use conditions using extrapolations based on the physically reasonable statistical models.

1.2 Concept of reliability

Due to increasing complexity of modern engineering systems, the concept of reliability has become a very important factor in the design of the any systems .Reliability is one of the most important factor because product design needs to work effectively both theoretically as well as practically. Reliability can be viewed as a measure of the successful performance of the system. The importance of reliability of components and systems is also recognized at all the stages of daily life, ranging from the consumer products to the larger systems such as trains and airlines.

(Rao et al.,1992) "Reliability is the probability of a device performing its intended function over a specified period of time and under the specified operating condition"

The need of reliable products was very first felt in both the commercial and military sectors in early 1950s. Since then the enormous and remarkable progress has been made in the area of reliability engineering. Before 1950s, the focus was either on the quality control or on machine maintenance problems. Before World War II reliability was intuitive in nature and basic concept of reliability aroused and gained importance during this time period. . Recently, due to increasing competition, complex product design and development, the use of increasingly popular manufacturing processes and equipments, particularly in the field of defense and the space technology, and increasing focus on customer satisfaction, the question of reliability has become a matter of interest. (Bhamare and Yadav, 2007)

System safety and the reliability evaluations are the main factors to ensure smooth functioning of system operations .Reliability design was very first used in the air and space industry. Product quality includes two aspects which are the functions and ability to keep the functions working normally. Therefore, reliability is a part of the quality, improvement of Reliability is an essential way to increase the value of some products, such as aircraft engine. (Suiran and Yang, 2011).

Reliability prediction based on the degradation modeling can be an effective method for evaluating the reliability of systems when observations of failures are very rare. The Current research shows that there has been an increasing interest in the application of degradation models for reliability prediction. Moreover, it illustrates that the significant progress has been achieved in applications of degradation models in various industrial areas.(Gorijan.N et al., 2009)

1.3 Research motivation

Traditionally, reliability assessment of new products is based on accelerated life tests which record failure and censoring times of products subjected to elevated stress or covariates. However, this approach may offer little help for highly reliable products. High quality reliable products are designed and manufactured to function for a sufficiently long time before they fail. Hence, it is a great challenge for manufacturers to obtain reliability statistics such as mean-time-to-failure (MTTF) with only a relatively short period of time available for performing internal life test.

Although there are helpful techniques, including censoring, and accelerating a products lifetime by testing at a higher level of stress, temperature etc. these techniques are less effective for highly reliable products because it is more difficult to obtain sufficient time-to-failure data for estimating their lifetime efficiently. To overcome this problem, accelerated degradation testing (ADT) has been proposed as a means to predict performance for highly reliable products. Usually, in order to facilitate observing the degradation phenomenon or shorten the degradation experiment under normal use condition, it is practical to gather the degradation data at higher levels of stress and, then, carry out the extrapolation in stress data for estimation of reliability under normal use conditions. Such an experiment is called an accelerated degradation test.

Accelerated degradation test (ADT) is an alternative to accelerated life test (ALT) with censoring to estimate reliability without waiting for actual failures to occur. Thus, test time is greatly shortened. Degradation analysis often yields more accurate estimates than those obtained from life data analysis, especially when a test is highly censored. In an ADT, an accurate reliability estimation prediction demands an appropriate degradation model.

1.4 Research objective

The research gaps presented in the form of prime focus area in the previous sections have been taken as motivational aspects to undertake the proposed study. The proposed study aims to develop a degradation model which can be useful in prediction of reliability .It also aims to incorporate into well established random models with their limitations and the study of proposed random drift model to ensure desired quality and reliability for predicting failure phenomenon or degradation behavior. Thus the ultimate goal of the study is the development of the random drift model with accelerated degradation test planning and demonstration of the proposed method through a numerical analysis.

1. To develop a model for accelerated degradation test planning using Inverse Gaussian process for random drift.
2. To develop an optimal plan for accelerated degradation test using proposed model.
3. To perform a numerical analysis for proposed model and to compare the results.

1.5 Research approach

Research question 1: Is it possible to develop an optimal plan for accelerated degradation test using random drift model?

Literature reveals that although there are quite large numbers of papers published in this field but most of them focus on the uniform data. A very few of the researchers have emphasized on the randomness of the data. Since in real life situation data is more random so there is more requirements to work on the random data. Different type of model developed with the passage of time but no one model takes into consideration unit to unit variation. So it is required to propose a plan which takes into effects randomness of data.

Research question 2: is it possible to combine inverse Gaussian and Weiner process for optimal ADT plan?

Since in the Weiner process both the negative and positive drift occurs with the time and since degradation phenomenon is continuously increasing in nature we requires only positive drift. So obtained the above condition we combine Weiner process with inverse Gaussian process to take into account monotone path of the degradation behavior.

1.6 Structure of thesis

The remainder of this thesis consists of 5 chapters. Chapter 2 presents a detailed review of the literature on the degradation.

Chapter 3 discusses about the methodology used for this research work in a systematic and step by step manner.

Chapter 4 Details out procedure for accelerated degradation test planning using inverse Gaussian process.

Chapter 5 A case study for the optimal test planning of the developed random drift model.

Chapter 6 concludes the thesis with final discussion including the future scope of the research work undertaken.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Degradation is the reduction in performance, reliability and life span of assets. Most assets degrade as they age or deteriorate due to some factors that are termed as covariates. Hence, reliability declines when assets degrade or deteriorate. Assets fail when their level of degradation reaches a specified failure threshold value.

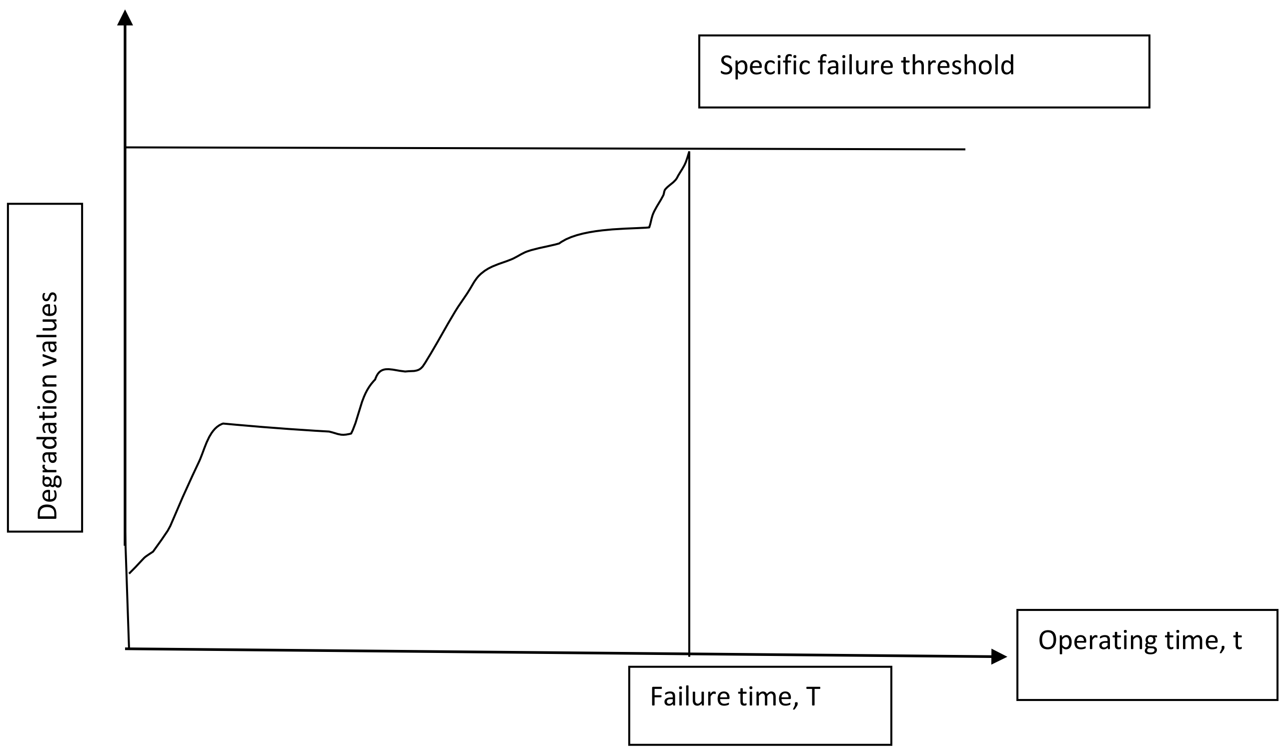


Figure 2.1: The degradation process

Degradation model in reliability analysis can be classified according to the figure given below:

2.2 Normal degradation model

Normal degradation model is used to estimate reliability of any product or component at normal operating conditions.

Normal degradation model can be classified into two type(normal degradation with stress factor and normal degradation without stress factor).while estimating the reliability of any component and the force(stress which resist the force) acting on the component is not taken into consideration then these models are called normal degradation model without stress factor. In these models reliability is estimated at a fixed stress level and if stress is taken into account then these are called normal degradation model with stress factor.

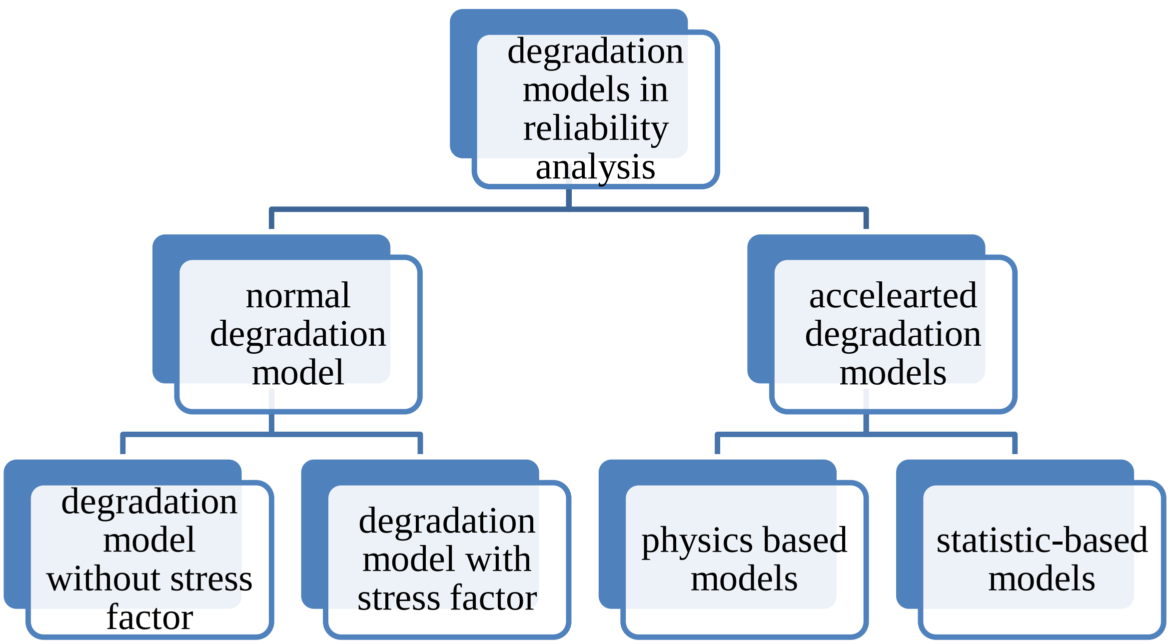


Figure 2.2: degradation model in reliability engineering

2.2.1 Normal degradation without stress factor

These can be classified into different types (general degradation path model, linear/nonlinear regression model, random process model, time series model).

2.2.1.1 General degradation path model

In the general degradation path model, the observed degradation path y is an asset's actual degradation path, a non-decreasing function of time that cannot be observed directly, plus measurement error ε. D is called threshold which denotes the critical level for the degradation path above which failure is assumed to have occurred.

2.2.1.2 Random process model

The random process model fits degradation measure at each observation time with a specific distribution with time dependent parameter.

2.2.1.3 Linear/nonlinear regression degradation model

The nonlinear regression model with a straight-line regression model can be generalized to:

$$Yi = (, \theta) + \varepsilon i \qquad i = 1,2, \ldots , n$$

Where, $(, \theta)$ explains a function of the vector of regression variables,X, and the vector of p model parameters, $\theta = \theta 1 , \theta 2 , \ldots , \theta pT$. A nonlinear regression model is the one in which at least one of its parameters appears nonlinearly.

2.2.1.4 Time series model

Lu et al. (2001) proposed a technique to predict individual system performance reliability in real-time considering multiple modes of failure. This technique unlike conventional reliability modeling approaches, which yield statistical results that reflect reliability characteristics of the population, includes on-line multivariate monitoring and forecasting of selected performance measures and conditional performance reliability estimates. The performance measures across time are treated as multivariate time series. The state-space approach is used for modeling the multivariate time series. The predicted mean vectors and covariance matrix of performance measures are applied to estimate system reliability with respect to the conditional performance reliability.

2.2.2 Normal degradation model with stress factor

In these models stress have been taken as a one of the important factor for estimating the reliability of any product. These models are discussed below:

2.2.2.1 Stress-strength interference model

The Stress-Strength Interference (SSI) model is an early and still popular representation of asset reliability. In this model, there is random dispersion in the stress, Y, which results from applied loads. The dispersion in the stress realized can be modeled by a distribution function (y). And (x) is random dispersion in inherent asset strength.

2.2.2.2 Damage accumulation model

The conceptual nature of the cumulative damage/shock model and SSI model is quite noticeable. In the SSI model, stress is treated as constant and strength as variable. However, in the cumulative damage/shock model, the strength (damage threshold) is treated as a constant quantity, and the stress (damage) is a variable parameter. Cumulative damage/shock model is based on the cumulative damage theory for a degradation process exposed to discrete stresses (e.g. temperature cyclic and random shock) and also the state of process is assumed discrete.

The cumulative damage/shock model is widely applied in the field of asset life prediction such as; fatigue failures in aircraft fuel age.

2.3 Accelerated degradation test model

Estimation of the failure-time distribution or long-term performance of the components of high-reliability products is somewhat a difficult task. Most of the modern products are designed to operate without failure for years, decades or longer. Thus very few units will fail to function or degrade appreciably in a test of practical length at the normal use conditions. For example, the design and construction of a communications satellite may allow only eight months of testing components that are expected to be in service for 10 or 15 years. For such applications, Accelerated Tests (ATs) are used in manufacturing industries for assessing or demonstrating component and subsystem reliability, for certifying components, for detecting failure modes so that they can be corrected, for comparing different manufacturers, and so forth.(Meeker and Hamda,1995) and(Meeker and Escorber,2005)

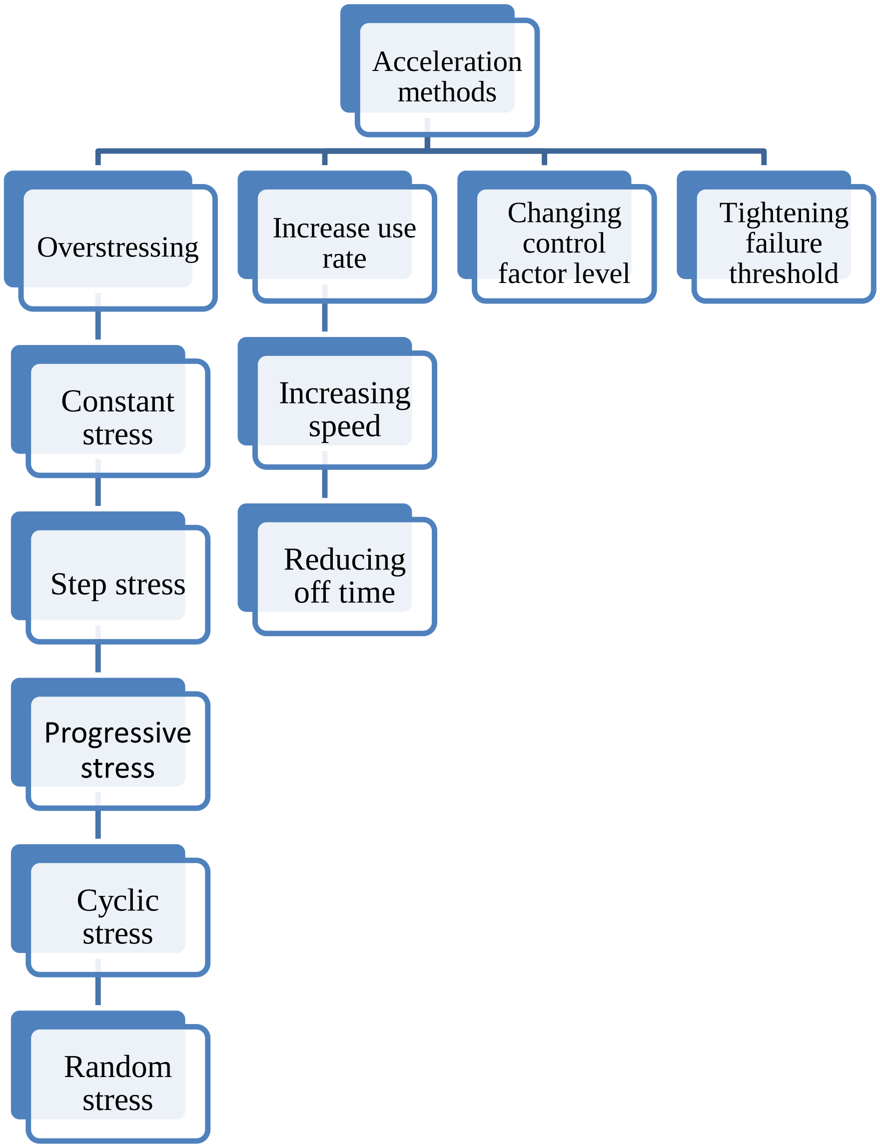


Figure 2.3: classification of acceleration methods (Adopted from Guangbin, 2007)

Accelerated degradation models make inferences about predicting the reliability at normal conditions using degradation data obtained at accelerated stress or time conditions. In real-life situations and industrial applications, a degradation process may occur very slowly at the normal stress level as well as time-to-failure may be comparatively higher (Yang et al., 1998; Tang and Shang, 1995) Estimating the failure time distribution or long term performance of components of high reliability products is particularly difficult (Yang.G et al., 2002; Meeker and Lu Valle, 1995). Therefore, in order to attain data quickly from a degradation test, it is often possible to use the accelerated life test (Shiau and Lin, 1999; Pham .H, 2006) .This test is applied by increasing the level of acceleration variables, such as vibration amplitude, temperature load, voltage, pressure.

However, the accelerated life test is rather a very costly approach and may take sufficiently longer time. Accelerated degradation models consist of: physics-based models and the statistics-based models. Physics-based models are: Arrhenius model, Eyring model, and Inverse Power model.

Arrhenius model is used when the damaging mechanism is caused by temperature (especially for dielectrics, semi-conductors, battery cells, lubricant, plastic, etc). Eyring model is used for accelerated life tests with respect to the thermal and non-thermal variable. Inverse power model is widely used to analyze accelerated life test data of many electronic and mechanical components such as insulating fluids, capacitors, bearings, and spindles in order to estimate their service lives when acceleration operating parameters are non thermal (e.g. speed, load, corrosive medium and vibration amplitude, etc). This model describes the damaging rate under a constant stress. (Meeker and Escobar, 2006)

Statisticians in manufacturing industries are often asked to become involved in planning or analyzing data from accelerated tests. At first glance, the statistics of accelerated testing appears to involve little more than regression analysis, perhaps with a few complicating factors, such as censored data. The very nature of ATs, however, always requires extrapolation in the accelerating variable(s) and often requires extrapolation in time. This implies critical importance of model choice. Relying on the common statistical practice of fitting curves to data can result in an inadequate model or even inappropriate results. Statisticians working on AT programs need to have awareness of general principles of AT modeling and current best practices. (Meeker and Escobar, 2006)

2.3.1 Types of accelerated test

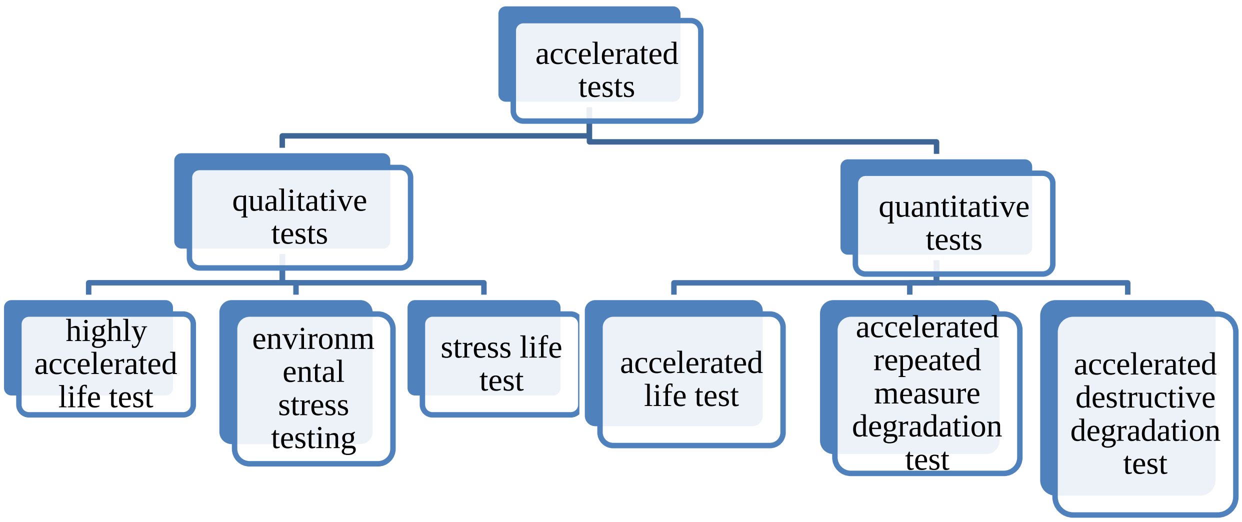


Figure 2.4: Types of accelerated tests (Modified from Meeker and Escobar, 2006)

2.3.1.1 Quantitative versus Qualitative

In reliability engineering, the term "accelerated test" is used to describe two completely different kind of useful, important tests that have completely different purposes. To distinguish between them, the terms "quantitative accelerated tests" and "qualitative accelerated tests" are being used. (Meeker and Escobar, 2006)

A Quantitative accelerated tests functions at combinations of higher than- usual levels of certain accelerating variables or covariates. The purpose of such a test is to obtain information about the failure-time distribution or degradation value at specified "use" levels of these variables. Generally failure modes of interest are known ahead of time, and there is some knowledge or prior data available that describes well the relationship between the failure mechanism and the accelerating variables (either based upon the physical/chemical theory or large amounts of previous experience with similar tests) that can be used for identifying a model that can be used to justify the extrapolation.

A Qualitative accelerated testing method tests units at higher-than-usual combinations of variables like temperature cycling and vibrations. Specific names of such tests include HALT (for highly accelerated life tests), STRIFE (stress-life) and EST (environmental stress testing). The basic purpose of such tests is to identify product weaknesses caused by flaws in the product's design or manufacturing process.

Nelson (1990), described such tests as "elephant tests" and brings into light some important issues related to Qualitative accelerated testing.

Benefits and Drawbacks of Qualitative tests:

Benefits

Increase reliability by revealing probable failure modes.

Provide valuable feedback in designing quantitative tests, and in many cases are a precursor to a quantitative test.

Drawback:

Do not quantify the reliability of the product at normal use conditions.

2.3.1.2 Methods of Acceleration

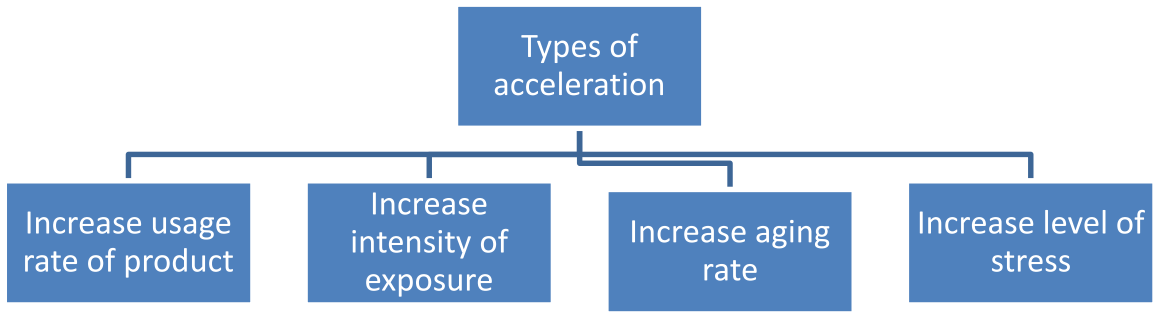


Figure 2.5: Types of acceleration, (Modified from Meeker and Escobar, 2006)

(A) Increase the use rate of the product

This method is suitable for the products that are ordinarily not in the continuous use. For example, the median life of a bearing of certain washing machine agitator is 12 years, based on an assumed usage rate of 8 loads per week. Now, if the machine is tested at the 112 loads per week (16 per day), the median life gets reduced to approx 10 months, under the assumption that the increased use rate does not change the cycles to the failure distribution. Also, because it is not essential to have all the units fail in a life test, useful reliability information can be obtained within few weeks instead of months.

(B) Increase the intensity of the exposure to radiation:

Different types of the radiations can lead to progressive degradation of product resulting in product failure. For example, organic materials (ranging from human skin to materials like epoxies and polyvinyl chloride or PVC) will degrade when they are exposed to the ultraviolet (UV) radiation. Electrical insulation exposed to the gamma rays in the nuclear power plants will degrade more rapidly than other similar insulation in similar environments in absence of radiation. Modeling and acceleration of degradation processes by increasing the intensity of radiation is commonly performed in a manner that is similar to acceleration by increasing the usage rate.(Meeker and Escorber,1998)

(C) Increase the aging rate of the product:

Increasing the level of experimental variables like temperature or humidity can rapidly accelerate the chemical processes of certain failure mechanisms such as the chemical degradation (resulting in eventual weakening and failure) of an adhesive mechanical bond or the growth of a conducting filament across an insulator which may eventually cause a short circuit leading to the failure.

(D) Increase the level of stress:

Covariates (e.g., amplitude in temperature cycling, voltage, or pressure) under which test units operate are enhance. A unit will fail when its strength drops below the applied stress. Thus a unit at a high stress value will generally fail more rapidly than it would have failed at low stress. (Meeker and Escobar, 2006)

2.3.1.3 Types of Responses:

(a) Accelerated Binary Tests (ABTs)

The response in an ABT is purely binary in nature. That is, whether the product has failed or not is the only reliability information we can obtain from the each unit in such test.

(b) Accelerated Life Tests (ALTs)

The response in an ALT is directly related to the lifetime of the product. Typically, ALT data are right-censored because the test is stopped before all the units fail. In other cases, the ALT response is an interval-censored because failures are discovered at particular inspection times.

(c) Accelerated Repeated Measures Degradation Tests (ARMDTs)

In an ARMDT, we measure degradation phenomenon on a sample of units at different points in time. In general, each unit provides several degradation measurements. The degradation response could be actual chemical or physical degradation or performance degradation (e.g., drop in power output).

(d) Accelerated Destructive Degradation Tests (ADDTs)

An ADDT is similar to an ARMDT, the only difference is that the measurements are destructive, so one can obtain only one observation per test unit and it incurs relatively high cost.

2.3.2 Statistical models for acceleration

Interpretation of accelerated test data often requires models that relate the covariates like temperature, voltage, pressure, size, etc. to time acceleration. To perform tests over some range of accelerating variables, one can fit a model to the data to show the effect that the variables have on the failure-causing processes. The general idea is to test at high levels of the accelerating variables or covariates to speed up failure processes and extrapolate to lower levels of the accelerating variables. For some situations, a physically reasonable statistical model may allow such extrapolation. Jensen (1995) and Klinger, Nakada and Menendez(1990) .

2.3.2.1 Physical acceleration models

For failure mechanisms, one may have a model based on physical/chemical theory which describes the failure causing process over the range of the data and provides extrapolation to the use conditions. The relationship between the covariates and the actual failure mechanism is usually an extremely complicated task. Often, one has a simple model that properly describes the process. For example, failure may result from a complicated chemical process with many steps, but there may be one or few rate-limiting (or dominant) steps and a thorough understanding of this part of the process may provide a model that may be adequate for extrapolation purpose.

2.3.2.2 Empirical acceleration models

When there is little understanding of the chemical or physical processes leading to failure, it may be impossible to develop a model based on physical/chemical theory. An empirical model may be the only alternative. An empirical model may provide an excellent fit to the available data, but provide improper extrapolations. In some situations there may be extensive empirical experience with particular combinations of variables and failure mechanisms and this experience may provide the needed justification for extrapolation to use conditions.

Table 2.1: Merits and limitations of model-based approaches

Merits	**Limitations**
These approaches provide technically comprehensive method that has been used traditionally to understand failure mechanism	These require specific mechanistic knowledge and theory relevant to monitored asset
These approaches provide a means to calculate damage to critical components as function of operating conditions	Model-based approaches need many assumptions about system and its operating conditions.
These require less data than data driven approaches	Physics-based models require estimating the physical parameters.
By integrating physical and stochastic modeling technique, the output model can be used for the evaluation of remaining useful component life.	These models sometime do not practically fit since fault is often unique from question to question.
Physics based models may be most suitable for the cost justified applications.	It is hard to identify the problem without any interrupting operation.

Table 2.2: General application of various degradation models (Adopted from Gorjian et al., 2009)

	Model name	Potential application
1	General degradation path Model	This model is suitable to fit the degradation observations by both linear and nonlinear regression models
2	Random process model	This model is suitable for reliability estimation with no assumption about degradation paths. This model is appropriate for multiple observations at certain time points.
3	Linear and nonlinear regression models	This model is more flexible than the above two models and is applied when observations are obtained at different time point.
4	Mixture model	This model provides extremely limited testing, thus further research is needed to analyze the properties of this model for modeling both soft and hard failures.
5	Time series model	This model is suitable for predicting individual system performance reliability with multiple performance measures in a dynamic environment.
6	SSI model	This model is appropriate for reliability estimation at random dispersion stress. This model is applied in a situation that external loading is higher than item strength
7	Cumulative damage/ shock Model	This model is applied for a degradation process exposed to discrete stress. The generalization of this model can be applied for continuous sample paths.
9	Weiner process model	This model is not suitable for modeling degradation which is monotone increasing; however, it can be effective to model the degradation process considering maintenance effects.
10	Gamma process model	This model is suitable for the stochastic modeling of monotonic and gradual degradation. This model can be applied to degradation process in maintenance optimization models.

2.4 Accelerated Degradation Testing review

To assess the reliability of newly designed products, engineers often resort to accelerated tests to shorten the life time of products, or hasten the degradation of their performance. During the test, the products are exposed to the harsh conditions, e.g., a combination of random vibration, higher temperature, voltage, or pressure. The main purpose of such accelerated testing is to obtain the reliability information quickly so as to save time and money.

Many new products are designed to be very reliable because of (a) rapid advancement in technology, (b) increasing customer expectations, and (c) enhanced global competition. For example, an electronic product may be considered as a complex system that comprises of many components. To maintain high reliability for the system, sub-system or components, we generally require that the individual components possess extremely high reliability. Traditional accelerated life test (ALT) methods are not suitable for such reliable products as extremely long test duration is required to yield the sufficient failures. On the other hand, we often observe that the failure of a product is associated with degradation of some quality characteristic (QC). Degradation of the product accumulates over time, and causes a failure when the degradation exceeds a failure threshold value. This threshold behavior naturally provides a linkage between the product degradation and reliability. (Meeker, 1998)

The degradation is most often hastened under the severe stresses. So, we can use accelerated degradation tests (ADTs) to quickly obtain the degradation information. In a simple constant-stress ADT experiment, a number of units are allocated to several stress levels, and the degradation levels of these units are measured, analyzed, and extrapolated to the failure threshold for predicting the life characteristics of interest under the use conditions. ADTs are able to greatly shorten the testing duration, and have gained much attention. There are two classes of models for ADT data. The first class is called general path models, proposed by (Lu and Meeker, 1995).

Some developments of models in this class can be found as in most of the literature degradation processes have been modeled by deterministic functions which lack robustness in specifications of the models. The stochastic models are used when the experimental data is lacking or when there is no prior knowledge about the product (Yu HF and Choao , 2002)

Shi and Meeker (2012) proposed Bayesian methods for ADDT planning under a class of non-linear degradation model with one accelerating variable. The authors used Bayesian criterion for setting optimal plans using ADDT. Accelerated destructive degradation tests (ADDT's) provides timely product reliability information in several practical applications. They provide quick information of the failure data. The other classes of models are called stochastic process models, which capture the time-dependent structure of the degradation with respect to time.

Two popular models are the Wiener process and the Gamma process. Tseng et al., (2000) proposed using step-stress accelerated degradation (SSADT) to assess the reliability of a light emitting diode by using the empirical regression method. The Optimal ADT settings were obtained by minimizing the estimated quantile of the product's lifetime distribution subject to a constraint on the total cost.

Following the research, some SSADT models have been developed based on the assumptions of Wiener processes .Tang and Yang(2004) proposed the means to predict the reliability of highly reliable product. They proposed the planning of ADT in which test stress was increased step by step ranging from lower stress value to higher stress value while conducting the test so that specimens are gradually conditioned to the stressed environment avoiding overstressing. Based on SSADT the model was developed and asymptotic variance was minimized for the minimum optimal plan.

Mao and Tsang(2006) showed that ADT does not apply well for accessing the life distribution of a newly developed or the expensive product which has very few available test units. Thus to overcome this difficulty SSADT was proposed, however the problem relating to choose the optimal settings of variables was not discussed such as sample size, termination time etc. The authors used the typical stochastic diffusion process for modeling SSADT problems and developed an optimal plan by putting constraints on the total cost. It was demonstrated by using an example of LED data.

Zhang and Jiang (2010) in their research used SSADT for estimating the reliability of the product most precisely. The drifts Brownian motion was used as the degradation model and in order to minimize the mean square error (MSE) for predicting the reliability of the product, the test plans of a SSADT that under the specified total test time and sample size are optimized using a Monte Carlo simulation method. The advantage of SSADT over other degradation methods is that it provides best results even with small sample size and short test duration, thus it is most commonly used in practice. Some assumption used here as:

1) Degradation process of the product performance is monotonic, i.e. the degradation damage is not able to reverse.

2) Degradation mode remains the same under different varying stresses.

3) The remaining life of product depends only on the cumulative fraction of the damage that has happened and the current stress level, but does not the cumulative mode.

4) The degradation process of the product performance is described using the drift Brownian motion.

5) The dispersion of the drift Brownian motion remains constant**.**

Chien and Tseng (2010) in his study used PSADT (Progressive Stress Accelerated Degradation Test) with a non-linear degradation path model.. Lifetime distribution of a product can be obtained analytically by the first passage time of its degradation path. Further, an exact relationship was developed between the lifetime distributions of the PSADT, and the conventional constant-stress degradation test (CSDT), which allowed extrapolating the lifetime distribution of the product under typical stress. Finally, the usage of the proposed model, and the efficiency of PSADT to reduce the product's life testing time were demonstrated through the proposed research.

Chen and Yuan, (2010) introduce proportional degradation hazards model (PDHM) . The PDHM is established with degradation data, then the probability density function and Likelihood function of the model are derived, and maximum likelihood estimate (MLE) and Newton-Raphson method were used for estimation of the model parameters. According to the relationship between failure threshold and degradation measure, the reliability model of the product was established, with which the reliability was assessed. The authors demonstrated and validated the processes by assessing the reliability of percent increase in resistance over time of carbon-film resistors.

Zhang et al., (2011) focused on the statistical analysis of ADT model with the assumption that the test stresses have the random errors and the degradation data from the same subject are being correlated. A minimum distance estimation method was being proposed. They compared the proposed estimation method with the traditional ones which ignores the random errors of test stresses and the correlation of degradation data from the same sample unit. The simulation results showed that the minimum distance estimation method provides better accuracy in the sense of MSE.

Shuen-Lin and Bei-Ying (2011) proposed to build a planning procedure for ADDTs when the degradation model distribution may be lognormal or Weibull. This study provided evaluation of the bias and variance of the ML estimators of the distribution quantile when we use the wrong distribution as the working model. Test plans are evaluated under the criterion of minimizing the large-sample approximate mean square error (AMSE). This criterion helped practitioners in choosing an appropriate ADDT plan. Assumption used in the study was that Lognormal and Weibull distributions are often used to describe the distribution of product characteristics in life and degradation tests. Random effect variant of wiener process can be found as- Peng and Tseng (2009) proposed a general linear degradation path where unit to unit variation of all tests units can be considered simultaneously with time dependent structure in the degradation paths. Through the use of lifetime distribution model, mean-time –to –failure was calculated .By using profile likelihood approach, maximum likelihood estimators, products MTTF as well as confidence interval was derived for predicting the reliability of the product or component.

Zhou and Yao (2011) aimed at the multiple degradation problems. Analyzed the multiple competition failure problems based the system machine which characterization by many more weak links. For two states, independence and relevant, study the reliability assessment method based on degradation quantity distribution, establishes the implementing procedures and methods. Joint probability density and variance covariance matrix used in the research have complicated structure, large amount of calculation, for simplified the model and accuracy is needed for further optimization design work. At present the application based on performance degradation most used in device level. Such as LED, satellite components, Gas laser etc.

Tang and Shu-Su (2008) instead of measuring the wiener degradation or performance process at predetermined time points to find the degradation of any process depicted the performance of the product in order to estimate the lifetime through the use of finding the first passage time of the process estimating the lifetime process over the certain non failure thresholds. Based on that minimum variance unbiased estimator was obtained for the estimation of the lifetime distribution function, maximum likelihood estimator was obtained for the reliability prediction. The advantage of using this method was that it was useful for highly reliable product when their failure times were difficult to obtain. The proposed new estimator of lifetime distribution was more accurate than the standard and modified most likelihood estimator. Finally, the author used the light –emitting diode for the demonstration and validation of the proposed method.

Esary and Marshall (1973) studied the model for the life distribution of device, system or any component occurring randomly in time governed by the Poisson process. The life distribution of device subjected to shock is considered to be the function of the probabilitiesP_k of not surviving the first k shocks. Damage results into failure when it exceeds the threshold value. Bagdonavicius and Nikulin (2000) modeled the effect of covariates on the degradation process. Degradation models with covariates can be used to model reliability when the governing conditions are dynamic (Singpurwalla ,1995). Degradation models are useful when optimal values of covariates, maximizing the reliability of the products are required. Taking an example of light emitting diode, its degradation is characterized by the decreasing luminosity whereas the rate of degradation depends upon such factors as type of silver, coating, lens material.

Xiao Wang (2010) studied Weiner process with random effects for degradation data. He basically studied maximum likelihood inference on the class of wiener process with random effects for the degradation data .Degradation data with independent subjects with a wiener process and drift parameter as well as diffusion parameters are observed at different times. Unit –to-unit variability is incorporated into the model by the random effects. The model was validated by the simulation method. The model was fitted to the bridge beam data and the goodness of fit was carried out.

Noortwijk (2009) surveyed the application of gamma process in maintenance. Since the introduction of gamma process in reliability analysis in 1975, it is increasingly used for the modeling of stochastic deterioration for optimizing maintenance. They have been used for determining optimal inspection and for taking maintenance decisions. He demonstrated the use of gamma process as a probabilistic stress-strength model. He showed that the gamma process is most appropriate for the modeling of the monotonic and gradual deterioration. Using statistical techniques gamma process were satisfactorily fitted to real-life data on creep of concrete fatigue crack growth , corrosion of steel protected through coatings, and for the longitudinal leveling of the railway tracks. Future scope involved the use of the gamma process for the entire system rather than for a component

Tseng and Balakrishnan (2009) proposed an optimal SSADT plan based on the assumption that the degradation path follows a Wiener process. The degradation model has been appropriately modeled by a gamma process which exhibits a monotone increasing pattern. In his study he introduced the SSADT model when the degradation path follows a gamma process. Now under the constraint that the total experimental cost should not exceed a pre-specified budget, the optimal settings such as sample size and termination time are obtained by minimizing the approximate variance of the estimated MTTF of the lifetime distribution for the product or component.

Li and Liang (2013) proposed an evaluation method of step-down stress accelerated degradation modeling based on Gamma process. The degradation path was portrayed by a Gamma process. Then using the Gamma process characteristics the parameters were evaluated. The high reliability and long life characteristics of electronic products result in long degradation test time and lower efficiency. Traditional degradation test depends on prior information. This method greatly simplifies the procedure of statistical analysis.

Anthony Desmond (1985) studied the stochastic models of failure considering the random effects. The failure producing stress environment is modeled considering it to be a stochastic process. By making use of the path of these processes, the failure time distribution was obtained. This method is commonly used in modeling of fatigue crack growth .In his example he considered the effect of mechanical stress, and the failure mode was due to the metal fatigue. All these models used finally lead to the failure –time distribution belonging to Birnbaum-Saunders family. Birnbaum and Saunder (1969) derived a two-parameter life distribution model based on finding the number of cycles necessary to force a fatigue crack to grow past some threshold or critical value.

Tang and Chang (1995) used Birnbaum –Saunder method for the evaluation of reliability of highly reliable products by using non-destructive accelerated degradation data. This method provides for the reliability testing without the labor of conventional life testing. It identifies the failure mechanism which degrades continuously. After identifying the degradation measure, modules are subjected to the statistically designed accelerated experiment. The failure time is then determined by the first passage time of the underlying stochastic process .The author demonstrated the application of the NADD (non-destructive accelerated degradation data) through the case of power supply DC unit in which the DC output was used as a degradation measure.

Although, ALT is widely used to quickly provide the life distribution of any product or to find the reliability at elevated stresses. The basic limitation with this method is that it results in very few failures and sometimes no failures at low stress levels. Thus Yang (2002) provided a method to estimate the life distribution by using the degradation data measurements. Since time to failure basically depends upon the critical values, life tests are accelerated by tightening the critical values and optimum test plan was chosen based on the degradation data by using MLE.

Fergusan and Klass (1972) in his study described a simple way of presenting processes with independent increments having no Gaussian components and no fixed points of discontinuity. It is similar to the way Weiner described the Brownian motion process on the interval as a countable sum. The limitations of these processes were that they did not consider the random environment.

Guo and Mettas (2007) proposed design of experiments(DOE) method of using the degradation process together with the observed failure data to improve the reliability .The advantage of this process is that by using DOE, all the factors are classified into two types. Amplification factor whose effect on the degradation is known based on the knowledge of past experience. The other one is classified as the control factors, whose effect are not known prior to performing experimentation. They are realized using the linear regression. The advantage of the proposed method was that it can be used in the case where the degradation cannot be easily measure, especially under the accelerated degradation testing. It does not require regular degradation measurements.

G.A. Whitmore (1986) introduced a family of Normal-gamma mixture of inverse –Gaussian distribution. For a Weiner diffusion process {W(t); t>0 with mean drift parameter δ and variance parameter λ , the first passage time possess inverse Gaussian distribution. Whitmore (1978) referred it as the defective inverse Gaussian distribution and its application was illustrated in modeling employee service times (Whitmore 1979) and equipment failure ages under the conditions of low stresses (Whitmore,1983).The proper form has been used extensively in data analysis and statistical modeling in connection with duration phenomenon.

Banarjee And Bhattacharya, (1974) presented a mixture in which δ and λ, followed a truncated normal distribution and a modified gamma distribution respectively. The major limitation of the mixture model was that it had complex mathematical form and was difficult to apply in real life problems. The future study should focus on dealing with simplified equations for solving real time application problems.

Sanchez and Pan (2009) used as a tool for improving product design. In particular, when external stress factors are not constants in a product's use environment, it is necessary to utilize ADT experiments to investigate the interaction between these stress factors and product design factors so that a robust design can be obtained. The authors have developed a methodology for achieving product robust design via ADTs. A degradation model was proposed that can be used for studying the effects of design and stress factors on degradation rate. Model parameter estimation is obtained by the maximum likelihood method and an optimization procedure.

Jerald and Crowder (2008) focused on models where the multiple units are in use, considering variation across units in usage rates. Thus, considering random effect model they dealt with joint models for the recurrent event and the usage processes, which analyzed their relationship and worked for the prediction of their failure. Data on the automobile warranty claims was used to illustrate the proposed models and the estimation methodology. Xiaoyang and jiang (2009) showed that the Accelerated Life tests (ALT) used for evaluation of lifetime and predicting the reliability has certain drawbacks. It is rather difficult to obtain enough failure time data to satisfy the requirement of ALT because of high reliable property of the products. Thus, ADT was preferred in most of the researches. The authors first used drift Brownian motion to model a typical step-stress ADT (SSADT) problem. Then, according to competing failure rule, reliability model of the product was established. Under the constraint that the total experimental cost does not exceed a predetermined budget, our objective is to minimize the asymptotic variance for determining reliability model of product.

Xiaobing and Jinzhong (2011) established methods on reliability estimation and mechanism consistency test for accelerated degradation data. The integrated degradation model was established using the Arrhenius relationship, which is commonly used in reliability engineering. Now, the confidence interval of percentile lifetime is provided through Fisher information matrix. Via the effective usage of data under different accelerated stress levels, the accuracy of reliability estimation and lifetime prediction for product are improved. Additionally, a mechanism consistency test method was proposed which validated the integrated model.

Shuen-Lin and Bei-Ying (2011) in their study built a planning procedure for ADDTs when the degradation model distribution may be lognormal or Weibull. This study provides evaluation of the bias and variance of the ML estimators of the distribution quantile when we use the wrong distribution as the working model. Test plans are evaluated under the criterion of minimizing the large-sample approximate mean square error (AMSE). This criterion will help practitioners to choose an appropriate ADDT plan.

Quan Sun and Huang (2012) predicted reliability of products with high reliability and long life, the Step Stress Accelerated Degradation Test (SSADT) is commonly applied. With the motivation of predicting product reliability most precisely, this paper discusses degradation process of products using Birnbaum-Saunders model, and proposed an optimal design method of SSADT. In order to minimize the mean square error (MSE) of product operation reliability, the test plans of SSADT under specified total test cost by using Monte Carlo simulation.

Hongliang Jia and Miao Cai (2013) proposed a multivariate system reliability estimation method based on step stress accelerated degradation testing. The method of step stress accelerated degradation testing (SSADT) was used to evaluate the reliability. The system reliability was calculated based on total probability theorem and Monte Carlo simulation. Method of step stress accelerated degradation testing (SSADT) is utilized to evaluate the reliability of LED luminaries.

Man Ho ling and Kwok Tsui (2014) proposed accelerated degradation analysis that characterizes health and quality of the systems with monotonic and bounded degradation. The MLE are derived based on gamma process using the power link function for associating the covariates. For the illustration of the proposed model, numerical example involving light intensity of light emitting diodes (LED) was analyzed.

Zhi-Sheng and Min Xie (2014) investigated the semi-parametric inference of simple Gamma-process model and examined the existence of random effects under semi-parametric scenario. The method is demonstrated using the fatigue-crack growth datasets.

Zhi-Sheng Ye and Min Xie (2014) proposed constant-stress accelerated degradation tests (ADT's) planning when the underlying degradation follows inverse-Gaussian process. The authors first considered the optimal plan settings without considering the random effects. The mathematical model has been developed for the random volatility model. The mathematical modeling and the optimal plan settings for the random drift model when the underlying degradation follows inverse-Gaussian process have been proposed incorporating the random effects.

Although the Wiener process and Gamma process have received intensive applications in degradation data analysis, it is obvious that the two models cannot handle all the degradation problems. Wang and Xu, (2010) in their study, proposed maximum likelihood estimation of a class of inverse Gaussian Process models for the degradation data. He incorporated both the heterogeneity factor and effect of covariates, thus including the random effect. The EM (Expected Maximization) algorithm was used to obtain the maximum likelihood estimators of the unknown parameters. The model was fitted to laser data. Failure time distributions in terms of degradation passages are calculated. He also found that neither models fits the Gas laser degradation data well. Recently proposed the IG process for degradation data, and investigated semi-parametric inference for this process.

Shen and Chan (2014) systematically investigated the IG process, and showed that, compared with the Gamma process the IG process has many superb properties when dealing with covariates and random effects. Therefore, this process can be an important method for degradation analysis.

The purpose of this thesis is to investigate the planning of ADT experiments using the IG process. Here we use an IG process without random effects (random drift model) which is described in next chapter of thesis.

The objective of ADT planning is to properly choose the stress levels, and the number of units allocated to each stress, to minimize the asymptotic variance of the -quantile under use conditions. Parameter estimation for random drift model is discussed, based on which the asymptotic variance of the -quantile can be derived. Then the optimal stress levels and the allocation scheme can be obtained. In reality, it is not difficult to observe a unit-to unit difference within a product population due to some unobserved factors, such as variations in the raw materials. Such heterogeneity is often modeled by a random-effect term. Random- effect degradation models are believed to be more realistic in modeling product degradation, and such models have found more applications recently.

In addition, our data analysis also shows that the random volatility model fits the stress relaxation data well. Therefore, we believe that ADT planning for this random-effects model is meaningful.

Table 2.3: Summary of the accelerated degradation model

Author name	**Year**	**Advantage**	**Limitation**	**Application**	**Future work**
Guo and Mettas	2007	Can be used in the case where the degradation cannot be easily measured, especially under accelerated testing circumstances.	Here noise effect is not considered	fractional factorial design, Plackett – Burman design	Develop model for soft failure using Weibull, Gumbel, and Birnbaum-Saunders distributions
Peng and Tseng	2010	The confidence interval obtained from the PSADT model is substantially wider than the true confidence interval	Here degradation paths of all tested subjects are assumed to be observed with the same measurement time but in real observations are taken at different times, and/or some measurements are missing	LED and other electronic component	PSADT plan with a Gamma degradation model

Chen and Yuan	2010	This method effectively apply to two kind of stress condition as temperature and current	the logarithm of the Degradation hazard function is a linear function of the stress covariates.	Highly reliable product	
Ma and Wang	2011	the model provides an accurate tool to predict the reliability in the normal operating conditions by using the accelerated degradation data		Mechanism consistency test is done	
Sun and Feng	2012	determine the optimal sample size, parameter measurement interval and measurement times of SSADT	The test data obtained under different stress obeys the same distribution which not meet in real application	Permanent magnet used in satellite and also in highly reliable product	Product performance describe other than Bimbaum-Saunders model
Shi and Meeker	2012	use a Bayesian criterion based on the estimation precision of a failure time distribution quantile at use conditions.	Here only one accelerating variable is considered for estimation as temperature	It used to formally incorporate prior information into estimation and test planning, providing test plans with better statistical precision	Develop ADDT planning for complicated degradation model such as models with multiple accelerating variables or nonlinear relationships between degradation and time
Ye and Chen	2013	the class of IG processes has similar		this conclude real world condition	applications of the IG

		properties to the Gamma process, but is much more exible.			process in burn-in tests, accelerated degradation tests, preventive maintenance scheduling and remaining useful life prediction are require extensive investigations
Li and Liang	2013	Gamma process based modeling used to simplify the statistic analysis methods and step-down stress test in improving the efficiency of accelerated degradation test		it can be applied into a variety of electronic products that the performance degradation processes can be described by Gamma process	
Hu and Lee	2014	the optimum plan is a simple plan using only the minimum and maximum stress levels under many commonly-used optimization criteria	Here considered that there is upper stress bound below which the failure mode is same as under normal stress level the applied stress should not be too high so that the underlying failure mechanism may become different	For highly reliable product as LED etc	Develop a robust SSADT plan so that it could provide reasonable efficiency to many objective of interest.
Ye and Chen	2014	An advantage of constant-stress ADT is that we can check	Here assume that the degradation rate parameter is an increasing function	to give optimal accelerating test planning	The optimal ADT plan for random drift and drift-volatility

		the assumed stress-degradation relationship by separately estimating the parameters under each stress level	of the stress. Due to this legitimate assumption because degradation is most often hastened under severe working conditions		can also be developed

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

Generally any research start with collection of particulars from the existing literature to get equipped with the latest development in the area of research. This helped us in building a theoretically background needed to propose a new research hypothesis. This hypothesis is then tested on the suitable platform, and the results are evaluated to prove the proposed work as a distribution in the concerned area. The main purpose of this chapter is to give an overview of research methodology used in this research in order to the research question and fulfill the research objectives.

3.2 Proposed methodology

Literature reveals that very few efforts have been made to develop an accelerated model for capturing the randomness of the product. Usually, degradation models are formulated to predict the reliability and future life of the product. Very few researchers have tried to consider the randomness of the degradation phenomenon considering the unit to unit variation or heterogeneity into account. Researchers have developed random volatility model which had certain limitations that it does not consider the variation in the data. This has motivated this study to develop a random drift accelerated degradation model which can be very useful in prediction of the reliability.

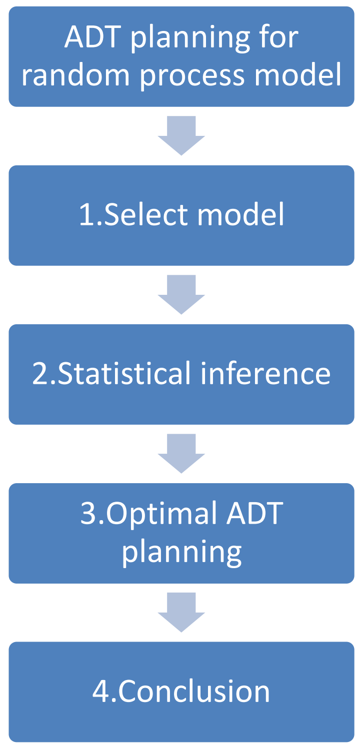


Figure 3.1: Outline of research methodology

3.2.1 Select the model

First step is to select the model on which we want to work. After going through the no. of research papers we have concluded that we have to select the random drift model for the optimization. This model has certain advantage compared to other models being referred in the same paper. This model takes into consideration unit to unit change since material properties vary from unit to unit. So this model will become an effective model for consideration in the future.

3.2.2 Statistical inference

In statistical inference we select the no of units to be tested and the value of the stress level and no of level at which the test is to be conducted. Then next step is to develop the log-likelihood function. And by taking the double derivative we find the elements of the fisher information matrix. And we also find the value of the estimated parameters by using the E-M (expected maximization) theorem. Now by putting the values of these estimators into the fisher matrix the elements of the matrix are calculated finally.

3.2.3 Optimal ADT planning

In this part of the research we estimate the quantile function. Then by using the fisher matrix and quantile function we find the asymptotic variance. Then optimize this asymptotic variance with the given time interval, no of units available and the no of stress levels available. Then optimize the variance function and the Q-Q quantile curve for the random drift models are plotted.

3.2.4 Conclusion

After optimizing the variance function we obtain how many units are to be tested at which stress level. And compare it with the simple IG process and also conclude how model presented in the thesis is better in comparison to previous model. And also future work can be described.

CHAPTER 4

OPTIMUM TEST PLANNING FOR RANDOM DRIFT MODEL

4.1 Introduction

Any component degrades with the passage of time. No of data has been collected with respect to time. These data show some properties if plotted with respect to time and give some useful information for the optimal setting of the no of units and stress level. These different type models are Weiner process and the Inverse Gaussian process are discussed below:

4.1.1Wiener Process as a Degradation Model

A Wiener process $\{X(t), t \geq 0\}$ has the following three defining properties (Karlin and Taylor, 1975).

(a). Every increment $X(t_i) - X(t_{i-1})$ for a time interval (t_{i-1}, t_i) is normally distributed with mean μ $(t_i - t_{i-1})$ and variance λ $(t_i - t_{i-1})$where λ> 0 is a fixed variance parameter and μ is a fixed drift parameter.

(b). The increments for any set of disjoint time intervals are independent random variables having the distributions described in property 1.

(c). $X(0) = 0$.

A Wiener process is a homogeneous process and has a continuous sample path with probability one. The increments $X(t_i) - X(t_{i-1})$ are independent of the past evolution of the process, that is,

$$P[X(t_i) \leq x_i \,|\, X(t_0) = x_0,\ X(t_1) = x_1, X(t_2) = x_2, .., X(t_{i-1}) = x_{i-1}]$$

$$= P[X(t_i) \leq x_i \,|\, X(t_{i-1}) = x_{i-1}], \qquad i = 1,2, \dots \dots, n,$$

For any 0= $t_0 < t_1 < t_3 < t_4 < t_n$

Given condition $X(0) = 0$, the probability density function of X (t) at t$>$ 0 is the normal density function

$$\phi(x; t) = \frac{1}{\sqrt{2\pi\lambda t}} \exp\left(-\frac{(x-\mu t)^2}{2\lambda t}\right) \tag{4.1}$$

Under the same condition, the joint PDF. of $X(t_1), X(t_2), \ldots, X(t_n),\ 0 < t_1 < t_2 < \cdots . t_n, is$

$$f(x_1, x_2, \ldots, x_n) = \phi(x_1; t_1)\ \phi(x_2 - x_1;\ t_2 - t_1) \ldots . \phi(x_n - x_{n-1};\ t_n - t_{n-1})$$

A Wiener process, $\{X$ (t)$\}$, is taken as the basic model of a degradation process. It is assumed that each item has its own degradation process which is independent of the others (for a given set of covariates). Items having the same design are assumed to have the same drift and variance parameters unless indicated otherwise. An item fails when its degradation process reaches a specified critical level for the first time. This critical level is referred to as a barrier and is denoted by a. The lifetime of the item corresponds to the first passage time, S, of the Wiener process to the barrier.

Wiener processes have been widely applied in both engineering and business situations. Many physical phenomena are described by the Wiener processes. The matter of whether a Wiener process is a suitable model for degradation processes, however, deserves a few comments on it.

Often, the degradation processes are monotonies, that is, degradation proceeds in only one direction, as in a wear-out process for example. The application of a Wiener process to this- kind of degradation process is the only approximate. However, when observed closely, the levels of many degradation processes vary bi-directionally over time as, for example, with the gain of a transistor or the extent of propagation delay. Other stochastic processes, such as the gamma processes, may be considered if it is essential to represent degradation by a strictly monotonies process. (Lu et al.,1995)

In this thesis, it is assumed that the degradation process of interest is a continuous process and, in many applications, this is a valid assumption. Where a degradation process is discrete and an approximation is not permitted, another type of stochastic process, such as a discrete-state Markov process, may also be considered.

A Wiener process is a time homogeneous process but not all degradation processes have this property. For example, in reliability engineering, acceleration tests are often used to obtain lifetime and the degradation data in a relatively short period of time. The stresses applied in these tests may be increased during the course of the testing in order to bring about rapid failure. Because the degradation parameters change as the stress level increases, the degradation process becomes time heterogeneous. As a second example, the physical mechanism that governs the deterioration may tend to accelerate or decelerate degradation, as in crack propagation for instance, producing time heterogeneity.

Modeling a degradation process by a Wiener process implies that degradation process, given its current state, evolves to a future state independently of its past behavior. This is referred to as its Markov property. While the Markov property is a valid assumption in many applications, it does not always hold.

4.1.2 Simple Inverse Gaussian process

An inverse Gaussian process $\{Y(t);\ t \geq 0\}$ with mean function $\Lambda(t)$ and scale parameter λ has the following properties:

(i) Y(t) has independent increments: for every pair of disjoint intervals $(t_1, t_2), (t_3, t_4)$ with $t_1 < t_2 < t_3 < t_4$, the random variables $Y(t_2) - Y(t_1)$ and $Y(t_4) - Y(t_3)$ are independent

(ii) Each increment Y(t)-Y(s) has an inverse Gaussian distribution $IG(\Delta\Lambda(t), \lambda\, \Delta\Lambda(t)^2)$ where $\Delta\Lambda = \Lambda(t) - \Lambda(s)$ and the PDF of an inverse Gaussian distribution random variable $IG(\mu, \lambda)$ with mean μ and variance $\frac{\mu^3}{\lambda}$ (Chikkara and Folks, 1989) is

$$ƒ(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2л}}\,x^{-\frac{3}{2}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \quad x>0 \tag{4.2}$$

(iii) Y(0)=0 with probability one

When the amount of degradation reaches a pre-specified critical level D, failure occurs. Let T=Inf{t:Y(t)=D} denote the failure time. Since the inverse Gaussian process has a monotone path, the failure time distribution by

P(T<t)= P(Y(t)>D)= 1-G($D;\ \Lambda(t), \lambda\,\Lambda(t)^2$)

$$=\Phi\left[\sqrt{\frac{\lambda}{D}}(\Lambda(t)-D)\right] - e^{2\lambda\Lambda(t)}\Phi\left[-\sqrt{\frac{\lambda}{D}}(\Lambda(t)+D)\right] \tag{4.3}$$

Where G (.; Λ, λ) is the cumulative distribution function (CDF) of IG(Λ, λ) and Φ is the standard normal cdf. From above equation we can write the CDF of the failure time distribution as

$$H_\lambda(t) = \Phi\left[\sqrt{\frac{\lambda}{D}}(t-D)\right] - e^{2\lambda t}\Phi\left[-\sqrt{\frac{\lambda}{D}}(t+D)\right] \tag{4.4}$$

It is an increasing function. Thus, within this class of models, there is a one to one relationship between $\Lambda(t)$ and the cdf of the failure time distribution $H_\lambda\Lambda(t)$ for a fixed scale parameter λ.

(iv) $$ƒ(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2л}}\,x^{-\frac{3}{2}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \tag{4.5}$$

Where μ>0 and λ>0 the parameter μ is the mean of the distribution and λ is a scale parameter. (Tweedie) gives three form of above pdf, which he obtained by replacing the set of parameters (μ, λ) by (∝, λ) or (μ, ϕ), or (ϕ, λ) using the relationship given by

$$\mu=\frac{\lambda}{\phi} = (2 \propto)^{\frac{-1}{2}} \tag{4.6}$$

Each of these forms was found useful by him in his investigation of Brownian motion for the colloid particles in a Turoila electrophoretic cell. Both μ and λ are of the same physical dimensions as the random variable X itself; but the parameter $\phi=\frac{\lambda}{\mu}$ is invariant under a scale transformation of X as can be seen from the following relationship:

$$f(x;\ \mu, \lambda) = \mu^{-1} f\left(\frac{x}{\mu}; 1, \phi\right) = \lambda^{-1} f\left(\frac{x}{\lambda};\ \phi, 1\right) \tag{4.7}$$

The probability density can be numerically computed using any of the three forms in above equation as shown above the cumulative distribution function depends essentially on only two variables, which might be taken as $\frac{x}{\mu}$ and ϕ. According, the case μ=1 for the (μ, ϕ) parametric form of above equation could be adopted as a standard form. This has also been obtained as a limiting form of the distribution of the sample size in a Wald's sequential probability ratio test and is sometimes referred to as the standard Wald's distribution.

The shape of the distribution depends on ϕ only; hence ϕ is the shape parameter. The inverse Gaussian density function represents a wide class of distribution, ranging from a highly skewed distribution to a symmetrical one as ϕ varies from 0 to ∞. The density curves shown in figures 2.1 and 2.2 illustrate this property.

These curves are obtained by specifying μ=1 and varying λ or ϕ in the figure 1 and by specifying λ=1 and varying μ or ϕ in fig 2.

The density function is un-imodal, with its mode at

$$\mu\left[\left(1+\frac{9}{4\phi^2}\right)^{\frac{1}{2}}-\frac{3}{2\phi}\right] \qquad (4.8)$$

Application of Inverse Gaussian distribution (Chikkara and Folks, 1989)

1. The first passage time of Brownian motion is distributed as inverse Gaussian, it is logical to use it as a life time model. It is useful in studying the life testing and reliability of a product, device or subcomponent.
2. Inverse Gaussian process is useful as a repair time model.
3. Besides the field of reliability, the inverse Gaussian distribution has been used in a wide range of applications which includes many diverse fields such as cardiology, hydrology, demography, linguistic and finance.
4. Other applications involving skewed distributions of wind energy and agricultural fields.

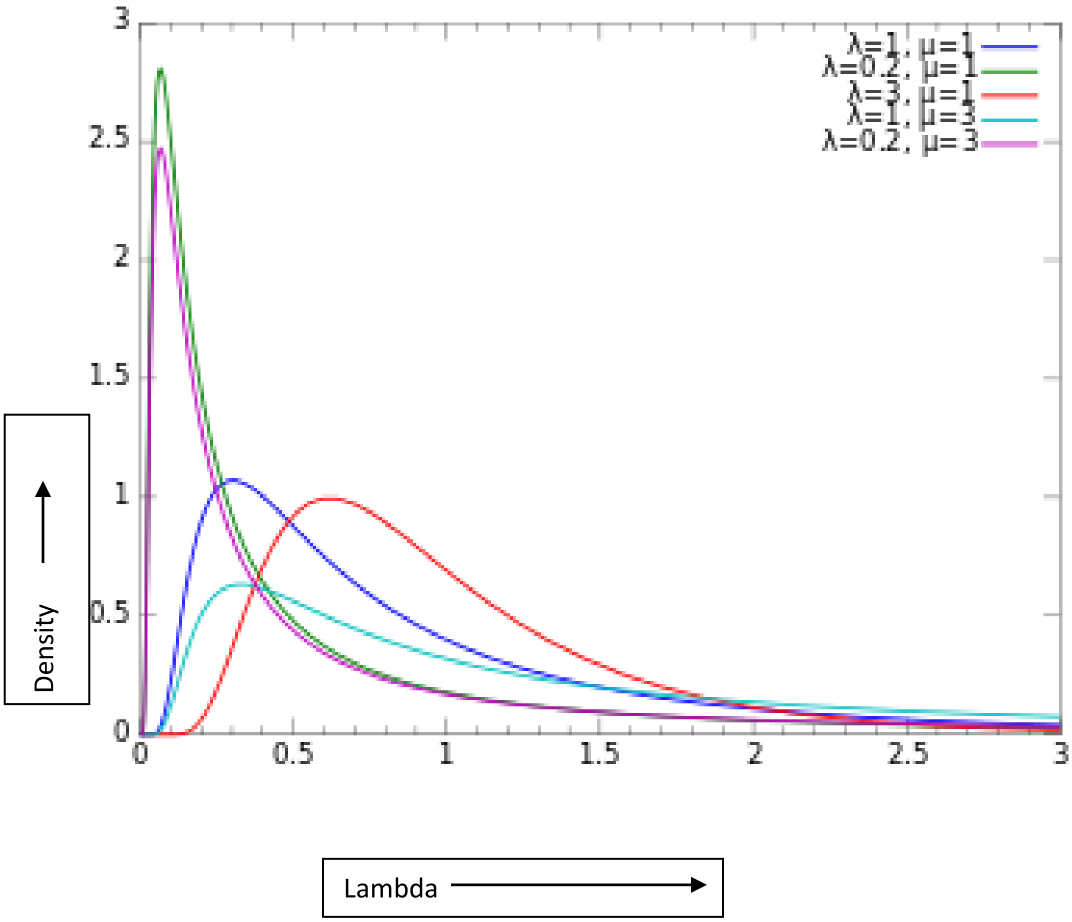


Figure 4.1: Inverse Gaussian Probability density function curve

To predict the reliability of newly developed product engineers adopt accelerated tests in order to shorten the life of the product or accelerate the degradation of their performance. During this test the products are exposed to extreme conditions such as combination of random vibrations, increases temperature, voltage or pressure.

The main purpose of performing such test is to gather reliability information quickly or to save time as well as money. The degradation process is most often hastened under several stresses therefore we can use accelerated degradation test (ADT) to quickly obtain degradation phenomenon. In a simple constant stress ADT experiment no of units are allocated to several stress level and the degradation level of these units are measured, analyzed, and extrapolated to the failure threshold so as to estimate the life characteristics of interest under use conditions.

ADTs are able to greatly shorten the testing duration, and have attracted much attention. These are two classes of models for ADT data.

4.2 Inverse Gaussian process with random effects

Random effects are needed in Inverse Gaussian process to account for unexplained heterogeneous degradation rates within the product population. By linking to the Weiner process this investigates different options to incorporate the random effects in the IG process model. Consider the wiener process W(x) =μ x +λ B(x) where μ>0 is the drift parameter and λ>0 is the volatility parameter and B(x) is the standard Brownian motion. Given a fix threshold Λ>0, it is well known that the first passage time T_A =inf$\{x > 0 \,|\, W(x) \geq \Lambda\}$ follows IG $\left(\frac{\Lambda}{\mu}, \frac{\Lambda^2}{\lambda^2}\right)$. going one step further, we consider a series of the thresholds Λ(t) indexed by t with Λ(0)=0 and Λ(t) increasing in t, and define the first passage time process Y(t)= $T_{\Lambda(t)}$. It is easily verified that the induced $\{Y(t); t > 0\}$ is an IG process with the mean function $\frac{\Lambda(t)}{\mu}$ and variance function $\frac{\Lambda(t)}{\lambda^2}$ by virtue of the stationary and independent increment property of the Wiener process W(x).

The inverse relation between the IG and the Wiener processes motivates investigation of the IG process from a new perspective. Existing results on the Wiener processes can lend support to the development of IG process model with the random effects. The random effect model is described below

4.2.1 Random volatility model

Consider a Wiener process W(x) = $\mu^{-1}x + \lambda^{\frac{-1}{2}}B(x)$ with the induced IG process .other way of introducing unit-specific random effects is to assume that each unit possess a distinct realization of the volatility parameter. Accordingly volatility parameter in the Inverse Gaussian process is random. With the random volatility parameter in the Inverse Gaussian process all units have the same mean degradation path, although they will have different variance functions. The Inverse Gaussian process with random volatility parameter was originally proposed by Wang and Xu(2010)

Shortcoming of random volatility model

It is uncommon to use the volatility parameter to control heterogeneity in the Weiner process thus application of random volatility model is limited. Thus random drift model was proposed which overcome shortcoming of random volatility model.

4.2.2 Random drift model

Consider a Weiner process W(x) = $\mu^{-1}x + \lambda^{\frac{-1}{2}}B(x)$ with the induced IG process Y(t)~ IG(μΛ(t), λ$\Lambda^2(t)$. A common practice to incorporate random effect in Wiener process is to let the drift parameter μ^{-1} vary randomly across units (Crowder and Lawless 2007; Peng and Tseng 2009). An effective way to incorporate random effect in the IG process is to let μ be a random variable. To avoid the negative values of μ (Whitmore 1986) and ensure mathematical tractability, we assume μ^{-1} follows a truncated normal distribution TN (ω, k^{-2}), $k > 0$ with PDF

$$g(\mu^{-1};\ \omega, k^{-2}) = \frac{k.\phi[k(\ \mu^{-1} - \omega)]}{1 - \Phi(-k\omega)} \qquad \mu>0 \tag{4.9}$$

Where (.) is the standard normal PDF.

In a degradation test, if the degradation of the i^th^ testing unit is observed at time $t_{io} < t_{i1} < \cdots . < t_{ini}$ with observations $Y_i\ (\ t_{ij}), j = 0,1,2, \ldots., n_i$ the joint PDF of $Y_i = [Y_i(t_{i1})$, $Y_i(t_{i2}), \ldots . Y_i(t_{ini})$] is computed by first conditioning on the random drift parameter μ_i and then marginalizing it, which yields the following equation

$$f_{IG}(Yi) = \frac{1-\phi(\widetilde{-\omega}_i\tilde{k}_i)}{1-\phi(-k\omega)}\frac{k}{\tilde{k}_i}\prod_{j=1}^{n_i}\sqrt{\frac{\lambda\Lambda_{ij}{}^2}{2\text{л}y_{ij}{}^3}}\frac{\widetilde{k_i}^2\,\widetilde{\omega}_i{}^2 - k^2\omega^2}{2} - \lambda\sum_{j=1}^{n_i}\frac{\Lambda_{ij}{}^2}{2y_{ij}} \tag{4.10}$$

Where y_{ij}=$Y_i\big(t_{ij}\big) - Y_i\big(t_{ij-1}\big)$ is the observed increment $\Lambda_{ij} = \Lambda\big(t_{ij}\big) - \Lambda\big(t_{ij-1}\big)$

$$\tilde{k}_{ij} = \sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} \tag{4.11}$$

$$\widetilde{\omega}_{ij} = \frac{\left[\lambda\Lambda\left(t_{ijK_j}\right)+k^2 \exp\left(\propto_0+\propto_1 x_j\right)\right]}{\tilde{k}_{ij}^{\;2}} \tag{4.12}$$

$$\widetilde{\omega}_{ij} = \frac{\left[\lambda\Lambda\left(t_{ijK_j}\right)+k^2 \exp\left(\propto_0+\propto_1 x_j\right)\right]}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)} \tag{4.13}$$

When the degradation path of N units are observed, the likelihood function is simply given by

$$\prod_{i=1}^{N} f_{IG}(Yi) \tag{4.14}$$

Now we have to taking the log of above function.

$$\text{Log}\left(\prod_{i=1}^{N} f_{IG}(Yi)\right) \tag{4.15}$$

Then the log-likelihood function is given by

$$\text{l}(\theta)=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[ln\frac{k}{\tilde{k}_{ij}}+\frac{\tilde{k}_{ij}^{\;2}\widetilde{\omega}_{ij}^{\;2}-k^2\exp(2\propto_0+2\propto_1 x_j)}{2}+\frac{1}{2}\sum_{K=1}^{K_j}\left[\ln\left(\lambda\partial\Lambda_{ijk}\right)-\frac{\lambda\Lambda_{ijk}^{\;2}}{y_{ijk}}\right]\right] \tag{4.16}$$

$$l(\theta)=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[ln\frac{k}{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2}}+\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)\frac{\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\exp(\propto_0+\propto_1 x_j)\right)^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}-k^2\exp\left(2\propto_0+2\propto_1 x_j\right)}{2}+\frac{1}{2}\sum_{K=1}^{K_j}\left(\ln\left(\lambda\partial\Lambda_{ijk}\right)-\frac{\lambda{\Lambda_{ijk}}^2}{y_{ijk}}\right)\right]$$

$$l(\theta)=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[ln\frac{k}{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2}}+\frac{\frac{\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\exp(\propto_0+\propto_1 x_j)\right)^2}{\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2}-k^2\exp\left(2\propto_0+2\propto_1 x_j\right)}{2}+\frac{1}{2}\sum_{K=1}^{K_j}\left(\ln\left(\lambda\partial\Lambda_{ijk}\right)-\frac{\lambda{\Lambda_{ijk}}^2}{y_{ijk}}\right)\right]$$

(4.17)

The log likelihood function up to a constant can be expressed by the above equation.

Where θ is a parameter vector including, $\propto_0$, $\propto_1$, $\lambda, \beta, and\ k$.

4.3 ADT settings and assumption

Let total N number of units is put into test. Suppose S_0 be the usage stress S_H be the maximum allowable stress. To collect the degradation data timely we allocate these units J stress level $S_1 <$

$S_2 < \cdots \ldots \ldots \ldots < S_J$ with $S_0 < S_1$ and $S_J = S_H$ consider N_j units to be allocated to jth stress level. j=1, 2, 3, ….. J. The degradation of these units is effected by the stress. Here, we have assumed μ_i= h(s), and λ is constant over s, where h(s) is a link function reflecting the effect of the stress on the degradation process. Due to the above assumption the degradation speed and drift changes with the stress (recall that the respective mean or variance for the random drift $Y(t)$ are μ_i Λ(t) and ${\mu_i}^3\Lambda(t)/\lambda$ which agree with the our daily assumption of the physical characteristic of the product).

Another alternative is that λ=h(s) while μ is constant which is not valid for random drift model since μ is changing from unit to unit.

For simplicity and without loss of generality, the additional assumptions are made as follows:

(a) The measurement time interval, and the number of measurement K_j under the j-th stress level, where j =1,2,………J, are pre-determined.

(b) The link function follows one of the following acceleration relations:

Power law relations h(s) = $\xi_0 . s^{\propto}$

Arrhenius relation h(s) = $\xi_0 . e^{\frac{-\propto}{s}}$

Exponential relation h(s) = $\xi_0 . e^{\propto s}$

In real time applications the time allowed for the test is often given by manager and time intervals at which the units are measured are predetermined because of the working time of experimenters. Thus, we assume that τ_j and K_j are given. In our model we treat these two variables as decision variables, and then we optimally determine their values.

When the assumed stress-degradation relation i.e., is correct we can use a two-stress ADT, i.e., J=2 in our model. But, in this minimum variance plan we are unable to check the validity of the assumed stress-degradation relationship.

Thus we prefer to use three-stress ADT planning taking J=3 to check the validity of the assumed model. In our settings, the purpose of ADT planning is to optimally determine the stress levels (S_j), and the number of samples for each stress level (N_j) are o be investigated in our proposed work.

4.4 Normalizing the stress

We standardize the stress levels depending on the acceleration relationship of the stress on the rate of degradation as follows:

$$x_j = \frac{lns_j - lns_0}{lns_H - lns_0}$$ For the power law relation

$$x_j = \frac{\frac{1}{s_0} - \frac{1}{s_j}}{\frac{1}{s_0} - \frac{1}{s_H}}$$ For the Arrhenius relation

$$x_j = \frac{s_j - s_0}{s_H - s_0}$$ For the exponential relation

From the above standardization, it is readily seen that x_0 =0, $x_j = 1$, and 0 <$x_j \leq 1$ for j=1,2….,J. then

$$h(x) = \exp(\propto_0 + \propto_1 x_j)$$

$$h(x) = exp\left[ln\xi_0 - \frac{\propto}{S_0} + \propto\left(\frac{1}{s_0} - \frac{1}{S}\right)\right]$$

$$h(x) = exp\left[ln\xi_0 - \frac{\propto}{s}\right]$$

$$h(x) = \xi_0 . e^{-\frac{\propto}{s}}$$

$$lnh(x) = ln\xi_0 - \frac{\propto}{s}$$

$$\omega_{j=} \exp(\propto_0 + \propto_1 x_j) \tag{4.18}$$

Where

$\propto_0 = ln\xi_0 - \frac{\propto}{s_0}$, $\propto_1 = \propto \left(\frac{1}{s_0} - \frac{1}{s_H}\right)$ For the Arrhenius function

$\propto_0 = ln\xi_0 + \propto lns_0$, $\propto_1 = \propto (lns_H - lns_0)$ For the power law function

$\propto_0 = ln\xi_0 + \propto s_0$, $\propto_1 = \propto (s_H - s_0)$ For the exponential function

4.5 Statistical inference

We suppose that the i-th unit under the j-th stress level is measured at time t_{ijk} =kτ_j with observations Y_{ij} (t_{ijk}), k=0,1,……..,k_j . Let y_{ijk} =Y_{ij} $(t_{ijk}) - Y_{ij}$ $(t_{ij,k-1})$ be the observed increments, and $\Lambda_{ijk} = \Lambda(t_{ijk}) - \Lambda(t_{ij,k-1})$. Now, the log-likelihood function up to a constant can be expressed by the equation above 1. The Fisher information matrix I(θ) for the element $\propto_0, \propto_1, k, \omega, \Lambda(.)$ can be developed as below. We assume nonlinear function for $\Lambda(.)$, i.e., Λ(t)= t^{β} and then θ=(k, ω, $\propto_0$, $\propto_1$, β)' detailed expression for the elements along with the Fisher information matrix can be found below

The elements of the fisher information matrix can be developed as follows:

$$\frac{\partial l(\theta)}{\partial \omega_j} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[0 + \frac{1}{2} \left\{ \frac{2\left(\lambda\Lambda\left(t_{ijK_j}\right) + k^2\omega_j\right)k^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} - 2k^2\omega_j \right\} + \frac{1}{2}\sum_{K=1}^{K_j}(0-0) \right]$$

$$\frac{\partial l(\theta)}{\partial \omega_j} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\frac{1}{2} \left\{ \frac{2\left(\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega_j\right) k^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} - 2k^2 \omega_j \right\} \right]$$

$$\frac{\partial l(\theta)}{\partial \omega_j} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\frac{k^2 (\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega_j)}{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} - k^2 \omega_j \right] \tag{4.19}$$

$$\frac{\partial^2 l(\theta)}{\partial {\omega_j}^2} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\left(\frac{-k^2 (0 + k^2)}{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} - k^2 \right) \right]$$

$$\frac{\partial^2 l(\theta)}{\partial {\omega_j}^2} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\left(\frac{k^4 - k^4 - k^2 \left(\lambda Y_{ij}\left(t_{ijK_j}\right)\right)}{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} \right) \right]$$

$$\frac{\partial^2 l(\theta)}{\partial {\omega_j}^2} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left(\frac{-k^2 \lambda Y_{ij}\left(t_{ijK_j}\right)}{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} \right) \tag{4.20}$$

$$\frac{\partial l(\theta)}{\partial k} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\left(\frac{1}{\frac{k}{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2}}} \frac{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} - k \frac{2k}{2\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2}}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} \right) + \right.$$

$$\left. \frac{1}{2} \left\{ \left(\frac{2\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)\left(\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega_j\right) k\omega - \left(\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega_j\right)^2 (0 + 2k)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)^2} \right) - 2k{\omega_j}^2 \right\} + 0 \right]$$

$$\frac{\partial l(\theta)}{\partial k}$$

$$= \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\left(\frac{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)}\left\{\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)-k^2}{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2}}\right\}\right)\right.$$

$$\left.+\frac{1}{2}\left\{\left(\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)\left(\lambda \Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)2k\omega-\left(\lambda \Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)^2 2k}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}\right)-2k{\omega_j}^2\right\}\right]$$

$$\frac{\partial l(\theta)}{\partial k} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{\lambda Y_{ij}\left(t_{ijK_j}\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)}+\right.$$

$$\left.\frac{\left(\lambda \Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)\left\{\left(2k\omega\lambda Y_{ij}\left(t_{ijK_j}\right)\right)+2k^3\omega_j-k\lambda\Lambda\left(t_{ijK_j}\right)-k^3\omega_j\right\}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}-k{\omega_j}^2\right] \quad (4.21)$$

$$\frac{\partial^2 l(\theta)}{\partial k^2} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\left(\frac{-2k\lambda Y_{ij}\left(t_{ijK_j}\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}\right)+\right.$$

$$\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2\left\{2\omega_j\lambda^2\Lambda\left(t_{ijK_j}\right)Y_{ij}\left(t_{ijK_j}\right)+6k^2{\omega_j}^2\lambda Y_{ij}\left(t_{ijK_j}\right)+6k^2\omega_j-\lambda\Lambda\left(t_{ijK_j}\right)-3k^2\omega_j\right\}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^4}+$$

$$\frac{3k^3+4k\lambda Y_{ij}\left(t_{ijK_j}\right)\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)(2k\omega_j\lambda)Y_{ij}\left(t_{ijK_j}\right)+2k^3\omega_j-k\lambda\Lambda\left(t_{ijK_j}\right)-k^3\omega_j}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^4} -$$

$$\left.\omega_j{}^2\right] \qquad (4.22)$$

$$\frac{\partial l(\theta)}{\partial\lambda} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{1}{\frac{k}{\sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2}}} - \frac{k}{2}\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^{-\frac{3}{2}}.Y_{ij}\left(t_{ijK_j}\right) + \right.$$

$$\frac{1}{2}\left\{\frac{2\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)\Lambda\left(t_{ijK_j}\right)-\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)^2 Y_{ij}\left(t_{ijK_j}\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}\right\} +$$

$$\left.\frac{1}{2}\sum_{K=1}^{K_j}\left(\frac{1}{\lambda\Lambda_{ijk}{}^2}\Lambda_{ijk}{}^2 - \frac{\Lambda_{ijk}{}^2}{y_{ijk}}\right)\right] \frac{\partial l(\theta)}{\partial\lambda} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{-1}{2}\frac{Y_{ij}\left(t_{ijK_j}\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)} + \right.$$

$$\frac{1}{2}\left\{\frac{\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)2\Lambda\left(t_{ijK_j}\right)-Y_{ij}\left(t_{ijK_j}\right)\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega_j\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}\right\}2 +$$

$$\left.\frac{1}{2}\sum_{K=1}^{K_j}\left(\frac{1}{\lambda} - \frac{\Lambda_{ijk}{}^2}{y_{ijk}}\right)\right] \qquad (4.23)$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda^2} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{1}{2}\frac{Y_{ij}\left(t_{ijK_j}\right).Y_{ij}\left(t_{ijK_j}\right)}{(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^2} + \right.$$

$$\frac{1}{2}\left\{\frac{2\Lambda\left(t_{ijK_j}\right)\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)\Lambda\left(t_{ijK_j}\right)-(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega)Y_{ij}\left(t_{ijK_j}\right)}{(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^2}\right\} -$$

$$\left\{\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2 Y_{ij}\left(t_{ijK_j}\right)\Lambda\left(t_{ijK_j}\right)-(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega)\left(2\lambda {Y_{ij}}^2\left(t_{ijK_j}\right)+2Y_{ij}\left(t_{ijK_j}\right)k^2\right)}{(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^4}\right\}+$$

$$\left.\frac{1}{2}\sum_{K=1}^{K_j}\left(\frac{-1}{\lambda^2}-0\right)\right]$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda^2}=\sum_{j=1}^{J}\sum_{i=1}^{N_J}\left[\frac{{Y_{ij}}^2(t_{ijK_j})}{2(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^2}+\right.$$

$$\frac{1}{2}\left\{\frac{2\Lambda^2\left(t_{ijK_j}\right)\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)-Y_{ij}\left(t_{ijK_j}\right)(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega)}{(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^2}\right\}-$$

$$\left\{\frac{Y_{ij}\left(t_{ijK_j}\right)\Lambda\left(t_{ijK_j}\right)(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^2-2\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega\right)(\lambda {Y_{ij}}^2\left(t_{ijK_j}\right)+(Y_{ij}\left(t_{ijK_j}\right)+k^2)}{(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2)^4}\right\}+$$

$$\left.\frac{1}{2}\sum_{K=1}^{K_J}-\frac{1}{\lambda^2}\right] \tag{4.24}$$

$$\frac{\partial l(\theta)}{\partial\beta}=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[0+\frac{1}{2}\left\{\frac{2\left(\lambda\Lambda_{ijk}+k^2\omega_j\right)}{\left(\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)\right.}\ \lambda\frac{\partial\Lambda_{ijk}}{\partial\beta}-0\right\}+\right.$$

$$\left.\frac{1}{2}\sum_{K=1}^{K_j}\left\{\frac{1}{\lambda{\Lambda_{ijk}}^2}2\lambda\Lambda_{ijk}\frac{\partial\Lambda_{ijk}}{\partial\beta}-\frac{2\lambda\Lambda_{ijk}}{y_{ijk}}\frac{\partial\Lambda_{ijk}}{\partial\beta}\right\}\right]$$

$$\frac{\partial l(\theta)}{\partial\beta}=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\left\{\lambda\frac{(\lambda\Lambda_{ijk}+k^2\omega_j)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)}\frac{\partial\Lambda_{ijk}}{\partial\beta}\right\}+\frac{1}{2}\sum_{K=1}^{K_j}\left(\frac{2}{\Lambda_{ijk}}\frac{\partial\Lambda_{ijk}}{\partial\beta}-\frac{2\lambda\Lambda_{ijk}}{y_{ijk}}\frac{\partial\Lambda_{ijk}}{\partial\beta}\right)\right]$$

$$\frac{\partial l(\theta)}{\partial\beta}=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\left(\frac{\partial\Lambda_{ijk}}{\partial\beta}\left\{\frac{\lambda(\lambda\Lambda_{ijk}+k^2\omega_j)}{\left(\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)\right)}\right\}\right)+\sum_{K=1}^{K_j}\left(\frac{1}{\Lambda_{ijk}}-\frac{2\lambda\Lambda_{ijk}}{y_{ijk}}\right)\frac{\partial\Lambda_{ijk}}{\partial\beta}\right] \tag{4.25}$$

$$\frac{\partial^2 l(\theta)}{\partial \beta^2} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\left\{ \frac{\partial \Lambda_{ijk}}{\partial \beta} \frac{\lambda\left(\lambda \frac{\partial \Lambda_{ijk}}{\partial \beta} + 0\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} + \left(\frac{\lambda\left(\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} \right) \frac{\partial^2 \Lambda_{ijk}}{\partial \beta^2} \right\} + \right.$$

$$\left. \sum_{K=1}^{K_j} \left\{ \frac{\partial^2 \Lambda_{ijk}}{\partial \beta^2} \left(\frac{1}{\Lambda_{ijk}} - \frac{2\lambda \Lambda_{ijk}}{y_{ijk}} \right) + \left(-\frac{1}{\Lambda_{ijk}{}^2} \frac{\partial \Lambda_{ijk}}{\partial \beta} - \frac{2\lambda}{y_{ijk}} \frac{\partial \Lambda_{ijk}}{\partial \beta} \right) \frac{\partial \Lambda_{ijk}}{\partial \beta} \right\} \right]$$

$$\frac{\partial^2 l(\theta)}{\partial \beta^2} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\left\{ \frac{\lambda^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} \left(\frac{\partial \Lambda_{ijk}}{\partial \beta} \right)^2 + \left(\frac{\lambda\left(\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} \right) \frac{\partial^2 \Lambda_{ijk}}{\partial \beta^2} \right\} + \right.$$

$$\left. \sum_{K=1}^{K_j} \left\{ \frac{\partial^2 \Lambda_{ijk}}{\partial \beta^2} \left(\frac{1}{\Lambda_{ijk}} - \frac{2\lambda \Lambda_{ijk}}{y_{ijk}} \right) + \left(-\frac{1}{\Lambda_{ijk}{}^2} - \frac{2\lambda}{y_{ijk}} \right) \left(\frac{\partial \Lambda_{ijk}}{\partial \beta} \right)^2 \right\} \right]$$

$$\frac{\partial^2 l(\theta)}{\partial \beta^2} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\left(\frac{\lambda \left(\frac{\partial \Lambda\left(t_{ijK_j}\right)}{\partial \beta} \right)^2 + \frac{\partial^2 \Lambda\left(t_{ijK_j}\right)}{\partial \beta^2} \left(\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \omega\right)}{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} \right) + \right.$$

$$\left. \sum_{K=1}^{K_j} \left\{ \left(\frac{\partial \Lambda_{ijk}}{\partial \beta} \right)^2 \left(-\frac{1}{\Lambda_{ijk}{}^2} - \frac{2\lambda}{y_{ijk}} \right) + \frac{\partial^2 \Lambda_{ijk}}{\partial \beta^2} \left(\frac{1}{\Lambda_{ijk}} - \frac{2\lambda \Lambda_{ijk}}{y_{ijk}} \right) \right\} \right]$$

(4.26)

$$\frac{\partial^2 l(\theta)}{\partial k \partial \beta} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\frac{\left(\lambda \frac{\partial \Lambda\left(t_{ijK_j}\right)}{\partial \beta} \right) \left\{ \left(2k\omega \lambda Y_{ij}\left(t_{ijK_j}\right) \right) + 2k^3 \omega_j - k\lambda \Lambda\left(t_{ijK_j}\right) - k^3 \omega_j \right\}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)^2} - \right.$$

$$\left. \frac{k\lambda \frac{\partial \Lambda\left(t_{ijK_j}\right)}{\partial \beta}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)} \right] \qquad (4.27)$$

$$\frac{\partial^2 l(\theta)}{\partial\propto_0 \partial\propto_1} = \sum_{j=1}^{J} \left[x_j \exp\left(\propto_0 + \propto_1\ x_j\right) \frac{\partial l(\theta)}{\partial\omega_j} + \exp\left(\propto_0 + \propto_1\ x_j\right) \frac{\partial^2 l(\theta)}{\partial{\omega_j}^2} x_j \right]$$

$$\frac{\partial^2 l(\theta)}{\partial\propto_0 \partial\propto_1} = \sum_{j=1}^{J} x_j \exp(\propto_0 + \propto_1\ x_j) \left(\frac{\partial l(\theta)}{\partial\omega_j} + \frac{\partial^2 l(\theta)}{\partial{\omega_j}^2}\right) \qquad (4.28)$$

$$\frac{\partial^2 l(\theta)}{\partial{\propto_0}^2} = \sum_{j=1}^{J} \left[exp\left(\propto_0 + \propto_1\ x_j\right) \frac{\partial l(\theta)}{\partial\omega_j} + exp\left(\propto_0 + \propto_1\ x_j\right) \frac{\partial^2 l(\theta)}{\partial{\omega_j}^2} \right]$$

$$\frac{\partial^2 l(\theta)}{\partial{\propto_0}^2} = \sum_{j=1}^{J} \exp\left(\propto_0 + \propto_1\ x_j\right) \left[\frac{\partial l(\theta)}{\partial\omega_j} + \frac{\partial^2 l(\theta)}{\partial{\omega_j}^2}\right] \qquad (4.29)$$

$$\frac{\partial^2 l(\theta)}{\partial\ {\propto_1}^2} = \sum_{j=1}^{J} \left[x_j exp\left(\propto_0 + \propto_1\ x_j\right) \frac{\partial l(\theta)}{\partial\omega_j} + x_j exp\left(\propto_0 + \propto_1\ x_j\right) \frac{\partial^2 l(\theta)}{\partial{\omega_j}^2} \right]$$

$$\frac{\partial^2 l(\theta)}{\partial{\propto_1}^2} = \sum_{i=1}^{J} {x_j}^2 \exp(\propto_0 + \propto_1\ x_j) \left[\frac{\partial l(\theta)}{\partial\omega_j} + \frac{\partial^2 l(\theta)}{\partial{\omega_j}^2}\right] \qquad (4.30)$$

$$\frac{\partial^2 l(\theta)}{\partial\propto_0 \partial\beta} = \sum_{j=1}^{J} \left[\exp\left(\propto_0 + \propto_1\ x_j\right) . \frac{\partial^2 l(\theta)}{\partial\omega_j \partial\beta}\right] \qquad (4.31)$$

$$\frac{\partial^2 l(\theta)}{\partial\propto_1 \partial\beta} = \sum_{i=1}^{J} x_j \exp(\propto_0 + \propto_1\ x_j) \frac{\partial^2 l(\theta)}{\partial\omega_j \partial\beta} \qquad (4.32)$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda \partial\beta} = \sum_{j=1}^{J} \sum_{i=1}^{N_j} \left[\frac{1}{2} \left\{ \frac{2\left(2\lambda\Lambda\left(t_{ijK_j}\right)\frac{\partial\Lambda\left(t_{ijK_j}\right)}{\partial\beta} + k^2\omega\frac{\partial\Lambda\left(t_{ijK_j}\right)}{\partial\beta}\right)}{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2} - \frac{Y_{ij}\left(t_{ijK_j}\right)\lambda\frac{\partial\Lambda\left(t_{ijK_j}\right)}{\partial\beta}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right) + k^2\right)^2} \right\} + \right.$$

$$\left. \frac{1}{2} \sum_{K=1}^{K_j} \left(-\frac{2\Lambda_{ijk}}{y_{ijk}} \frac{\partial\Lambda_{ijk}}{\partial\beta}\right) \right] \qquad (4.33)$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda\partial\omega_j} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{1}{2}\left(\frac{2\Lambda\left(t_{ijK_j}\right)k^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)} - \frac{Y_{ij}\left(t_{ijK_j}\right)k^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}\right)\right] \tag{4.34}$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda\partial\propto_0} = \sum_{j=1}^{J}\exp\left(\propto_0+\propto_1\ x_j\right)\frac{\partial^2 l(\theta)}{\partial\lambda\partial\omega_j}$$

(4.35)

$$\frac{\partial^2 l(\theta)}{\partial\lambda\partial\propto_1} = \sum_{j=1}^{J}x_j\exp\left(\propto_0+\propto_1\ x_j\right)\frac{\partial^2 l(\theta)}{\partial\lambda\partial\omega_j}$$

(4.36)

$$\frac{\partial^2 l(\theta)}{\partial k\partial\omega_j} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{\left(2k\lambda^2\Lambda\left(t_{ijK_j}\right)Y_{ij}\left(t_{ijK_j}\right)+4k^3\omega_j\lambda Y_{ij}\left(t_{ijK_j}\right)+k^3\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2} - 2k\omega_j\right]$$

(4.37)

$$\frac{\partial^2 l(\theta)}{\partial k\partial\lambda} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\left(\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)Y_{ij}\left(t_{ijK_j}\right)-\lambda Y_{ij}\left(t_{ijK_j}\right)^2}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^2}\right)+\right.$$
$$\left(\frac{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^4\left\{4k\omega\lambda\Lambda\left(t_{ijK_j}\right)Y_{ij}\left(t_{ijK_j}\right)+2k^3\omega_j{}^2Y_{ij}\left(t_{ijK_j}\right)-k\Lambda\left(t_{ijK_j}\right)\right\}}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^4}\right)-$$
$$\left.\left(\frac{\left\{\left(\lambda\Lambda\left(t_{ijK_j}\right)+k^2\omega\right)2k\omega_j\lambda Y_{ij}\left(t_{ijK_j}\right)+k^3\omega_j-k\lambda\Lambda\left(t_{ijK_j}\right)\right\}\left(2\lambda Y_{ij}\left(t_{ijK_j}\right)^2+2Y_{ij}\left(t_{ijK_j}\right)k^2\right)}{\left(\lambda Y_{ij}\left(t_{ijK_j}\right)+k^2\right)^4}\right)\right] \tag{4.38}$$

$$\frac{\partial^2 l(\theta)}{\partial k \partial \propto_0} = \sum_{j=1}^{J} \exp\left(\propto_0 + \propto_1 \, x_j\right) \frac{\partial^2 l(\theta)}{\partial k \partial \omega_j} \tag{4.39}$$

$$\frac{\partial^2 l(\theta)}{\partial k \partial \propto_1} = \sum_{j=1}^{J} x_j \exp\left(\propto_0 + \propto_1 \, x_j\right) \frac{\partial^2 l(\theta)}{\partial k \partial \omega_j} \tag{4.40}$$

$$\frac{\partial l(\theta)}{\partial \propto_0} = \sum_{j=1}^{J} \exp\left(\propto_0 + \propto_1 \, x_j\right) \frac{\partial l(\theta)}{\partial \omega_j} \tag{4.41}$$

$$\frac{\partial l(\theta)}{\partial \propto_1} = \sum_{j=1}^{J} x_j \exp\left(\propto_0 + \propto_1 \, x_j\right) \frac{\partial l(\theta)}{\partial \omega_j} \tag{4.42}$$

And then the fisher information matrix can be developed as given below:

.

$$\begin{matrix}
E\left[-\frac{\partial^2 l(\theta)}{\partial {\propto_0}^2}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial \propto_1}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial k}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial \lambda}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial \beta}\right] \\
E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial \propto_1}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial {\propto_1}^2}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_1 \partial k}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_1 \partial \lambda}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_1 \partial \beta}\right] \\
E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial k}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_1 \partial k}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial K^2}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial k \partial \lambda}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial k \partial \beta}\right] \\
E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial \lambda}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_1 \partial \lambda}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial k \partial \lambda}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \lambda^2}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \lambda \partial \beta}\right] \\
E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_0 \partial \beta}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \propto_1 \partial \beta}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial k \partial \beta}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \lambda \partial \beta}\right] & E\left[-\frac{\partial^2 l(\theta)}{\partial \beta^2}\right]
\end{matrix} \tag{4.43}$$

The log-likelihood function can be maximized to obtain maximum likelihood estimator MLEs. The direct maximization of log-likelihood function gives equations which are computationally difficult to solve. Under the truncated normal distribution, direct maximization of the likelihood function often yields a solution far away from the MLE.

In the following we use the EM algorithm (Dempster, Laird, and Rubin, 1977). Expectation–maximization (EM) algorithm is an iterative method for finding the maximum likelihood estimates of parameters in statistical models, where the model depends on unobserved latent variables The It is used to find (locally) the maximum likelihood parameters of a statistical model in cases where the equations cannot be solved directly. Typically these models involve latent variables in addition to the unknown parameters and known data observations. That is, either there are missing values among the data, or the model can be formulated more simply by assuming the existence of the additional unobserved data points. For example, a mixture model can be described more simply by assuming that each observed data point has a corresponding unobserved data point, or latent variable, specifying the mixture component that each data point belongs to.

The EM iteration alternates between performing an expectation (E) step, which creates a function for the expectation of the log-likelihood evaluated using the current estimate for the parameters, and maximization (M) step, which computes the parameters maximizing the expected log-likelihood found on the *E* step. These parameter-estimates are then used to determine the distribution of the latent variables in the next E step.

4.5.1 EM algorithm for parameter estimation

For simplicity we consider parametric Λ (t) in this part. We denote t_{ik} , k=0,1,1,…k_j. As the observation time of the degradation path I, for i=1,2,…….N. Consequently we, denote the degradation increment as $y_{ijk} = Y_i(t_{ijk}) - Y_i(t_{ijk-1})$ and the increment of the shape function as $\Lambda_{ijk} = \Lambda(t_{ijk}) - \Lambda\ (t_{ijk-1}\ )$ .The shape function has unknown parameter p. In the EM framework, Y=$\{\, y_{ijk},\ i = 1,2 \ldots N,\ j = 1, \ldots J,\ k = 1 \ldots \ldots K_j \}$ is the observed data, and the realization of the random parameters in each degradation path are considered as random data.

4.5.1.1 EM algorithm for random drift model

In the random drift model, we denote μ= $\{\mu_1, \mu_2, \ldots, \mu_N$ as the unobserved random effects realization .Given the complete data including Y and μ, log-likelihood (up to a constant) can be expressed as

$$E(l\theta)=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[\frac{ln\lambda}{2}-\frac{\lambda}{2}\left(v_{ij}y_{ijk}-2u_{ij}\Lambda_{ijk}+\frac{\Lambda_{ijk}{}^2}{2y_{ij}}\right)+ ln\Lambda_{ijk}\right]+\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[lnk-\ln(1-\phi(-\omega k)-\frac{k^2}{2}\left(v_{ij}-2u_{ij}\omega+\omega^2\right)\right] \quad (4.44)$$

The unknown parameters in l(θ) in the shape function λ, p and ω, k of the random effects distribution . We use Θ to denote all the parameters. And Θ^k denotes the estimated parameters at the k^{th} iteration.

In the E-step, we want to compute the expectation of given the observed data and the current estimated parameters. In particular we need $\mathrm{E}\left(\mu_i^{-1} \middle| Y, \Theta^k\right)$ and $\mathrm{E}\left(\mu_i^{-2} \middle| Y, \Theta^k\right)$. We know that $\tilde{k}_{ij} = \sqrt{\lambda Y_{ij}\left(t_{ijK_j}\right) + k^{(k)^2}}$ follows truncated normal distribution with parameter and $\widetilde{\omega}_{ij} = \frac{\left[\lambda \Lambda\left(t_{ijK_j}\right) + k^2 \exp(\propto_0 + \propto_1 x_j)\right]}{\tilde{k}_{ij}^{\ 2}}$.

Therefore, we have

$$u_{ij} = \widetilde{\omega}_{ij} + \frac{\phi(-\widetilde{\omega}_{ij}\tilde{k}_{ij})}{[1-\phi(-\widetilde{\omega}_{ij}\tilde{k}_{ij})]\tilde{k}_{ij}} \tag{4.45}$$

$$v_{ij} = \widetilde{\omega}_{ij}^{\ 2} + 2\widetilde{\omega}_{ij}\frac{\phi(-\widetilde{\omega}_{ij}\tilde{k}_{ij})}{[1-\phi(-\widetilde{\omega}_{ij}\tilde{k}_{ij})]\tilde{k}_{ij}} + \left[1 - \frac{\widetilde{\omega}_{ij}\tilde{k}_{ij}\phi(-\widetilde{\omega}_{ij}\tilde{k}_{ij})}{[1-\phi(-\widetilde{\omega}_{ij}\tilde{k}_{ij})]}\right]\frac{1}{\tilde{k}_{ij}^{\ 2}} \tag{4.46}$$

Where $\widetilde{\omega}_{ij}, \widetilde{k}_{ij}$ are updated at each iteration k.

$$\mathrm{E(l\theta)} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[\frac{ln\lambda}{2} - \frac{\lambda}{2}\left(v_{ij}Y_{ijk} - 2u_{ij}\Lambda_{ijk} + \frac{\Lambda_{ijk}^{\ 2}}{2y_{ij}}\right) + ln\Lambda_{ijk}\right] + \sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[lnk - \ln\left(1 - \phi(-\omega_j k)\right) - \frac{k^2}{2}\left(v_{ij} - 2u_{ij}\omega_j + \omega_j^{\ 2}\right)\right] \tag{4.47}$$

Where the first step only depends on λ, p and the second term only depends on ω, k. Therefore, in the M-step, we can find the updated parameter estimates separately. In particular from the first term, we can obtain λ^{k+1} , $p^{(k+1)}$ by solving

For first part differentiating with respect to λ and Λ_{ijK}

Differentiating with respect to λ

$$\sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[\frac{1}{2\lambda}-\frac{1}{2}\left(v_{ij}Y_{ijK}-2u_{ij}\Lambda_{ijK}+\frac{\Lambda_{ijK}{}^{2}}{y_{ijK}}\right)\right]=0$$

$$\sum_{i=1}^{N_j}K_j\frac{1}{2\lambda}=\sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[\frac{1}{2}\left(v_{ij}Y_{ijK}-2u_{ij}\Lambda_{ijK}+\frac{\Lambda_{ijK}{}^{2}}{y_{ijK}}\right)\right]$$

$$\frac{1}{\lambda}=\frac{\sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[v_{ij}Y_{ijK}-2u_{ij}\Lambda_{ijK}+\frac{\Lambda_{ijK}{}^{2}}{y_{ijK}}\right]}{\sum_{i=1}^{N_j}Kj} \quad (4.48)$$

Now differentiating with respect to Λ_{ijk} and putting it equal to zero we get the value of estimatorΛ_{ijk} as follows

$$\sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[\Lambda_{ijk}{}'\frac{1}{\Lambda_{ijk}}+\frac{u_{ij}}{\lambda}\Lambda_{ijk}{}'-\lambda\Lambda_{ijk}\Lambda_{ijk}{}'\frac{1}{y_{ijK}}\right]=0$$

$$\sum_{j=1}^{J}\sum_{i=1}^{N_j}\sum_{K=1}^{K_j}\left[\frac{\partial\Lambda_{ijk}}{\partial\beta}\frac{1}{\Lambda_{ijk}}+\frac{u_{ij}}{\lambda}\frac{\partial\Lambda_{ijk}}{\partial\beta}-\lambda\Lambda_{ijk}\frac{\partial\Lambda_{ijk}}{\partial\beta}\frac{1}{y_{ijK}}\right]=0 \quad (4.49)$$

Now, by using the second part of the logarithmic function l(θ);

Differentiating with respect to k and (-kω)

First differentiating with respect to k

$$\frac{1}{k} - k\left(v_{ij} - 2u_{ij}\omega_j + \omega_j\right) = 0$$

$$\frac{1}{k^2} = \sum_{j=1}^{J}\sum_{i=1}^{N_j}\left[\frac{v_{ij}-u_{ij}\omega_j}{N_j}\right] \tag{4.50}$$

Now differentiating with respect to $\left(-\omega_j k\right)$

$$0-\frac{1}{1-\phi\left(-\omega_j k\right)}.-\phi\left(-\omega_j k\right).-k-\frac{k^2}{2}\left(-2u_{ij}+2\omega\right)=0$$

$$-\frac{\phi\left(-\omega_j k\right)}{1-\phi\left(-\omega_j k\right)} = k(-u_{ij}+\omega)$$

$$\frac{\phi\left(-\omega_j k\right)}{1-\phi\left(-\omega_j k\right)} = k(u_{ij}-\omega)$$

$$\phi\left(-\omega_j k\right) = k(u_{ij}-\omega)\left[1-\phi\left(-\omega_j k\right)\right]$$

$$N_j\phi\left(-\omega_j k\right) = \left(\sum_{j=1}^{J}\sum_{i=1}^{N_j}\left(u_{ij}-\omega_j N_j\right)\right)k\left[1-\phi\left(-\omega_j k\right)\right] \tag{4.51}$$

Equation 4.44 is divided into two parts. First part contains the elements of λ and β and the second part contains k and ω of the truncated normal distributions. Then we have separately optimized both the parts to find the value of the estimator in an iterative manner.

4.6 Optimal ADT planning

An ADT experiment is characterized by the total number of test units available, the number of stress levels used in the test, as well as the stress value of each level, the allocation scheme of the testing units to each stress level, the test duration, and the measurement time interval. Following the assumptions, we have assumed that the number of units, the test duration, and the measurement time interval are already provided. Therefore, the objective of the ADT planning is to determine the optimal stress levels, as well as the proportion of units allocated to each level based on some optimization criterion. Usually, we are concerned with a small quantile (ξ_p) of the time to failure under normal use conditions. Therefore, our objective here is to minimize Avar($\hat{\xi}_p$) the asymptotic variance of $\hat{\xi}_p$.

For the simple inverse Gaussian process p-quantile is given as

$$\xi_p = \Lambda^{-1}\left[\frac{\mu}{4\lambda}\left(z_p + \sqrt{{z_p}^2 + \frac{4D\lambda}{\mu^2}}\right)^2\right] \tag{4.52}$$

Where μ is the mean of the process λ is the scale parameter of the process. z_p is the standard normal p-quantile and Λ^{-1} is inverse function of Λ(.) .

Since for the random drift model mean and variance is specified as μ_i Λ(t) and ${\mu_i}^3\Lambda(t)/\lambda$. To compute the distribution of first passage time to failure threshold D, we use the normal distribution to approximate then the CDF of T_D is given by

$$F_{T_D}(t) = F_{z_p}\left[\frac{\mu_i\,\Lambda(\mathrm{t}) - \mathrm{D}}{\sqrt{{\mu_i}^3\Lambda(t)/\lambda}}\right] \tag{4.53}$$

Where z_p is the standard normal p-quantile

When Λ (t)= t^β the p-quantile (ξ_p) under the specified condition for random drift can be described as below:

$$\xi_p = \left[\frac{\exp(\propto_0 + \propto_1 x_j)}{4\lambda}\left(z_p + \sqrt{{z_p}^2 + 4D\lambda}\exp -2\left(\propto_0 + \propto_1 x_j\right)\right)^2\right]^{\frac{1}{\beta}} \tag{4.54}$$

The estimate of ξ_p can be obtained by substituting the MLE of θ into the equation (3.54). The asymptotic variance of $\hat{\xi}_p$ can be obtained based on the delta method as

$$\mathrm{Avar}(\hat{\xi}_p) = (\nabla\xi_p)'\, I^{-1}(\theta)\, (\nabla\xi_p) \tag{4.55}$$

Where $\nabla\xi_p$ is the first derivative of ξ_p with respect to the θ, which is given by

$$\nabla\xi_p = \left(\frac{\partial\xi_p}{\partial\propto_0}, \frac{\partial\xi_p}{\partial\propto_1}, \frac{\partial\xi_p}{\partial k}, \frac{\partial\xi_p}{\partial\lambda}, \frac{\partial\xi_p}{\partial\beta}\right)' \tag{4.56}$$

The element of $\nabla\xi_p$ can be derived as below:

$$\frac{\partial\xi_p}{\partial\propto_0} = \frac{1}{2\beta}\left(\frac{1}{4}\left(\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}}\right)\right)^{\frac{2}{\beta}-1} \left[\frac{1}{2}\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \frac{1}{2}\left(\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}\right)^{\frac{-1}{2}} \left(\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 - \frac{4D}{\exp(\propto_0+\propto_1 x_j)}\right)\right] \tag{4.57}$$

$$\frac{\partial\xi_p}{\partial\propto_1} = \frac{1}{2\beta}\left(\frac{1}{4}\left(\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}}\right)\right)^{\frac{2}{\beta}-1} \left[\frac{1}{2}x_j\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \frac{1}{2}\left(\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}\right)^{\frac{-1}{2}} \left(\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 x_j - \frac{4D}{\exp(\propto_0+\propto_1 x_j)} x_j\right)\right] \tag{4.58}$$

$$\frac{\partial\xi_p}{\partial k} = 0 \tag{4.59}$$

$$\frac{\partial \xi_p}{\partial \lambda} = \frac{1}{2\beta}\left(\frac{1}{4}\left(\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}}\right)\right)^{\frac{2}{\beta}-1} \left[\frac{1}{2\lambda}\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\frac{1}{2}\left(\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}\right)^{\frac{-1}{2}}\left(-\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda^2} {z_p}^2\right)\right] \qquad (4.60)$$

$$\frac{\partial \xi_p}{\partial \beta} = \frac{-2}{\beta^2}\left(\frac{1}{4}\left(\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}}\right)\right)\left(\frac{1}{4}\left(\sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda}}\ z_p + \sqrt{\frac{\exp(\propto_0+\propto_1 x_j)}{\lambda} {z_p}^2 + \frac{4D}{\exp(\propto_0+\propto_1 x_j)}}\right)\right)^{\frac{2}{\beta}} \qquad (4.61)$$

With the asymptotic variance in hand, the optimization problem can be formulated as follows

Minimize Avar($\hat{\xi}_p$)

Subjected to $0 \leq x_j \ll 1,\ j = 1,2, \dots, J$

$x_M = x_H$ and $x_0 \leq x_j \leq x_H, j = 1,2, \dots, J$

$\sum_{j=1}^{J} N_j = N,$

$0 < N_j \leq N, j = 1,2, \dots, J$ (4.62)

To solve the above mixed integer programming problem, some software package such as Mat lab can be used.

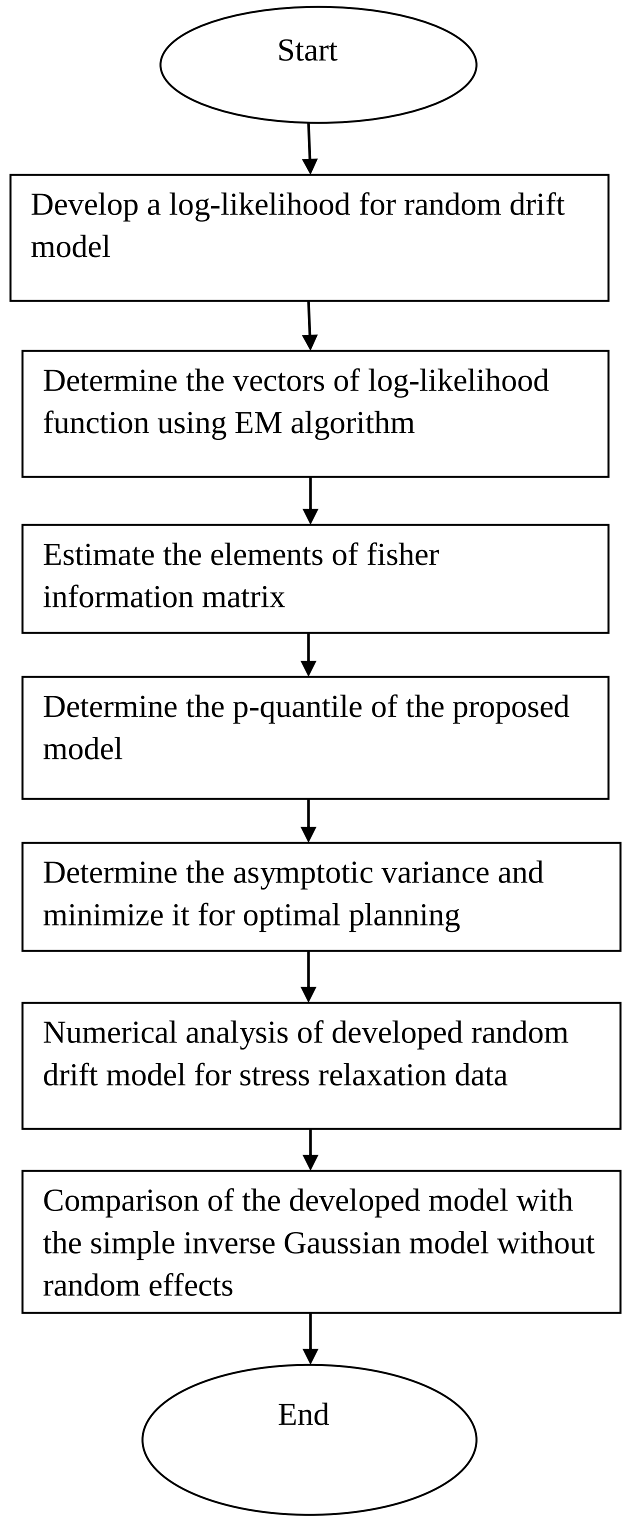


Figure 4.2: Flow chart for optimal plan

CHAPTER 5

NUMERICAL ANALYSIS

5.1 Introduction

In the preceding sections we described techniques for degradation analysis at single stress level, which may be a use stress level or an elevated level. In many situations, testing a sample at a use stress level yields a small amount of degradation in a reasonable length of time. Insufficient degradation inevitably provides biased reliability estimates. To overcome this problem, as in accelerated life testing, two or more groups of test units are subjected to higher-than-use stress levels. The degradation data so obtained are extrapolated to estimate the reliability at a use stress level. In this section we present methods for accelerated degradation analysis.

5.2 Numerical example

The stress relaxation data in (G.Yang et al., 2007) (example 8.7 pp.351) are used here to illustrate the proposed procedure. The stress relaxation is the loss of stress in a component subjected to a constant strain over the time. E.g. The contact of electrical connectors often fails due to the excessive stress relaxation. The electrical connector is said to have failed if the stress relaxation exceed 30%, i.e., D=30. Data are collected under three temperature levels: 65℃ , 85℃ , 100℃ . The time interval between measurements is tabulated in table no. 5.1. The 7th point of the second unit under 65℃ (labeled blank in (Yang et al., 2007) to preserve the monotone behavior of the stress relaxation. Table 3 lists the measurement approaches under each temperature level.

Table 5.1: Stress relaxation data under the temperature level (G.Yang, 2007)

Temperature	I.D	Stress loss
65℃	1	2.12, 2.7, 3.52, 4.25, 5.55, 6.12, 6.75, 7.22, 7.68, 8.46, 9.46
	2	2.29, 3.24, 4.16, 4.86, 5.74, 6.85, 0, 7.4, 8.14, 9.25, 10.55
	3	2.4, 3.61, 4.35, 5.09, 5.5, 7.03, 8.24, 8.81, 9.629, 10.27, 11.11

	4	2.31, 3.48, 5.51, 6.2, 7.31, 7.96, 8.57, 9.07, 10.46, 11.48, 12.31
	5	3.14, 4.33, 5.92, 7.22, 8.14, 9.07, 9.44, 10.09, 11.2, 12.77, 13.51
	6	3.59, 5.55, 5.92, 7.68, 8.61, 10.37, 11.11, 12.22, 13.51, 14.16, 15
85℃	**7**	2.77, 4.62, 5.83, 6.66, 8.05, 10.61, 11.2, 11.98, 13.33, 15.64
	8	3.88, 4.37, 6.29, 7.77, 9.16, 9.9, 10.37, 12.77, 14.72, 16.8
	9	3.18, 4.53, 6.94, 8.14, 8.79, 10.09, 11.11, 14.72, 16.47, 18.66
	10	3.61, 4.37, 6.29, 7.87, 9.35, 11.48, 12.4, 13.7, 15.37, 18.51
	11	3.42, 4.25, 7.31, 8.61, 10.18, 12.03, 13.7, 15.27, 17.22, 19.25
	12	5.27, 5.92, 8.05, 9.81, 12.4, 13.24, 15.83, 17.59, 20.09, 23.51
100℃	**13**	4.25, 5.18, 8.33, 9.53, 11.48, 13.14, 15.55, 16.94, 18.05, 19.44
	14	4.81, 6.16, 7.68, 9.25, 10.37, 12.4, 15, 16.2, 18.24, 20.09
	15	5.09, 7.03, 8.33, 10.37, 12.22, 14.35, 16.11, 18.7, 19.72, 21.66
	16	4.81, 7.5, 9.16, 10.55, 13.51, 15.55, 16.57, 19.07, 20.27, 22.24
	17	5.64, 6.57, 8.61, 12.5, 14.44, 16.57, 18.7, 21.2, 22.59, 24.07

	18	4.72, 8.14, 10.18, 12.4, 15.09, 17.22, 19.16, 21.57, 24.35, 26.2

Table 5.2: Measurement time under three temperature (G.Yang, 2007)

Temperature	**Measurement time epochs (in hours)**
65°C	108, 241, 534, 839, 1074, 1350, 1637, 1890, 2178, 2513, 2810
85°C	46, 108, 212, 408, 632, 764, 1011, 1333, 1517, 2586
100°C	46, 108, 212, 344, 446, 626, 829, 1005, 1218

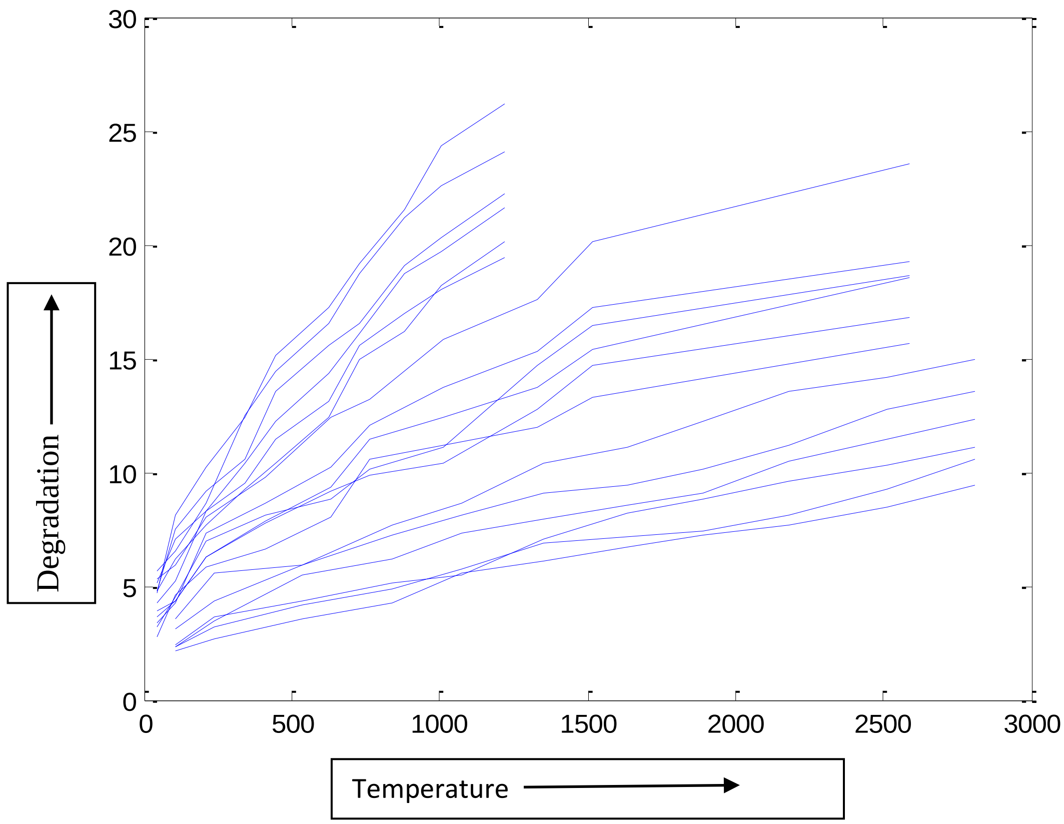


Figure 5.1: Degradation in unit with temperature

Above figure shows the variation of degradation for each unit with the time. In the figure from lower end to upper end shows the degradation behavior of unit 1 to 18.

G.Yang (2007) Regression is used to fit each degradation path, and extrapolated to the failure threshold to obtain the pseudo failure time for each unit. He then used the lognormal distribution to fit the pseudo failure times under each temperature level, and link the Arrhenius relationship to link the failure time distribution. Here we used the stochastic approach. In keeping with G.Yang (2007), we use the normal use stress is s_0 is 40℃ , the highest allowable stress s_H is 100℃ and the transformed stress is

$$x_j = \frac{\frac{1}{s_0} - \frac{1}{s_j}}{\frac{1}{s_0} - \frac{1}{s_H}}$$

For 65℃ x_1= 0.641

For 85℃ x_2= 0.8823

For 100℃ x_3= 1

The mean degradation path is given by direct averaging over the sample under each stress level. First we take the direct average of one stress level at a particular time interval. By using this we provide the regression fit of all the data points. It shows how the average data varies with the measurement time interval.

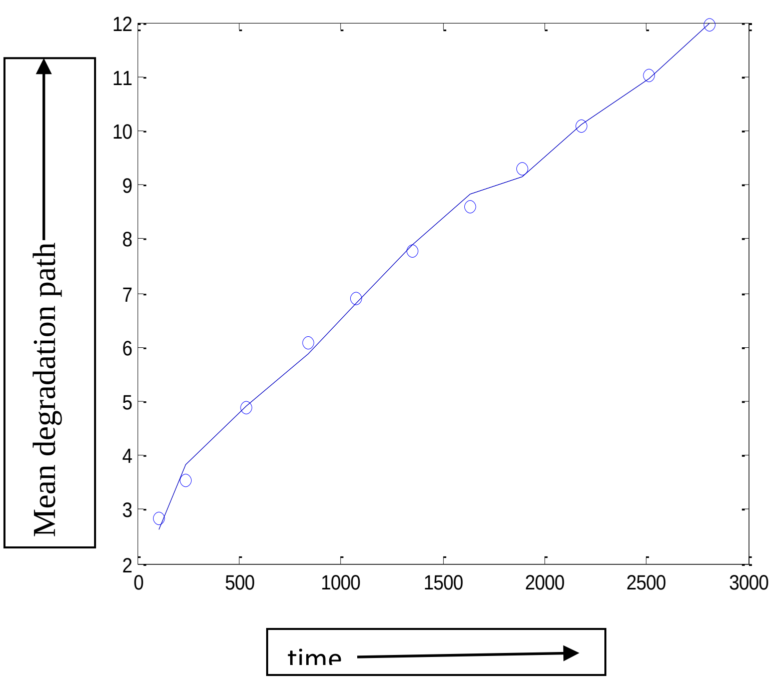


Figure5.2**:** For 65°C regression fit

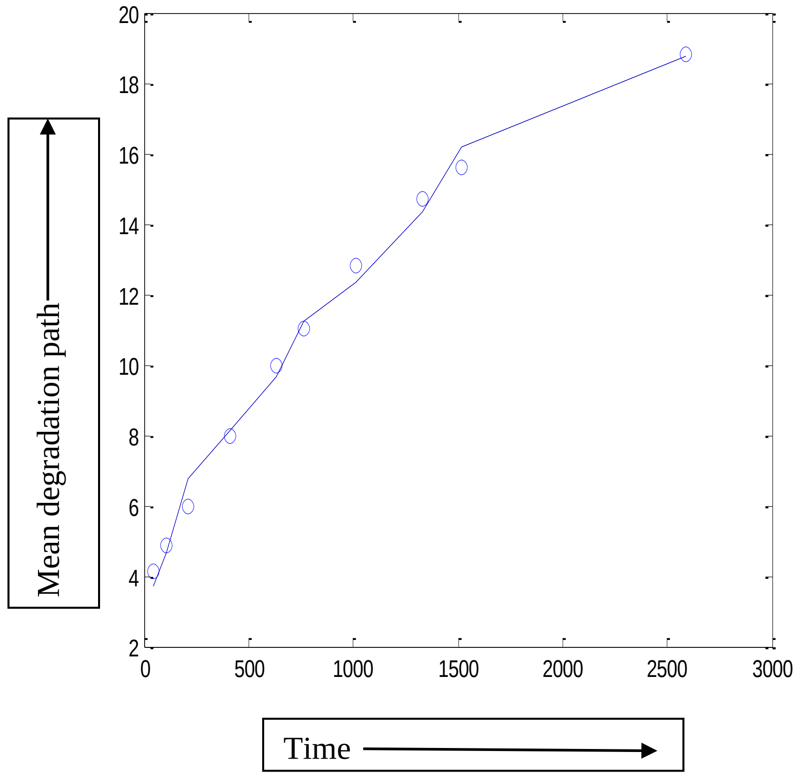


Figure 5.3: For 85°C regression fit

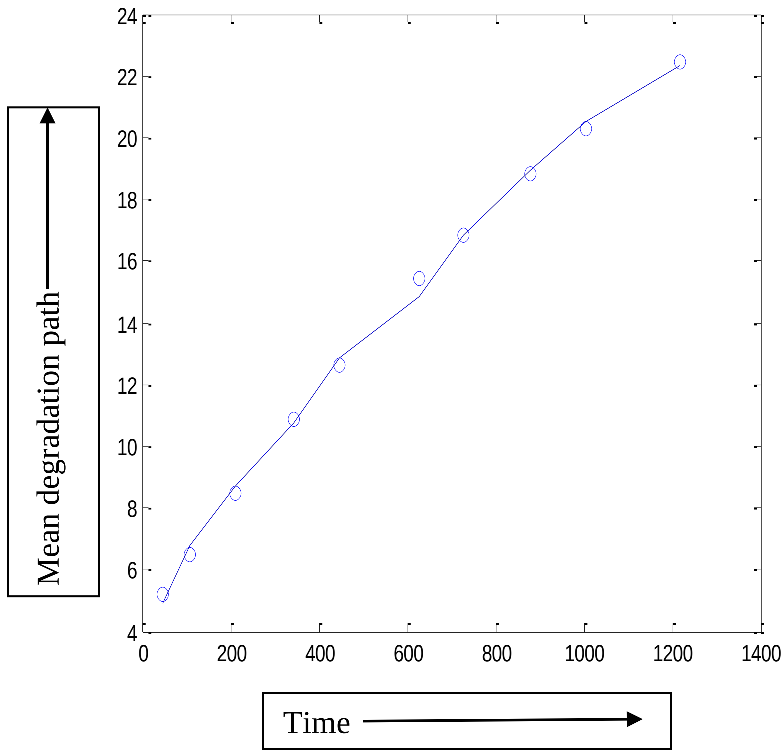


Figure 5.4: For 100°C

The inverse Gaussian process is used to fit the data for each stress level and also for combining the degradation data. The standard normal PDF and CDF for each stress level is given below:

For 65°C the standard PDF and CDF curve is given below:

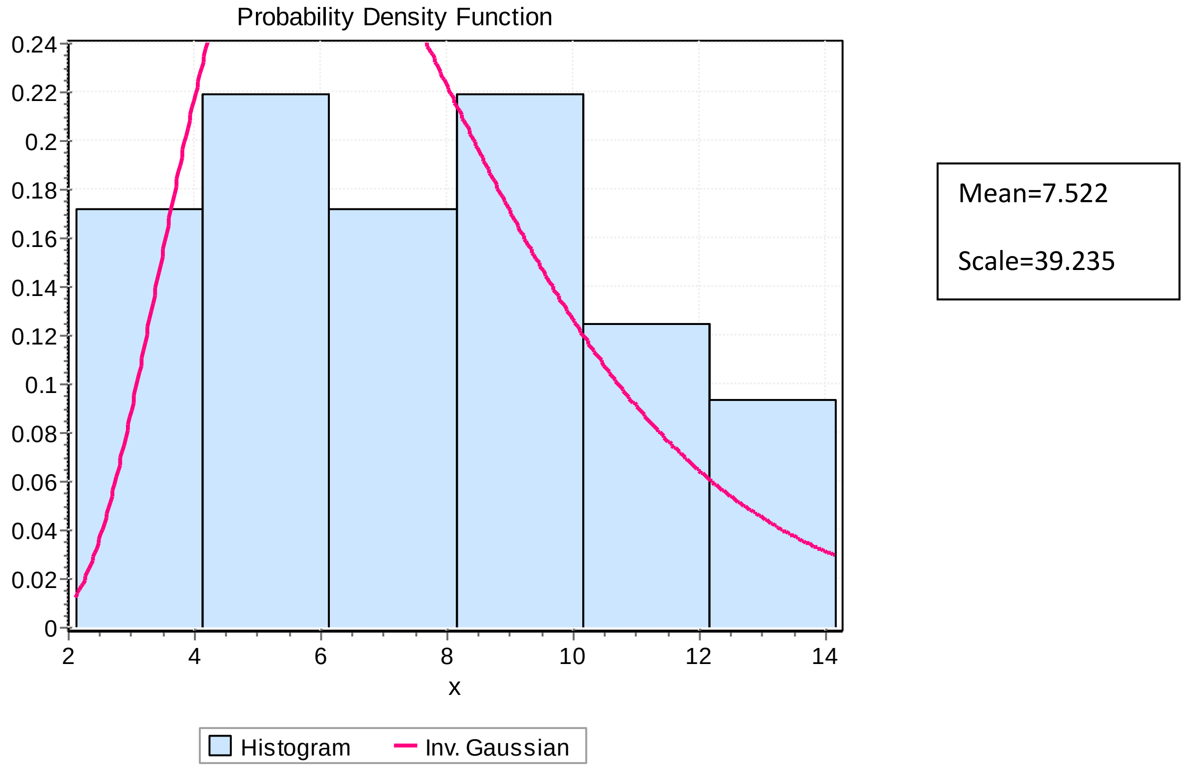


Figure 5.5: PDF for inverse Gaussian at 65 degree

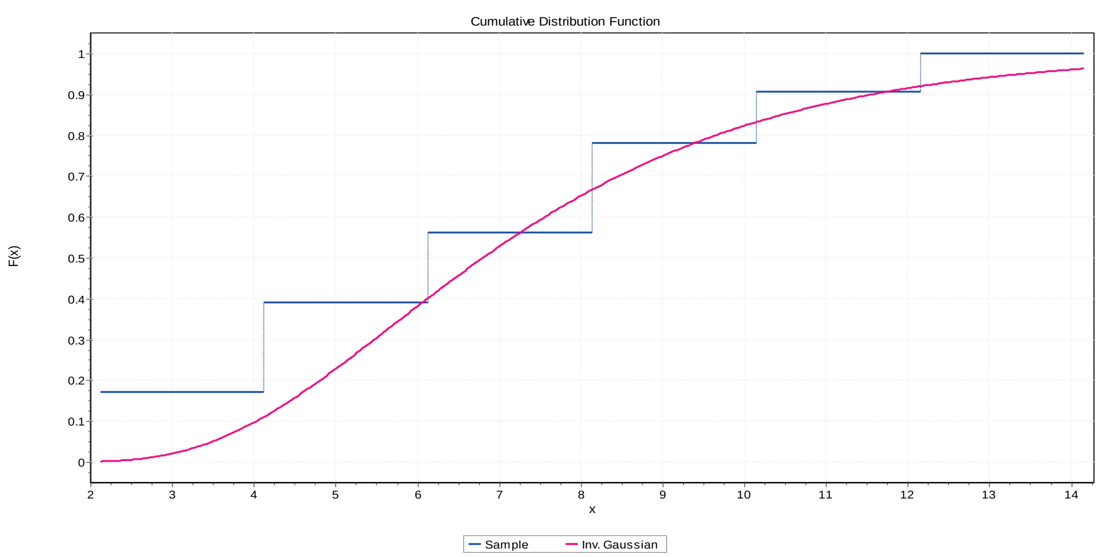
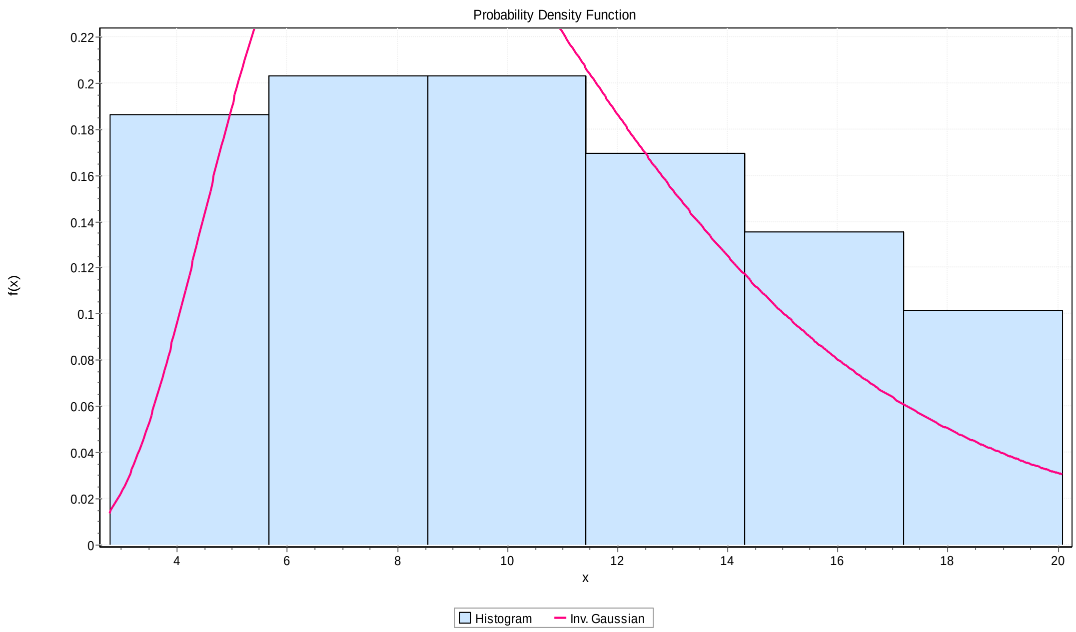


Figure 5.6: CDF for inverse Gaussian at 65 degree

For 85°C PDF and CDF for inverse Gaussian



Mean =10.368

Scale=50.284

Figure 5.7: PDF for inverse Gaussian at 85 degree

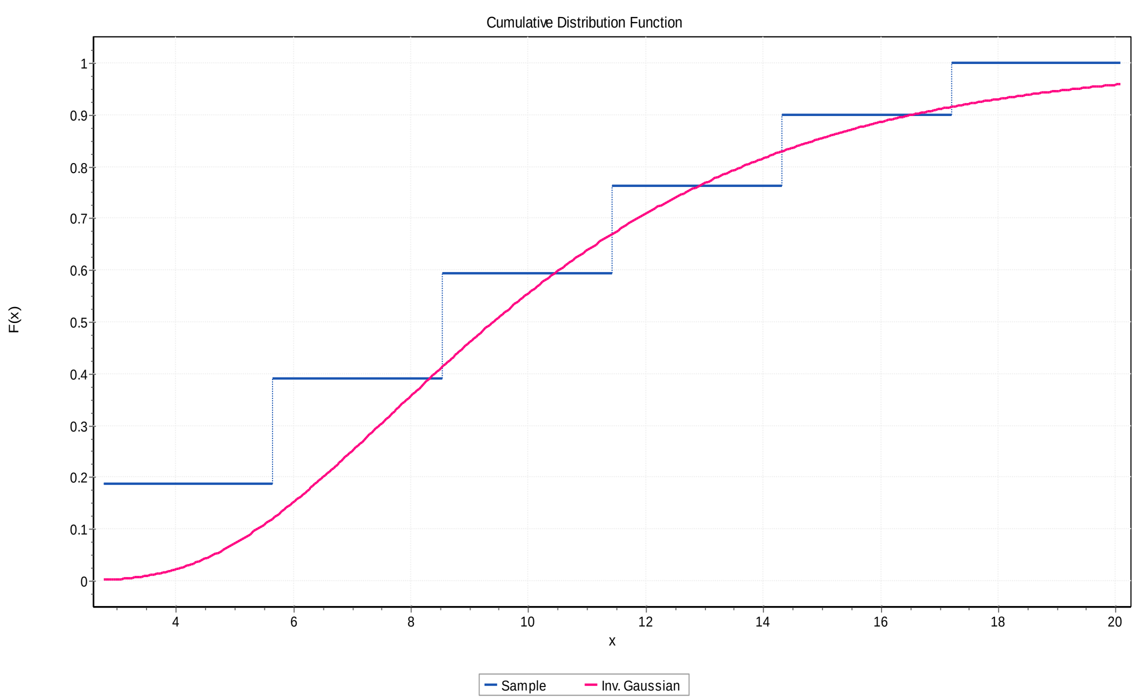
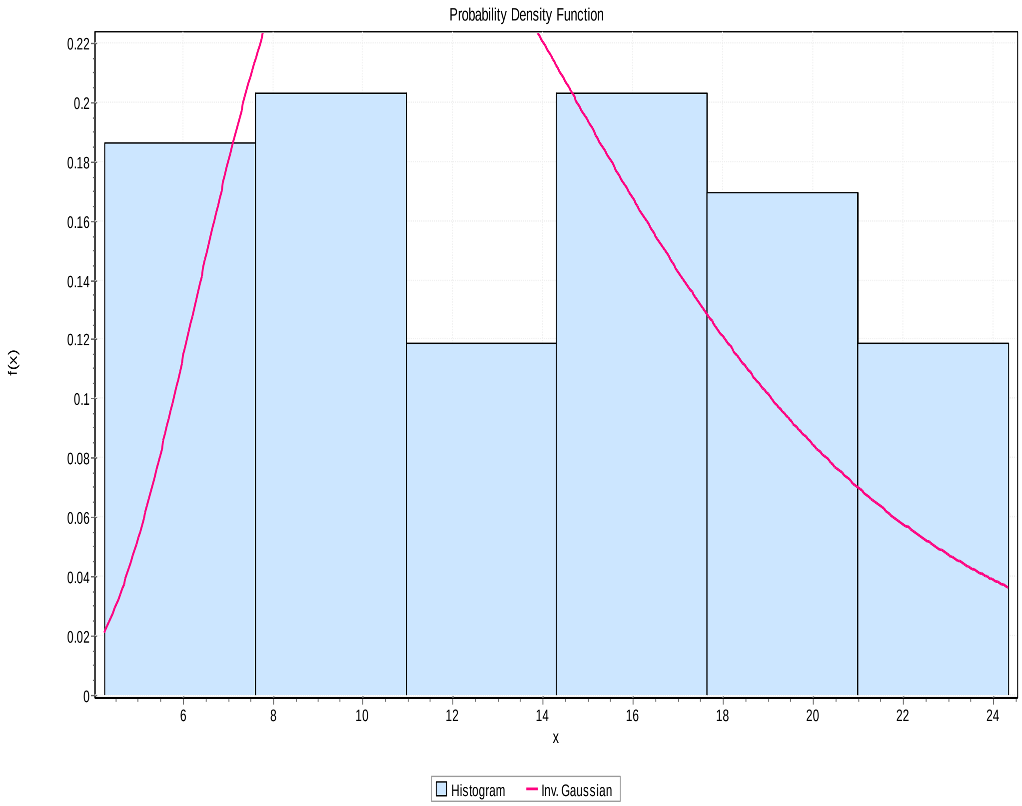


Figure 5.8: CDF for inverse Gaussian at 85 degree

For 100℃ PDF and CDF of inverse Gaussian



Mean =13.539

Scale=75.138

Figure 5.9: PDF for inverse Gaussian at 100 degree C

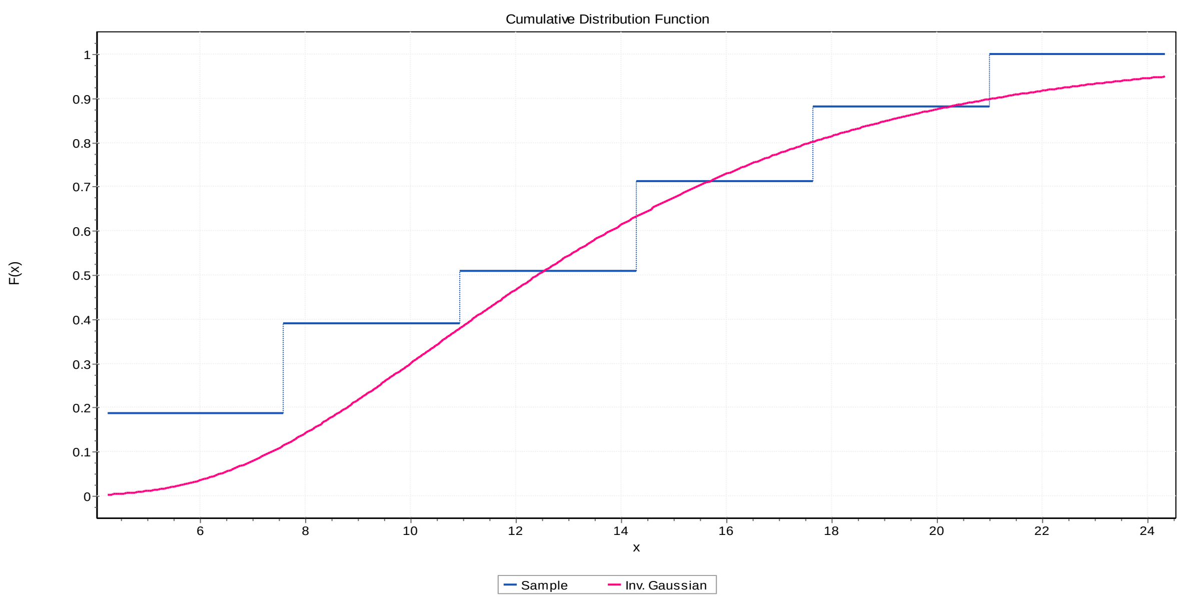


Figure 5.10: CDF for inverse Gaussian at 100 degree C

Combine plot for all stress level PDF and CDF of inverse Gaussian:

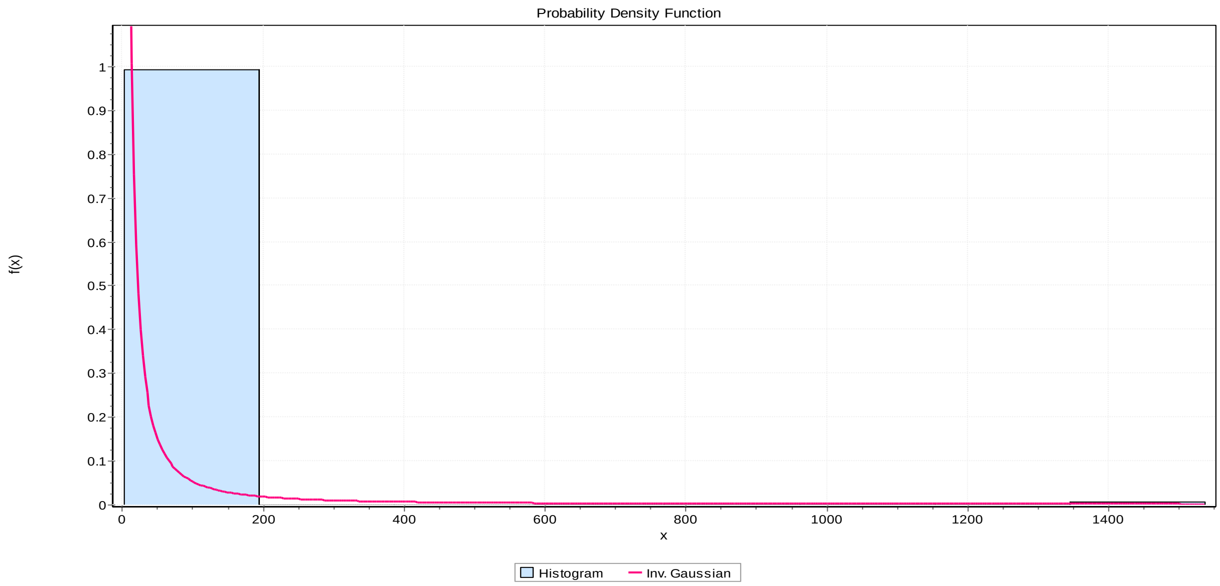


Figure 5.11: PDF for inverse Gaussian for combine all level

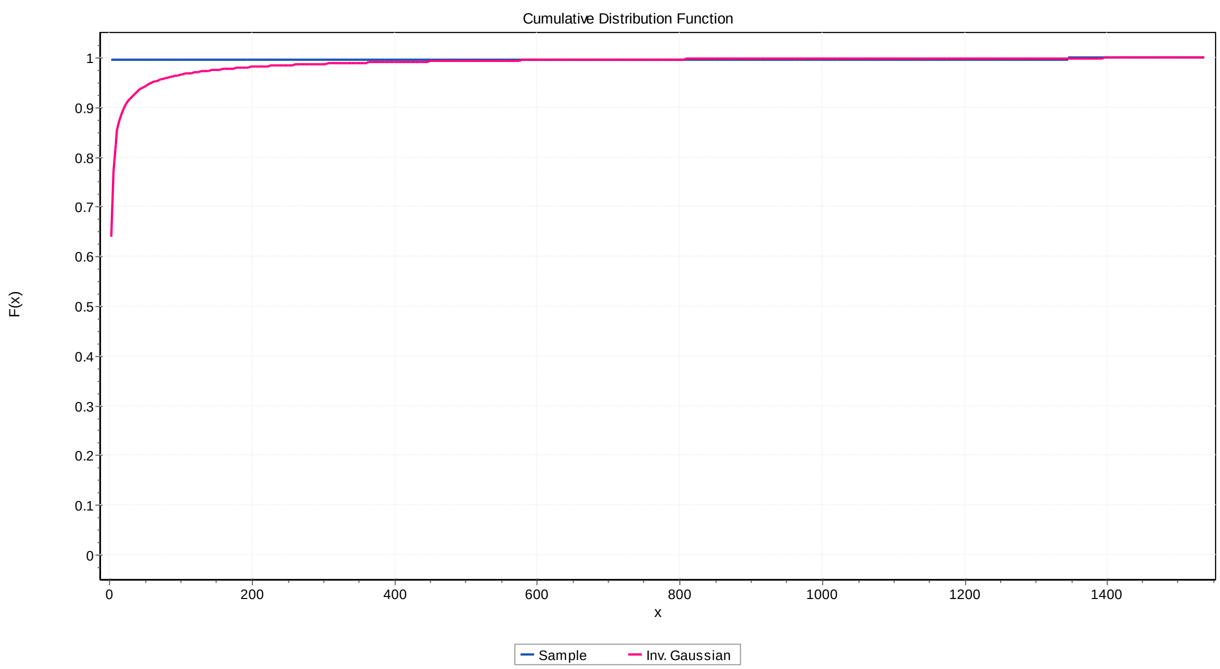


Figure 5.12: CDF for inverse Gaussian for combine all level

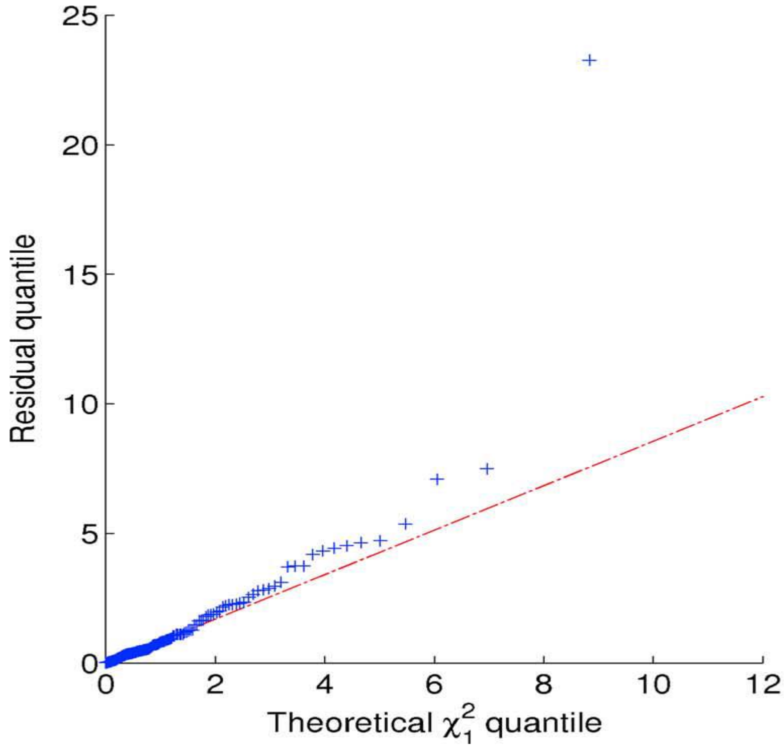


Figure 5.13: Q-Q plot for simple model

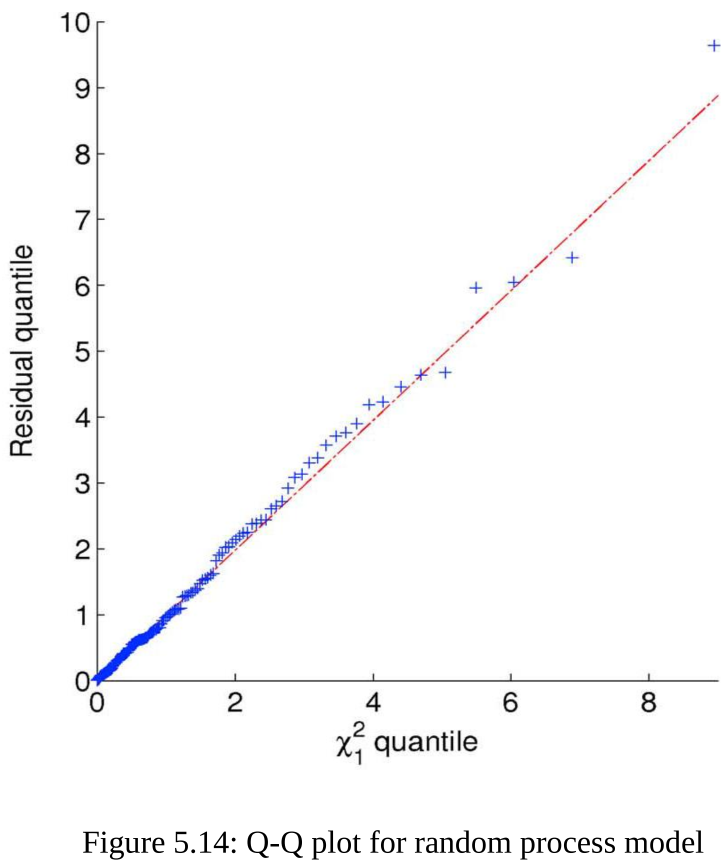


Figure 5.14: Q-Q plot for random process model

The estimated parameters for the random drift model are:

$$\propto_0 = -2.423$$

$$\propto_1 = 2.3932$$

$$k = 2.1954$$

$$\lambda = 0.7307$$

$$\beta = 0.53$$

The maximum log likelihood value is -192 since the log likelihood value is more compare to the simple model (-222) so this model provides a better fit.

In addition, G.Yang(2007) proposed using the Q-Q plot to test the goodness-of-fit. The Q-Q plot is given in Fig. 5.13. As can be seen from the Q-Q plot, there seems to be one outlier in the data. Except for this outlier, the Q-Q plot tends to be satisfactory. The outlier may be attributed to a random effect, and is incorporated into the analysis. The Q-Q plot for random effect is given in figure 5.14. Figure shows that simple model is also acceptable.

In the following, we will determine the optimal ADT plans based on both models.

Suppose 10 units are available for the ADT test. In the ADT, we set $\tau_{j=}24$, and $K_j = 14$for all j=1,2,….J. This setting means that we measure the degradation level once every day, and the test lasts two weeks. Our planning involves selecting the stress level, $(x_1, x_2, \dots, x_{J-1})$, and the proportion of samples allocated to each testing level, $(N_1, N_2, \dots . N_{J-1})$. Consider a two-level ADT plan. Suppose we are interested in minimizing the asymptotic variance of B10, the 0.1-quantile of the failure time distribution at use conditions. Solving the optimization problem equation (4.63) when J=2 yields the optimal ADT design.

The elements of fisher matrix by solving through mat lab are:

$$\begin{bmatrix} -1.363 \times 10^8 & -1.063 \times 10^8 & -1.6345 \times 10^9 & -20.08 \times 10^8 & -1.36 \times 10^5 \\ -1.03 \times 10^8 & -8.7208 \times 10^7 & -1.2830 \times 10^9 & -15.7626 \times 10^8 & -1.067 \times 10^8 \\ -1.63454 \times 10^9 & -1.2830 \times 10^9 & -6.986 \times 10^9 & -8.235 \times 10^{12} & 1.448 \times 10^9 \\ -20.08 \times 10^8 & -15.7626 \times 10^8 & -8.235 \times 10^{12} & -4.9076 \times 10^9 & -1.46 \times 10^7 \\ -1.36 \times 10^5 & -1.067 \times 10^8 & 1.448 \times 10^9 & -1.46 \times 10^7 & 5.55 \times 10^7 \end{bmatrix}$$

Minimize (50*((1442898714464025*n*((1000*exp((5983*n)/2500 - 2423/1000))/753)^(1/2))/2251799813685248 - (120*n*exp(2423/1000 - (5983*n)/2500)….

Subjected to:

$$0 \leq x_j \leq 1 \;\; j = 1,2$$

$$N_1 + N_2 = 10$$

$$0 < N_j \leq 10 \;\; j = 1,2$$

Table 5.3: Optimization table for random drift model

Process	x_1	x_2	N_1	N_2	Std($\hat{\xi}_p$)
Random drift model	0	1	1	9	4352

The optimal ADT design is shown in the above table. It is interesting to observe that optimal lower stress value is 0. This result is true because the degradation under the normal use condition is fast enough so that the error caused by extrapolation to the failure threshold is small, even if we test the unit under use conditions.

Comparison to simple IG process model:

Table 5.4: Optimization table for simple IG process

Process	x_1	x_1	N_1	N_2	Std($\hat{\xi}_p$)
Simple IG process	0	1	8	2	16604

(Ye and Chan,2014)

5.3 Conclusion

This thesis has investigated the optimal constant stress ADT plan based on the inverse Gaussian process. The objective is to minimize the asymptotic variance of the p-quantile under the use condition by properly specifying the stress levels. The inverse Gaussian process with the random effect is considered in this thesis. Random drift model is considered for investigation. Here we assume that random drift parameter μ_i is variable with unit to unit since there is change in material properties from unit to unit. This assumption is a legitimate assumption because degradation is most often hastened under severe working conditions. We applied the IG process model to fit the stress-relaxation data of a component, and used the methods developed here to help with the ADT planning.

This study has considered the constant-stress ADT planning. An advantage of the constant-stress ADT is that we can check the assumed stress-degradation relationship by separately estimating the parameters under each stress level. When the number of samples available for testing is extremely small, SSADT can be a good choice, as long as it is believed that the stress-degradation acceleration relation is correct. MLEs of the model parameters and the asymptotic variance of the -quantile can be derived similar to the procedures presented in this thesis.

This model is more useful in comparison to simple IG process and random volatility model. Since the maximum value of log-likelihood function obtained is more compared to other two models. And also it takes into account unit to unit variation (due to the change of properties of material at an lower or molecular level or due to some defects in the material) since here value of drift parameter is changing with unit.

The no of units required at higher stress level is more due to the random effect compared to the simple IG process. And at lower stress level no of units required is less (since random change occurs in the degradation) compare to the simple IG process. And the standard value of asymptotic variance is also less compare to the simple IG process it indicates that here variation at different stress level in degradation is less comparatively.

The Q-Q quantile curve also shows that random drift model provides a better fits for the random degradation data since maximum probability value lies on the line compare to the simple IG process in which more no of point lies outlier.

Thus by this thesis we can conclude that random drift model is a good choice for the manufacture to select no of optimum units at an optimum stress level for acceleration degradation testing. This model can be easily used in various types of industries as electronic, aerospace, to test different type of electronic component.

And this model is very useful tool of Gas-Laser degradation testing and also to study the degradation behavior of other components or system.

CHAPTER 6

SUMMARY AND FUTURE SCOPE

6.1 SUMMARY

In the present scenario of highly competitive world every industry wants to develop highly reliable product. So there is need to test the units for reliability. Different methods are developed with the time to test the product. But in the electronic industry accelerated degradation test gets more utility compared to the other methods. Since company produces large sample of similar products so there is need to test the product in short duration. So accelerated degradation test is more suitable and effective for studying the degradation behavior since in this testing we increase the value of stress to fail the component quickly and collect the degradation data for predicting the reliability of the product. Accelerating degradation tool has become highly popular throughout the world in different type of industry as electronic, aerospace, and electrical.

Different type of accelerating degradation models have developed with the time and can be used in different types of situations. However, it has become necessary for the manager to test how many no of units should be tested at a particular stress level so that the cost of testing is less. SSADT method has been developed by considering various criterion required such as robustness of design, optimality of design, tightened the value of constrained etc. While considering Gas-Laser problem gamma process does not provide best results. So, inverse Gaussian process is used for the optimization of no of units and stress value.

Inverse Gaussian use different type of models as simple IG process and IG process with random effect as-random drift model, random volatility model and random drift- volatility model. In the present thesis we have considered random drift model for the study since this model takes into effect unit to unit variation of the sample of product. While random volatility model does not consider this factor so random drift model is more useful compared to the simple IG process and random volatility model.

So the proposed model provides estimation of the no of units required for optimum stress level by minimizing the value of asymptotic variance. Fisher information matrix is a useful tool for estimating value of vectors used for finding the asymptotic variance. A number of researchers are working presently on the acceleration degradation testing and different types of criterion have been considered from time to time. So many research objectives have emerged as the competition increases so that more optimal solutions can be found.

Thus acceleration degradation testing has become very popular and a wide area for the future research. Since every day new challenge arises in the industry. To face different situations it has become necessary to develop new methods from time to time. In the present thesis random drift model is used which can be used in number of industries where products are developed in lots to find the optimum stress level and the optimum value of units.

6.2 Future work

The random drift model considers the unit to unit variation while performing the optimization of the stress level with the no of units. Except it there are few things which can be considered in the future as

1. The sensitivity analysis of the model can be done to know how the asymptotic variance varies on changing the value of different parameters $\propto_0$, $\propto_1$, λ, k etc. By performing the sensitivity analysis a range can be found for each parameter.
2. Similar procedure can be developed for random drift-volatility model for the optimization of no of unit with the stress value.
3. Sensitivity analysis for random drift-volatility model can be done in future.
4. Validation of the above model can be done though simulation methods in the future studies.

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